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Infinite games and chain conditions

by

Santi Spadaro (São Paulo)

Abstract. We apply the theory of infinite two-person games to two well-known problems in topology: Suslin's Problem and Arhangel'skii's problem on the weak Lindelöf number of the G_δ topology on a compact space. More specifically, we prove results of which the following two are special cases: 1) every linearly ordered topological space satisfying the game-theoretic version of the countable chain condition is separable, and 2) in every compact space satisfying the game-theoretic version of the weak Lindelöf property, every cover by G_δ sets has a continuum-sized subcollection whose union is G_δ-dense.

1. Introduction. Infinite games have been exploited in recent years to give partial answers to various important problems in general topology, including van Douwen's D-space problem (see [1]), Arhangel'skii's problem on the cardinality of Lindelöf spaces with G_δ points (see [14], [2]) and Bell, Ginsburg and Woods's problem on the cardinality of weakly Lindelöf first-countable regular spaces (see [6], [4]). We use them to give partial ZFC answers to Suslin's Problem and Arhangel'skii's question of whether in every compact space, a cover by G_δ sets has a continuum-sized subfamily with a G_δ-dense union.

It was already known to Cantor that the real line is the unique complete dense linear order without endpoints which is separable. In the first issue of Fundamenta Mathematicae, Suslin asked whether in this result separability could be replaced with the countable chain condition. Any counterexample to this assertion came to be known as a Suslin Line. The problem turned out to be independent of the usual axioms of ZFC: under MA_{ω_1} there are no Suslin Lines, and thus the answer to Suslin's question is yes. However, Suslin Lines can be found in certain models of set theory, for example under

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$V = L$. Various mathematicians have wondered whether there is a natural strengthening of the ccc which implies a positive answer to Suslin's Problem. Following this line, Knaster proved that every ordered continuum with the Knaster property is separable, and Shapirovskiĭ proved that every compact space with countable tightness and Shanin's condition is separable (see [15]).

Another strengthening of the ccc was suggested by Scheepers [13] and involves a two-person game in countably many moves: at inning $n < \omega$ player I chooses a maximal family of pairwise disjoint open sets $\mathcal{U}_n$, and player II picks an open set $U_n \in \mathcal{U}_n$. Player II wins if $\overline{\bigcup\{U_n : n < \omega\}} = X$. Let us use the term *playful ccc* for the property that player II has a winning strategy in this game. The name is justified by the fact that if X contains an uncountable (maximal) pairwise disjoint family of non-empty open sets, all player I has to do to win is choose that family at every inning. So player I has a winning strategy in every space which does not have the countable chain condition. Hence the playful ccc implies the usual ccc.

Daniels, Kunen and Zhou [7] proved that, unlike the ccc, the playful ccc is productive in ZFC. We show that every complete dense linear order with the playful ccc is separable.

The *weak Lindelöf number* of a topological space X (denoted $\mathrm{wL}(X)$) is defined as the minimum cardinal κ such that every open cover has a κ-sized subfamily with a dense union. A space is called *weakly Lindelöf* if it has countable weak Lindelöf number. Every Lindelöf space is clearly weakly Lindelöf, and it is not hard to prove that every space with the countable chain condition is weakly Lindelöf. Woods [16] used the weak Lindelöf property to characterize the C^*-embedded subsets of the Stone–Čech compactification of the integers under CH, and Bell, Ginsburg and Woods [5] exploited it in their elegant generalization of Arhangel'skii's theorem on the cardinality of compact first-countable spaces.

Given a topological space X, we indicate with X_δ the space whose underlying set is X and whose topology is generated by the G_δ sets of X. Arhangel'skii asked (see [10]) whether $\mathrm{wL}(X_\delta) \le 2^{\aleph_0}$ for every compact space X. This problem remains open.

Juhász [10] gave a partial positive answer by proving that $\mathrm{wL}(X_\delta) \le 2^{c(X)}$ for every compact space X (a related result was given in [11] for another chain-condition type cardinal invariant known as Noetherian type). His result is a consequence of the Erdős–Rado theorem from infinite combinatorics. In particular, Arhangel'skii's question has a positive answer for compact ccc spaces. Here we prove another partial positive result: if X is a countably compact space where player II has a winning strategy in the *weak Lindelöf game of length* ω_1 (see below for the definition) then $\mathrm{wL}(X_\delta) \le 2^{\aleph_0}$. Juhász's result follows from ours, but our proof uses no infinite combinatorics other than elementary counting arguments.

Let us recall some standard notation regarding games.

Given collections $\mathcal{A}$ and $\mathcal{B}$ of families of subsets of a topological space X, we indicate with $G_1^\kappa(\mathcal{A},\mathcal{B})$ (respectively $G_{\mathrm{fin}}^\kappa(\mathcal{A},\mathcal{B})$) the two-player game in κ many innings where at inning α player I plays $\mathcal{A}_\alpha \in \mathcal{A}$ and player II plays $A_\alpha \in \mathcal{A}_\alpha$ (respectively $\mathcal{F}_\alpha \in [\mathcal{A}]^{<\omega}$), and player II wins if $\{A_\alpha : \alpha < \kappa\} \in \mathcal{B}$ (respectively $\bigcup \mathcal{F}_\alpha \in \mathcal{B}$).

We denote by $\mathcal{C}^X$ the collection of all maximal families of pairwise disjoint non-empty open sets of X, and by $\mathcal{O}_D^X$ the collection of all open families with dense union. Obviously $\mathcal{C}^X \subset \mathcal{O}_D^X$. Moreover, $\mathcal{O}^X$ is the collection of all open covers of X. When there is no danger of ambiguity we will omit X from the superscript.

We call the game $G_1^\kappa(\mathcal{O},\mathcal{O}_D)$ the *weak Lindelöf game of length* κ.

In our proofs we will often use elementary submodels of the structure $(H(\mu),\epsilon)$. Dow's survey [8] on this topic is enough to read our paper, and we give a brief informal refresher here. Recall that $H(\mu)$ is the set of all sets whose transitive closure has cardinality smaller than μ. When μ is regular uncountable, $H(\mu)$ is known to satisfy all axioms of set theory except the power set axiom. We say, informally, that a formula is satisfied by a set S if it is true when all existential quantifiers are restricted to S. A set $M \subset H(\mu)$ is said to be an *elementary submodel* of $H(\mu)$ (and we write $M \prec H(\mu)$) if a formula with parameters in M is satisfied by $H(\mu)$ if and only if it is satisfied by M.

The downward Löwenheim–Skolem theorem guarantees that for every $S \subset H(\mu)$, there is an elementary submodel $M \prec H(\mu)$ such that $|M| \leq |S| \cdot \omega$ and $S \subset M$. This theorem is sufficient for many applications, but it is often useful (especially in cardinal bounds for topological spaces) to have the following closure property. We say that M is κ*-closed* if for every $S \subset M$ such that $|S| \leq \kappa$ we have $S \in M$. For large enough regular μ and for every countable set $S \subset H(\mu)$ there is always a κ-closed elementary submodel $M \prec H(\mu)$ such that $|M| = 2^\kappa$ and $S \subset M$.

The following theorem is also used frequently: Let $M \prec H(\mu)$ be such that $\kappa + 1 \subset M$, and $S \in M$ be such that $|S| \leq \kappa$. Then $S \subset M$.

Undefined notions can be found in [9] for topology and [12] for set theory.

2. Arhangel'skii's problem about G_δ covers in compact spaces. A game-theoretic version of the weak Lindelöf property can be obtained by considering the game $G_1^\kappa(\mathcal{O},\mathcal{O}_D)$. At inning $\alpha < \kappa$ player I chooses an open cover $\mathcal{U}_\alpha$, and player II chooses an open set $U_\alpha \in \mathcal{U}_\alpha$. Player II wins if $\overline{\bigcup\{U_\alpha : \alpha < \kappa\}} = X$. If player II has a winning strategy in $G_1^\kappa(\mathcal{O},\mathcal{O}_D)$, then $\mathrm{wL}(X) \leq \kappa$. But there are even compact spaces where player I has a winning strategy in $G_1^{\omega_1}(\mathcal{O},\mathcal{O}_D)$ (and hence player II cannot have a winning strategy in that game, see [3]).

In [4] we proved that the weak Lindelöf game is the dual of the *open-picking game*. It will be convenient to exploit this duality in the proof of our partial solution to Arhangel'skii's problem.

DEFINITION 2.1. The game $G_o^p(\kappa)$ is the two-player game in κ many innings defined as follows: at inning $\alpha < \kappa$, player I picks a point $x_\alpha \in X$ and player II chooses an open set U_α such that $x_\alpha \in U_\alpha$. Player I wins if $\overline{\bigcup\{U_\alpha : \alpha < \kappa\}} = X$.

LEMMA 2.2 ([4]).

(1) *Player I has a winning strategy in* $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$ *if and only if player II has a winning strategy in* $G_o^p(\kappa)$.
(2) *Player II has a winning strategy in* $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$ *if and only if player I has a winning strategy in* $G_o^p(\kappa)$.

Proof. We prove only the direct implication of (2), because it is the only one we will need in our proof of Theorem 2.3 below, and we refer the reader to [4] for the other implications.

Let σ be a winning strategy for player II in $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$ on some space X.

CLAIM. *Let* $(\mathcal{O}_\alpha : \alpha < \beta)$ *be a sequence of open covers, where* $\beta < \kappa$. *Then there is a point* $x \in X$ *such that for every neighbourhood* U *of* x *there is an open cover* $\mathcal{U}$ *with* $U = \sigma((\mathcal{O}_\alpha : \alpha < \beta)^\frown(\mathcal{U}))$.

Proof. Recalling that $\mathcal{O}$ denotes the set of all open covers of X, let $\mathcal{V} = \{V \text{ open}: (\forall \mathcal{U} \in \mathcal{O})(V \neq \sigma((\mathcal{O}_\alpha : \alpha < \beta)^\frown(\mathcal{U}))\}$. Its definition easily implies that $\mathcal{V}$ cannot be an open cover, and hence there is a point $x \in X \setminus \bigcup \mathcal{V}$. By definition of $\mathcal{V}$, for every neighbourhood U of x there is an open cover $\mathcal{U}$ such that $U = \sigma((\mathcal{O}_\alpha : \alpha < \beta)^\frown(\mathcal{U}))$, and hence we are done. $\triangle$

We are now going to define a winning strategy τ for player I in $G_o^p(\kappa)$.

Use the Claim to choose a point x_0 such that for every neighbourhood U of x_0 there is an open cover $\mathcal{U}$ with $\sigma((\mathcal{U})) = U$ and let $\tau(\emptyset) = x_0$.

Suppose we have defined τ for the first α many innings. Let now $\{V_\beta : \beta \leq \alpha\}$ be a sequence of open sets and $\{\mathcal{O}_\beta : \beta < \alpha\}$ be a sequence of open covers such that $V_\beta = \sigma((\mathcal{O}_\gamma : \gamma \leq \beta))$ for every $\beta < \alpha$. Use the Claim to choose a point x_α such that for every open neighbourhood U of x_α there is an open cover $\mathcal{O}$ with $U = \sigma((\mathcal{O}_\beta : \beta < \alpha)^\frown(\mathcal{O}))$ and let $\tau((V_\beta : \beta \leq \alpha)) = x_\alpha$.

We now claim that τ is a winning strategy for player I in $G_o^p(\kappa)$. Indeed, let $(x_0, V_0, x_1, V_1, \dots x_\alpha, V_\alpha, \dots)$ be a play where player I uses τ. Then there must be a sequence $\{\mathcal{O}_\alpha : \alpha < \kappa\}$ of open covers such that $V_\beta = \sigma((\mathcal{O}_\alpha : \alpha \leq \beta))$ for every $\beta < \kappa$. Since σ is a winning strategy for player II in $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$, we see that $\bigcup\{V_\alpha : \alpha < \kappa\}$ is dense in X, and this proves that τ is a winning strategy for player I in $G_o^p(\kappa)$. ∎

THEOREM 2.3. *Let κ be a regular uncountable cardinal and X be a countably compact regular space where player II has a winning strategy in $G_1^\kappa(\mathcal{O}, \mathcal{O}_D)$. Then* $\mathrm{wL}(X_\delta) \leq 2^{<\kappa}$.

Proof. Denote by ρ the set of all open subsets of X. Fix a winning strategy τ for player I in $G_o^p(\kappa)$ and let $\mathcal{U}$ be an open cover of X_δ. Since X is regular, we can assume without loss of generality that for every $U \in \mathcal{U}$ there are $\{U_n : n < \omega\} \subset \rho$ such that $\overline{U_{n+1}} \subset U_n$ for every $n < \omega$ and $U = \bigcap\{U_n : n < \omega\} = \bigcap\{\overline{U_n} : n < \omega\}$. Let M be a $<\kappa$-closed elementary submodel of $H(\theta)$ for large enough regular θ such that $X, \rho, \tau, \mathcal{U} \in M$, $|M| = 2^{<\kappa}$ and $2^{<\kappa} + 1 \subset M$. We claim that $\mathcal{U} \cap M$ is dense in X_δ. Suppose this is not the case and let V be an open subset of X_δ such that $V \cap \bigcup(\mathcal{U} \cap M) = \emptyset$. We can assume that $V = \bigcap\{V_n : n < \omega\}$, where $\{V_n : n < \omega\}$ is a family of open subsets of X such that $\overline{V_{n+1}} \subset V_n$ for every $n < \omega$.

CLAIM. *$X \cap M$ is countably compact.*

Proof. Let $A \subset X \cap M$ be a countable set. By countable compactness of X, the set A must have an accumulation point $x \in X$. In other words,

$$H(\theta) \models (\exists x \in X)(\forall U \in \rho)(x \in U \Rightarrow U \cap A \neq \emptyset).$$

By $<\kappa$-closedness of M we have $A \in M$, and hence, by elementarity,

$$M \models (\exists x \in X)(\forall U \in \rho)(x \in U \Rightarrow U \cap A \neq \emptyset).$$

This means that we can fix a point $p \in X \cap M$ such that for every neighbourhood U of p with $U \in M$ we have $U \cap A \neq \emptyset$. It follows that

$$M \models (\forall U \in \rho)(p \in U \Rightarrow U \cap A \neq \emptyset).$$

Hence, by elementarity,

$$H(\theta) \models (\forall U \in \rho)(p \in U \Rightarrow U \cap A \neq \emptyset).$$

So p is actually an accumulation point for A in the topology of X, and since $p \in X \cap M$, the Claim is proved. $\triangle$

For all $x \in X \cap M$ there is $B_x \in \mathcal{U} \cap M$ such that $x \in B_x$, and hence $B_x \cap V = \emptyset$. Since $B_x \in M$, there are $\{B_n^x : n < \omega\} \subset M \cap \rho$ such that $B_x = \bigcap_{n<\omega} B_n^x$ and $\overline{B_{n+1}^x} \subset B_n^x$ for every $n < \omega$. Fix $x \in X \cap M$. By countable compactness, taking into account that B_x and V are closed G_δ's and that $B_x \cap V = \emptyset$, we can find a positive integer $m(x)$ such that $B_{m(x)}^x \cap V_{m(x)} = \emptyset$. Let $\mathcal{B}_n = \{B_{m(x)}^x : m(x) = n\}$. Set $B_n = \bigcup \mathcal{B}_n$. Then $\{B_n : n < \omega\}$ is an open cover of $X \cap M$, and hence by the Claim there is an integer $k < \omega$ such that $\{B_n : n \leq k\}$ covers $X \cap M$. Let $\mathcal{B}' = \bigcup\{\mathcal{B}_n : n \leq k\} \subset M$ and note that $\mathcal{B}'$ is an open cover of $X \cap M$.

We are going to play a game of $G_0^p(\kappa)$ where player I uses τ and player II picks their moves inside $\mathcal{B}'$.

More precisely, in the first inning player I plays the point $x_0 = \tau(\emptyset)$. Since $\tau \in M$, we have $x_0 \in X \cap M$. Hence there is an open set $B_0 \in \mathcal{B}'$ such that $x_0 \in B_0$. Let $\alpha < \kappa$ and $B_\beta \in \mathcal{B}'$ be the open set played by player II at inning β, for every $\beta < \alpha$. Since $\alpha < \kappa$ and M is $<\kappa$-closed, we have $\{B_\beta : \beta < \alpha\} \in M$ and hence $x_\alpha = \tau((B_\beta : \beta < \alpha)) \in M$. So there is $B_\alpha \in \mathcal{B}'$ such that $x_\alpha \in B_\alpha$. Since τ is a winning strategy for player I, we must have $\overline{\bigcup\{B_\alpha : \alpha < \kappa\}} = X$, but this contradicts the fact that $B_\alpha \cap V_k = \emptyset$ for every $\alpha < \kappa$. ∎

Corollary 2.4. *Let X be a countably compact regular space where player II has a winning strategy in $G_1^{\omega_1}(\mathcal{O}, \mathcal{O}_D)$. Then* $\mathrm{wL}(X_\delta) \leq 2^{\aleph_0}$.

Let us now see how Juhász's result from [10] follows from Theorem 2.3.

Recall that $\hat{c}(X)$ is the minimal cardinal κ such that X does not have a κ-sized pairwise disjoint family of non-empty open sets.

Lemma 2.5. *Let X be any space. Then player II has a winning strategy in $G_1^{\hat{c}(X)}(\mathcal{O}, \mathcal{O}_D)$.*

Proof. Let $\hat{c}(X) = \kappa$. We describe the strategy by induction. Let $\beta < \kappa$ and suppose player II has picked the open set U_α at inning α for every $\alpha < \beta$. Suppose we have chosen open sets $\{V_\alpha : \alpha < \beta\}$ such that $\{U_\alpha \cap V_\alpha : \alpha < \beta\}$ is a pairwise disjoint family of non-empty open sets. If $\bigcup\{U_\alpha : \alpha < \beta\}$ is dense in X then player II has won, otherwise let V_β be a non-empty open set such that $V_\beta \cap \bigcup\{U_\alpha : \alpha < \beta\} = \emptyset$. Suppose at inning β player I chooses the open cover $\mathcal{O}_\beta$. Let $U_\beta \in \mathcal{O}_\beta$ be an open set such that $U_\beta \cap V_\beta \neq \emptyset$ and let player II pick U_β at inning β. If this could be carried on for κ many moves then $\{U_\alpha \cap V_\alpha : \alpha < \kappa\}$ would be a pairwise disjoint family of non-empty open sets having size κ, which contradicts $\hat{c}(X) = \kappa$. Therefore there must be $\beta < \kappa$ such that $\overline{\bigcup\{U_\alpha : \alpha < \beta\}} = X$. ∎

Corollary 2.6 (Juhász, [10]). *Let X be a compact Hausdorff space. Then* $\mathrm{wL}(X_\delta) \leq 2^{c(X)}$.

Question 2.7. *Let X be a countably compact regular space such that player II has a winning strategy in $G_1^{\omega_1}(\mathcal{C}, \mathcal{O}_D)$. Is then $c(X_\delta) \leq 2^{\aleph_0}$?*

We note that consistently, the above question has a positive answer.

Proposition 2.8. *Assume $2^{\aleph_0} = 2^{\aleph_1}$. Let X be a countably compact regular space such that player II has a winning strategy in $G_1^{\omega_1}(\mathcal{C}, \mathcal{O}_D)$. Then $c(X_\delta) \leq 2^{\aleph_0}$.*

Proof. It is easy to see that if $c(X) > \aleph_1$, then player I has a winning strategy in $G_1^{\omega_1}(\mathcal{C}, \mathcal{O}_D)$. It follows that $c(X) \leq \aleph_1$. Now, Juhász [10] proved that $c(Y_\delta) \leq 2^{c(Y)}$ for every countably compact regular space Y, and so $c(X_\delta) \leq 2^{\aleph_1} = 2^{\aleph_0}$. ∎

QUESTION 2.9. *Let X be a countably compact regular space such that player II has a winning strategy in $G^{\kappa}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}_D)$, where κ is an infinite regular cardinal. Is* $\mathrm{wL}(X_\delta) \le 2^{<\kappa}$*?*

A playful version of the weak Lindelöf property alternative to the one considered by us is the property that player I has no winning strategy in $G^{\omega}_1(\mathcal{O}, \mathcal{O}_D)$. Of course this is weaker than player II having a winning strategy in that game. We do not know whether it is enough to provide a positive answer to Arhangel'skii's question.

QUESTION 2.10. *Let X be a countably compact regular space such that player I has no winning strategy in $G^{\omega}_1(\mathcal{O}, \mathcal{O}_D)$. Is* $\mathrm{wL}(X_\delta) \le 2^{\aleph_0}$*?*

3. The Suslin Problem for the playful ccc. Recall that a *π-base* is a family $\mathcal{P}$ of non-empty open sets such that for every non-empty open set $U \subset X$ there is $P \in \mathcal{P}$ such that $P \subset U$. The *π-weight* of X (denoted $\pi w(X)$) is defined as the minimal size of a π-base for X.

A *local π-base* at $x \in X$ is a family $\mathcal{P}$ of non-empty open subsets of X such that for every open neighbourhood U of x there is $P \in \mathcal{P}$ such that $P \subset U$. The *local π-character* of x (denoted $\pi\chi(x, X)$) is defined as the minimum cardinality of a local π-base at x.

We are going to prove the following theorem.

THEOREM 3.1. *Let X be a regular space with a dense set of points of countable π-character. If player II has a winning strategy in $G^{\omega}_1(\mathcal{C}, \mathcal{O}_D)$ then X has a countable π-base.*

Before going to the proof, let us see how the announced partial ZFC solution to Suslin's Problem follows as a corollary.

LEMMA 3.2. *Let L be a complete dense linear order. Then L contains a dense set of points of countable π-character.*

Proof. Let (a, b) be a non-empty open interval. We claim that (a, b) contains a point of countable π-character. Let $c \in (a, b)$ and suppose we have constructed $\{x_i : i \le n\} \subset (a, c)$. Choose $x_{n+1} \in (x_n, c)$. Then $y = \sup\{x_n : n < \omega\} \in (a, b)$ and $\{(x_n, y) : n < \omega\}$ is a local π-base at y. ∎

COROLLARY 3.3. *Let X be a continuous linearly ordered topological space without isolated points such that player II has a winning strategy in the game $G^{\omega}_1(\mathcal{C}, \mathcal{O}_D)$. Then X has a countable π-base.*

Although we could give a more direct proof of Theorem 3.1, we think that the duality between $G^{\kappa}_1(\mathcal{C}, \mathcal{O}_D)$ and the open-open game makes this result a lot more transparent. We believe this duality has been known for some time, but since we could not find a proof in the literature, we provide one in Lemma 3.5 for the reader's convenience.

The *open-open game* of length κ (denoted $G_o^o(\kappa)$) is the two-player game where at inning $\alpha < \kappa$ player I chooses a non-empty open set $U_\alpha \subset X$ and player II chooses a non-empty open set $V_\alpha \subset U_\alpha$. Player I wins if $\overline{\bigcup\{V_\alpha : \alpha < \kappa\}} = X$ (see [7]).

First of all we note that the game $G_1^\kappa(\mathcal{C}, \mathcal{O}_D)$ is equivalent to the game $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$. The proof of the following proposition is routine.

PROPOSITION 3.4. *Player I* (*resp. player II*) *has a winning strategy in* $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$ *if and only if player I* (*resp. player II*) *has a winning strategy in* $G_1^\kappa(\mathcal{C}, \mathcal{O}_D)$.

Finally, we prove that the latter game is the dual of the open-open game.

LEMMA 3.5.

(1) *Player I has a winning strategy in* $G_o^o(\kappa)$ *if and only if player II has a winning strategy in* $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$.
(2) *Player II has a winning strategy in* $G_o^o(\kappa)$ *if and only if player I has a winning strategy in* $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$.

Proof. To prove the direct implication of (1), let τ be a winning strategy for player I in $G_o^o(\kappa)$. Given $\mathcal{O} \in \mathcal{O}_D$, let $\sigma((\mathcal{O}))$ be any open set $O \in \mathcal{O}$ such that $O \cap \tau(\emptyset) \neq \emptyset$.

Now, suppose we have defined σ for sequences of order type $\leq \alpha$ and let $(\mathcal{O}_\beta : \beta \leq \alpha)$ be a sequence of members of $\mathcal{O}_D$. Then just let $\sigma((\mathcal{O}_\beta : \beta \leq \alpha))$ be any open set $O \in \mathcal{O}_\alpha$ such that $\tau((\sigma((\mathcal{O}_\beta : \beta \leq \gamma)) : \gamma < \alpha)) \cap O \neq \emptyset$. We claim that σ is a winning strategy for player II in $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$. Indeed, let $(\mathcal{O}_0, O_0, \mathcal{O}_1, O_1, \ldots, \mathcal{O}_\alpha, O_\alpha, \ldots)$ be a play of $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$ where player II plays according to σ. Then the set $V_\alpha = \tau((\sigma((\mathcal{O}_\gamma : \gamma \leq \beta)) : \beta < \alpha)) \cap O_\alpha$ is non-empty. But since $V_\alpha \subset \tau((\sigma((\mathcal{O}_\gamma : \gamma \leq \beta)) : \beta < \alpha))$ and τ is a winning strategy for player I in $G_o^o(\kappa)$, the union $\bigcup_{\alpha<\kappa} V_\alpha$ must be dense. Hence $\bigcup_{\alpha<\kappa} O_\alpha$ is dense too, and we are done.

Conversely, let σ be a winning strategy for player II in $G_1^\kappa(\mathcal{O}_D, \mathcal{O}_D)$.

CLAIM. *Let* $\beta < \kappa$ *and* $\{\mathcal{O}_\alpha : \alpha < \beta\} \subset \mathcal{O}_D$. *Then there is an open set* V *such that for every* $U \subset V$ *there is* $\mathcal{O} \in \mathcal{O}_D$ *with* $U = \sigma((\mathcal{O}_\alpha : \alpha < \beta)^\frown(\mathcal{O}))$.

Proof. Let $\mathcal{U} = \{U \text{ open: } (\forall \mathcal{O} \in \mathcal{O}_D)(U \neq \sigma((\mathcal{O}_\alpha : \alpha < \beta)^\frown(\mathcal{O}))\}$. By definition $\mathcal{U} \notin \mathcal{O}_D$, so there must be a non-empty open set V such that $V \cap \bigcup \mathcal{U} = \emptyset$. By definition of $\mathcal{U}$, for every $U \subset V$ open there must also be an $\mathcal{O} \in \mathcal{O}_D$ such that $U = \sigma((\mathcal{O}_\alpha : \alpha < \beta)^\frown(\mathcal{O}))$, as we wanted. ■

Now, use the Claim to choose an open set V_0 such that for all $U \subset V_0$ there is $\mathcal{O} \in \mathcal{O}_D$ such that $\sigma((\mathcal{O})) = U$, and let $\tau(\emptyset) = V_0$.

Suppose we have defined τ for every sequence of order type $\leq \alpha$. Let $\{U_\beta : \beta \leq \alpha\}$ be a sequence of open sets and $\{\mathcal{O}_\beta : \beta \leq \alpha\}$ be a sequence of elements of $\mathcal{O}_D$ such that $U_\beta = \sigma((\mathcal{O}_\gamma : \gamma \leq \beta))$. Use the Claim to choose an open set V_α such that for every open set $U \subset V_\alpha$ there is $\mathcal{O} \in \mathcal{O}_D$ such that $U = \sigma((\mathcal{O}_\beta : \beta < \alpha)^\frown(\mathcal{O}))$, and define $\tau((U_\beta : \beta \leq \alpha))$ to be V_α. We claim that τ is a winning strategy for player I in $G^o_o(\kappa)$.

Indeed, let $(V_0, U_0, V_1, U_1, \ldots, V_\alpha, U_\alpha, \ldots)$ be a play of $G^o_o(\kappa)$ where player I plays according to τ. Then there must be a sequence $(\mathcal{O}_\alpha : \alpha < \kappa)$ of elements of $\mathcal{O}_D$ such that $U_\beta = \sigma((\mathcal{O}_\alpha : \alpha \leq \beta))$ for all $\beta < \kappa$. Since σ is a winning strategy for player II in $G^\kappa_1(\mathcal{O}_D, \mathcal{O}_D)$, we see that $\bigcup_{\beta<\kappa} U_\beta$ is dense and this proves that τ is a winning strategy for player I in $G^o_o(\kappa)$.

To prove the direct implication of (2), let τ be a winning strategy for player II in $G^o_o(\kappa)$. Denote by ρ the set of all open sets of X. We first let $\sigma(\emptyset) = \{\tau(O) : O \in \rho\}$. Now suppose we have defined σ for sequences of order type $\leq \alpha$ in such a way that if U_β is the open set played by player II in the βth inning, then there are open sets $\{V_\beta : \beta \leq \alpha\}$ with $U_\beta = \tau((V_\gamma : \gamma \leq \beta))$. We simply define $\sigma((U_\beta : \beta \leq \alpha))$ to be $\{\tau((V_\gamma : \gamma \leq \alpha)^\frown(O)) : O \in \rho\}$. We now check that σ is a winning strategy for player I in $G^\kappa_1(\mathcal{O}_D, \mathcal{O}_D)$. Let $\{\mathcal{O}_0, U_0, \mathcal{O}_1, U_1, \ldots, \mathcal{O}_\alpha, U_\alpha, \ldots\}$ be a game of $G^\kappa_1(\mathcal{O}_D, \mathcal{O}_D)$ where player I uses the strategy σ. So we can find a sequence $\{V_\beta : \beta < \kappa\}$ of open sets such that $U_\alpha = \tau((V_\beta : \beta \leq \alpha))$ and hence $\bigcup\{U_\alpha : \alpha < \kappa\}$ is not dense, since τ is a winning strategy for player II in $G^o_o(\kappa)$. So σ is a winning strategy for player I in $G^\kappa_1(\mathcal{O}_D, \mathcal{O}_D)$.

To prove the converse implication of (2), let σ be a winning strategy for player I in $G^\kappa_1(\mathcal{O}_D, \mathcal{O}_D)$. Given an open set U, let $V \in \sigma(\emptyset)$ be any open set such that $U \cap V \neq \emptyset$ and set $\tau((U)) = U \cap V$. Suppose τ has been defined for all sequences of open sets of order type $\leq \alpha$. Given $(U_\beta : \beta \leq \alpha) \subset \rho$, let V be any open set $V \in \sigma((\tau((U_\gamma : \gamma \leq \beta)) : \beta < \alpha))$ such that $U_\alpha \cap V \neq \emptyset$ and set $\tau((U_\beta : \beta \leq \alpha)) = U_\alpha \cap V$. We claim that τ thus defined is a winning strategy for player II in $G^o_o(\kappa)$. Indeed, let $(V_0, U_0, V_1, U_1, \ldots, V_\alpha, U_\alpha, \ldots)$ be a play where player II plays according to τ. Then, for every $\alpha < \kappa$, there is $G_\alpha \in \sigma((\tau((U_\gamma : \gamma \leq \beta)) : \beta < \kappa))$ such that $U_\alpha \subset G_\alpha$. Now $\bigcup_{\alpha<\kappa} G_\alpha$ is not dense because σ is a winning strategy for player I in $G^\kappa_1(\mathcal{O}_D, \mathcal{O}_D)$ and hence $\bigcup\{U_\alpha : \alpha < \kappa\}$ cannot be dense either. Therefore τ is a winning strategy for player II in $G^o_o(\kappa)$, and we are done. ∎

THEOREM 3.6. *Let κ be an infinite regular cardinal and X be a regular space with a dense set of points of π-character $\leq 2^{<\kappa}$ where player I has a winning strategy in $G^o_o(\kappa)$. Then $\pi\mathrm{w}(X) \leq 2^{<\kappa}$.*

Proof. Denote by ρ the set of all open sets of X. Fix a winning strategy τ for player I in $G^o_o(\kappa)$ and let $D = \{x \in X : \pi\chi(x, X) \leq 2^{<\kappa}\}$. By assumption, the set D is dense in X. Let θ be a large enough regular cardinal and M be a $<\kappa$-closed elementary submodel of $H(\theta)$ such that $X, \rho, \tau, D \in M$, $|M| = 2^{<\kappa}$ and $2^{<\kappa} + 1 \subset M$.

We claim that $D \cap M$ is dense in X. Suppose this is not the case. Then there is an open set $G \subset X$ such that $\overline{G} \cap D \cap M = \emptyset$. Note that nevertheless $D \cap M$ is dense in the possibly coarser topology generated by $\rho \cap M$. Now, the first move of player I, $\tau(\emptyset)$, is an open set belonging to M, and thus we can fix a point $x_1 \in \tau(\emptyset) \cap D \cap M$. Let U_1 be an open neighbourhood of x_1 such that $U_1 \cap G = \emptyset$. Let $\mathcal{P}(x_1) \in M$ be a local π-base at x_1 having size $2^{<\kappa}$. We actually have $\mathcal{P}(x_1) \subset M$, so we can find $P_1 \in M$ such that $P_1 \subset U_1 \cap \tau(\emptyset)$. We let player II choose P_1 in their first move.

Let $\beta < \kappa$ and suppose that player II has picked open sets $\{P_\alpha : \alpha < \beta\} \subset M$ with $P_\alpha \cap G = \emptyset$. By $<\kappa$-closedness of M we have $\{P_\alpha : \alpha < \beta\} \in M$, and since $\tau \in M$ we have $\tau((P_\alpha : \alpha < \beta)) \in M$. Hence we can find a point $x_\beta \in D \cap M \cap \tau((P_\alpha : \alpha < \beta))$. Let U_β be an open neighbourhood of x_β disjoint from G. Let $\mathcal{P}(x_\beta) \in M$ be a local π-base at x_β having size $2^{<\kappa}$. We actually have $\mathcal{P}(x_\beta) \subset M$, and hence we can fix an open set $P_\beta \in M$ such that $P_\beta \subset \tau((P_\alpha : \alpha < \beta)) \cap U_\beta$. We let player II pick P_β in the βth inning.

Since τ is a winning strategy for player I in $G^o_o(\kappa)$, it must be the case that $\bigcup\{P_\alpha : \alpha < \kappa\}$ is dense in X; this contradicts the fact that $P_\alpha \cap G = \emptyset$ for every $\alpha < \kappa$.

Therefore $D \cap M$ is dense in X, and hence X has a $2^{<\kappa}$-sized dense set of points of π-character $2^{<\kappa}$. Now, putting together the $2^{<\kappa}$-sized π-bases at all points of $D \cap M$ one gets a $2^{<\kappa}$-sized (global) π-base for our space. ∎

COROLLARY 3.7. *Let X be a regular space with a dense set of points of countable π-character where player I has a winning strategy in $G^o_o(\omega)$. Then X has a countable π-base.*

QUESTION 3.8. *Let X be a space with a dense set of points of countable π-character were player II has a winning strategy in $G^\omega_{\rm fin}(\mathcal{C}, \mathcal{O}_D)$. Is it true that X has a countable π-base?*

We note that the property that player I has no winning strategy in $G^\omega_1(\mathcal{O}_D, \mathcal{O}_D)$ is not strong enough to imply a positive ZFC answer to Suslin's Problem. Indeed, Scheepers noted in [13] that a Suslin Line has the property that for every sequence $\{\mathcal{U}_n : n < \omega\}$ of maximal pairwise disjoint families of non-empty open sets we can choose $U_n \in \mathcal{U}_n$, for every $n < \omega$, such that $\bigcup\{U_n : n < \omega\}$ is dense in X, and that this is equivalent to player I not having a winning strategy in $G^\omega_1(\mathcal{C}, \mathcal{O}_D)$.

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Santi Spadaro
Instituto de Matemática e Estatística (IME-USP)
Universidade de São Paulo
Rua do Matão, 1010 – Cidade Universitária
05508-090 São Paulo – SP, Brazil
E-mail: santidspadaro@gmail.com