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On the unstable directions and Lyapunov exponents of Anosov endomorphisms

by

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Abstract. Unlike in the invertible setting, Anosov endomorphisms may have infinitely many unstable directions. Here we prove, under the transitivity assumption, that an Anosov endomorphism of a closed manifold M is either special (that is, every $x \in M$ has only one unstable direction), or for a typical point in M there are infinitely many unstable directions. Another result is the semi-rigidity of the unstable Lyapunov exponent of a $C^{1+\alpha}$ codimension one Anosov endomorphism that is C^1-close to a linear endomorphism of $\mathbb{T}^n$ for $(n \geq 2)$.

1. Introduction. In the 1970s, the works [8] and [6] generalized the notion of Anosov diffeomorphism to non-invertible maps, introducing the notion of Anosov endomorphism. Let M be a closed C^∞ manifold.

Definition 1.1 ([8]). Let $f : M \to M$ be a C^1 local diffeomorphism. We say that f is an *Anosov endomorphism* if there are constants $C, \lambda > 1$ such that for every f-orbit $(x_n)_{n\in\mathbb{Z}}$ there is a splitting

$$T_{x_i}M = E^s_{x_i} \oplus E^u_{x_i}, \qquad \forall i \in \mathbb{Z},$$

which is preserved by Df, and for all $n \geq 0$ we have

$$\|Df^n(x_i)\cdot v\| \geq C^{-1}\lambda^n\|v\| \quad \text{for all } v \in E^u_{x_i} \text{ and } i \in \mathbb{Z},$$
$$\|Df^n(x_i)\cdot v\| \leq C\lambda^{-n}\|v\| \quad \text{for all } v \in E^s_{x_i} \text{ and } i \in \mathbb{Z}.$$

Anosov endomorphisms can be defined in an equivalent way [6]:

Definition 1.2 ([6]). A C^1 local diffeomorphism $f : M \to M$ is called an *Anosov endomorphism* if Df contracts uniformly a continuous subbundle $E^s \subset TM$ into itself, and the action of Df on TM/E^s is uniformly expanding.

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Sakai [10] proved that, in fact, Definitions 1.1 and 1.2 are equivalent.

A contrast between Anosov diffeomorphisms and Anosov endomorphisms is the non-structural stability of the latter. Indeed, C^1-close to any linear Anosov endomorphism A of the torus, Przytycki [8] constructed an Anosov endomorphism which has infinitely many unstable directions for some orbit, and consequently A is not structurally stable. However, it is curious to observe that topological entropy is locally constant among Anosov endomorphisms. Indeed, take the lift of an Anosov endomorphism to the inverse limit space (see preliminaries for the definition). At the level of the inverse limit space, two nearby Anosov endomorphisms are conjugate [8], [2], and lifting to the inverse limit space does not change entropy.

Two endomorphisms (permitting singularities) f_1, f_2 are C^1-*inverse limit conjugate* if there exists a homeomorphism $h : M^{f_1} \to M^{f_2}$ such that $h \circ \tilde{f}_1 = \tilde{f}_2 \circ h$ where $\tilde{f}_i$ are the lifts of f_i to the orbit space (see preliminaries).

Denote by p the natural projection $p : \overline{M} \to M$, where $\overline{M}$ is the universal covering. Note that an unstable direction $E^u_{\overline{f}}(y)$ projects onto an unstable direction of $T_x M$, $x = p(y)$, following Definition 1.1, that is, $Dp(y) \cdot (E^u_{\overline{f}}(y)) = E^u(\tilde{x})$, where $\tilde{x} = p(\mathcal{O}(y))$.

Proposition 1.3 ([6]). *A local diffeomorphism f is an Anosov endomorphism of M if and only if the lift $\overline{f} : \overline{M} \to \overline{M}$ is an Anosov diffeomorphism of $\overline{M}$, the universal cover of M.*

An advantage to work with the latter definition is that in $\overline{M}$ we can construct invariant foliations $\mathcal{F}^s_{\overline{f}}$ and $\mathcal{F}^u_{\overline{f}}$.

Given an Anosov endomorphism and an f-orbit $\tilde{x} = (x_n)_{n\in\mathbb{Z}}$ we denote by $E^u(\tilde{x})$ the unstable bundle subspace of $T_{x_0}(M)$ corresponding to the orbit $(x_n)_{n\in\mathbb{Z}}$. In [8] there are examples of Anosov endomorphisms such that $E^u(\tilde{x}) \neq E^u(\tilde{y})$ with $x_0 = y_0$, but $(x_n)_n \neq (y_n)_n$. In fact, it is possible that $x_0 \in M$ has uncountably many unstable directions [8]. An Anosov endomorphism for which $E^u(\tilde{x})$ just depends on x_0 (a unique unstable direction for each point) is called a *special* Anosov endomorphism. A linear Anosov endomorphism of the torus is an example of a special Anosov endomorphism.

A natural question is whether it is possible to find an example of a (non-special) Anosov endomorphism such that each $x \in M$ has a finite number of unstable directions. It is also interesting to understand the structure of points with infinitely many unstable directions. For transitive Anosov endomorphisms we prove the following dichotomy:

Theorem 1.4. *Let $f : M \to M$ be a transitive Anosov endomorphism. Then either*

(1) *f is a special Anosov endomorphism, or*

(2) *there exists a residual subset $\mathcal{R} \subset M$ such that every $x \in \mathcal{R}$ has infinitely many unstable directions.*

Observe that when M is the torus $\mathbb{T}^n$, $n \geq 2$, all Anosov endomorphisms of $\mathbb{T}^n$ are transitive (see [1]).

Analysing the unstable Lyapunov exponents of an Anosov endomorphism, similarly to [7] we can prove the following result for conservative systems (preserving a probability measure equivalent to Lebesgue measure).

THEOREM 1.5. *Let $A : \mathbb{T}^n \to \mathbb{T}^n$, $n \geq 2$, be a linear Anosov endomorphism with* $\dim E^u_A = 1$. *Then there is a C^1-open set $\mathcal{U}$, containing A, such that for every $C^{1+\alpha}$, $\alpha > 0$, conservative Anosov endomorphism $f \in \mathcal{U}$, we have $\lambda^u_f(x) \leq \lambda^u(A)$ for m-almost every $x \in \mathbb{T}^n$, where m is the Lebesgue measure of $\mathbb{T}^n$.*

REMARK 1.6. To prove Theorem 1.5, the neighbourhood $\mathcal{U}$ can be chosen very small, such that every $f \in \mathcal{U}$ has its lift conjugate to A in $\mathbb{R}^n$. Then we can consider a priori that also $\dim E^u_f = 1$.

2. General preliminaries. In this section we present some classical results on Anosov endomorphisms that will be important for the rest of this work.

2.1. The inverse limit space. Let (X, d) be a compact metric space and $f : X \to X$ a continuous map. We define a new compact metric space, called the *inverse limit space* for f or the *natural extension* of f, by

$$X^f := \Big\{(x_n)_{n\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} X_i : X_i = X \ \forall i \in \mathbb{Z} \text{ and } f(x_i) = x_{i+1} \ \forall i \in \mathbb{Z}\Big\}.$$

In this text we denote X^f by $\widetilde{X}$. Also we denote by $\tilde{x}$ the element $(x_n)_{n\in\mathbb{Z}}$ of $\widetilde{X}$. We introduce a metric $\tilde{d}$ in $\widetilde{X}$ by setting

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{i\in\mathbb{Z}} \frac{d(x_i, y_i)}{2^{|i|}}.$$

It is easy to see that $(\widetilde{X}, \tilde{d})$ is a compact metric space. Let $\pi : \widetilde{X} \to X$ be the projection on the zero coordinate, that is, if $\tilde{x} = (x_n)_{n\in\mathbb{Z}}$, then $\pi(\tilde{x}) = x_0$. One can verify that π is continuous.

DEFINITION 2.1. A *prehistory* of x is a sequence of type

$$\tilde{x}_- = (\ldots, x_{-2}, x_{-1}, x_0 = x)$$

such that $f(x_{-i}) = x_{-i+1}$, $i = 1, 2, \ldots.$

Denote by X^f_- or $\widetilde{X}_-$ the space of all the prehistories with $x_0 \in X$. The space $(\widetilde{X}_-, \tilde{d})$ is also compact and the distance between two prehistories of the same point $x_0 \in X$ is $\sum_{i=0}^{\infty} d_M(x_{-i}, y_{-i})/2^i$.

In the Anosov endomorphism context, $E^u(\tilde{x})$ depends only on $\tilde{x}_-$, and this is why in this work we often deal only with prehistories.

2.2. Some nice properties of Anosov endomorphisms. The set of C^1 Anosov endomorphisms is open, like that of Anosov diffeomorphisms. However, structural stability in the usual sense does not hold for Anosov endomorphisms (see the correct context for structural stability of Anosov endomorphisms in Berger–Rovella [2]).

THEOREM 2.2 (Przytycki [8], Mañé–Pugh [6]). *The set of Anosov endomorphisms of a manifold M is an open set in the C^1 topology.*

THEOREM 2.3 ([8]). *Let $f : M \to M$ be an Anosov endomorphism. Then the map $\tilde{x} \mapsto E^u(\tilde{x})$ is continuous.*

DEFINITION 2.4. Let $f : M \to M$ be an Anosov endomorphism, Denote by $\mathcal{E}^u_f(x) := \bigcup_{\tilde{x}:\, \pi(\tilde{x})=x} E^u(\tilde{x})$ the union of all unstable directions at x.

Considering Definitions 1.2 and 1.1 a natural question arises: What is the relation between $\mathcal{E}^u_f(x)$ and $\bigcup_{y\in p^{-1}(x)} Dp(E^u_{\overline{f}}(y))$?

Observe that $\mathcal{E}^u_f(x)$ is not necessarily $\bigcup_{\pi(y)=x} Dp(y)\cdot(E^u_{\overline{f}}(y))$. Indeed, the latter is a countable union and the former may be uncountable (see [8]).

PROPOSITION 2.5. *Let $f : M \to M$ be an Anosov endomorphism. Then*

$$\mathcal{E}^u_f(x) = \overline{\bigcup_{p(y)=x} Dp(y)\cdot(E^u_{\overline{f}}(y))}.$$

Proof. First of all, $\mathcal{E}^u_f(x)$ is a closed subset of the u-dimensional grassmannian of T_xM. This is an immediate corollary of Theorem 2.3. Clearly $\bigcup_{\pi(y)=x} Dp(y)\cdot(E^u_{\overline{f}}(y)) \subset \mathcal{E}^u_f(x)$, so

$$\overline{\bigcup_{\pi(y)=x} Dp(y)\cdot(E^u_{\overline{f}}(y))} \subseteq \mathcal{E}^u_f(x).$$

Now for the opposite inclusion, let $E^u(\tilde{x})$ be an unstable direction at $x \in M$. We want to prove that $E^u(\tilde{x}) \in \overline{\bigcup_{p(y)=x} Dp(y)\cdot(E^u_{\overline{f}}(y))}$.

We claim that given any finite prehistory $(x_{-k},\dots,x_{-2},x_{-1},x=x_0)$, there is a finite piece of an $\overline{f}$-orbit, $(y_{-k},\dots,\overline{f}^k(y_{-k}))$, which projects onto $(x_{-k},\dots,x_{-2},x_{-1},x)$, that is,

$$\pi(\overline{f}^j y_{-k}) = x_{-k+j}, \quad j \in \{1,\dots,k\}.$$

Indeed, choose any $y_{-k} \in \overline{M}$ such that $p(y_{-k}) = x_{-k}$. As $p\circ\overline{f} = f\circ p$, the piece of the orbit of y_{-k} under $\overline{f}$ projects onto $(x_{-k},\dots,x_{-2},x_{-1},x)$.

Now for each k consider $\mathcal{O}(y_{-k})$, the full orbit of y_{-k} under $\overline{f}$. It is clear that $p(\mathcal{O}(y_{-k}))$ converges to $\tilde{x}$ in M^f. Recall that

$$E^u(p(\mathcal{O}(y_{-k}))) = Dp(E^u(\overline{f}^k(y_{-k}))). \tag{2.1}$$

By continuity [8], we have

$$E^u(\pi(\mathcal{O}(y_{-k}))) \to E^u(\tilde{x}),$$

and using (2.1) we obtain

$$Dp(E^u(\overline{f}^k(y_{-k}))) \to E^u(\tilde{x}),$$

which completes the proof. ∎

The next lemma is useful for the rest of this paper.

LEMMA 2.6. *Suppose that $f : M \to M$ is an Anosov endomorphism such that there are two different unstable directions $E_1^u(x)$ and $E_2^u(x)$ at x. Then the angle $\angle(Df^n(x)(E_1^u(x)), Df^n(x)(E_2^u(x)))$ goes to zero as $n \to \infty$.*

Proof. In fact, suppose that $\dim E^s = k$, $\dim E^u = n$, and $E_1^u(x) \neq E_2^u(x)$ for all $x \in M$. Let $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_n\}$ be bases for $E_1^u(x)$ and $E_2^u(x)$ respectively. Since $E_1^u(x) \neq E_2^u(x)$, there is u_i, say u_1, such that $B = \{u_1, v_1, \ldots, v_n\}$ is a linearly independent set.

Let $E := \langle u_1, v_1, \ldots, v_n\rangle$ with $\dim E = n + 1$ be the subspace generated by B. Observe that $\dim E + \dim E^s = n + k + 1 > n + k = \dim T_xM$. This implies that $E \cap E^s$ is non-trivial. Let $0 \neq v_s \in E \cap E^s$. Then

$$v_s = cu_1 + v$$

for some $c \neq 0$ and $v \in E_1^u(x) \setminus \{0\}$.

Considering the following properties of vectors in stable and unstable bundles:

$$\|Df^n(x)v_s\| \to 0, \quad \|Df^n(x)u_1\| \to \infty, \quad \|Df^n(x)v\| \to \infty,$$

it follows that $\angle([Df^n(x)u_1], Df^n(x)E_1^u(x)) \to 0$. In fact the same argument shows that $\angle([Df^n(x)u_i], Df^n(x)E_1^u(x)) \to 0$ for all u_i not in $E_1^u(x)$. Thus

$$\lim_{n\to\infty} \angle\big(Df^n(x)(E_1^u(x)), Df^n(x)(E_2^u(x))\big) = 0. \ ∎$$

3. Proof of Theorem 1.4. In the course of the proof we need to analyse the number of unstable directions as a function of $x \in M$. Let $u : M \to \mathbb{N} \cup \{\infty\}$ be defined as

$$u(x) := \#(\mathcal{E}_f^u(x)),$$

which assigns to each x the"number" of all possible unstable directions in T_xM.

A simple and useful remark is the following:

LEMMA 3.1. *$u(x)$ is non-decreasing along the forward orbit of x.*

Proof. It is enough to use the fact that f is a local diffeomorphisms and $Df(x)$ is injective. However, we emphasize that it is not clear whether $u(x)$ is constant or not along the orbit. This is because all the prehistory of x is included in the prehistory of $f(x)$. ∎

PROPOSITION 3.2. *Let $f : M \to M$ be a transitive Anosov endomorphism. Then either there is $x \in M$ such that $u(x) = \infty$, or u is uniformly bounded on M; in fact, in the latter case, f is a special Anosov endomorphism.*

Proof. Suppose that $u(x) < \infty$ for all $x \in M$. Define

$$\Lambda_k = \{x \in M \mid u(x) \leq k\}.$$

The sets Λ_k are closed. Indeed, by continuity (Theorem 2.3) the set $M \setminus \Lambda_k$ is open. Now observe that

$$M = \bigcup_{k=1}^{\infty} \Lambda_k,$$

so by the Baire category theorem, there is $k_0 \geq 1$ such that $\operatorname{int} \Lambda_{k_0} \neq \emptyset$.

Now we claim that

$$M = \Lambda_{k_0}.$$

To prove this, take any x in M with l unstable directions, and a small neighbourhood V_x of x such that each point in V_x has at least l unstable directions. Consider a point with dense orbit in V_x and take an iterate of it that belongs to Λ_{k_0}. By Lemma 3.1 we conclude that $l \leq k_0$, which yields $M = \Lambda_{k_0}$.

Finally, we prove that $M = \Lambda_1$, implying that f is a special Anosov endomorphism. Suppose that there is $x \in M$ such that $u(x) \geq 2$ and choose two different unstable directions $E_1^u(x), E_2^u(x)$ in T_xM. Let $\alpha > 0$ be the angle between $E_1^u(x)$ and $E_2^u(x)$.

Let U_x be a small neighbourhood of x such that every $y \in U_x$ has at least two unstable directions, say $E_1^u(y)$ and $E_2^u(y)$, with $\angle(E_1^u(y), E_2^u(y)) > \alpha/2$.

Let x_0 be a point with dense orbit. Let n_1 be a large number satisfying

- $f^{n_1}(x_0) \in U_x$,
- $\angle(Df^{n_1}(x_0) \cdot E, Df^{n_1}(x_0) \cdot F) < \alpha/3$ for any $E, F \in \mathcal{E}_f^u(x_0)$.

The choice of n_1 is possible thanks to density of the forward orbit of x_0 and Lemma 2.6. By definition of U_x, the above two properties imply that either $E_1^u(f^{n_1}(x_0))$ or $E_2^u(f^{n_1}(x_0))$ is not contained in $Df^{n_1}(x_0) \cdot \mathcal{E}_f^u(x_0)$. So, we obtain

$$u(f^{n_1}(x_0)) > u(x_0) + 1.$$

By repeating this argument, we to obtain an infinite sequence $f^{n_k}(x_0)$ such that

$$u(f^{n_{k+1}}(x_0)) \geq u(f^{n_k}(x_0)) + 1,$$

contradicting $M = \Lambda_{k_0}$. ■

3.1. Ending the proof of Theorem 1.4. To complete the proof of Theorem 1.4 it remains to show that $u(x) = \infty$ for a residual set $\mathcal{R} \subset M$, whenever f is not a special Anosov endomorphism. In fact, suppose that there is $x \in M$ such that $u(x) = \infty$. Given $k > 0$, fix exactly k different unstable directions at x, and a neighbourhood U_x^k of x such that $u(y) \geq k$ for every $y \in U_x^k$. Now, since f is transitive, the open set $U_k = \bigcup_{i \geq 0} f^i(U_x^k)$ is dense in M. Finally, consider

$$\mathcal{R} := \bigcap_{k \geq 1} U_k,$$

which is a residual set. By construction, given $x \in \mathcal{R}$ we have $u(x) \geq k$ for every $k > 1$, which implies $u(x) = \infty$. This completes the proof of Theorem 1.4.

4. Proof of Theorem 1.5. Given an Anosov endomorphism $f : \mathbb{T}^n \to \mathbb{T}^n$, by Proposition 1.3, the lift $\overline{f} : \mathbb{R}^n \to \mathbb{R}^n$ is an Anosov diffeomorphism.

Let $f_* : \mathbb{T}^n \to \mathbb{T}^n$ be the linearization of f, by which we mean the unique linear endomorphism of the torus, homotopic to f. By [1, Theorem 8.1.1], the linearization map is hyperbolic.

Although $\mathbb{R}^n$ is not compact, since $\overline{f}$ preserves $\mathbb{Z}^n$, the derivatives of $\overline{f}$ are periodic in fundamental compact domains of $\mathbb{T}^n$. This periodicity allows us to prove, in the $\mathbb{R}^n$ setting, results analogous to those for Anosov diffeomorphisms in the compact case.

Lemma 4.1. *Let $f : \mathbb{T}^n \to \mathbb{T}^n$ be a $C^{1+\alpha}$ Anosov endomorphism. Then for $\overline{f} : \mathbb{R}^n \to \mathbb{R}^n$ there exist transversally absolutely continuous foliations $\mathcal{F}^u_{\overline{f}}$ and $\mathcal{F}^s_{\overline{f}}$ tangent to $E^u_{\overline{f}}$ and $E^s_{\overline{f}}$ respectively.*

Proof. Similar to the compact case [5]. ■

Definition 4.2. A foliation W of $\mathbb{R}^n$ is *quasi-isometric* if there exist positive constants Q and b such that for all x, y in a common leaf of W we have

$$d_W(x, y) \leq Q^{-1}\|x - y\| + b.$$

Here d_W denotes the Riemannian metric on W and $\|x - y\|$ is the Euclidean distance.

Remark 4.3. Observe that if $\|x - y\|$ is large enough, we can take $b = 0$ in the above definition.

LEMMA 4.4. *Let A be as Theorem* 1.5. *If f is an Anosov endomorphism sufficiently C^1-close to A, then $\mathcal{F}^{s,u}_{\overline{f}}$ are quasi-isometric foliations.*

This lemma follows directly from a proposition due to Brin [3]:

PROPOSITION 4.5. *Let W be a k-dimensional foliation on $\mathbb{R}^m$. Suppose that there is an $(m-k)$-dimensional plane Δ such that $T_xW(x) \cap \Delta = \{0\}$ and $\angle(T_xW(x), \Delta) \geq \beta > 0$ for every $x \in \mathbb{R}^m$. Then W is quasi-isometric.*

Proof of Lemma 4.4. Let U be a C^1-open set containing A such that for every $f \in U$, $\overline{f}$ and $\overline{A}$ are C^1-close in the universal cover $\mathbb{R}^n$.

The C^1-neighbourhood U is taken such that

$$|\angle(E^u_{\overline{f}}(x), E^u_A)| < \alpha, \tag{4.1}$$

$$|\angle(E^s_{\overline{f}}(x), E^s_A)| < \alpha, \tag{4.2}$$

for any $x \in \mathbb{R}^n$, where α is a small number less than $\frac{1}{2}\angle(E^u_A, E^s_A)$. For the foliation $\mathcal{F}^u_{\overline{f}}$ take $\Delta := E^s_A$, and for $\mathcal{F}^s_{\overline{f}}$, $\Delta := E^u_A$. Applying Proposition 4.5 completes the proof. ∎

COROLLARY 4.6 (Nice properties). *For any Anosov endomorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ close to its linearization A, the following properties hold in the universal covering:*

(1) *For each $k \in \mathbb{N}$ and $C > 1$ there is M such that*

$$\|x - y\| > M \;\Rightarrow\; \frac{1}{C} \leq \frac{\|\overline{f}^k x - \overline{f}^k y\|}{\|A^k x - A^k y\|} \leq C.$$

(2) $$\lim_{\substack{\|y-x\|\to\infty \\ y \in \mathcal{F}^\sigma_{\overline{f}}(x)}} \frac{y-x}{\|y-x\|} = E^\sigma_A,\ \sigma \in \{s,u\},\ \textit{uniformly}.$$

Proof. The proof follows the lines of [4]; we repeat it for completeness. Let K be a fundamental domain of $\mathbb{T}^d$ in $\mathbb{R}^d$, $d \geq 2$. On K we have

$$\|\overline{f}^k - A^k\| < \infty.$$

For $\overline{x} \in \mathbb{R}^d$, there are $x \in K$ and $\vec{n} \in \mathbb{Z}^d$ such that $\overline{x} = x + \vec{n}$. Since $f_* = A$, we obtain

$$\begin{aligned}\|\overline{f}^k(\overline{x}) - A^k(\overline{x})\| &= \|\overline{f}^k(x + \vec{n}) - A^k(x + \vec{n})\| \\ &= \|\overline{f}^k(x) + A^k\vec{n} - A^k x - A^k\vec{n}\| < \infty.\end{aligned}$$

Now, for all $x, y \in \mathbb{R}^d$,

$$\|\overline{f}^k x - \overline{f}^k y\| \leq \|A^k x - A^k y\| + 2\|\overline{f}^k - A^k\|_0,$$
$$\|A^k x - A^k y\| \leq \|\overline{f}^k x - \overline{f}^k y\| + 2\|\overline{f}^k - A^k\|_0,$$

where

$$\|\overline{f}^k - A^k\|_0 = \max_{x \in K} \|\overline{f}^k(x) - A^k(x)\|.$$

Since A is non-singular, if $\|x-y\| \to \infty$, then $\|A^k x - A^k y\| \to \infty$. So dividing both expressions by $\|A^k x - A^k y\|$ and letting $\|x-y\| \to \infty$ we obtain the proof of the first item.

For the second item, we just consider the case of E^s_A; for E^u just take A^{-1} and $(\overline{f})^{-1}$, and the same proof holds.

Let $|\theta^s| = \max\{\,|\theta| : \theta$ is an eigenvalue of A and $0 < |\theta| < 1\}$. Fix a small $\varepsilon > 0$ and consider $\delta > 0$ such that $0 < (1+2\delta)|\theta^s| < 1$. If f is sufficiently C^1-close to A, then $\overline{f}$ is an Anosov diffeomorphism of $\mathbb{R}^d$ with contracting constant less than $(1+\delta)|\theta^s|$.

By hyperbolic splitting, there is $k_0 \in \mathbb{N}$ such that if $v \in \mathbb{R}^d$, $k > k_0$ and

$$\|A^k v\| < (1+2\delta)^k |\theta^s|^k \|v\|,$$

then

$$\|\pi^u_A(v)\| < \varepsilon \|\pi^s_A(v)\|.$$

Pick $k > k_0$ and M sufficiently large, satisfying the first item with $C = 2$ and in accordance with Remark 4.3.

Take $y \in \mathcal{F}^s_{\overline{f}}(x)$ and $\|x-y\| > M$. Let d^s denote the Riemannian distance on stable leaves of $\mathcal{F}^s_{\overline{f}}$. Since $\mathcal{F}^s_{\overline{f}}$ is quasi-isometric, we get

$$\begin{aligned}
&d^s(\overline{f}^k x, \overline{f}^k y) < ((1+\delta)|\theta^s|)^k d^s(x,y) \;\Rightarrow\\
&\|\overline{f}^k x - \overline{f}^k y\| < ((1+\delta)|\theta^s|)^k (Q^{-1}\|x-y\|) \;\Rightarrow\\
&\|A^k x - A^k y\| < 2((1+\delta)|\theta^s|)^k (Q^{-1}\|x-y\|).
\end{aligned}$$

Finally, for large k we have

$$2Q^{-1}((1+\delta)|\theta^s|)^k \le ((1+2\delta)|\theta^s|)^k.$$

So,

$$\|\pi^u_A(x-y)\| < \varepsilon \|\pi^s_A(x-y)\|. \;\blacksquare$$

LEMMA 4.7 ([7]). *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be an Anosov endomorphism close to $A : \mathbb{T}^d \to \mathbb{T}^d$ such that $\dim E^u_A = 1$. Then for all $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists M such that for x, y with $y \in \mathcal{F}^u_{\overline{f}}(x)$ and $\|x-y\| > M$ we have*

$$(1-\varepsilon)e^{n\lambda^u_A}\|y-x\| \le \|A^n(x) - A^n(y)\| \le (1+\varepsilon)e^{n\lambda^u_A}\|y-x\|,$$

where λ^u is the Lyapunov exponent of A corresponding to E^u_A.

Proof. Denote by E^u_A the eigenspace corresponding to λ^u_A and $|\mu| = e^{\lambda^u_A}$, where μ is the eigenvalue of A in the E^u_A direction.

Let $N \in \mathbb{N}$ and choose $x, y \in \mathcal{F}^u_f(x)$ such that $\|x-y\| > M$. By Corollary 4.6, we have

$$\frac{x-y}{\|x-y\|} = v + e_M,$$

where $v = v_{E^u_A}$ is a unitary eigenvector of A in the E^u_A direction and e_M is a correction vector that converges to zero uniformly as $M \to \infty$. We have

$$A^N\left(\frac{x-y}{\|x-y\|}\right) = \mu^N v + A^N e_M = \mu^N\left(\frac{x-y}{\|x-y\|}\right) - \mu^N e_M + A^N e_M.$$

This implies that

$$\begin{aligned}\|x-y\|(|\mu|^N - |\mu|^N\|e_M\| - \|A\|^N\|e_M\|) &\le \|A^N(x-y)\| \\ &\le \|x-y\|(|\mu|^N + |\mu|^N\|e_M\| + \|A\|^N\|e_M\|).\end{aligned}$$

Since N is fixed, we can choose $M > 0$ such that

$$|\mu|^N\|e_M\| + \|A\|^N\|e_M\| \le \varepsilon|\mu|^N,$$

and the lemma is proved. ∎

Remark 4.8. By the multiplicative ergodic theorem for endomorphisms [9] the unstable Lyapunov exponent for a typical point is independent of the unstable direction. We denote by $\lambda^u(x) = \lambda^u(\tilde{x})$ the unique unstable Lyapunov exponent of x in our context where $\dim E^u = 1$.

Theorem 4.9 (Theorem 1.5). *Let $A : \mathbb{T}^n \to \mathbb{T}^n$, $n \ge 2$, be a conservative linear Anosov endomorphism with* $\dim E^u_A = 1$. *Then there is a C^1-open set $\mathcal{U}$, containing A, such that for every $C^{1+\alpha}$, $\alpha > 0$, conservative Anosov endomorphism $f \in \mathcal{U}$, we have $\lambda^u_f(x) \le \lambda^u_A$ for m-almost every $x \in \mathbb{T}^n$, where m is the Lebesgue measure of $\mathbb{T}^n$.*

Proof. Suppose for contradiction that there is a positive measure set $Z \subset \mathbb{T}^n$ such that for every $x \in Z$ we have $\lambda^u_{\overline{f}}(x) > (1+5\varepsilon)\lambda^u_A$ for a small $\varepsilon > 0$. Since $\overline{f}$ is $C^{1+\alpha}$, the unstable foliation $\mathcal{F}^u_{\overline{f}}$ is absolutely continuous. So, there is a positive measure set $B \subset \mathbb{R}^n$ such that for every $x \in B$,

$$m^u_x(\mathcal{F}^u_{\overline{f}}(x) \cap Z) > 0, \tag{4.3}$$

where m^u_x is the Lebesgue measure of the leaf $\mathcal{F}^u_{\overline{f}}(x)$. Choose a $p \in B$ satisfying (4.3) and consider an interval $[x,y]_u \subset \mathcal{F}^u_{\overline{f}}(p)$ satisfying $m^u_p([x,y]_u \cap Z) > 0$ such that the length of $[x,y]_u$ is greater than M as required in Lemma 4.7 and Corollary 4.6. We can choose M such that

$$\|Ax - Ay\| < (1+\varepsilon)e^{\lambda^u_A}\|y-x\|$$

and

$$\frac{\|\overline{f}(x) - \overline{f}(y)\|}{\|Ax - Ay\|} < 1 + \varepsilon.$$

whenever $d^u(x,y) \geq M$, where d^u denotes the Riemannian distance in unstable leaves. The above implies that

$$\|\overline{f}(x) - \overline{f}(y)\| < (1+\varepsilon)^2 e^{\lambda_A^u} \|y - x\|.$$

Inductively, we assume that for $n \geq 1$ we have

$$\|\overline{f}^n(x) - \overline{f}^n(y)\| < (1+\varepsilon)^{2n} e^{n\lambda_A^u} \|y - x\|. \tag{4.4}$$

Since f expands uniformly in the u direction, we have $d^u(\overline{f}^n(x), \overline{f}^n(y)) > M$, and consequently

$$\begin{aligned} \|\overline{f}(\overline{f}^n x) - \overline{f}(\overline{f}^n y)\| &< (1+\varepsilon)\|A(\overline{f}^n x) - A(\overline{f}^n y)\| \\ &< (1+\varepsilon)^2 e^{\lambda_A^u} \|\overline{f}^n x - \overline{f}^n y\| < (1+\varepsilon)^{2(n+1)} e^{(n+1)\lambda_A^u}. \end{aligned}$$

For each $n > 0$, let

$$A_n = \{x \in Z \colon \|D\overline{f}^k(x)|E^u_{\overline{f}}(x)\| > (1+2\varepsilon)^{2k} e^{k\lambda_A^u} \text{ for any } k \geq n\}.$$

We have $m(Z) > 0$ and $Z_n := A_n \cap Z \uparrow Z$, as $1 + 5\varepsilon > (1+2\varepsilon)^2$ for small $\varepsilon > 0$.

Define the number $\alpha_0 > 0$ so that

$$\frac{m_p^u([x,y]_u \cap Z)}{m_p^u([x,y]_u)} = 2\alpha_0.$$

Since $Z_n \cap [x,y]_u \uparrow Z \cap [x,y]_u$, there is $n_0 \in \mathbb{N}$, such that if $n \geq n_0$, then

$$m_p^u([x,y]_u \cap Z_n) = \alpha_n \cdot m_p^u([x,y]_u)$$

with $\alpha_n > \alpha_0$. Thus, for $n \geq n_0$ we have

$$\begin{aligned} \|\overline{f}^n x - \overline{f}^n y\| &> Q \int\limits_{[x,y]_u \cap Z_n} \|Df^n(z)\|\, dm_p^u(z) \\ &> Q(1+2\varepsilon)^{2n} e^{n\lambda_A^u} m_p^u([x,y]_u \cap Z_n) \\ &> \alpha_0 Q^2 (1+2\varepsilon)^{2n} e^{n\lambda_A^u} \|x - y\|. \end{aligned} \tag{4.5}$$

The inequalities (4.4) and (4.5) give a contradiction. ∎

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