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Discrepancy estimates for some linear generalized monomials

by

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1. Introduction. A natural way to extend the family of real-valued polynomials is to add the integer part operation, denoted by $[\cdot]$, to the arithmetical operations of addition and multiplication. Then functions like $p_1(x) = [[b_1x] + b_2x^2]x + b_3$ and $p_2(x) = [[b_1x][b_2[x]b_3x^2]]$ can be obtained. We call such functions *generalized polynomials*. The question whether a sequence $(p(n))_{n\geq 0}$, where p is a generalized polynomial, is uniformly distributed modulo one was for instance studied by Håland [9, 10], Bergelson and Leibman [4], and Leibman [13], using either the Weyl criterion or dynamical properties.

A sequence $(x_n)_{n\geq 0}$ of real numbers is said to be *uniformly distributed modulo one* (u.d. mod 1) if

$$\lim_{N\to\infty} \frac{\#\{n : 0 \leq n < N,\, a \leq \{x_n\} < b\}}{N} = b - a$$

for all real numbers a, b satisfying $0 \leq a < b \leq 1$. Here and in what follows, $\{x\}$ denotes the fractional part of x.

For example the sequence $([n\alpha]\beta)_{n\geq 0}$ is known to be u.d. mod 1 when α is a nonzero rational real if and only if β is irrational. If α is zero then the sequence is u.d. mod 1 for no real number β. And when α is irrational, the uniform distribution modulo one of $([n\alpha]\beta)_{n\geq 0}$ is equivalent to the condition, $1, \alpha, \alpha\beta$ being linearly independent over the rationals. (For a proof see [12, Chapter 5, Theorem 1.8].)

We define (and use) the notion of the *discrepancy of a sequence* in the multi-dimensional setting. The discrepancy of the first N terms of the

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s-dimensional sequence $(x_n^{(1)},\dots,x_n^{(s)})_{n\ge 0}$ of real numbers is given by

$$\sup_{\substack{0\le a_i<b_i\le 1\\ i=1,\dots,s}}\left|\frac{\#\{n : 0\le n<N,\, a_i\le \{x_n^{(i)}\}<b_i,\, i=1,\dots,s\}}{N}-\prod_{i=1}^{s}(b_i-a_i)\right|,$$

denoted by $D_N((x_n^{(1)},\dots,x_n^{(s)}))$ and most often abbreviated to D_N when the relevant sequence is clear from the context.

So far the discrepancy of sequences of the form $(p(n))_{n\ge 0}$, where p is a generalized polynomial, has only been studied when the polynomial p is nongeneralized, i.e., no brackets $[\cdot]$ occur. The most basic example is of course given by $(n\alpha)_{n\ge 0}$. It has been amply studied, and the Diophantine properties of α play an important role (we refer to [7] for an excellent overview).

We turn our attention to the first nontrivial generalized example, namely the sequence $([n\alpha]\beta)_{n\ge 0}$. This sequence is also of interest because it is a subsequence of $(n\beta)_{n\ge 0}$. Such subsequences are actively studied (see for instance the recent work [1]). When α equals p/q with nonzero integers p,q and $q>0$, one can easily fragment the sequence $([n\alpha]\beta)_{n\ge 0}$ into q sequences of the form $(mp\beta+[rp/q]\beta)_{m\ge 0}$ with $r=0,1,\dots,q-1$. These sequences coincide with the most basic case $(n\alpha)_{n\ge 0}$, which—as already mentioned—is well studied. We thus focus on the case when $1,\alpha,\alpha\beta$ are linearly independent over the rationals. To state our main result we follow the terminology of Niederreiter (cf. for example [14]).

Definition 1. Let t be a real number greater than or equal to 1. We say the pair of real numbers (γ,δ) is of *finite type* t if for any $\varepsilon>0$ there exists a positive constant c such that for any pair (m,n) of rational integers, not both zero, we have

$$\big(\max(1,|m|)\big)^{t+\varepsilon}\big(\max(1,|n|)\big)^{t+\varepsilon}\|m\gamma+n\delta\|\ge c.$$

The constant c may depend on ε, γ, and δ but not on m or n. (Here and below, $\|x\|$ stands for the-distance-to-the-nearest-integer function.)

Example 1. (1) Let γ and δ be real algebraic numbers such that $1,\gamma,\delta$ are linearly independent over the rationals. Then by [18, Theorem 2] we know that the pair (γ,δ) is of finite type 1.

(2) Let q_1 and q_2 be different nonzero rational numbers. Then the pair (e^{q_1},e^{q_2}) is of finite type 1 (see [2] and [15]).

(3) By, e.g., [3, equation (2.9)] almost all pairs $(\gamma,\delta)\in\mathbb{R}^2$ in the sense of Lebesgue measure are of finite type 1. In this case an even stronger estimate holds: for every $\varepsilon>0$ there exists a positive constant c such that

$$(\operatorname{Log} m \operatorname{Log} n)^{1+\varepsilon}\max(1,|m|)\max(1,|n|)\|m\gamma+n\delta\|\ge c$$

for any pair of rational integers (m, n) not both zero. The constant may depend on γ, δ, and ε. (Throughout this paper, we use $\mathrm{Log}(x)$ for $\max(1, \log(|x|))$ if $x \neq 0$, and we set $\mathrm{Log}(0) = 1$).

In the following definition we introduce a linear independence measure, related to the notion of finite type, and studied for instance in [11] or [20].

Definition 2. Let t', γ, and δ be real numbers and let t' be greater than or equal to 1. We say that 1, γ, and δ have *linear independence measure* t' if for any $\varepsilon > 0$ there exists a positive constant c such that for any pair (m, n) of rational integers, not both zero, we have

$$\max(|m|, |n|)^{t'+\varepsilon} \|m\gamma + n\delta\| \geq c.$$

The constant c may depend on ε, γ, and δ but not on m or n.

Note that $\max(|m|, |n|) \leq \max(1, |m|) \max(1, |n|)$. Hence, a pair that together with 1 has linear independence measure t' admits also t' as a finite type.

The corresponding notion for a single real number γ is the one of irrationality measure. We say that $t + 1$ is an irrationality measure of γ when for any nonzero integer m and any $\varepsilon > 0$, one has

$$|m|^{t+\varepsilon} \|m\gamma\| \gg_{\varepsilon,\gamma} 1.$$

It is well known that if γ has irrationality measure $t+1$, then $D_N(n\gamma) \ll_{\gamma,\varepsilon} N^{-1/t+\varepsilon}$ (see, e.g., [12, Theorem 3.2]). This bound is, up to the ε, best possible in N (note the lower bound $\Omega(N^{-1/t-\varepsilon})$ for infinitely many N in [15, Theorem 2]). In two dimensions, it is known that the discrepancy of the sequence $(n\beta_1, n\beta_2)_{n\geq 0}$, where (β_1, β_2) is of finite type t, satisfies for every $\varepsilon > 0$,

$$\Omega(N^{-1/t-\varepsilon}) = D_N(n\beta_1, n\beta_2) = \mathcal{O}_{\beta_1,\beta_2,\varepsilon}(N^{-1/(2t-1)+\varepsilon}). \tag{1}$$

(See [15, Theorem 2] for the lower bound, which holds for infinitely many N, and [14, Lemma 6] for the upper bound.)

Theorem 1.1. *Let $t \geq 1$ and α, β be real numbers such that $1, \alpha, \alpha\beta$ are linearly independent over the rationals. Assume that both pairs $(\alpha, \alpha\beta)$ and $(\beta, 1/\alpha)$ are of finite type t. Then, for every $\varepsilon > 0$,*

$$D_N([n\alpha]\beta) \ll_{\alpha,\beta,\varepsilon} N^{-1/(3t-2)+\varepsilon}.$$

Corollary 1.2. *Let α and β be algebraic real numbers such that 1, α, and $\alpha\beta$ are linearly independent over the rationals. Then for every $\varepsilon > 0$,*

$$D_N([n\alpha]\beta) \ll_{\alpha,\beta,\varepsilon} N^{-1+\varepsilon}.$$

Corollary 1.3. *Let q_1 and q_2 be nonzero rational numbers satisfying $q_1 + q_2 \neq 0$. Then for every $\varepsilon > 0$,*

$$D_N([ne^{q_1}]e^{q_2}) \ll_{q_1,q_2,\varepsilon} N^{-1+\varepsilon}.$$

Corollary 1.4. *For almost all pairs of real numbers (α, β) in the sense of Lebesgue measure, for every $\varepsilon > 0$, we have $D_N([n\alpha]\beta) \ll_{\alpha,\beta,\varepsilon} N^{-1+\varepsilon}$.*

Corollary 1.5. *Let $t \geq 1$ and α, β be real numbers such that $1, \alpha, \alpha\beta$ have linear independence measure t. Then for every $\varepsilon > 0$,*

$$D_N([n\alpha]\beta) \ll_{\alpha,\beta,\varepsilon} N^{-1/(3t-2)+\varepsilon}.$$

Example 2. Let us note that [11, Theorem 2.1] proves the following:

$$\max(|m|, |n|)^{7.0161} \|m\pi + n \log 2\| \gg 1$$

for all $m, n \in \mathbb{Z}$ not both zero. Or, on employing our Definition 2, the triple $(1, \pi, \log 2)$ admits 7.0161 as a linear independence measure. By using Corollary 1.5 together with Lemma 2.5, we derive that the discrepancy of both sequences $([n/\pi] \log 2)_{n\geq 0}$ and $([n/\log 2]\pi)_{n\geq 0}$ is $\mathcal{O}(N^{-0.052498})$.

The remaining part of the paper is organized as follows. In Section 2 we collect several auxiliary results that will be used to prove our main Theorem 1.1 and its corollaries. Section 3 elaborates the main tool to handle the specific exponential sums occurring when using the Erdős–Turán inequality for the discrepancy of the sequence $([n\alpha]\beta)_{n\geq 0}$. Theorem 1.1 and its corollaries are proved in Section 4. Finally, Section 5 collects some interesting unresolved problems and future research tasks.

2. Auxiliary results. To estimate the discrepancy of our sequence we use the celebrated Erdős–Turán inequality (see for example [7, 12, 16]).

Lemma 2.1. *For the discrepancy D_N of N points $x_0, x_1, \ldots, x_{N-1}$ we have*

$$D_N \leq \frac{2}{H+1} + 2\sum_{h=1}^{H} \frac{1}{h} \bigg| \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i h x_k} \bigg|,$$

where H is an arbitrary positive integer usually chosen smaller than N.

Lemma 2.2 (The Gap Lemma). *Let $I \in \mathbb{N}$, let $x_1, \ldots, x_I$ be real numbers, and let f be a nonnegative nonincreasing function over $[0, 1]$. Furthermore, let $\delta \in (0, 1/2]$ be such that for all distinct $i, j \in \{1, \ldots, I\}$ we have*

$$\|x_i\| \geq \delta \quad \text{and} \quad \|x_i - x_j\| \geq \delta.$$

Then for every $i \in \{1, \ldots, I\}$ there are at most two values $j \in \{1, \ldots, I\}$ such that $\big|\|x_i\| - \|x_j\|\big| < \delta$ and

$$\sum_{1\leq i\leq I} f(\|x_i\|) \leq 2 \sum_{1\leq j\leq (2\delta)^{-1}} f(j\delta).$$

Proof. The statement on $\big|\|x_i\| - \|x_j\|\big|$ follows by the special properties of the function $x \mapsto \|x\|$ and the fact that different indices i, j yield $\|x_i - x_j\| \geq \delta$. This implies that each of the intervals

$$[0,\delta),\ [\delta,2\delta),\ \ldots,\ [k\delta,(k+1)\delta)$$

with $1/2-\delta < k\delta \leq 1/2$ contains at most two points $\|x_i\|$, and since $\|x_i\| \geq \delta$ the first interval is empty. Note that $k = [1/(2\delta)]$. Finally, the fact that f is nonincreasing yields the desired inequality. ∎

LEMMA 2.3. *Let $D \geq 1$. Assume $(\beta, 1/\alpha)$ is of finite type t. If $|m\beta - n| \leq D$ for some integers m and n not both zero, then*

$$\|(m\beta - n)\alpha\| \gg_{\alpha,\beta,\varepsilon} \max(1,|m|)^{-t-\varepsilon} D^{-t-\varepsilon}.$$

Proof. Let m, n not both zero be such that $|m\beta - n| \leq D$. Note that the assumption that $(\beta, 1/\alpha)$ is of finite type t guarantees that $1, \alpha, \alpha\beta$ are linearly independent over the rationals. Hence, there is a unique integer ℓ such that

$$0 < \|(m\beta - n)\alpha\| = |m\beta\alpha - n\alpha - \ell| < 1/2.$$

Obviously, $|\ell| \leq |\alpha| D + 1/2$. We derive

$$\begin{aligned}\frac{1}{|\alpha|}\|(m\beta - n)\alpha\| &= \frac{1}{|\alpha|}|m\beta\alpha - n\alpha - \ell| = |m\beta - l/\alpha - n| \\ &\geq \|m\beta - \ell/\alpha\| \gg_{\alpha,\beta,\varepsilon} \max(1,|m|)^{-t-\varepsilon} \max(1,|\ell|)^{-t-\varepsilon}.\end{aligned}$$

Finally, $\max(1,|\ell|)^{-t-\varepsilon} \gg_{\alpha,\varepsilon} D^{-t-\varepsilon}$. ∎

LEMMA 2.4. *When α is an algebraic real number of degree greater than* 1, *we have $D_N(n\alpha) \ll_{\alpha,\varepsilon} N^{-1+\varepsilon}$.*

Proof. Note that by the famous Thue–Siegel–Roth theorem an algebraic real number of degree greater than 1 has irrationality measure 2 (cf., e.g., [5, p. 248]). ∎

LEMMA 2.5. *Let α and β be nonzero real numbers. Then the triple $(1, \alpha, \alpha\beta)$ has linear independence measure t' if and only if $(1, \beta, 1/\alpha)$ has linear independence measure t'.*

Proof. Let $\varepsilon > 0$. By definition, we have

$$\min_{\substack{m,n\in\mathbb{Z}\\ \text{not both zero}}} \min_{\ell\in\mathbb{Z}} \max(|m|,|n|)^{t'+\varepsilon}|m\alpha + n\alpha\beta + \ell| \gg_{\alpha,\beta,\varepsilon} 1,$$

which is equivalent to

$$\min_{\substack{m,n,\ell\in\mathbb{Z}\\ \text{not all zero}}} \max(1,|m|,|n|)^{t'+\varepsilon}|m\alpha + n\alpha\beta + \ell| \gg_{\alpha,\beta,\varepsilon} 1.$$

This inequality immediately yields

$$\min_{\substack{m,n,\ell\in\mathbb{Z}\\ \text{not all zero}}} \max(|m|,|n|,|\ell|)^{t'+\varepsilon}|m\alpha + n\alpha\beta + \ell| \gg_{\alpha,\beta,\varepsilon} 1. \tag{2}$$

The latter inequality also implies the former one: either $|m\alpha+n\alpha\beta+\ell| \le 1$, and therefore $|\ell| \ll_{\alpha,\beta} \max(1,|m|,|n|)$, or $|m\alpha+n\alpha\beta+\ell| \ge 1$ and the inequality is obvious. Now that we have a definition that is symmetrical in m, n, and ℓ, we simply divide by α in (2) to show that any linear independence measure for $(1,\alpha,\alpha\beta)$ is also valid for $(1,\beta,1/\alpha)$, and conversely. ∎

Lemma 2.6. *Let λ_2 be the Lebesgue measure on $\mathbb{R}^2$. Set $X = (\mathbb{R}\setminus\{0\})^2$. Let $f : X \to X$ be a continuously differentiable bijection. Suppose the Jacobian matrix Df of f satisfies $|\det Df| > 0$ everywhere in X. Furthermore, let $g : X \to \{0,1\}$ be a Lebesgue measurable function satisfying $\lambda_2(g^{-1}(\{1\})) = 0$. Then*

$$\lambda_2\big((g\circ f)^{-1}(\{1\})\big) = 0.$$

Proof. By the change-of-variable theorem (see for example [17, Theorem 7.26]) we obtain

$$\int_X g\,d\lambda_2 = \int_X (g\circ f)|\det Df|\,d\lambda_2.$$

From this equality together with the fact that $\lambda_2(g^{-1}(\{1\})) = 0$ we find that both integrals above are 0. We observe that $(g\circ f)|\det Df| \ge 0$ everywhere in X. Using $|\det Df| > 0$ we conclude $\lambda_2((g\circ f)^{-1}(\{1\})) = 0$. ∎

3. An approximation and its Fourier transform. In the following we abbreviate $e^{2\pi iy}$ to $e(y)$. We consider the 1-periodic function

$$f_\tau(x) = e(\{x\}\tau)$$

where τ is some nonzero real number (in our application, τ will be $-h\beta$ for some possibly large integer h and an irrational β). We need to expand f_τ in a Fourier series. This is not a priori possible since this function is not regular enough for the expansion to converge pointwise. We have to recourse to a smooth approximation. We select a positive real parameter δ, a positive integer r, and look at

$$g_\tau(x;\delta) = \int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta} e(\{x+u_1+\cdots+u_r\}\tau)\,\frac{du_1\cdots du_r}{(2\delta)^r}.$$

This is an approximation of f_τ in the following sense.

Lemma 3.1. *For any real number α,*

$$\sum_{0\le n<N} \big|f_\tau(n\alpha) - g_\tau(n\alpha;\delta)\big| \ll r^2\delta(1+|\tau|)N + ND_N(n\alpha).$$

Proof. We first check that if $r\delta < \{x\} < 1 - r\delta$, then

$$g_\tau(x;\delta) - f_\tau(x) = \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} \big(e(\{x + u_1 + \cdots + u_r\}\tau) - e(\{x\}\tau)\big) \frac{du_1 \cdots du_r}{(2\delta)^r}$$
$$\ll \int_{-r\delta}^{r\delta} |\{x+u\} - \{x\}|\,|\tau| \frac{du}{\delta} \ll r^2\delta|\tau|.$$

Thus

$$\sum_{\substack{0\le n<N,\\ \|\alpha n\|>r\delta}} \big|f_\tau(n\alpha) - g_\tau(n\alpha;\delta)\big| \ll r^2 N\delta|\tau|.$$

The contribution of the remaining terms is easily bounded above by $cr\delta N + cND_N(n\alpha)$ with an absolute positive constant c. Note that

$$\sum_{\substack{0\le n<N,\\ \|\alpha n\|\le r\delta}} \big|f_\tau(n\alpha) - g_\tau(n\alpha;\delta)\big| \ll \sum_{\substack{0\le n<N,\\ \|\alpha n\|\le r\delta}} 2$$
$$\ll \#\{0 \le n < N : \{\alpha n\} \in [0, r\delta]\} + \#\{0 \le n < N : \{\alpha n\} \in [1 - r\delta, 1]\}$$
$$\ll r\delta N + ND_N(n\alpha). \blacksquare$$

The Fourier coefficients of g_τ are easy to obtain by replacing $x + u_1 + \cdots + u_r$ by y and invoking the periodicity of $y \mapsto \{y\}$:

$$\int_0^1 g_\tau(x;\delta)e(kx)\,dx$$
$$= \int_0^1 \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} e(\{x + u_1 + \cdots + u_r\}\tau + kx) \frac{du_1 \cdots du_r}{(2\delta)^r}\,dx$$
$$= \int_{-\delta}^{\delta} \cdots \int_{-\delta}^{\delta} e(-ku_1 - \cdots - ku_r) \int_0^1 e(\{y\}\tau + ky)\,dy \frac{du_1 \cdots du_r}{(2\delta)^r}$$
$$= \left(\frac{\sin 2\pi k\delta}{2\pi k\delta}\right)^r \frac{e(\tau + k) - 1}{2\pi i(k + \tau)} = \hat{g_\tau}(k;\delta).$$

For $r \ge 2$, the function g_τ is at least C^1 and we have

$$g_\tau(x;\delta) = \sum_{k\in\mathbb{Z}} \hat{g_\tau}(k;\delta)e(-kx)$$

(pointwise, for every x).

In the following we always assume that $r \ge 2t + 1$.

4. Proof of Theorem 1.1 and its corollaries. Using Lemma 2.1, for the discrepancy D_N of the first N terms of the sequence $(\{[n\alpha]\beta\})_{n\ge1}$ we obtain the bound

$$D_N \le \frac{2}{H+1} + 2\sum_{h=1}^{H} \frac{1}{h}\left|\frac{1}{N}\sum_{n=0}^{N-1} e(h[n\alpha]\beta)\right|. \tag{3}$$

We notice that $h[n\alpha]\beta = hn\alpha\beta - h\{n\alpha\}\beta$, approximate $e(-h\{n\alpha\}\beta) = f_{-h\beta}(n\alpha)$ by $g_{-h\beta}(n\alpha;\delta)$, and rely on Lemma 3.1 to bound the inner sum:

$$\begin{aligned}\Big|\sum_{0\le n<N} e(h[n\alpha]\beta)\Big| &= \Big|\sum_{0\le n<N} e(-h\{n\alpha\}\beta)e(hn\alpha\beta)\Big| \\ &\le \Big|\sum_{k\in\mathbb{Z}} \hat{g}_{-h\beta}(k;\delta) \sum_{0\le n<N} e(hn\alpha\beta - kn\alpha)\Big| \\ &\quad + \mathcal{O}_r(\delta(1+|h\beta|)N) + \mathcal{O}(ND_N(n\alpha)).\end{aligned}$$

We concentrate on the sum of the right hand side and make use of the special form of $\hat{g}_{-h\beta}(k;\delta)$. The trivial fact that $|\sin(2\pi x)/x|^r \ll_r \min(1, 1/|x|^r)$ yields

$$\begin{aligned}&\Big|\sum_{k\in\mathbb{Z}} \hat{g}_{-h\beta}(k;\delta) \sum_{0\le n<N} e(hn\alpha\beta - kn\alpha)\Big| \\ &\qquad \ll_r \sum_{k\in\mathbb{Z}} \frac{\|h\beta-k\|}{|h\beta-k|}\min(1, 1/(|k|\delta)^r)\frac{1}{\|(h\beta-k)\alpha\|}\end{aligned}$$

where we have also employed, for any irrational κ, the elementary estimates

$$\Big|\sum_{n=0}^{N-1} e(\kappa n)\Big| \le \frac{2}{|e(\kappa)-1|}$$

and

$$0 < \|\kappa\| \ll |e(\kappa)-1| = 2|\sin(\pi\kappa)| \ll \|\kappa\|.$$

We select $\delta^{-1} = hN^\theta$ for some $\theta \in (0, 1/t]$, which we will choose later. With this choice of δ, the terms $\mathcal{O}_r(\delta(1+|h\beta|)N) + \mathcal{O}(ND_N(n\alpha)))$ in the inequality above, when summed over h, give a contribution to (3) that is

$$\mathcal{O}_{r,\beta}\left(\frac{1}{N^\theta}\operatorname{Log}(H) + \operatorname{Log}(H)D_N(n\alpha)\right) = \mathcal{O}_{r,\alpha,\beta,\varepsilon_1}(N^{-\theta+\varepsilon_1})$$

for every $\varepsilon_1 > 0$. Note that the assumptions on α and β ensure that α has irrationality measure $t+1$, and therefore $D_N(n\alpha) \ll_{\alpha,\varepsilon} N^{-1/t+\varepsilon} \le N^{-\theta+\varepsilon}$.

Restricting the range in k. We first consider the case when $|k| \ge |h|^{\rho-1}\delta^{-1} = h^\rho N^\theta$ for some $\rho \in [1,2]$ that we will choose later (very close to 1). For N larger than $(2|\beta|)^{1/\theta}$, we have $k \ge 2|h\beta|$ and thus $|h\beta - k| \ge |k|/2$, while, of course, $\|h\beta - k\| \le 1$. We set $K := K_h = |h|^\rho N^\theta$ and apply dyadic subdivision to get

$$\sum_{|k|\geq K}\frac{\|h\beta-k\|}{|h\beta-k|}\min(1,1/(|k|\delta)^r)\frac{1}{\|(h\beta-k)\alpha\|}\ll\sum_{|k|\geq K}\frac{h^rN^{r\theta}}{|k|^{r+1}}\frac{1}{\|(h\beta-k)\alpha\|}$$
$$\ll h^rN^{r\theta}\sum_{\mu=0}^{\infty}(2^\mu K)^{-r-1}\sum_{2^\mu K\leq|k|<2^{\mu+1}K}\frac{1}{\|(h\beta-k)\alpha\|}.$$

Using the assumption that $(\alpha,\alpha\beta)$ is of finite type t we obtain, for any real $\varepsilon_2>0$ and any integer $\mu\geq 0$, a constant $\gamma=\gamma_{\alpha,\beta,\varepsilon_2}(\mu,K,h)$ of the form $c_{\alpha,\beta,\varepsilon_2}2^{\mu(-t-\varepsilon_2)}(hK)^{-t-\varepsilon_2}$ for some constant $c_{\alpha,\beta,\varepsilon_2}>0$, independent of μ, K, and h, such that for any $|k|,|k_1|,|k_2|<2^{\mu+1}K$ we have both

$$\|h\alpha\beta-k\alpha\|\geq\gamma\quad\text{and}\quad\|h\alpha\beta-k_1\alpha-(h\alpha\beta-k_2\alpha)\|\geq\gamma$$

whenever $k_1\neq k_2$. Therefore the Gap Lemma applies. Hence,

$$\sum_{2^\mu K\leq|k|<2^{\mu+1}K}\frac{1}{\|(h\beta-k)\alpha\|}\leq 2\sum_{j=1}^{[(2\gamma)^{-1}]}\frac{1}{j\gamma}\leq\frac{2}{\gamma}(1+\log(1/(2\gamma)))$$
$$\ll_{\alpha,\beta,\varepsilon_2}2^{\mu(t+\varepsilon_2)}h^{t+\varepsilon_2}K^{t+\varepsilon_2}\operatorname{Log}(2^\mu hK)$$
$$\ll 2^{\mu(t+\varepsilon_2)}h^{t+\varepsilon_2}K^{t+\varepsilon_2}(\mu+\operatorname{Log}(hK)).$$

The summation over μ can be bounded by an absolute constant when $r>t+\varepsilon_2$. When summed over h, this gives a contribution to (3) that is

$$\mathcal{O}_r\Big(N^{r\theta-1}\sum_{h=1}^{H}h^{r+t+\varepsilon_2-1}K^{-r-1+t+\varepsilon_2}\operatorname{Log}(hK)\Big)$$
$$\ll_{\alpha,\beta,r,\rho,\theta,\varepsilon_2}(\operatorname{Log}N)N^{(t-1)\theta-1+\varepsilon_2\theta}\sum_{h=1}^{H}h^{r+t-(r+1-t)\rho+(1+\rho)\varepsilon_2-1}$$
$$\ll_{\alpha,\beta,r,\rho,\theta,\varepsilon_2}(\operatorname{Log}N)N^{(t-1)\theta-1+\varepsilon_2\theta}$$

provided that

$$r+t-(r+1-t)\rho+(1+\rho)\varepsilon_2\leq -1. \tag{4}$$

Treating the remaining terms: Summing over h in a dyadic box and localizing $h\beta-k$. Now that k is bounded above in absolute value, we majorize $\min(1,(hN^\theta/|k|)^r)$ by 1. It remains to bound

$$\varSigma=\frac{1}{N}\sum_{h=1}^{H}\frac{1}{h}\sum_{|k|<|h|^\rho N^\theta}\frac{\|h\beta-k\|}{|h\beta-k|}\frac{1}{\|(h\beta-k)\alpha\|}.$$

We use a dyadic decomposition for $h = 1, \dots, H$ and obtain

$$(5) \qquad \Sigma \ll \frac{1}{N} \sum_{\nu=0}^{[\log_2 H]} \frac{1}{2^\nu} \sum_{h=2^\nu}^{\min(2^{\nu+1}-1,H)} \sum_{|k|<|h|^\rho N^\theta} \frac{\|h\beta - k\|}{|h\beta - k|} \frac{1}{\|(h\beta-k)\alpha\|}.$$

The next step is to localize $h\beta - k$. Note that

$$|h\beta - k| \le 2\min(2^{\nu+1}-1, H)^\rho N^\theta$$

when N is large enough with respect to β, and for given h and D there are at most two values for k such that $D \le |h\beta - k| < D+1$. We thus write

$$\sum_{h=2^\nu}^{\min(2^{\nu+1}-1,H)} \sum_{|k|<|h|^\rho N^\theta} \frac{\|h\beta - k\|}{|h\beta - k|} \frac{1}{\|(h\beta-k)\alpha\|} \le \Sigma_B + \sum_{\mu=0}^{\theta \log_2 N + \rho \min(\nu+1, \log_2 H)+1} \left(\frac{1}{2^\mu} \Sigma_\mu \right)$$

where

$$\Sigma_B := \sum_{h=2^\nu}^{\min(2^{\nu+1}-1,H)} \sum_{\substack{k\in\mathbb{Z} \\ |h\beta-k|\le 1}} \frac{1}{\|(h\beta-k)\alpha\|},$$

$$\Sigma_\mu := \sum_{h=2^\nu}^{\min(2^{\nu+1}-1,H)} \sum_{\substack{k\in\mathbb{Z} \\ 2^\mu \le |h\beta-k| < 2^{\mu+1}}} \frac{1}{\|(h\beta-k)\alpha\|}.$$

Since $(\beta, 1/\alpha)$ is of finite type t, we can apply Lemma 2.3. For every $\varepsilon_3 > 0$ we have a $\xi := \xi_{\alpha,\beta,\varepsilon_3}(\nu,\mu)$ in the style of $c_{\alpha,\beta,\varepsilon_3} 2^{\nu(-t-\varepsilon_3)} 2^{\mu(-t-\varepsilon_3)}$ that satisfies the following: for any h, h' in $[2^\nu, 2^{\nu+1})$ and any k, k' such that $|h\beta - k|$ and $|h'\beta - k'|$ in $[2^\mu, 2^{\mu+1})$ we have both

$$\|(h\beta-k)\alpha\| \ge \xi \quad \text{and} \quad \|(h\beta-k)\alpha - (h'\beta - k')\alpha\| \ge \xi.$$

The Gap Lemma applies, and we obtain

$$\Sigma_\mu \ll_{\alpha,\beta,\rho,\theta,\varepsilon_3} (\log N) 2^{\nu(t+\varepsilon_3)} 2^{\mu(t+\varepsilon_3)}.$$

We also derive, quite analogously,

$$\Sigma_B \ll_{\alpha,\beta,\varepsilon_3} (\log N) 2^{\nu(t+\varepsilon_3)}.$$

Inserting these results into (5), we get

$$\begin{aligned} \Sigma &\ll_{\alpha,\beta,\rho,\theta,\varepsilon_3} \frac{\log N}{N} \sum_{\nu=0}^{[\log_2 H]} 2^{\nu(t-1+\varepsilon_3)} \sum_{\mu=0}^{\theta \log_2 N + \rho \min(\nu+1, \log_2 H)+1} 2^{\mu(t-1+\varepsilon_3)} \\ &\ll_{\alpha,\beta,\rho,\theta,\varepsilon_3} (\log N) H^{(\rho+1)(t-1+\varepsilon_3)} N^{\theta(t-1+\varepsilon_3)-1} \end{aligned}$$

with an additional dependence of the constant on r in the Erdős–Turán inequality.

Balancing all terms occurring in the discrepancy bound. We summarize all terms, which we obtained for the upper bound of the discrepancy D_N^* for the first N terms of the sequence $([n\alpha]\beta)_{n\geq 1}$,

$$\begin{aligned}D_N^* = \mathcal{O}(1/H) + \mathcal{O}_{r,\alpha,\beta,\varepsilon_1}(N^{-\theta+\varepsilon_1}) + \mathcal{O}_{\alpha,\beta,r,\rho,\theta,\varepsilon_2}((\log N)N^{(t-1)\theta-1+\varepsilon_2\theta}) \\ + \mathcal{O}_{\alpha,\beta,r,\rho,\theta,\varepsilon_3}\big((\log N)H^{(\rho+1)(t-1+\varepsilon_3)}N^{\theta(t-1+\varepsilon_3)-1}]\big).\end{aligned}$$

We take $H=[N^\theta]$ to balance the first two terms. Let $\varepsilon_4, \varepsilon_5 > 0$. We choose $r = r(t,\varepsilon_2,\varepsilon_4)$ as the minimal value such that $\rho = 1+\varepsilon_4$ is an admissible choice for (4), we use $\log N \ll_{\varepsilon_5} N^{\varepsilon_5}$, and arrive at

$$\begin{aligned}D_N^* = \mathcal{O}_{\alpha,\beta,\varepsilon_1,\varepsilon_2,\varepsilon_4}(N^{-\theta+\varepsilon_1}) + \mathcal{O}_{\alpha,\beta,\theta,\varepsilon_2,\varepsilon_4,\varepsilon_5}(N^{(t-1)\theta-1+\varepsilon_2\theta+\varepsilon_5}) \\ + \mathcal{O}_{\alpha,\beta,\theta,\varepsilon_2,\varepsilon_4,\varepsilon_3,\varepsilon_5}(N^{(2+\varepsilon_4)\theta(t-1+\varepsilon_3)+\theta(t-1+\varepsilon_3)-1+\varepsilon_5}).\end{aligned}$$

The latter exponent can be rewritten as

$$3\theta(t-1) - 1 + 3\theta\varepsilon_3 + \varepsilon_4\theta(t-1+\varepsilon_3) + \varepsilon_5.$$

We equate the first and the third exponent, ignore the epsilons, and obtain

$$3\theta(t-1) - 1 = -\theta.$$

By setting $\theta = 1/(3t-2)$ we arrive at

$$\begin{aligned}D_N^* = \mathcal{O}_{\alpha,\beta,\varepsilon_1,\varepsilon_2,\varepsilon_4}(N^{-1/(3t-2)+\varepsilon_1}) + \mathcal{O}_{\alpha,\beta,\varepsilon_2,\varepsilon_4,\varepsilon_5}(N^{-1/(3t-2)+\varepsilon_2/(3t-1)+\varepsilon_5}) \\ + \mathcal{O}_{\alpha,\beta,\varepsilon_2,\varepsilon_4,\varepsilon_3,\varepsilon_5}(N^{-1/(3t-2)+3\varepsilon_3/(3t-2)+\varepsilon_4(t-1+\varepsilon_3)/(3t-2)+\varepsilon_5}).\end{aligned}$$

Obviously, we can now choose for any $\varepsilon > 0$ a set of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$ depending on ε such that the right hand side of the equation above is

$$\mathcal{O}_{\alpha,\beta,\varepsilon}(N^{-1/(3t-2)+\varepsilon}),$$

and the proof of Theorem 1.1 is complete.

Proof of the corollaries. Corollary 1.2 easily follows from Theorem 1.1 and Example 1(1). The fact that $1, \alpha, \alpha\beta$ are linearly independent over $\mathbb{Q}$ immediately implies the linear independence of $1, 1/\alpha, \beta$. By Example 1(1) both pairs $(\alpha, \alpha\beta)$ and $(1/\alpha, \beta)$ are of finite type 1. The result follows from Theorem 1.1.

For Corollary 1.3, note that for nonzero rationals q_1 and q_2 satisfying $q_1+q_2 \neq 0$, Example 1(2) ensures that both pairs $(e^{q_1}, e^{q_1+q_2})$ and (e^{-q_1}, e^{q_2}) are of finite type 1. By Theorem 1.1 the desired discrepancy bound is valid.

Corollary 1.5 is an immediate consequence of Theorem 1.1, Lemma 2.5, and the trivial fact that the pair (γ, δ) of real numbers is of finite type t if the triple $(1, \gamma, \delta)$ has linear independence measure t.

In order to prove Corollary 1.4 we define a function $g : \mathbb{R}\setminus\{0\}\times\mathbb{R}\setminus\{0\} \to \{0,1\}$ by setting $g(\tau,\sigma) = 1$ when (τ,σ) is not of finite type 1 and $g(\tau,\sigma) = 0$ otherwise. From Example 1(2) we know that g is a Lebesgue measurable function and that $\lambda_2(g^{-1}(\{1\})) = 0$. We define two more functions from X to X with $X = \mathbb{R}\setminus\{0\}\times\mathbb{R}\setminus\{0\}$ by setting $f_1(\alpha,\beta) = (\alpha,\alpha\beta)$ and $f_2(\alpha,\beta) = (1/\alpha,\beta)$. It is easily verified that both f_1 and f_2 satisfy all conditions required in Lemma 2.6, which then yields

$$\lambda_2\big((g\circ f_1)^{-1}(\{1\})\big) = \lambda_2\big((g\circ f_2)^{-1}(\{1\})\big) = 0.$$

These equalities imply that for almost all $(\alpha,\beta)\in X$ in the sense of Lebesgue measure both $(\alpha,\alpha\beta)$ and $(1/\alpha,\beta)$ are of finite type 1. Such α and $\alpha\beta$ together with 1 are also linearly independent over the rationals. An application of Theorem 1.1 concludes the proof of Corollary 1.4.

5. Future research and unresolved questions. This section collects several unsolved problems which emerged during the investigation of the sequence $([n\alpha]\beta)_{n\ge 0}$.

Problem 1. *Improve* (1) *and find sharp estimates for the discrepancy of the sequence* $(n\beta_1, n\beta_2)_{n\ge 0}$, *where* (β_1,β_2) *is of finite type* t.

Problem 2. *Improve Corollary* 1.4 *and find for almost all* (α,β) (*in the sense of Lebesgue measure*) *a discrepancy bound for the sequence* $([n\alpha]\beta)_{n\ge 0}$ *for all* $\varepsilon > 0$ *of the form*

$$D_N \ll_{\alpha,\beta,\varepsilon} \frac{\mathrm{Log}^{d+\varepsilon} N}{N}$$

where d *is a positive number.*

Problem 3. *Investigate the discrepancy of further examples of generalized polynomials.*

Investigations concerning other arithmetical aspects, besides distribution modulo one, of the sequence $([n\alpha]\beta)_{n\ge 0}$ can be found in [8], where in particular it is shown that such a sequence is orthogonal to the Möbius function. This can be seen as the sequel of [6] where, following the breakthrough result of Vinogradov on primes [19], Davenport showed that

$$\sum_{n\le N} \mu(n)e(n\alpha) = o(N)$$

and, by some further analysis, inferred that the series $\sum_{n\ge 1}\mu(n)\{n\alpha\}/n$ converges. Correspondingly, we may investigate the convergence of the series $\sum_{n\ge 1}\mu(n)\{[n\alpha]\beta\}/n$ but we keep such an investigation for a later paper.

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