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A NOTE ON GROUPS WITH FEW ISOMORPHISM CLASSES OF SUBGROUPS

BY

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Abstract. The structure of infinite groups in which any two (proper) subgroups of the same cardinality are isomorphic is described within the universe of locally graded groups. The corresponding problem for finite groups was considered by R. Armstrong (1958).

1. Introduction. A group G is said to have the *C-property* (or to be a *C-group*) if any two subgroups of G of the same cardinality are isomorphic, while G is called a *C_0-group* if all proper subgroups of G of the same cardinality lie in the same isomorphism class. The properties C and C_0 are obviously equivalent within the universe of finite groups, and the structure of finite soluble C-groups was investigated by R. Armstrong in her doctoral thesis, written under the supervision of P. Hall (see [1]). Among other results, she proved that finite soluble groups with the C-property have derived length at most 4 and Fitting length at most 3, and that these bounds are best possible. A complete description of finite unsoluble C-groups has been given more recently by J. Zhang [10], who showed in particular that they must contain a subgroup which is isomorphic either to $\mathrm{SL}(2,5)$ or to $\mathrm{SL}(2,2^n)$ for some $n \geq 2$. Moreover, Ya. G. Berkovich [2] studied finite unsoluble groups in which soluble subgroups of the same order are isomorphic.

The aim of this short paper is to study infinite groups in the classes C and C_0. Although there exist finite simple non-abelian C-groups, like for instance the alternating group of degree 5, it turns out that infinite locally finite groups with the C_0-property are close to be abelian, and actually they are at least metabelian; this phenomenon essentially depends on the famous theorem of Hall–Kulatilaka and Kargapolov on the existence of infinite abelian subgroups in any infinite locally finite group. On the other hand, *Tarski groups* (i.e. infinite simple groups whose proper non-trivial subgroups have the same prime order) obviously have the C-property. In order to avoid

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Tarski groups and other similar pathologies, our main results will be proved in the case of locally graded groups; here, a group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Locally graded groups form a large class of generalized soluble groups, containing in particular all locally (soluble-by-finite) groups.

Most of our notation is standard and can be found in [8].

2. Statements and proofs. Recall that a group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this property. In particular, a group has rank 1 if and only if it is a locally cyclic non-trivial group.

Our first result gives a strong restriction on the structure of groups with the C_0-property containing elements of infinite order.

LEMMA 2.1. *Let G be a non-periodic C_0-group. Then G is a 2-generator group whose infinite proper subgroups are cyclic.*

Proof. Let a be an element of infinite order of G, and assume for a contradiction that G cannot be generated by two elements. If x is any element of G, then $\langle a, x\rangle$ is a proper countable subgroup of G, so that $\langle a, x\rangle \simeq \langle a\rangle$ is cyclic; in particular, x has infinite order and a belongs to the centre of G. Then G is a torsion-free abelian group. Moreover, $G/\langle a\rangle$ is periodic, so that G has rank 1 and hence it is isomorphic to a subgroup of the additve group of rational numbers. As all proper non-trivial subgroups of G are isomorphic, it follows that G is cyclic. This contradiction proves that G is a 2-generator group. In particular, G is countable and so all its infinite proper subgroups are isomorphic to $\langle a\rangle$. ∎

As finitely generated groups are countable, Lemma 2.1 has the following obvious consequence.

COROLLARY 2.2. *Let G be a non-periodic C-group. Then G is infinite cyclic.*

Observe that A. Yu. Ol'shanskiĭ [7] constructed a simple group G whose proper non-trivial subgroups are infinite cyclic, and of course G is a torsion-free C_0-group. Moreover, it follows from a result of V. N. Obraztsov [6] that if p is a sufficiently large prime number (for instance $p > 10^{75}$), there exists a non-periodic simple group in which every proper subgroup either is infinite cyclic or has order p, and also this group has the C_0-property. On the other hand, within the universe of locally graded groups, the above statement can be extended to groups with the C_0-property.

Corollary 2.3. *Let G be a locally graded non-periodic C_0-group. Then G is infinite cyclic.*

Proof. It follows from Lemma 2.1 that G is a 2-generator group whose infinite proper subgroups are cyclic. As G is locally graded, it contains a proper normal subgroup H of finite index, and the factor group $G/C_G(H)$ has order at most 2. If $|G : C_G(H)| = 2$, the centralizer $C_G(H)$ is likewise infinite cyclic, and so G is infinite dihedral, which is clearly a contradiction. Then H lies in the centre $Z(G)$, and it follows from a celebrated theorem of Schur that the commutator subgroup G' of G is finite (see [8, Part 1, Theorem 4.12]). If a is an element of infinite order of G, the subgroup $\langle a^2, G'\rangle$ is properly contained in G, and hence it is cyclic. Therefore $G' = \{1\}$, and so the finitely generated abelian group G is infinite cyclic. ∎

It is easy to prove that there are only few infinite abelian groups enjoying one of the properties C and C_0.

Corollary 2.4. *An infinite abelian group G has the C_0-property if and only if it satisfies one of the following conditions:*

(a) *G is infinite cyclic;*
(b) *G is of prime exponent;*
(c) *G is of type p^∞ for some prime number p;*
(d) *$G = P \times E$, where P is a group of type p^∞ for some prime number p and E has prime order $q \neq p$.*

Moreover, the group in (d) *is the unique abelian C_0-group for which the C-property does not hold.*

Proof. Clearly, all groups described in (a), (b) and (c) have the C-property, while the group in (d) has the C_0-property but it is not a C-group.

Suppose that G is an infinite abelian C_0-group. By Corollary 2.3, it can be assumed that G is periodic, so that it follows from the C_0-property that G has only finitely many non-trivial primary components. Then there exists a unique prime number p such that the p-component P of G is infinite. Moreover, we have $G = P \times E$, where the p'-component E of G either is trivial or has prime order $q \neq p$. Suppose first that $E = \{1\}$, i.e. G is a p-group. Then either G has exponent p or it is a group of type p^∞. Finally, if $E \neq \{1\}$, the subgroup P cannot contain infinite proper subgroups, and so P is a group of type p^∞. ∎

The structure of infinite non-abelian groups with the C-property is described by the following result.

Theorem 2.5. *Let G be an infinite non-abelian C-group. Then G is a 2-generator group satisfying the maximal condition, and it has no proper subgroups of finite index.*

Proof. The group G is periodic by Corollary 2.2. Suppose first that G is locally finite, so that it contains a countably infinite abelian subgroup A by the well-known theorem of Hall–Kulatilaka and Kargapolov (see [8, Part 1, Theorem 3.43]). If x and y are arbitrary elements of G, the subgroup $\langle x, y, A\rangle$ is likewise countable, so that $\langle x, y, A\rangle \simeq A$ is abelian and hence $xy = yx$. This contradiction shows that G cannot be locally finite, therefore it contains a finitely generated infinite subgroup E. Application of the C-property yields that all countable subgroups of G are isomorphic to E, and so they are finitely generated. It follows that the group G satisfies the maximal condition on subgroups; in particular, G is countable and hence it is isomorphic to all its infinite subgroups. Then G has no infinite abelian subgroups, and it follows from a result of Strunkov [9] that there exists in G an infinite 2-generator subgroup U. Therefore $G \simeq U$ is a 2-generator group.

Let N be any normal subgroup of finite index of G, and let X be a subgroup of G containing N. Obviously X is infinite, so that $X \simeq G$ can be generated by two elements. Thus every finite homomorphic image of G has rank at most 2. If J is the finite residual of G, it follows from a result of Mann and Segal [5] that G/J contains a soluble subgroup of finite index. But G is a finitely generated periodic group, so that G/J is finite and hence $G \simeq J$ has no proper subgroups of finite index. ∎

Corollary 2.6. *Let G be an infinite locally graded C-group. Then G is abelian.*

Although the consideration of Tarski groups shows that the above statement does not hold for an arbitrary group with the C-property, some more information can be obtained in the case of groups with involutions.

Corollary 2.7. *Let G be an infinite C-group containing an involution. Then G has non-trivial centre.*

Proof. The group G can be assumed to be periodic by Corollary 2.2. Then it follows from a result of Shunkov that G contains an infinite subgroup H with non-trivial centre (see [8, Part 1, Theorem 3.42]). As G is countable by Theorem 2.5, it is isomorphic to H, and so $Z(G) \neq \{1\}$. ∎

A result of Held shows that every infinite 2-group contains an infinite abelian subgroup (see [8, Part 1, Theorem 3.41]), and hence it also follows from Theorem 2.5 that all infinite 2-groups with the C-property are abelian.

Our next result completes the description of the structure of infinite locally graded groups with the C_0-property.

Theorem 2.8. *Let G be an infinite locally graded C_0-group. Then either G is abelian or $G = \langle x\rangle \ltimes P$, where P is a group of type p^∞ for some odd prime number p, and x has prime order q dividing $p - 1$.*

Proof. Assume first that G is uncountable. Then G cannot be finitely generated, and hence it contains a countable subgroup which is not finitely generated. It follows from the C_0-property that every finitely generated subgroup of G is finite, i.e. the group G is locally finite, and so G contains an abelian countably infinite subgroup A. If x and y are arbitrary elements of G, the subgroup $\langle x, y, A\rangle$ is likewise countable, so that it is abelian and $xy = yx$. Therefore G is abelian in this case.

Suppose now that G is not abelian, so that it must be countable. Moreover, G is periodic by Corollary 2.3, and it follows from Corollary 2.6 that all infinite proper subgroups of G are isomorphic and abelian. Observe that G cannot be finitely generated, because otherwise G would have a proper subgroup of finite index and hence it would be finite. It follows that G is locally finite.

Assume for a contradiction that G is not a Chernikov group. Then all non-abelian subgroups of G are normal (see [4, Corollary 3.3]), and hence the commutator subgroup G' of G is finite (see for instance [3, Theorem 2.2.6]). In particular, G/G' is a periodic abelian group which is not Chernikov, and so there exists an infinite subgroup H of G such that $G' \leq H$ and G/H is infinite. If a and b are arbitrary elements of G, then $\langle a, b, H\rangle$ is an infinite proper subgroup of G, so that it is abelian and $ab = ba$. This contradiction shows that G is a Chernikov group.

Since all infinite proper subgroups of G are isomorphic, it follows that the finite residual P of G is a group of type p^∞ for some prime number p. Moreover, P must be a maximal subgroup of G, and so the index $|G : P|$ is a prime number q. Clearly, the C_0-property does not hold for the locally dihedral 2-group, and hence q is a divisor of $p-1$. Therefore $G = \langle x\rangle \ltimes P$, where x is any element of order q of G. ∎

As we mentioned in the introduction, Armstrong [1] obtained the best possible bounds for the derived length and the Fitting length of finite soluble C-groups. In the general case, our Theorem 2.8 shows in particular that every infinite locally graded C_0-group is metabelian, and so we can state the following result.

Corollary 2.9. *Let G be a locally soluble C_0-group. Then G has derived length at most* 4 *and Fitting length at most* 3.

The structure of finite C-groups of prime-power order was studied by Armstrong [1, Lemmas 2.1 and 2.2] and Zhang [10, Lemmas 2]. Our last result provides a more elementary proof of their descriptions, and also characterizes the C-property for finite primary groups in terms of their Frattini properties.

Theorem 2.10. *Let G be a finite p-group* (*where p is a prime number*). *Then the following statements are equivalent:*

(a) *G has the C-property.*
(b) *If X and Y are subgroups of G and $|X| = |Y|$, then $|\Phi(X)| = |\Phi(Y)|$.*
(c) *G is one of the following:*
  (c_1) *a cyclic group;*
  (c_2) *an abelian group of exponent p;*
  (c_3) *the quaternion group Q_8;*
  (c_4) *a non-abelian group of order p^3 and exponent p.*

Proof. Clearly, every C-group satisfies condition (b), and all groups described in (c) have the C-property. Assume that the finite p-group G has the Frattini property (b). If G is abelian of exponent larger than p, we observe that G contains a cyclic subgroup of order p^2, and so it has no subgroups of exponent p and order p^2. In particular, the socle of G has order p, and hence G is cyclic.

Suppose now that G is not abelian, and assume first that G contains a maximal subgroup M which is abelian. It follows from the first part of the proof that M either is cyclic or has exponent p, so that all maximal subgroups of G are isomorphic to M. If M is cyclic, we deduce that all proper subgroups of G are cyclic, and hence G is isomorphic to Q_8. Suppose now that M has exponent p. Then G has exponent p, and all its proper subgroups are abelian. In particular, the Frattini subgroup $\Phi(G)$ is contained in $Z(G)$, and G is a 2-generator group, so that $G/\Phi(G)$ has order p^2. Then G' has order p, and hence G has order p^3.

Assume finally for a contradiction that all maximal subgroups of G are non-abelian. It follows from the above argument that G contains a subgroup H of order p^4 in which every maximal subgroup is non-abelian. Let A be a normal subgroup of H of order p^2. Then $C_H(A) = A$, and so H/A is isomorphic to a subgroup of the full automorphism group $\mathrm{Aut}(A)$ of A. This is of course impossible, as the order of $\mathrm{Aut}(A)$ is not divisble by p^2, and this contradiction completes the proof of the theorem. ∎

COROLLARY 2.11. *Let p be a prime number. A finite p-group G has the C-property if and only if $X/\Phi(X) \simeq Y/\Phi(Y)$ whenever X and Y are subgroups of G of the same order.*

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