

Point networks for special subspaces of \mathbb{R}^n

by

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Abstract. Uniform characterizations of certain special subspaces of products of lines are presented. The characterizations all involve a collection of subsets (base, almost sub-base, network or point network) organized by a directed set. New characterizations of Eberlein, Talagrand and Gul'ko compacta follow.

Introduction. The purpose of this paper is to give uniform characterizations of certain special subspaces of products of lines that arise in analysis. The characterizations all involve two objects: first an order, and second a collection of subsets of the space 'organized' by the order. For the collection of subsets we take bases, almost subbases, networks and, most importantly, point networks. As a result we derive some new (and re-derive some old) characterizations of Eberlein, Gul'ko and Talagrand compacta. The characterizations in terms of point networks yield new clean proofs that the continuous image of a compact space which is Eberlein, Gul'ko or Talagrand has the same respective property. Rudin's original proof [BRW] that the continuous image of an Eberlein compact space is Eberlein compact is widely acknowledged to be involved. (The referee notes that a non-trivial bridge from the definition of Eberlein still relies on the Amir–Lindenstrauss theorem [AL].)

An important connection between topology and functional analysis was made by Gruenhage [G] when he characterized Eberlein compacta as being those compacta X such that $X^2 \setminus \Delta$ is σ -metacompact. Order theory was added to the mix by Garcıa, Oncina and Orihuela [GOO] when they characterized Gul'ko and Talagrand compacta in a similar fashion. Let P be a directed set and X a space (all spaces are assumed to be Tychonoff,

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unless otherwise stated). A collection \mathcal{C} of subsets of X is *P-point finite* if $\mathcal{C} = \bigcup\{\mathcal{C}_p : p \in P\}$ where (i) if $p \leq p'$ then $\mathcal{C}_p \subseteq \mathcal{C}_{p'}$, and (ii) each \mathcal{C}_p is point finite. Then X is *P-metacompact* if every open cover has a *P-point finite* open refinement. The results of Gruenhage and of García et al. can then be phrased: a compact space X is (i) Eberlein, (ii) Talagrand, or (iii) Gul'ko if and only if $X^2 \setminus \Delta$ is, respectively, (i)' \mathbb{N} -metacompact, (ii)' $\mathcal{K}(\omega^\omega)$ -metacompact, or (iii)' $\mathcal{K}(M)$ -metacompact for some separable metrizable space M . Here $\mathcal{K}(M)$ is the set of all compact subsets of M ordered by reverse inclusion. (The second author and Morgan have introduced and studied *P-paracompact* and *P-metrizable* spaces (see [FGM, GM1]). A typical result is that if X is compact and $X^2 \setminus \Delta$ is *P-paracompact*, where P is a $\mathcal{K}(M)$, or more generally of calibre (ω_1, ω) , then X is metrizable. A directed set has *calibre* (ω_1, ω) if every uncountable subset contains an infinite unbounded subset.)

We define a space X to be *P-Eberlein compact* if it is compact and $X^2 \setminus \Delta$ is $(P \times \mathbb{N})$ -metacompact. In this terminology '1-Eberlein compact' (and ' \mathbb{N} -Eberlein compact') is equivalent to 'Eberlein compact', while ' $\mathcal{K}(\omega^\omega)$ -Eberlein compact' is 'Talagrand compact', and a space is 'Gul'ko compact' if and only if it is ' $\mathcal{K}(M)$ -Eberlein compact' for some $\mathcal{K}(M)$, where M is separable metrizable (see [GMa] for more on the order structure of $\mathcal{K}(M)$'s).

In Section 1 we develop a cycle of implications which results in the characterization of *P-Eberlein compacta*—for completely arbitrary P —in terms of almost subbases, bases, networks and point networks. These specialize, when taking $P = \mathbb{N}$, $P = \mathcal{K}(\omega^\omega)$ and $P = \mathcal{K}(M)$, to characterizations of Eberlein, Talagrand and Gul'ko compacta—some known, some new. The cycle starts with a topological characterization of certain function spaces via *P-point finite* almost subbases. From these almost subbases we extract a base. This base has a more complex order property (*P-additively Noetherian*) than the corresponding almost subbase. We generalize from bases to (expandable) networks, and then to point networks. Next we show that point networks (with certain order properties) are pleasantly stable under finite products, subspaces and perfect images. We also show that spaces with a point network, with suitable order conditions, are *P-metacompact*. This allows us to complete the cycle—given a compact space X with a *P-additively Noetherian* point network, we deduce that $X^2 \setminus \Delta$ is *P-metacompact* (in other words, is *P-Eberlein compact*), and modify the argument of García et al. to show that such an X embeds in a suitable function space, namely one with a *P-point finite* almost subbase.

Attempting to characterize Corson compacta in similar terms (via a base, network etc.) we note that much of the above cycle works when 'compact' is weakened to 'Lindelöf'. Unfortunately the relevant order conditions are not

(necessarily) productive, and so characterizations can only be obtained by putting conditions on X^2 rather than X . We also explain why, even though our results hold for *all* directed sets P , when P is calibre (ω_1, ω) then P -Eberlein compact spaces are Corson compact, but when P is not calibre (ω_1, ω) then any compact space of weight no more than ω_1 is P -Eberlein compact. It follows that ' P -Eberlein compact for P calibre (ω_1, ω) ' form an interesting class of Corson compacts naturally extending the familiar classes of Eberlein, Talagrand and Gul'ko compacta. The second author and Morgan [GM2] have shown that there is a wide variety of directed sets with calibre (ω_1, ω) which are not of the form $\mathcal{K}(M)$ for some separable metrizable M . In particular it is shown that arbitrary Σ -products of $\mathcal{K}(M)$'s have calibre (ω_1, ω) .

As noted above, in our cycle of results in Section 1 the move from 'almost subbase' to 'base' results in an increase of complexity in the relevant order condition. We see no way of avoiding this increase in general, but, in Section 2, we show how to reduce the complexity of the associated networks and point networks in certain cases. These results give new characterizations of Gul'ko and Talagrand compacta in terms of small networks and point networks. We conclude with a brief discussion of the results in [GOO] pertaining to networks for Gul'ko and Talagrand compacta, explaining that there is a critical error in those results, and the authors of that paper have withdrawn their claim to characterizing Talagrand and Gul'ko compacta in terms of networks.

Partial order preliminaries. All our partially ordered sets, P , will be directed (given p, q in P there is an r in P such that $r \geq p$ and $r \geq q$). It turns out that what is important about our directed sets happens cofinally. So we compare directed sets P and Q via the Tukey order. A map $\phi : P \rightarrow Q$ is a *Tukey quotient* if ϕ is order-preserving and $\phi(P)$ is cofinal in Q . Write $P \geq_T Q$ if there is a Tukey quotient of P to Q . Then P and Q are *Tukey equivalent*, denoted $P =_T Q$, if $P \geq_T Q$ and $Q \geq_T P$. We note that $P \geq_T P \times \mathbb{N}$ unless P is countably directed (every countable subset of P has an upper bound).

The results in the first section of this paper apply to *all* directed sets P . In the second section we improve the characterizations obtained in the first part by using specific properties of the partial orders associated with Eberlein, Talagrand and Gul'ko compacta—respectively: the natural numbers, \mathbb{N} , the product order on $\mathbb{N}^{\mathbb{N}}$, and the set of compact subsets, $\mathcal{K}(M)$, ordered by inclusion, of a separable metrizable space M . It is now well-known that $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) =_T \mathbb{N}^{\mathbb{N}} \geq_T \mathbb{N} \geq_T 1$; for compact M , $\mathcal{K}(M) =_T 1$; for locally compact separable metrizable M , $\mathcal{K}(M) \geq_T \mathbb{N}$; and for any non-locally compact separable metrizable M , we have $\mathcal{K}(M) \geq_T \mathcal{K}(\mathbb{N}^{\mathbb{N}})$.

As stated above, given a directed set P , a collection \mathcal{C} of subsets of a space X is said to be P -ordered if we can write $\mathcal{C} = \bigcup\{\mathcal{C}_p : p \in P\}$ so that $p \leq p'$ implies $\mathcal{C}_p \subseteq \mathcal{C}_{p'}$. Observe that if ϕ is a Tukey quotient from Q to P , then \mathcal{C} is also a Q -ordered cover; indeed, we can write $\mathcal{C} = \bigcup\{\mathcal{C}'_q : q \in Q\}$ where $\mathcal{C}'_q = \mathcal{C}_{\phi(q)}$. Further, if \mathcal{Q} is a property that a collection of subsets of X might have, then we say that the collection \mathcal{C} is P - \mathcal{Q} if it can be P -ordered as $\mathcal{C} = \bigcup\{\mathcal{C}_p : p \in P\}$ so that every subcollection \mathcal{C}_p has property \mathcal{Q} .

Topological preliminaries. Our topological notation is standard (see [E] for example) except we introduce the following uniform method for dealing with compactness and the Lindelöf property. Let κ be a cardinal. We say that a space Y is $<\kappa$ -compact if every open cover of Y has a subcover of size $<\kappa$. Then ' $<\aleph_0$ -compact' is standard compactness and ' $<\aleph_1$ -compact' is the Lindelöf property. Other less well known terms (such as 'almost subbase' and 'point network') will be defined below when it becomes appropriate. As mentioned above, all spaces are assumed to be Tychonoff.

1. Almost subbases, bases, networks and point networks

Almost subbases. Almost subbases were introduced by Dimov [D1] who used them to characterize the subspaces of Eberlein compacta. We say that a family α of subsets of a space X is an *almost subbase* of X , with respect to a family $\{f_V : X \rightarrow [0, 1]\}_{V \in \alpha}$ of continuous functions, if $V = f_V^{-1}(0, 1]$ for $V \in \alpha$ and the family $\alpha \cup \{X \setminus f_V^{-1}[1/n, 1] : V \in \alpha, n \in \mathbb{N}\}$ is a subbase of X . An almost subbase of X is a family α for which such a family of functions exists. In constructing these functions the following lemma is helpful.

LEMMA 1.1 ([D1]). *Let $(C_n)_{n=1}^\infty$ be a sequence of zero sets such that for each n there exists a cozero set U_n with $C_n \subseteq U_n \subseteq C_{n+1}$. Then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $C_n = f^{-1}[1/n, 1]$ for $n \in \mathbb{N}$.*

A family α of subsets of a space X is F -separating if whenever x and y are distinct points in X then there is a V in α such that x is in V but y is not in \overline{V} , or vice versa ($y \in V$ but $x \notin \overline{V}$). A given almost subbase need not be F -separating. The next lemma allows us to find one which is—and to do so in a manner which respects order.

LEMMA 1.2. *Let α be a P -point- $<\kappa$ almost subbase of X with respect to $\{f_V\}_{V \in \alpha}$. Then:*

- (a) *The family $\alpha' = \{f_V^{-1}(r, 1] : V \in \alpha, r \in \mathbb{Q} \cap [0, 1]\}$ is F -separating and is a $(P \times \mathbb{N})$ -point- $<\kappa$ almost subbase of X .*
- (b) *Given a subspace A of X , the family $\{V \cap A, V \in \alpha\}$ is a P -point- $<\kappa$ almost subbase of A with respect to $\{f_{V|_A}\}_{V \in \alpha}$.*

Proof. Part (b) is clear. For (a) note that $\alpha = \{f_V^{-1}(0, 1] : V \in \alpha\}$, so α is a subset of α' . Hence α' is an almost subbase. It is straightforward to check (and observed in [D2]) that α' is F -separating. We know α can be P -ordered, say as $\alpha = \bigcup\{\alpha_p : p \in P\}$, where every α_p is point- $<\kappa$. Fix an enumeration $\{q_n\}_{n \in \mathbb{N}}$ of $[0, 1] \cap \mathbb{Q}$. For p in P and n in \mathbb{N} , set $\alpha'_{p,n} = \{f_V^{-1}(q_i, 1] : i = 1, \dots, n, V \in \alpha_p\}$. It is clear that $\alpha' = \bigcup\{\alpha'_{p,n} : (p, n) \in P \times \mathbb{N}\}$, and $\alpha'_{p,n} \subseteq \alpha'_{p',n'}$ when $(p, n) \leq (p', n')$. So we have a $(P \times \mathbb{N})$ -ordering of α' . Further, for a fixed p , we know that α'_p is point- $<\kappa$. So for a fixed i the collection $\{f_V^{-1}(q_i, 1] : V \in \alpha_p\}$ is also point- $<\kappa$. Hence, for any p in P and n in \mathbb{N} , $\alpha'_{p,n}$ is point- $<\kappa$, as a finite union of point- $<\kappa$ collections. ■

Almost subbases characterize embeddings. For any free filter \mathcal{F} on an infinite set X , write $X(\mathcal{F})$ for the space with underlying set $X \cup \{*\}$, and topology where the points of X are isolated and neighborhoods of $*$ are $\{*\} \cup U$ for U in \mathcal{F} . We allow the possibility that $\mathcal{F} = \mathbb{P}(X)$, in which case $X(\mathcal{F})$ has the discrete topology. Write $A(\kappa)$ for the case $X = \kappa$, an infinite cardinal, and $\mathcal{F} = \{\kappa \setminus F : F \text{ is a finite subset of } \kappa\}$. It is the one-point compactification of a discrete space of size κ . Write $L(\kappa)$ for the case $X = \kappa$ and $\mathcal{F} = \{\kappa \setminus C : C \text{ is a countable subset of } \kappa\}$. It is the one-point Lindelöfication of a discrete space of size κ .

Recall that for any space X , $C_p(X)$ is the subspace of \mathbb{R}^X consisting of all continuous $f : X \rightarrow \mathbb{R}$. When $X = X(\mathcal{F})$, write $C_p(X, *)$ for all continuous $f : X \rightarrow \mathbb{R}$ such that $f(*) = 0$. It is easy to check that $C_p(X(\mathcal{F}))$ and $C_p(X(\mathcal{F}), *)$ are homeomorphic. Then it is clear that the standard Σ -product of κ many lines, $\Sigma\mathbb{R}^\kappa$, is homeomorphic to $C_p(L(\kappa))$, while the standard Σ_* -product of κ many lines is homeomorphic to $C_p(A(\kappa))$. By definition a compact space is Corson compact if and only if it embeds in some Σ -product, and so if and only if it embeds in some $C_p(L(\kappa))$. It is well known that a compact space is Eberlein compact if and only if it embeds in some Σ_* -product, and so if and only if it embeds in some $C_p(A(\kappa))$. Mercourakis [M] and Sokolov [S] have shown that a compact space is Talagrand (respectively, Gul'ko) compact if and only if it embeds in some $C_p(X(\mathcal{F}))$ where $X(\mathcal{F})$ has an $\mathbb{N}^{\mathbb{N}}$ -ordered cover by compact sets (respectively, where $X(\mathcal{F})$ has a $\mathcal{K}(M)$ -ordered compact cover for some separable metrizable space M). Note that if K is a compact subset of $X(\mathcal{F})$ and $f \in C_p(X(\mathcal{F}))$ then $f \upharpoonright K$ is in $c_0(K)$, where $c_0(K)$ has the pointwise topology.

THEOREM 1.3. *Let P be a directed set and κ an infinite cardinal.*

- (1) *Every space Y which embeds in some $C_p(X(\mathcal{F}))$ where $X(\mathcal{F})$ has a P -ordered cover by $<\kappa$ -compact sets has a $(P \times \mathbb{N})$ -point- $<\kappa$ almost subbase.*

- (2) Every space Y with a P -point- $<\kappa$ almost subbase embeds in some $C_p(X(\mathcal{F}))$ where $X(\mathcal{F})$ has a $(P \times \mathbb{N})$ -ordered cover by $<\kappa$ -compact sets.

If P is not countably directed then ‘ $P \times \mathbb{N}$ ’ can be replaced simply by ‘ P ’ in both (1) and (2).

Proof. For claim (1) suppose $\mathcal{C} = \{C_p : p \in P\}$ is a P -ordered cover of $X(\mathcal{F})$ by $<\kappa$ -compact sets. By Lemma 1.2 it will then suffice to show that $C_p(X(\mathcal{F}), *)$ has a $(P \times \mathbb{N})$ -point- $<\kappa$ almost subbase.

Let $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ be a countable base of $\mathbb{R} \setminus \{0\}$ consisting of open intervals such that $B_n \subseteq \mathbb{R} \setminus [-1/n, 1/n]$. For points x_1, \dots, x_m in $X(\mathcal{F})$ and subsets S_1, \dots, S_m of \mathbb{R} , let $O(x_1, \dots, x_m; S_1, \dots, S_m) = \{f \in C_p(X(\mathcal{F}), *) : f(x_i) \in S_i \text{ for } i = 1, \dots, m\}$. Then $O(x_1, \dots, x_m; S_1, \dots, S_m)$ is a cozero set (resp. a zero set) in $C_p(X(p), *)$ if all the intervals S_i are open (resp. closed) in \mathbb{R} .

For any $p \in P$ and $n \in \mathbb{N}$, define $\mathcal{O}_{p,n} = \{O(x_1, \dots, x_m; S_1, \dots, S_m) : x_i \in C_p \text{ and each } S_i \text{ is either one of } B_1, \dots, B_n \text{ or } (-\infty, -1/j) \text{ or } (1/j, \infty) \text{ for } j = 1, \dots, n\}$. Note that if (p, n) and (p', n') are in $P \times \mathbb{N}$ and $(p, n) \leq (p', n')$ (that is, $p \leq p'$ and $n \leq n'$) then $\mathcal{O}_{p,n} \subseteq \mathcal{O}_{p',n'}$. We will verify that every family $\mathcal{O}_{p,n}$ is point- $<\kappa$, and hence $\mathcal{O} = \bigcup \{\mathcal{O}_{p,n} : p \in P \text{ and } n \in \mathbb{N}\}$ is $(P \times \mathbb{N})$ -point- $<\kappa$.

Suppose, for contradiction, $\mathcal{O}_{p,n}$ is not point- $<\kappa$ for some $p \in P$ and $n \in \mathbb{N}$, in other words there is an $f \in C_p(X(\mathcal{F}), *)$ and $\{O_\beta : \beta < \kappa\} \subseteq \mathcal{O}_{p,n}$ such that $f \in O_\beta$ for each $\beta < \kappa$. Since $\kappa \geq \aleph_0$, there must be a κ -sized subset $\{x_\gamma : \gamma < \kappa\} \subseteq C_p$ such that $f(x_\gamma) \in B$ for some B from either B_1, \dots, B_n , or $(-\infty, -1/j)$ or $(1/j, \infty)$ with $j = 1, \dots, n$. Hence $|f(x_\gamma)| > 1/n$ for each $\gamma < \kappa$, which contradicts the fact that C_p is $<\kappa$ -compact.

We now show that \mathcal{O} is an almost subbase of $C_p(X(\mathcal{F}), *)$. Take any $V = O(x_1, \dots, x_m; S_1, \dots, S_m)$ in \mathcal{O} , and represent each $S_i = (a_i, b_i)$ as a union of a family of intervals $S_{i,n} = [a_i + 1/n, b_i - 1/n]$ (where $-\infty + 1/n = -\infty$ and $\infty - 1/n = \infty$). Clearly, $S_{i,n} \subseteq U_n \subseteq S_{i,n+1}$ for some open interval U_n , and hence by Lemma 1.1 there exists a continuous function f_V on $C_p(X(\mathcal{F}), *)$, with values in $[0, 1]$, such that $f_V^{-1}[1/n, 1] = O(x_1, \dots, x_m; S_{1,n}, \dots, S_{m,n})$ for all $n \in \mathbb{N}$.

Let us verify that $\mathcal{O} \cup \{C_p(X(\mathcal{F}), *) \setminus f_V^{-1}[1/n, 1] : V \in \mathcal{O}, n \in \mathbb{N}\}$ is a subbase of $C_p(X(\mathcal{F}), *)$. Take any $f \in C_p(X(\mathcal{F}), *)$ and basic open neighborhood of f , namely, $O(x_1, \dots, x_k : I_1, \dots, I_k)$ with I_1, \dots, I_k being open intervals. Then there exists $p \in P$ such that $x_1, \dots, x_k \in C_p$. Relist the points $\{x_1, \dots, x_k\}$ as $\{x_1, \dots, x_{\ell_1}\} \cup \{y_1, \dots, y_{\ell_2}\}$ such that $f(x_i) \neq 0$ for $i = 1, \dots, \ell_1$ and $f(y_j) = 0$ for $j = 1, \dots, \ell_2$. Relist the intervals I_1, \dots, I_k as I_1, \dots, I_{ℓ_1} and $\hat{I}_1, \dots, \hat{I}_{\ell_2}$ so that $f(x_i) \in I_i$ and $0 = f(y_j) \in \hat{I}_j$ for $i = 1, \dots, \ell_1$ and $j = 1, \dots, \ell_2$.

Choose N large enough that: (i) there exist $B_{i_1}, \dots, B_{i_{\ell_1}} \in \{B_1, \dots, B_N\}$ with $x_j \in B_{i_j} \subseteq I_j$ for $j = 1, \dots, \ell_1$; and (ii) $\mathbb{R} \setminus ((-\infty, -1/N - 1/N] \cup [1/N + 1/N, \infty)) \subseteq \hat{I}_j$ for $j = 1, \dots, \ell_2$. Let $V = O(x_1, \dots, x_{\ell_1}; B_{i_1}, \dots, B_{i_{\ell_1}})$, $V_j = O(y_j; (1/N, \infty))$ for $j = 1, \dots, \ell_2$ and $V_j = O(y_j; (-\infty, -1/N))$ for $j = \ell_2 + 1, \dots, 2\ell_2$. Then we see that $f \in V \cap \bigcap \{C_p(X(\mathcal{F})) \setminus f_V^{-1}[1/N, 1] : j = 1, \dots, 2\ell_2\} \subseteq O(x_1, \dots, x_k; I_1, \dots, I_k)$, as required.

For claim (2) suppose Y is a space with a P -point- $<\kappa$ almost subbase. Let $Q = P \times \mathbb{N}$. By Lemma 1.2 there is an F -separating Q -point- $<\kappa$ almost subbase for Y , say $\alpha = \bigcup \{\alpha_q : q \in Q\}$, with respect to $\{f_V\}_{V \in \alpha}$. We show Y can be embedded in a $C_p(X(\mathcal{F}))$ where $X(\mathcal{F})$ has a $(Q \times \mathbb{N})$ -ordered compact cover. Since $\mathbb{N} =_T \mathbb{N} \times \mathbb{N}$, we see that $Q \times \mathbb{N} =_T P \times \mathbb{N}$. So $X(\mathcal{F})$ has a $(P \times \mathbb{N})$ -ordered compact cover, as desired.

For any $n \in \mathbb{N}$ and $V \in \alpha$ there is a continuous function $a_n^V : Y \rightarrow [0, 1]$ with $a_n^V(y) = 1$ for $y \in f_V^{-1}[1/n, 1]$, and $a_n^V(y) = 0$ for $y \notin f_V^{-1}(1/(n+1), 1]$. By Lemma 1.1, we can assume that $a_n^V(x) \neq 1$ for all $x \notin f_V^{-1}[1/n, 1]$. Let $X = \{a_n^V : n \in \mathbb{N} \text{ and } V \in \alpha\}$. Let $C_{q,n} = \{a_i^V : i \leq n \text{ and } V \in \alpha_q\}$ for $n \in \mathbb{N}$ and $q \in Q$. Then we can see that if $(q, n) \leq (q', n')$, then $C_{q,n} \subseteq C_{q',n'}$. Define a collection \mathcal{F} of subsets of X by $F \in \mathcal{F}$ if and only if $C_{q,n} \setminus F$ has cardinality $<\kappa$ for each $q \in Q$ and $n \in \mathbb{N}$. For each $q \in Q$ and $n \in \mathbb{N}$, we can see that $C_{q,n}$ is a $<\kappa$ -compact subset of $X(\mathcal{F})$. Also, we can see that: (i) if F_1 and F_2 are in \mathcal{F} , then $C_{q,n} \setminus (F_1 \cap F_2) = (C_{q,n} \setminus F_1) \cup (C_{q,n} \setminus F_2)$ also has cardinality $<\kappa$ for each q and n , in other words, $F_1 \cap F_2$ is in \mathcal{F} ; and (ii) if $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $C_{q,n} \setminus F_2$ has size $<\kappa$ for each q, n , so F_2 is in \mathcal{F} . The space $X(\mathcal{F})$ evidently has a $(Q \times \mathbb{N})$ -ordered cover, $\{C_{q,n} : q \in Q, n \in \mathbb{N}\}$, of $<\kappa$ -compact sets. It remains to show that Y can be embedded in $C_p(X(\mathcal{F}))$.

We define a mapping $H : Y \rightarrow C_p(X(\mathcal{F}))$ by $H(y)(a_n^V) = a_n^V(y)$ for each $y \in Y$. Since α_q is point- $<\kappa$ for each $q \in Q$, $H(y)(a_n^V) \neq 0$ only at $<\kappa$ many a_n^V for each $q \in Q$ and $n \in \mathbb{N}$. So $H(y)$ is continuous on the space $X(\mathcal{F})$ for each $y \in Y$. Hence H is a well-defined map from Y into $C_p(X(\mathcal{F}))$.

We will show that the mapping H is injective, continuous, and open onto its image. We take distinct $y_1, y_2 \in Y$. Since α is F -separating (after interchanging y_1 and y_2 if necessary) there exists a $V \in \alpha$ such that $y_1 \in V$ and $y_2 \notin \bar{V}$. Then there exists $N \in \mathbb{N}$ such that $y_1 \in f_V^{-1}[1/N, 1]$, and $H(y_1) \neq H(y_2)$ because $H(y_1)(a_N^V) = 1$ while $H(y_2)(a_N^V) = 0$. Thus H is injective.

Choose a set $\{I_1, \dots, I_k\}$ of open intervals in the real line \mathbb{R} . The set $O = O(a_{n_1}^{V_1}, \dots, a_{n_k}^{V_k}; I_1, \dots, I_k)$ is a basic open neighborhood of $C_p(X(\mathcal{F}))$. Then $H^{-1}(O) = \bigcap \{(a_{n_i}^{V_i})^{-1}(I_i) : i = 1, \dots, k\}$. By the continuity of $a_{n_i}^{V_i}$ for $i = 1, \dots, k$, $H^{-1}(O)$ is an open subset of Y . Therefore, H is continuous.

Since H is injective and α is an almost subbase, to show H is onto its image it is enough to show that $H(V)$ and $H(Y \setminus f_V^{-1}[1/n, 1])$ are open for every $V \in \alpha$ and $n \in \mathbb{N}$. By the definition of H , we can see that for each $n \in \mathbb{N}$, $V \in \alpha$ and $y \in Y$ we have $H(y)(a_n^V) > 0 \Leftrightarrow a_n^V(y) > 0$, and hence

$$(\star) \quad y \in f_V^{-1}(1/(n+1), 1] \subset V,$$

while $H(y)(a_n^V) < 1 \Leftrightarrow a_n^V(y) < 1$ and thus

$$(*) \quad y \notin f_V^{-1}[1/n, 1].$$

Take any $\mathbf{y} \in H(V)$. Let $y = H^{-1}(\mathbf{y})$. Choose N with $y \in f_V^{-1}[1/N, 1]$. Then $a_N^V(y) = 1$. Let $W = O(a_N^V; (1/2, +\infty)) \cap H(Y)$, which is open in $H(Y)$. For each $\mathbf{z} \in W$, let $z = H^{-1}(\mathbf{z})$. We see that $H(z)(a_N^V) > 1/2$, so $z \in V$ by (\star) . Hence $H(z) \in H(V)$. Therefore, $\mathbf{y} \in W \subseteq H(V)$. So $H(V)$ is an open set in $H(Y)$.

Finally, take any $\mathbf{y} \in H(Y \setminus f_V^{-1}[1/n, 1])$. Let $y = H^{-1}(\mathbf{y})$, and note $a_n^V(y) < 1$. Let $W = O(a_n^V; (-\infty, 1)) \cap H(Y)$, which is also open in $H(Y)$. For each $\mathbf{z} \in W$, let $z = H^{-1}(\mathbf{z})$. We see that $H(z)(a_n^V) < 1$, and so $z \notin f_V^{-1}(1/(n+1), 1]$ by $(*)$. Hence $H(z) \in H(Y \setminus f_V^{-1}[1/n, 1])$. Therefore, $\mathbf{y} \in W \subseteq H(X \setminus f_V^{-1}[1/n, 1])$. So $H(X \setminus f_V^{-1}[1/n, 1])$ is also open in $H(Y)$. ■

Almost subbases give bases. Let \mathcal{C} be a family of sets. For any x , write $(\mathcal{C})_x = \{C \in \mathcal{C} : x \in C\}$. The collection \mathcal{C} is said to be $<\kappa$ -Noetherian if every subcollection of \mathcal{C} which is well-ordered by \subseteq has cardinality $<\kappa$. The family is said to be *additively $<\kappa$ -Noetherian* if the collection of all unions of members of the family is $<\kappa$ -Noetherian. We extend the notation to say that \mathcal{C} is κ -Noetherian (resp. *additively κ -Noetherian*) if \mathcal{C} is $<\kappa^+$ -Noetherian (resp. additively $<\kappa^+$ -Noetherian). Instead of (additively) $<\aleph_0$ -Noetherian we simply say (*additively*) *Noetherian*.

LEMMA 1.4. *Let \mathcal{C} be a family of sets. The following are equivalent:*

- (1) \mathcal{C} is additively $<\kappa$ -Noetherian,
- (2) the collection of all $<\kappa$ -unions of members of \mathcal{C} is $<\kappa$ -Noetherian,
- (3) every $\geq\kappa$ -subcollection, say \mathcal{C}_2 , of \mathcal{C} contains a κ -sized subcollection \mathcal{C}_1 with a $<\kappa$ -sized subcollection \mathcal{C}_0 such that $\bigcup \mathcal{C}_0 \supseteq \bigcup \mathcal{C}_1$,
- (4) it is not possible to find, for all $\alpha < \kappa$, sets C_α from \mathcal{C} and points x_α such that for all $\beta < \alpha < \kappa$ the point x_α is in C_α but not in C_β .

Proof. It is easy to verify that (1) \Rightarrow (2).

(2) \Rightarrow (3). Take any subcollection \mathcal{C}_2 of \mathcal{C} of size at least κ . Take any (exactly) κ -sized subcollection $\mathcal{C}_1 = \{C_\alpha : \alpha < \kappa\}$ of \mathcal{C}_2 . For each α in κ , let $\hat{C}_\alpha = \bigcup \{C_\beta : \beta \leq \alpha\}$. Then each \hat{C}_α is a $<\kappa$ -union of elements of \mathcal{C} , and if $\alpha' < \alpha$ then $\hat{C}_{\alpha'} \subseteq \hat{C}_\alpha$. So by (2), there must be an α_0 such that for all

$\alpha \geq \alpha_0$ we have $\hat{C}_{\alpha_0} = \hat{C}_\alpha$. Thus $\mathcal{C}_0 = \{C_\beta : \beta \leq \alpha_0\}$ is a $<\kappa$ -subcollection of \mathcal{C}_1 such that $\bigcup \mathcal{C}_0 \supseteq \bigcup \mathcal{C}_1$, as required for (3).

$\neg(4) \Rightarrow \neg(3)$. We know that there is a family $\{(C_\alpha, x_\alpha) : \alpha \in \kappa\}$ of elements of \mathcal{C} and points as in the statement of (4). Let $\mathcal{C}_2 = \{C_\alpha : \alpha \in \kappa\}$. Take any κ -sized subcollection \mathcal{C}_1 of \mathcal{C}_2 . Write $\mathcal{C}_1 = \{C_\alpha : \alpha \in S_1\}$ where S_1 is some κ -sized subset of κ . For any κ -sized subset S_0 of S_1 , pick an α_0 in S_1 such that $\alpha_0 > \max S_0$. Then $x_{\alpha_0} \notin \bigcup \{C_\beta : \beta \in S_0\}$. Thus the negation of (3) holds.

$\neg(1) \Rightarrow \neg(4)$. Suppose we have a κ -sized subfamily \mathcal{C}_1 of \mathcal{C} which is well-ordered by (strict) inclusion. By transfinite induction we can easily find, for each $\alpha < \kappa$, a C_α in \mathcal{C}_1 such that $D_\alpha = C_\alpha \setminus \bigcup \{C_\beta : \beta < \alpha\} \neq \emptyset$. Picking $x_\alpha \in D_\alpha$ for each α in κ yields the sets C_α and points x_α required to negate (4). ■

THEOREM 1.5. *Every productively $<\kappa$ -compact space with a P -point- $<\kappa$ almost subbase has a $(P \times \mathbb{N})$ -point additively $<\kappa$ -Noetherian base.*

Proof. Let X be a productively $<\kappa$ -compact space which has a P -point- $<\kappa$ almost subbase α with respect to $\{f_V\}_{V \in \alpha}$. So $\alpha = \bigcup \{\alpha_p : p \in P\}$ where each α_p is point- $<\kappa$, $\alpha_p \subseteq \alpha_{p'}$ if $p \leq p'$, and the collection $\mathcal{S} = \alpha \cup \{X \setminus f_V^{-1}[1/n, 1] : V \in \alpha, n \in \mathbb{N}\}$ is a subbase for X . Let \mathcal{B} be the base for X consisting of all finite non-empty intersections of members of \mathcal{S} . Then naturally we write $\mathcal{B} = \bigcup \{\mathcal{B}_{p,m} : m \in \mathbb{N} \text{ and } p \in P\}$ where B is $\mathcal{B}_{p,m}$ if and only if $B = \bigcap_{i=1}^{k_1} V_i' \cap \bigcap_{j=1}^{k_2} (X \setminus f_{V_j}^{-1}[1/n_j, 1])$ where $k_1, k_2 \in \mathbb{N}$, each $k_1, k_2, n_j \leq m$, and each V_i' and V_j is in α_p . Clearly \mathcal{B} is $(P \times \mathbb{N})$ -ordered.

We show that $\mathcal{B}_{p,m}$ is point additively $<\kappa$ -Noetherian. To this end, fix x in X and any $\geq \kappa$ -sized subset \mathcal{B}' of $(\mathcal{B}_{p,m})_x$. Take any subcollection \mathcal{B}_1 with exactly κ many elements. Since α_p is point- $<\kappa$, we can suppose there is a fixed finite subset α' of $(\alpha_p)_x$ and a fixed $k \in \mathbb{N}$ such that every B in \mathcal{B} has the form $B = \bigcap \alpha' \cap \bigcap_{i=1}^k (X \setminus A(B, i))$, where each $A(B, i)$ is a closed set of the form $f_V^{-1}[1/n, 1]$ for some n and V in α_p .

Since X is productively $<\kappa$ -compact, the space X^k is $<\kappa$ -compact. We will show there is a $<\kappa$ -sized collection \mathcal{B}_0 contained in \mathcal{B}_1 such that $\bigcup \mathcal{B}_0 \supseteq \bigcup \mathcal{B}_1$, thereby verifying that $\mathcal{B}_{p,m}$ satisfies claim (3) of the preceding lemma.

By definition of $\mathcal{B}_{p,m}$, the collection $\{A(B, i) : B \in \mathcal{B}_1\}$ is point- $<\kappa$ for each $i = 1, \dots, k$. Then $\mathcal{C} = \{\prod_{i=1}^k A(B, i) : B \in \mathcal{B}_1\}$ is a point- $<\kappa$ family of closed subsets in X^k . Since \mathcal{B}_1 has cardinality κ , so does \mathcal{C} , hence it must have empty intersection. By $<\kappa$ -compactness of X^k , \mathcal{C} cannot have the κ -Intersection Property. So there is a $<\kappa$ -sized subcollection \mathcal{C}_0 of \mathcal{C} with $\bigcap \mathcal{C}_0 = \emptyset$. For each $C \in \mathcal{C}_0$, choose B_C from \mathcal{B}_1 such that $C = \prod_{i=1}^k A(B_C, i)$. Then let $\mathcal{B}_0 = \{B_C : C \in \mathcal{C}_0\}$. Hence \mathcal{B}_0 has cardinality $< \kappa$.

Take any y in some element of \mathcal{B}_1 , say

$$B' = \bigcap \alpha' \cap \bigcap_{i=1}^k (X \setminus A(B', i)).$$

Then the vector (x_1, \dots, x_k) with $x_i = y$ for each $i = 1, \dots, k$ is not in C^* for some $C^* \in \mathcal{C}_0$. Hence $y \notin A(B_{C^*}, i)$ for each $i = 1, \dots, k$, and so we have $y \in B_{C^*}$. Thus $\bigcup \mathcal{B}_0 \supseteq \bigcup \mathcal{B}_1$, as required. ■

Expandable networks and point networks. Let X be a space. An *expandable network* for X is a collection \mathcal{N} of pairs (N, V) of subsets of X , where $N \subseteq V$ and V is open, such that if a point x of X is in an open set U then there is an element (N, V) of \mathcal{N} such that $x \in N \subseteq U$. Clearly if \mathcal{N} is an expandable network for X , then $\mathcal{N}' = \{N : (N, V) \in \mathcal{N}\}$ is a (standard) network for X , and if \mathcal{N}' is a network, then $\mathcal{N} = \{(N, X) : N \in \mathcal{N}'\}$ is an expandable network. The interest in expandable networks (as with almost subbases) comes when they are structured as a P - \mathcal{Q} family.

Extending our previous notation and definition, let \mathcal{P} be a family of pairs of subsets of a set X ; then for any x , write $(\mathcal{P})_x = \{(P_1, P_2) \in \mathcal{P} : x \in P_2\}$, and we say that \mathcal{P} is *point- $<\kappa$* if for every x in X the set $(\mathcal{P})_x$ has size $<\kappa$. Dow, Junnila and Pelant [DJP] showed that a compact space is Eberlein compact if and only if it has an ω -point finite expandable network.

Point networks are the natural local version of expandable networks. A *point network* (respectively, *strong point network*) for X is a collection $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where each $\mathcal{W}(x)$ is a collection of subsets of X containing x such that whenever $x \in U$, U open, then there is an open V with $x \in V \subseteq U$ such that, whenever $y \in V$ then $x \in W \subseteq U$ (respectively, $x \in W \subseteq V$) for some $W \in \mathcal{W}(y)$. We note that we can take U and V to be basic. Point networks are also known as ‘condition (F)’, and as the ‘Collins–Roscoe structuring mechanism’ after the authors who introduced them [CR]. The term ‘point network’ was suggested by Gruenhage.

If \mathcal{Q} is a property that can be reasonably applied to a family of subsets of a space then we say a space X has a \mathcal{Q} (*strong*) *point network* if it has a (strong) point network \mathcal{W} such that for each $\mathcal{W}(x)$ the property \mathcal{Q} holds. Let \mathcal{W} be a (strong) point network for a space X . We can further structure the (strong) point network \mathcal{W} analogously to P - \mathcal{Q} expandable networks. Formally, let P be a directed set, and \mathcal{Q} be as above. We say that \mathcal{W} for X is a P - \mathcal{Q} (*strong*) *point network* if for each point x in X we can write $\mathcal{W}(x) = \bigcup_{p \in P} \mathcal{W}_p(x)$, where every $\mathcal{W}_p(x)$ has property \mathcal{Q} , if $p \leq p'$ then $\mathcal{W}_p(x) \subseteq \mathcal{W}_{p'}(x)$, and if some point x is in an open set U , then there is an open set $V = V(x, U)$ containing x and contained in U and a $p = p(x, U)$ from P such that if $y \in V$ then there is a $W \in \mathcal{W}_p(y)$ such that $x \in W \subseteq U$

(respectively, $x \in W \subseteq V$). Note that a ‘1- \mathcal{Q} (strong) point network’ is exactly the same as a ‘ \mathcal{Q} (strong) point network’ from above.

Bases which give expandable networks, give strong point networks. The proof of the next lemma demonstrating how P - \mathcal{Q} bases give P - \mathcal{Q} expandable networks can safely be left to the reader.

LEMMA 1.6. *Let \mathcal{B} be a base for a space X . Then $\mathcal{N} = \{(B, B) : B \in \mathcal{B}\}$ is an expandable network for X . Further, if $\mathcal{B} = \bigcup_{p \in P} \mathcal{B}_p$, where $\mathcal{B}_p \subseteq \mathcal{B}_{p'}$ if $p \leq p'$, and each \mathcal{B}_p has an order or cardinality property \mathcal{Q} , then $\mathcal{N} = \bigcup_p \mathcal{N}_p$, $\mathcal{N}_p \subseteq \mathcal{N}_{p'}$ if $p \leq p'$, and each \mathcal{N}_p has property \mathcal{Q} . Thus a space with a P - \mathcal{Q} base has a P - \mathcal{Q} expandable network.*

Now we show that P - \mathcal{Q} expandable networks give P - \mathcal{Q} strong point networks. It should be evident that a P - \mathcal{Q} strong point network is a P - \mathcal{Q} point network.

LEMMA 1.7. *Suppose a space X has a P - \mathcal{Q} expandable network \mathcal{N} , say $\mathcal{N} = \bigcup_{p \in P} \mathcal{N}_p$, where $\mathcal{N}_p \subseteq \mathcal{N}_{p'}$ if $p \leq p'$, and each \mathcal{N}_p has property \mathcal{Q} . Then $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, where for each x the space $\mathcal{W}(x) = \bigcup_p \mathcal{W}_p(x)$ with $\mathcal{W}_p(x) = \{\{x\} \cup N : (N, V) \in \mathcal{N}_p \text{ and } x \in V\}$ is a P - \mathcal{Q} strong point network for X .*

Proof. Fix x in X and $p \leq p'$ from P . We verify $\mathcal{W}_p(x) \subseteq \mathcal{W}_{p'}(x)$. Take any W from $\mathcal{W}_p(x)$. Then $W = \{x\} \cup N$ for some (N, V) in \mathcal{N}_p such that $x \in V$. Since $(N, V) \in \mathcal{N}_p$ and $\mathcal{N}_p \subseteq \mathcal{N}_{p'}$, we see that $W = \{x\} \cup N$ is in $\mathcal{W}_{p'}(x)$.

Now take any open set U containing a point x . Since $\mathcal{N} = \bigcup\{\mathcal{N}_p : p \in P\}$ is an expandable point network, there is a $p = p(x, U)$ such that $x \in N \subseteq U$ for some (N, V') in \mathcal{N}_p . Let $V(x, U) = V' \cap U$, and note $x \in V \subseteq U$. Take any y in V . Then, as $V \subseteq V'$, by definition, $W = \{y\} \cup N$ is a member of $\mathcal{W}_p(y)$, and since $y \in V$, $x \in N \subseteq U$ and $N \subseteq V'$, we clearly have $x \in W \subseteq V$. ■

Combining the previous two lemmas we know that if a space X has a P - \mathcal{Q} base $\mathcal{B} = \bigcup\{\mathcal{B}_p : p \in P\}$ then it has a P - \mathcal{Q} strong network. Indeed, we can take $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where $\mathcal{W}(x) = \bigcup_{p \in P} \mathcal{W}_p(x)$ and $\mathcal{W}_p(x) = (\mathcal{B}_p)_x$.

Stability of point networks. Let \mathcal{Q} be a property that can be reasonably applied to families of subsets of a topological space. We say that \mathcal{Q} is *hereditary* if whenever A is a subspace of X and \mathcal{C} is a family of subsets of X satisfying \mathcal{Q} , then their traces onto A , $\mathcal{C}_A = \{C \cap A : C \in \mathcal{C}\}$, form a family of subsets of A also satisfying \mathcal{Q} . We say that \mathcal{Q} is (*finitely*) *productive* if whenever \mathcal{C} is a family of subsets of X satisfying \mathcal{Q} and \mathcal{C}' is a family of subsets of Y satisfying \mathcal{Q} , then the collection of subsets of $X \times Y$ given by $\{C \times C' : C \in \mathcal{C}, C' \in \mathcal{C}'\}$ satisfies \mathcal{Q} . Finally we say \mathcal{Q} is *preserved by images* if whenever \mathcal{C} is a collection of subsets of a space X which satisfies \mathcal{Q} ,

and f is a continuous map of X into a space Y , then $f(\mathcal{C}) = \{f(C) : C \in \mathcal{C}\}$ has \mathcal{Q} in Y .

The next proposition is straightforward. Note for claim (2) that for any directed set P we have $P \geq_T P \times P$, which is why the natural $(P \times P)$ - \mathcal{Q} (strong) point network for $X \times Y$ is in fact a P - \mathcal{Q} (strong) point network.

PROPOSITION 1.8.

- (1) *Let \mathcal{Q} be hereditary. If a space X has a P - \mathcal{Q} (strong) point network, then so does every subspace of X .*
- (2) *Let \mathcal{Q} be productive. If spaces X and Y have a P - \mathcal{Q} (strong) point network, then $X \times Y$ also has a P - \mathcal{Q} (strong) point network.*

Invariance under maps requires a little more work.

PROPOSITION 1.9. *Let κ be an infinite cardinal. Let $f : X \rightarrow Y$ be a closed surjection with $<\kappa$ -compact fibers. Let P be a $<\kappa$ -directed set, and \mathcal{Q} be a property preserved by taking images. Then Y has a P - \mathcal{Q} point network provided that X has a P - \mathcal{Q} point network.*

Proof. Let $\mathcal{W}(x) = \bigcup_p \mathcal{W}_p(x)$ be a P - \mathcal{Q} point network of X . For each $y \in Y$, pick $x_y \in f^{-1}(y)$. Then for each p in P define $\mathcal{W}_p(y) = \{f(W) : W \in \mathcal{W}_p(x_y)\}$, and note it has property \mathcal{Q} . Set $\mathcal{W}(y) = \bigcup_p \mathcal{W}_p(y)$ and $\mathcal{W}_Y = \{\mathcal{W}(y) : y \in Y\}$.

We verify that \mathcal{W}_Y is a P - \mathcal{Q} point network of Y . Take $y \in U$ where U is an open subset in Y . By the definition of P - \mathcal{Q} point network, for any $x \in f^{-1}(y)$, there exist $V(x, f^{-1}(U))$ and $p(x, f^{-1}(U))$ such that, for any $\hat{x} \in V(x, f^{-1}(U))$, there exists $W \in \mathcal{W}_{p(x, f^{-1}(U))}(\hat{x})$ such that $x \in W \subseteq U$. By hypothesis, $f^{-1}(y)$ is $<\kappa$ -compact. Hence, there is a $\tau < \kappa$ such that for each $\alpha < \tau$ there exists $x_\alpha \in f^{-1}(y)$ such that $f^{-1}(y) \subseteq \bigcup_{\alpha < \tau} V(x_\alpha, f^{-1}(U))$. Since, by hypothesis, P is $<\kappa$ -directed and $\tau < \kappa$, we can pick $\hat{p} = \hat{p}(y, U)$ satisfying $\hat{p} \geq p(x_\alpha, f^{-1}(U))$ for $\alpha < \tau$. As f is a closed mapping, we can pick $\hat{V} = V(y, U)$ such that $f^{-1}(y) \subseteq f^{-1}(\hat{V}) \subseteq \bigcup_{\alpha < \tau} V(x_\alpha, f^{-1}(U))$.

Take any $\hat{y} \in \hat{V}$. Then $x_{\hat{y}} \in f^{-1}(\hat{V}) \subseteq \bigcup_{\alpha < \tau} V(x_\alpha, f^{-1}(U))$. Hence $x_{\hat{y}} \in V(x_\alpha, f^{-1}(U))$ for some α . Therefore, there exists $W \in \mathcal{W}_{p(x_\alpha, f^{-1}(U))}(x_{\hat{y}})$ such that $x_\alpha \in W \subseteq f^{-1}(U)$. So we have $W \in \mathcal{W}_{\hat{p}}(x_{\hat{y}})$ and $y = f(x_\alpha) \in f(W) \subseteq U$ where $f(W) \in \mathcal{W}_{\hat{p}}(\hat{y})$. ■

LEMMA 1.10. *For any cardinal κ :*

- (1) *the property $\mathcal{Q} = \text{'<\kappa'}$ (has size strictly less than κ) is hereditary, productive and preserved by images,*
- (2) *the property $\mathcal{Q} = \text{'additively <\kappa\text{-Noetherian'}$ is hereditary and preserved by images, while*
- (3) *the property $\mathcal{Q} = \text{'additively Noetherian'}$ is productive.*

Proof. Claim (1) is immediate, as is the fact that additively $<\kappa$ -Noetherian is hereditary. Claim (3) was established by Nyikos [N]. We indicate why additively $<\kappa$ -Noetherian is preserved by images.

Fix spaces X and Y , a collection \mathcal{C} of subsets of X , and a map f of X into Y . Applying clause (4) of Lemma 1.4 it is simple to see that if $f(\mathcal{C})$ is not additively $<\kappa$ -Noetherian then neither is \mathcal{C} . ■

Recall that in this paper all partial orders P are directed (in other words, $<\aleph_0$ -directed), and note that the trivial partial order $\mathbf{1}$ is (trivially!) $<\kappa$ -directed for all κ . Thus we deduce:

COROLLARY 1.11.

- (1) *If $f : X \rightarrow Y$ is a closed surjection and X has a \mathcal{Q} point network then so does Y , whenever \mathcal{Q} is closed under images.*
- (2) *If $f : X \rightarrow Y$ is a perfect surjection and X has a P -additively $<\kappa$ -Noetherian point network then so does Y .*
- (3) *If $f : X \rightarrow Y$ is a perfect surjection and X has a P -finite point network then so does Y .*

Covering properties from point networks. A space is said to be P -meta- $<\kappa$ -compact if every open cover has a P -point- $<\kappa$ open refinement. Let us abbreviate ‘ P -meta- $<\aleph_0$ -compact’ to ‘ P -metacompact’. Note that a space is metacompact if and only if it is 1-metacompact, σ -metacompact if and only if it is \aleph -metacompact, and is meta-Lindelöf if and only if it is 1-meta- $<\aleph_1$ -compact.

THEOREM 1.12. *If a space has a P -additively $<\kappa$ -Noetherian point network then it is P -meta- $<\kappa$ -compact.*

Proof. Let $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, where $\mathcal{W}(x) = \bigcup_{p \in P} \mathcal{W}_p(x)$, each $\mathcal{W}_p(x)$ is $<\kappa$ -additively Noetherian, if $p \leq q$ then $\mathcal{W}_p(x) \subseteq \mathcal{W}_q(x)$, and this whole structure is a point network for X .

We show X is P -meta- $<\kappa$ -compact. To this end take any open cover of X , say $\mathcal{U} = \{U_\alpha : \alpha < \tau\}$. For each α and $p \in P$ define $V_{\alpha,p} = \bigcup\{V(x, U_\alpha) : x \in U_\alpha \setminus \bigcup\{U_\beta : \beta < \alpha\}, p(x, U_\alpha) = p\}$. Set $\mathcal{V}_{\leq p} = \{V_{\alpha,p'} : \alpha < \tau, p' \leq p\}$ and $\mathcal{V} = \bigcup_{p \in P} \mathcal{V}_{\leq p}$.

Then \mathcal{V} clearly is an open refinement of \mathcal{U} . It is also clear that if $p \leq q$ then $\mathcal{V}_{\leq p} \subseteq \mathcal{V}_{\leq q}$. So it remains to show that each $\mathcal{V}_{\leq p}$ is point- $<\kappa$. Suppose not. Then there is a point y in X such that for each $\gamma < \kappa$, y is in some $V(x_\gamma, U_{\alpha_\gamma})$, where $x_\gamma \in U_{\alpha_\gamma} \setminus \bigcup\{U_\beta : \beta < \alpha_\gamma\}$, $p(x_\gamma, U_{\alpha_\gamma}) = p_\gamma \leq p$ and $\delta < \gamma$ implies $\alpha_\delta < \alpha_\gamma$. Hence, for each $\gamma < \kappa$, there is a W_γ in $\mathcal{W}_{p_\gamma}(y) \subseteq \mathcal{W}_p(y)$ such that $x_\gamma \in W_\gamma \subseteq U_{\alpha_\gamma}$. Thus, for each $\delta < \gamma < \kappa$ we see that $x_\gamma \in W_\gamma$, but $x_\gamma \notin U_{\alpha_\delta} \supseteq W_\delta$. But by Lemma 1.4(4) this explicitly contradicts the fact that $\mathcal{W}_p(y)$ is $<\kappa$ -additively Noetherian. ■

From Proposition 1.8 and Lemma 1.10 we deduce:

COROLLARY 1.13.

- (1) For any κ , a space with a P -additively $<\kappa$ -Noetherian point network is hereditarily P -meta- $<\kappa$ -compact.
- (2) A finite product of spaces each with a P -additively Noetherian point network is (hereditarily) P -metacompact.

For compacta, off-diagonal covering properties give almost subbases. For a compact space X we have shown that if X has a suitable almost subbase then it is hereditarily P -meta- $<\kappa$ -compact for some directed set P and infinite cardinal κ . We now attempt to close the loop and prove:

THEOREM 1.14. *Let X be a compact space. If $X^2 \setminus \Delta$ is P -meta- $<\kappa$ -compact then X has a $(P \times \mathbb{N})$ -point- $<\kappa$ almost subbase.*

The proof is essentially the same as that given by García et al. [GOO] for the case when $P = \mathcal{K}(M)$, which in turn was a straightforward extension of Gruenhage’s original argument for the $P = \mathbb{N}$ case. Consequently, we only sketch the proof for general directed sets P . We start with a lemma which is the P -analogue of [GOO, Proposition 23].

LEMMA 1.15. *Let X be a P -meta- $<\kappa$ -compact, locally compact space, and let \mathcal{B} be a basis for X . Then X has a subcover $\mathcal{B}' \subseteq \mathcal{B}$ such that $\{\overline{B} : B \in \mathcal{B}'\}$ is a P -point- $<\kappa$ family.*

The proof of this lemma is identical to that of [GOO, Proposition 8]; it suffices to replace references in that argument of ‘point finite’ with ‘point- $<\kappa$ ’ and ‘ $K \in \mathcal{K}(M)$ ’ with ‘ $p \in P$ ’, as nothing specific about the partial order of set inclusion on $\mathcal{K}(M)$ is used.

Our Theorem 1.14 is essentially the content of the part ‘(ii) implies (i)’ of [GOO, Theorem 24]. We follow their argument making the necessary adjustments.

Proof of Theorem 1.14. Since $X^2 \setminus \Delta$ is P -meta- $<\kappa$ -compact, applying Lemma 1.15 to $\mathcal{B} = \{U \times V : U, V \text{ are cozero subsets of } X \text{ such that } \overline{U} \cap \overline{V} = \emptyset\}$ and tidying, we obtain a cover $\mathcal{P} = \{U_\gamma \times V_\gamma : \gamma \in A\}$ of $X^2 \setminus \Delta$ with the following properties: (a) U_γ and V_γ are cozero sets of X , (b) $\overline{U_\gamma} \cap \overline{V_\gamma} = \emptyset$, (c) $\{\overline{U_\gamma} \times \overline{V_\gamma} : \gamma \in A\}$ is P -point- $<\kappa$, and (d) if $U \times V$ is in \mathcal{P} then so is $V \times U$.

Suppose $X = \overline{\{p_\alpha : \alpha < \mu\}}$ where $\mu = d(X)$ (the minimal size of a dense set of X). Set, for each $\alpha < \mu$, $X_\alpha = \overline{\{p_\beta : \beta < \alpha\}}$ and $\mathcal{U}_\alpha = \{\bigcap_{\gamma \in F} U_\gamma : F \subseteq A \text{ and } \{\overline{V_\gamma} : \gamma \in F\} \text{ is a finite minimal cover of } X_\alpha\}$.

Note that \mathcal{U}_α covers $X \setminus X_\alpha$. Then the family $\mathcal{U} = \bigcup \{\mathcal{U}_\beta : \beta < \mu\}$ is T_0 -separating as in [G, Theorem 2.2, Claim 2]. Since the elements of \mathcal{U} are cozero sets, and X is compact, we deduce that \mathcal{U} is an almost subbase for X .

It remains to show that \mathcal{U} is a $(P \times \mathbb{N})$ -point- $<\kappa$ family. Well, we know that $A = \bigcup\{A_p : p \in P\}$ where $A_p \subseteq A_q$ whenever $p \leq q$ and each family $\{\overline{U}_\gamma \times \overline{V}_\gamma : \gamma \in A_p\}$ is point- $<\kappa$.

Fix p in P and n in \mathbb{N} . For any α , let $\mathcal{U}_{\alpha,n}^p$ be all members of \mathcal{U}_α whose index set F has size $\leq n$ and is contained in A_p . Let $\mathcal{U}_n^p = \bigcup\{\mathcal{U}_{\alpha,n}^p : \alpha < \mu\}$. Then $\mathcal{U} = \bigcup\{\mathcal{U}_n^p : p \in P, n \in \mathbb{N}\}$ and clearly if $p \leq p', n \leq n'$ then $\mathcal{U}_n^p \subseteq \mathcal{U}_{n'}^{p'}$.

To complete the proof we will show that \mathcal{U}_n^p is point- $<\kappa$. Suppose, for a contradiction, that for some point x in X and each $\rho < \kappa$ there are $\beta_\rho < \mu$ and sets $F_\rho \subseteq A_p$ such that $|F_\rho| \leq n, x \in \bigcap\{U_\gamma : \gamma \in F_\rho\}, X_{\beta_\rho} \subseteq \bigcup\{\overline{V}_\gamma : \gamma \in F_\rho\}$ and $F_\rho \neq F_\sigma$ if $\rho \neq \sigma$. Tidying we can assume $\beta_\rho \leq \beta_\sigma$ if $\rho \leq \sigma$.

Since for every $\rho < \kappa$ we have $|F_\rho| \leq n$ and all the F_ρ 's are different, we may assume that $\{F_\rho : \rho < \kappa\}$ form a Δ -system with root R . Pick $y \in X_{\beta_0} \setminus \bigcup\{\overline{V}_\gamma : \gamma \in R\}$. Then for each ρ there is a $\delta(\rho) \in F_\rho \setminus R$ with $y \in \overline{V}_{\delta(\rho)}$. But then $(x, y) \in \bigcap\{U_{\delta(\rho)} \times \overline{V}_{\delta(\rho)} : \rho < \kappa\}$ and, for all $\rho, \delta(\rho)$ is in A_p , which contradicts $\{\overline{U}_\gamma \times \overline{V}_\gamma : \gamma \in A_p\}$ being point- $<\kappa$. ■

Characterizations and an application. Recall that a directed set is not countably directed (every countable subset has an upper bound) if and only if $P \geq_T \mathbb{N}$. In particular $\mathbb{N} \geq_T \mathbb{N} \times \mathbb{N}$. Hence if P is not countably directed then $P =_T P \times \mathbb{N}^n$ for every n in \mathbb{N} . Applying this fact and 1.3, 1.5, 1.6, 1.7, 1.13, and 1.14, we establish the equivalence of claims (1) and (2) of the following theorem, and then the implications (2) \Rightarrow (3), (3) \Rightarrow (4), (4) \Rightarrow (5), (6) \Rightarrow (0) and (0) \Rightarrow (2) ((5) \Rightarrow (6) being trivial).

THEOREM 1.16. *Let X be compact. Let P be a directed set which is not countably directed. Then the following are equivalent:*

- (0) X is P -Eberlein compact,
- (1) X embeds in a $C_p(X(\mathcal{F}))$ where $X(\mathcal{F})$ has a P -ordered compact cover,
- (2) X has a P -point finite almost subbase,
- (3) X has a P -point additively Noetherian base,
- (4) X has a P -point additively Noetherian expandable network,
- (5) X has a P -additively Noetherian strong point network,
- (6) X has a P -additively Noetherian point network.

Essentially, only one directed set P is not covered by the above theorem. To see this take any countably directed P and suppose (1) above holds. Since P is countably directed, any countable subset of $X(\mathcal{F})$ must be contained in one of the elements of the P -ordered compact cover provided by (1). It follows that $X(\mathcal{F})$ is countably compact. As $X(\mathcal{F})$ is paracompact, we see that $X(\mathcal{F})$ is compact. So we might as well take $P = 1$, and observe that $X(\mathcal{F}) = A(\kappa)$ for some κ . Let us deal with this remaining case of $P = 1$.

THEOREM 1.17. *Let X be compact. Then the following are equivalent:*

- (0) X is Eberlein compact,
- (1) X embeds in a $C_p(A(\kappa))$,
- (2) X has an \mathbb{N} -point finite almost subbase,
- (3) X has an \mathbb{N} -point additively Noetherian base,
- (4) X has an \mathbb{N} -point additively Noetherian expandable network,
- (5) X has an \mathbb{N} -additively Noetherian strong point network,
- (6) X has an \mathbb{N} -additively Noetherian point network.

Proof. The equivalence of (0) and (1)—taking the definition of an Eberlein compact space to be one homeomorphic to a weakly compact subspace of a Banach space—is due to [AL]. Taking $P = 1$ and recalling that $\mathbb{N} \geq_T \mathbb{N} \times \mathbb{N}$, by the same argument as above, we have (1) \Rightarrow (2), (2) \Rightarrow (3), (3) \Rightarrow (4), (4) \Rightarrow (5), (5) \Rightarrow (6), and (6) \Rightarrow (2). Theorem 1.3 tells us that X embeds in a $C_p(X(\mathcal{F}))$ where $X(\mathcal{F})$ is σ -compact (has an \mathbb{N} -ordered compact cover). But it is well-known and not difficult to verify that $C_p(X(\mathcal{F}))$ is then homeomorphic (not linearly) to a subspace of some $C_p(A(\kappa))$. So we get back to (1). ■

In addition to Amir and Lindenstrauss' proof of the equivalence of (0) and (1) in Theorem 1.17, we should also mention that the equivalence of (1) and (2) is essentially Rosenthal's theorem [R], and Junnila [J] established equivalence of (2) and (3).

For completeness we also highlight the cases of Theorem 1.16 for $P = \mathbb{N}^{\mathbb{N}}$ and $P = \mathcal{K}(M)$ for some separable metrizable M .

THEOREM 1.18. *Let X be compact. Then the following are equivalent:*

- (0) X is Talagrand compact,
- (1) X embeds in a $C_p(X(\mathcal{F}))$ where $X(\mathcal{F})$ has an $\mathbb{N}^{\mathbb{N}}$ -ordered compact cover,
- (2) X has an $\mathbb{N}^{\mathbb{N}}$ -point finite almost subbase,
- (3) X has an $\mathbb{N}^{\mathbb{N}}$ -point additively Noetherian base,
- (4) X has an $\mathbb{N}^{\mathbb{N}}$ -point additively Noetherian expandable network,
- (5) X has an $\mathbb{N}^{\mathbb{N}}$ -additively Noetherian strong point network,
- (6) X has an $\mathbb{N}^{\mathbb{N}}$ -additively Noetherian point network.

The equivalence of (0) and (1) above is due to [M, S].

THEOREM 1.19. *Let X be compact. Then the following are equivalent:*

- (0) X is Gul'ko compact,
- (1) X embeds in a $C_p(X(\mathcal{F}))$ where $X(\mathcal{F})$ has a $\mathcal{K}(M)$ -ordered compact cover for some non-compact, separable metrizable M .

Further, for a fixed non-compact, separable metrizable space M , the following are equivalent:

- (2) X has a $\mathcal{K}(M)$ -point finite almost subbase,
- (3) X has a $\mathcal{K}(M)$ -point additively Noetherian base,
- (4) X has a $\mathcal{K}(M)$ -point additively Noetherian expandable network,
- (5) X has a $\mathcal{K}(M)$ -additively Noetherian strong point network,
- (6) X has a $\mathcal{K}(M)$ -additively Noetherian point network.

The equivalence of (0) and (1) above is again due to [M, S]. We observe that, for a separable metrizable space M , the ordered set $\mathcal{K}(M)$ is countably directed if and only if M is compact (in which case $\mathcal{K}(M) =_T 1$).

The equivalence of condition (6) with the others in the theorems above has a pleasant application. Recall that P -additively Noetherian networks are preserved by perfect images (Corollary 1.11) to deduce:

THEOREM 1.20. *Let X be compact. Let P be a directed set which is either not countably directed or $P =_T 1$. If X satisfies any of the equivalent conditions of Theorem 1.16, and Y is a continuous image of X , then Y satisfies the same conditions.*

COROLLARY 1.21. *A continuous image of an Eberlein (respectively, Talagrand or Gul'ko) compact space is again Eberlein (respectively, Talagrand or Gul'ko).*

We note there is nothing new about this result, but emphasize the uniform nature of the proofs for Eberlein, Talagrand and Gul'ko compacta, and mention again the simplicity of the proof in contrast to Rudin's argument for Eberlein compacta.

A Corson counter-example and theorem. All of the ingredients used in the proof of Theorem 1.16 remain true when we replace 'compact' with ' $<\kappa$ -compact' and 'finite' with ' $<\kappa$ '. Except one. In Lemma 1.10(3) we only assert the productivity of 'additively Noetherian' and *not* the productivity of 'additively $<\kappa$ -Noetherian' for uncountable κ .

We give here an example of a compact space which shows that no analogue of Theorems 1.17, 1.18 or 1.19 holds for Corson compact spaces. Thus the request from [GOO] for a network characterization of Corson compacta remains unfulfilled. Noting that a compact space X is Corson compact if and only if it embeds in some $C_p(L(\kappa))$, and $L(\kappa)$ has a 1-ordered cover by $<\aleph_1$ -compact sets, we see that the weakest conjecture is that a compact space is Corson if and only if it has a point additively \aleph_0 -Noetherian base.

EXAMPLE 1.22. *Let X be the Double Arrow space. Then X is compact, has a (point) additively \aleph_0 -Noetherian base but is not Corson compact.*

Proof. The Double Arrow space X is hereditarily separable but non-metrizable, so far from Corson compact. It is also hereditarily Lindelöf. Let

\mathcal{B} be the collection of all open subsets of X . It is a base for X . The hereditary Lindelöf property immediately shows that condition (3) of Lemma 1.4 holds for \mathcal{B} with $\kappa = \aleph_1$, so \mathcal{B} is additively \aleph_0 -Noetherian, and *a fortiori* \mathcal{B} is point additively \aleph_0 -Noetherian. ■

Of course, since Corson compactness is evidently finitely productive, we can bypass the problem by looking at the square of a space rather than the space itself.

THEOREM 1.23. *Let X be compact. Then the following are equivalent:*

- (0) X is Corson compact (i.e. X embeds in a Σ -product of lines),
- (1) X has a point countable almost subbase,
- (2) X^2 has a point additively \aleph_0 -Noetherian base,
- (3) X^2 has a point additively \aleph_0 -Noetherian expandable network,
- (4) X^2 has a point additively \aleph_0 -Noetherian strong point network,
- (5) X^2 has a point additively \aleph_0 -Noetherian point network.

Why calibre (ω_1, ω) is critical. Theorem 1.16 applies to *all* directed sets P which are not countably directed (and Theorem 1.17 essentially covers all the remaining directed sets). A natural question is: for which P is Theorem 1.16 ‘interesting’? It is striking that this admittedly vague question has a clear answer: Theorem 1.16 is interesting if and only if P has calibre (ω_1, ω) .

A directed set P has *calibre* (κ^+, κ) if for every κ^+ -sized subcollection A of P there is a κ -sized subset A_0 of A which has an upper bound in P . We note that $\mathcal{K}(M)$ (and hence $\mathbb{N}^{\mathbb{N}} =_T \mathcal{K}(\mathbb{N}^{\mathbb{N}})$) has calibre (ω_1, ω) . Informally, the following theorem says that in the case when P does not have calibre (ω_1, ω) then far too many spaces satisfy the conditions of Theorem 1.16 for it to be of interest. But—provided we agree that Corson compacta are ‘nice’, and surely we do—the next result also says that when P does have calibre (ω_1, ω) then compact spaces satisfying the conditions of Theorem 1.16 are ‘nice’.

THEOREM 1.24. *Let P be a directed set.*

- (1) *If P is not calibre (ω_1, ω) then every space of weight $\leq \omega_1$ has a P -point finite base (and hence a P -point finite almost subbase, a P -point additively Noetherian base/expandable network, and a P -additively Noetherian (strong) point network).*
- (2) *If P has calibre (ω_1, ω) and X is a compact space with a P -finite additively Noetherian point network (or a P -point finite almost subbase et cetera) then X is Corson compact.*

Proof (1). Take any space X with weight no more than ω_1 . Fix a base \mathcal{B} for X where $|\mathcal{B}| \leq \omega_1$. List \mathcal{B} , with repeats if necessary, as $\mathcal{B} = \{B_\alpha :$

$\alpha < \omega_1$. Then writing $\mathcal{B} = \bigcup\{\mathcal{B}_F : F \in [\omega_1]^{<\omega}\}$ where $\mathcal{B}_F = \{B_\alpha : \alpha \in F\}$ demonstrates that X has a $[\omega_1]^{<\omega}$ -(point) finite base.

But recall that a directed set P does not have calibre (ω_1, ω) if and only if $P \geq_T [\omega_1]^{<\omega}$. Hence if P does not have calibre (ω_1, ω) then every space of weight $\leq \omega_1$ has a P -point finite base, as claimed. ■

To prove the second claim we will use the following lemma of independent interest.

LEMMA 1.25. *Let P be a directed set with calibre (κ^+, κ) , and let $\mathcal{C} = \{\mathcal{C}_p : p \in P\}$ be a P -ordered collection.*

- (a) *If each \mathcal{C}_p has size $< \kappa$, then $\bigcup \mathcal{C}$ has size $\leq \kappa$.*
- (b) *If each \mathcal{C}_p is additively $< \kappa$ -Noetherian, then $\bigcup \mathcal{C}$ is additively $\leq \kappa$ -Noetherian.*

Proof. First suppose, for a contradiction, that $\bigcup \mathcal{C}$ has size $> \kappa$ but each \mathcal{C}_p has size $< \kappa$. Then by transfinite induction we can find, for every $\alpha < \kappa^+$, points x_α and p_α from P such that $C_\alpha \in \mathcal{C}_{p_\alpha}$ but $C_\alpha \notin \mathcal{C}_{p_\beta}$ for any $\beta < \alpha$. By calibre (κ^+, κ) applied to $\{p_\alpha : \alpha < \kappa^+\}$, there is a κ -sized subset S_0 of κ^+ and a p_0 in P such that p_0 is an upper bound of $\{p_\alpha : \alpha \in S_0\}$. But now we see that for every α in S_0 the element C_α is in \mathcal{C}_{p_α} , which is a subset of \mathcal{C}_{p_0} , and so \mathcal{C}_{p_0} must have size at least κ , contradicting our assumption on the size of the \mathcal{C}_p 's.

To prove (b), suppose, for a contradiction, that $\bigcup \mathcal{C}$ is not additively $\leq \kappa$ -Noetherian. By Lemma 1.4(4), we can find, for all $\alpha < \kappa^+$, sets C_α from $\bigcup \mathcal{C}$ and x_α such that for all $\beta < \alpha < \kappa^+$ the point x_α is in C_α but not in C_β . For each $\alpha < \kappa^+$, we choose $p_\alpha \in P$ such that $C_\alpha \in \mathcal{C}_{p_\alpha}$. Similarly, by calibre (κ^+, κ) applied to $\{p_\alpha : \alpha < \kappa^+\}$, there is a κ -sized subset S_0 of κ^+ and a p_0 in P such that p_0 is an upper bound of $\{p_\alpha : \alpha \in S_0\}$. But now we see that, for every α in S_0 , the element x_α is in \mathcal{C}_{p_α} , which is a subset of \mathcal{C}_{p_0} , and also, for all $\alpha, \beta \in S_0$ with $\beta < \alpha$, the point x_α is in C_α but not in C_β . Applying Lemma 1.4(4) again, we deduce \mathcal{C}_{p_0} is not additively $< \kappa$ -Noetherian, contradicting our assumption on the \mathcal{C}_p 's. ■

Proof of Theorem 1.24(2). Suppose P has calibre (ω_1, ω) , and X is a compact space with a P -additively Noetherian point network. Then $Z = X^2$ also has a P -additively Noetherian point network, say $\mathcal{W} = \{\mathcal{W}(z) : z \in Z\}$ where $\mathcal{W}(z) = \bigcup\{\mathcal{W}_p(z) : p \in P\}$. So for each z in Z we see $\mathcal{W}(z)$ is a P -ordered collection of additively Noetherian sets, and hence, by Lemma 1.25, is \aleph_0 -additively Noetherian. Hence $Z = X^2$ is hereditarily meta-Lindelöf. And a compact space is Corson compact if and only if its square is hereditarily meta-Lindelöf. ■

2. Constructing small expandable networks, and point networks

The problem. Our key theorem (Theorem 1.16) states that for a compact space X and directed set $P \geq_T \mathbb{N}$, the space X has a P -point finite almost subbase if and only if it has a P -point additively Noetherian base (or expandable network), if and only if it has a P -additively Noetherian (strong) point network. There is an asymmetry here between the situation with almost subbases compared with bases, expandable networks and (strong) point networks. While finite families are additively Noetherian, the converse is not true. The natural problem is whether every compact space with a P -point finite almost subbase has a P -point finite base or expandable network, and a P -finite (strong) point network.

The problem has a negative solution for the ‘base’ version. Any space with an \mathbb{N} -point finite base (or even point countable base) is clearly first countable. But $A(\omega_1)$ has an \mathbb{N} -point finite almost subbase but is not first countable. In this section we give solutions to the remaining parts of the problem, which, while incomplete, cover the most important cases (when $P = \mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$ or $\mathcal{K}(M)$).

We first show that, provided P has calibre (κ^+, κ) , a space has a P -point- $<\kappa$ expandable network if and only if it has a P -point- $<\kappa$ strong point network. So only two questions remain. For general directed sets P it is not clear to the authors that having a P -point finite almost subbase implies the existence of a P -point finite expandable network (or even a point network). But the partial orders motivating us, namely \mathbb{N} , $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{K}(M)$, are not only partial orders but also have natural topologies (discrete, product and Vietoris, respectively), which interact nicely with their orders. So we briefly move away from point networks structured by a partial order, and look at point networks structured by a topological space. We then discuss topological directed sets, and their connection with topological point networks. These preliminaries combine in the proof of the main result of this section, Theorem 2.7, which says that, for suitable topological directed sets P (including our motivating examples), every space with a P -point finite almost subbase has a $(P \times \mathbb{N})$ -point finite expandable network.

Strong point networks give expandable networks

THEOREM 2.1. *Let X be a space. Let P be a directed set with calibre (κ^+, κ) . Then the following are equivalent:*

- (1) X has a P -point- $<\kappa$ expandable network,
- (2) X has a P - $<\kappa$ strong point network.

Proof. (1) \Rightarrow (2). This is an instance of Lemma 1.7 (and does not require the calibre restriction on P).

(2) \Rightarrow (1). By hypothesis X has a P - $<\kappa$ strong point network $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, so we can write $\mathcal{W}(x) = \bigcup_{p \in P} \mathcal{W}_p(x)$ for each x , where every \mathcal{W}_p has size $<\kappa$, if $p \leq p'$ then $\mathcal{W}_p(x) \subseteq \mathcal{W}_{p'}(x)$, and if a point x is in an open set U , then there is an open set $V = V(x, U)$ containing x and contained in U , and a $p = p(x, U)$ in P such that whenever $y \in V$ then there is a W in $\mathcal{W}_p(y)$ with $x \in W \subseteq V$.

We construct by transfinite induction for all $\alpha \in \kappa^+$, p in P and points x of X : collections \mathcal{A}_α and $\mathcal{A}_{\alpha, \leq p}$ of pairs of points and open neighborhoods, subsets X_α of X and closed subsets C_α^x of X as follows. Let $C_\alpha^x = \overline{\{y : \exists(y, V) \in \bigcup_{\beta < \alpha} \mathcal{A}_\beta, x \in V\}}$, $X_\alpha = \{x : x \in C_\alpha^x\}$, \mathcal{A}_α be a maximal subcollection of $\{(x, V(x, X \setminus C_\alpha^x)) : x \notin X_\alpha\}$ such that if (x_1, V_1) and (x_2, V_2) are in the collection then either $x_1 \notin V_2$ or $x_2 \notin V_1$, and $\mathcal{A}_{\alpha, \leq p} = \{(x, V) \in \mathcal{A}_\alpha : p(x, X \setminus C_\alpha^x) \leq p\}$. (Note: $C_0^x = \emptyset$, $X_0 = \emptyset$ et cetera.) Let $\mathcal{A}_{\leq p} = \bigcup_\alpha \mathcal{A}_{\alpha, \leq p}$ and $\mathcal{A} = \bigcup_p \mathcal{A}_{\leq p}$.

Now set $\mathcal{N}_p = \{(V \cap W, V) : \text{there is } (x', V) \in \mathcal{A}_{\leq p} \text{ and } W \in \mathcal{W}_p(x')\}$ and let $\mathcal{N} = \bigcup_p \mathcal{N}_p$. Clearly, if $p \leq q$ then $\mathcal{N}_p \subseteq \mathcal{N}_q$. We will show that \mathcal{N} is an expandable network, and each \mathcal{N}_p is point- $<\kappa$.

Fix p . Observe that \mathcal{N}_p is point- $<\kappa$ provided $\mathcal{A}_{\leq p}$ is point- $<\kappa$, in the sense that for any point y the set $(\mathcal{A}_{\leq p})_y := \{(x', V) \in \mathcal{A}_{\leq p} : y \in V\}$ has size $<\kappa$. To establish $\mathcal{A}_{\leq p}$ point- $<\kappa$ we define an injection from $(\mathcal{A}_{\leq p})_y$ into $\mathcal{W}_p(y)$, which we know has size $<\kappa$. Take any (x', V) in $(\mathcal{A}_{\leq p})_y$. So $y \in V$, and by definition of a P -ordered strong point network there is a $W_{(x', V)}$ in $\mathcal{W}_p(y)$ such that $x' \in W_{(x', V)} \subseteq V$.

Let us show that $(x', V) \mapsto W_{(x', V)}$ is the desired injection. Well, suppose (x_1, V_1) and (x_2, V_2) are in $(\mathcal{A}_{\leq p})_y$. Let $W_i = W_{(x_i, V_i)}$ for $i = 1, 2$. Two cases arise. Suppose first both pairs are in some $\mathcal{A}_{\alpha, \leq p}$. Without loss of generality we can suppose $x_1 \notin V_2$. This gives $W_1 \not\subseteq W_2$, and in particular $W_1 \neq W_2$. Now suppose $(x_1, V_1) \in \mathcal{A}_{\alpha, \leq p}$ and $(x_2, V_2) \in \mathcal{A}_{\beta, \leq p}$ where $\alpha < \beta$. If $W_1 = W_2$ then $x_2 \in V_1$ and $x_1 \in V_2$. But this leads to a contradiction because $x_2 \in V_1 = V(x_1, X \setminus C_\alpha^{x_1})$ and $C_\beta^{x_2} \supseteq \{z : \text{there is } (z, V) \in \mathcal{A}_\alpha \text{ and } x_2 \in V\} \ni x_1$, so $x_1 \notin V(x_2, X \setminus C_\beta^{x_2}) = V_2$. Thus $W_1 \neq W_2$ in the second case as well as the first, and our map is indeed injective.

It remains to show that \mathcal{N} is an expandable network. To this end take any point x in an open set U . We will show that x is in some X_α . If so then, by definition of X_α and C_α^x , the neighborhood $V(x, U)$ of x must meet $\{x' : \exists(x', V) \in \bigcup_{\beta < \alpha} \mathcal{A}_\beta, x \in V\}$, and there is a point x' in $V(x, U)$, an element p_0 , and (x', V) in $\mathcal{A}_{\alpha, \leq p_0}$ with $x \in V$. Let p be an upper bound of p_0 and $p(x, U)$. Then (x', V) is in $\mathcal{A}_{\leq p}$ and there is a W in $\mathcal{W}_p(x')$ such that $x \in W \subseteq U$. Now $x \in W \cap V \subseteq U$ and $(W \cap V, V)$ is in \mathcal{N}_p , as desired.

Suppose, for a contradiction, that for all $\alpha < \kappa^+$ the point x is not in X_α . Then for all α the point x is not in C_α^x . As \mathcal{A}_α is a maximal subcollection

of $\{(x', V(x', X \setminus C_\alpha^{x'}) : x' \notin C_\alpha^{x'})\}$, there must be a pair (x', V) in \mathcal{A}_α such that $x \in V$. However $\mathcal{A} = \bigcup_p \mathcal{A}_{\leq p}$, and for each p in P the set $(\mathcal{A}_{\leq p})_x$ has size $< \kappa$, so recalling that P has calibre (κ^+, κ) , applying Lemma 1.25 we see that there can only be $\leq \kappa$ many \mathcal{A}_α containing a pair (x', V) such that $x \in V$, a contradiction. ■

COROLLARY 2.2.

- $(P = \mathbb{N}, \kappa = \aleph_0)$ A space has a σ -point finite expandable network if and only if it has a σ -finite strong point network.
- $(P = \mathcal{K}(M), \kappa = \aleph_0)$ A space has a $\mathcal{K}(M)$ -point finite expandable network if and only if it has a $\mathcal{K}(M)$ -finite strong point network.
- $(P = \mathbb{N}^{\mathbb{N}}, \kappa = \aleph_0)$ A space has an $\mathbb{N}^{\mathbb{N}}$ -point finite expandable network if and only if it has an $\mathbb{N}^{\mathbb{N}}$ -finite strong point network.
- $(P = 1, \kappa = \aleph_1)$ A space has a point countable expandable network if and only if it has a countable strong point network.

Topological point networks. Let X be a space. Given a space Z and property \mathcal{Q} , we say $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ is a Z - \mathcal{Q} (strong) point network (in the topological sense) if for each x in X we can write $\mathcal{W}(x) = \bigcup_{z \in Z} \mathcal{W}_z(x)$, and:

- (1) if x is in an open U , then there is an open $V = V(x, U)$ containing x and contained in U , and a $z = z(x, U)$ in Z such that if $y \in V$ then there is a W in $\mathcal{W}_z(y)$ such that $x \in W \subseteq U$ (respectively, $W \subseteq V$ for the strong version),
- (2) for all x and z there is an open T around z such that $\mathcal{W}_T(x) = \bigcup\{\mathcal{W}_{z'}(x) : z' \in T\}$ has \mathcal{Q} .

LEMMA 2.3. *Let a space X have a Z -finite (strong) point network where Z is a space with P -ordered compact cover. Then X has a P -finite (strong) point network.*

Proof. Let $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ where $\mathcal{W}(x) = \bigcup_{z \in Z} \mathcal{W}_z(x)$ is as in the definition of a Z -finite (strong) point network, and let $Z = \bigcup_{p \in P} K_p$ be a P -ordered compact cover of Z . Fix x in X . For each z in Z fix the open set T_z given by condition (1). Fix p in P . As K_p is compact, some finite subcollection of $\{T_z : z \in Z\}$ covers K_p . By condition (2) we see that $\mathcal{W}_{K_p}(x)$ is finite (as a subset of a finite union of finite sets). Let $\hat{\mathcal{W}} = \{\hat{\mathcal{W}}(x) : x \in X\}$ where $\hat{\mathcal{W}}(x) = \bigcup_p \hat{\mathcal{W}}_p(x)$ and $\hat{\mathcal{W}}_p(x) = \mathcal{W}_{K_p}(x)$. We have just shown that each $\hat{\mathcal{W}}_p(x)$ is finite, and then it is easy to check $\hat{\mathcal{W}}$ is the desired P -finite (strong) point network. ■

Topological directed sets. Topological point networks are relevant here because the directed sets we are most interested in carry natural topologies which interact with the order. We call a directed set with a topology a

topological directed set. Any directed set becomes topological with the discrete topology. More usefully, $\mathbb{N}^{\mathbb{N}}$ with the product topology and any $\mathcal{K}(M)$ with the Vietoris topology (where M is separable metrizable) are separable metrizable topological directed sets in which all down sets are compact and every convergent sequence has an upper bound.

In the next two lemmas we see how the topology of a topological directed set can influence the order, and vice versa.

LEMMA 2.4. *Let P be a topological directed set which is sequential, has countable extent, and every convergent sequence has an upper bound. Then P has calibre (ω_1, ω) .*

Proof. Take any uncountable subset S of P . Since P has countable extent and is sequential, S is not a closed discrete subset, so there is a sequence on S converging to some p in P which is in the closure of $S \setminus \{p\}$. By hypothesis on P , this *infinite* sequence has an upper bound. Thus some infinite subset of S has an upper bound, as required for calibre (ω_1, ω) to hold. ■

LEMMA 2.5. *Let P be a topological directed set which is first countable, and every convergent sequence has an upper bound. Let \mathcal{C} be a P -finite collection of subsets of a space X . Then for any $p \in P$ there exists an open neighborhood T of p such that $\bigcup\{\mathcal{C}_q : q \in T\}$ is finite.*

Proof. Fix p in P . Since P is first countable we can fix $\{B_n : n \in \mathbb{N}\}$ a local base at p . Suppose, for contradiction, that for any open neighborhood T of p the set $\bigcup\{\mathcal{C}_q : q \in T\}$ is infinite. Then for each n , we can find $p_n \in B_n$ and $C_n \in \mathcal{C}_{p_n} \setminus \{C_i : i < n\}$. Note that the sequence $\{p_n : n \in \mathbb{N}\}$ converges to p . By hypothesis, there is $p_0 \in P$ such that $p_n \leq p_0$ for all n . Since \mathcal{C} is P -ordered, $\{C_n : n \in \mathbb{N}\}$ is an infinite subcollection of \mathcal{C}_{p_0} , which is a contradiction. ■

Now we connect topological directed sets and the topological point networks of the previous section.

LEMMA 2.6. *Let P be a topological directed set in which down sets are compact. Then if a space X has a P -finite (strong) point network in the topological sense then it also has a P -finite (strong) point network in the order sense.*

Proof. Since the down sets are compact and $P = \bigcup_{p \in P} (\downarrow p)$, the space P has a P -ordered compact cover, so we can apply Lemma 2.3. ■

Some almost subbases give small expandable networks

THEOREM 2.7. *Let P be a topological directed set which is separable metrizable, down sets are compact and every convergent sequence has an upper bound. If a space X has a P -point finite almost subbase then it has a $(P \times \mathbb{N})$ -point finite expandable network.*

Proof. Let \mathcal{B} be a countable base for P . Give $Q = [\mathcal{B}]^{<\omega}$ the discrete topology, and order it by \subseteq . Note that $P \times \mathbb{N} \times Q$ is a topological directed set (with product partial order and topology) with the same properties hypothesized for P . We will show that X has a $(P \times \mathbb{N} \times Q)$ -finite topological strong point network. Then, by Lemma 2.6, X has a $(P \times \mathbb{N} \times Q)$ -finite strong point network (in the order sense). Lemma 2.4 permits us to apply Theorem 2.1, and, after noting that $P \times \mathbb{N} \times Q =_T P \times \mathbb{N}$, the proof is complete.

Let $\alpha = \bigcup\{\alpha_p : p \in P\}$ be an almost subbase for X , with respect to $\{f_V\}_{V \in \alpha}$, where each α_p is point finite and $\alpha_p \subseteq \alpha_{p'}$ if $p \leq p'$. We assume, without loss of generality, that each α_p is closed under finite intersections. For any $S \subseteq P$ write $\alpha_S = \bigcup\{\alpha_p : p \in S\}$. Fix an enumeration of \mathcal{B} . Because P has the relevant order and topological properties, we know (Lemma 2.5) that for every p in P and x in X there is a B in \mathcal{B} containing p such that $(\alpha_B)_x$ is finite. Let $B(p, x)$ be the first member in the enumeration of \mathcal{B} such that $p \in B(p, x)$ and $(\alpha_{B(p,x)})_x$ is finite.

Fix $x \in X$. Fix $p \in P$, $m \in \mathbb{N}$ and \mathcal{B}_0 in Q . Let $\mathcal{W}_{p,m,\mathcal{B}_0}(x)$ be all (finite) intersections of:

- (i) $A \in (\alpha_{B(p,x)})_x$,
- (ii) sets of the form $A \setminus f_{A'}^{-1}[1/n, 1]$ where A and A' are in $(\alpha_{B(p,x)})_x$, $n \leq m$ and $x \notin f_{A'}^{-1}[1/n, 1]$, and
- (iii) $W_{B,x} = X \setminus \bigcup\{A \in \alpha_B : x \notin A\}$ for B in \mathcal{B}_0 .

Note that $\mathcal{W}_{p,m,\mathcal{B}_0}(x)$ is finite. Let $\mathcal{W}(x) = \bigcup\{\mathcal{W}_{p,m,\mathcal{B}_0}(x) : p \in P, m \in \mathbb{N} \text{ and } \mathcal{B}_0 \in Q\}$.

We show that $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ is a $(P \times \mathbb{N} \times Q)$ -finite topological strong point network. To do so we need to check (1) and (2) in the definition.

For (1): Take any point x in an open set T . Since $\alpha \cup \{X \setminus f_A^{-1}[1/n, 1] : A \in \alpha\}$ is a subbase and P is directed, there is a $p_0 \in P$, $M \in \mathbb{N}$, $A_0 \in \alpha_{p_0}$, A_1, \dots, A_k in α_{p_0} , $n_1, \dots, n_k \leq M$ such that $x \in A_0 \cap \bigcap_{i=1}^k (X \setminus f_{A_i}^{-1}[1/n_i, 1]) \subseteq T$.

Let $p(x, T) = (p_0, M, \{B(p_0, x)\})$ and $V = V(x, T) = \bigcap (\alpha_{B(p_0,x)})_x \cap \bigcap_{i=1}^k (X \setminus f_{A_i}^{-1}[1/n_i, 1])$. Then x is in V , which is open, and $V \subseteq T$.

Take any y in V . Then y is in $(\alpha_{B(p_0,x)})_x$, so $(\alpha_{B(p_0,x)})_y \supseteq (\alpha_{B(p_0,x)})_x$. It follows that $x \in W_{B(p_0,x),x} \subseteq W_{B(p_0,x),y}$. Let $V_0 = \bigcap (\alpha_{B(p_0,x)})_x$. Relabelling if necessary, we can let $V_1 = \bigcap_{i=1}^p (A_0 \setminus f_{A_i}^{-1}[1/n_i, 1])$ and $V_2 = \bigcap_{i=p+1}^k (X \setminus f_{A_i}^{-1}[1/n_i, 1])$, where y is in A_i for $i \leq p$ but $y \notin A_i$ for $i > p$. So $V = V_0 \cap V_1 \cap V_2$. Let $W = \bigcap (\alpha_{B(p_0,x)})_x \cap \bigcap_{i=1}^p (A_0 \setminus f_{A_i}^{-1}[1/n_i, 1]) \cap W_{B(p_0,x),y}$. Then each element in this (finite) intersection is as required for W to be in $\mathcal{W}_{p(x,T)}(y) = \mathcal{W}_{p_0,M,\{B(p_0,x)\}}(y)$. Each element in the intersection contains x , so x is in W .

It remains to show that W is contained in V . Clearly $W = V_0 \cap V_1 \cap W_{B(p_0,x),y}$. So it suffices to see that $W_{B(p_0,x),y} \subseteq V$. But for $i > p$, as $y \notin A_i$, by definition we see that $W_{B(p_0,x),y} \subseteq X \setminus A_i$ and $X \setminus A_i \subseteq X \setminus f_{A_i}^{-1}[1/n_i, 1]$. Hence $W_{B(p_0,x),y} \subseteq \bigcap_{i=p+1}^k (X \setminus f_{A_i}^{-1}[1/n_i, 1])$.

For (2): Fix (p, m, \mathcal{B}_0) in $P \times \mathbb{N} \times Q$. We need to show that there is a neighborhood T of it such that $\mathcal{W}_T(x)$ is finite. Indeed, we claim $\mathcal{W}_{B(p,x) \times \{(m, \mathcal{B}_0)\}}(x)$ is finite. Take any $p' \in B(p, x)$. Elements of $\mathcal{W}_{p',m,\mathcal{B}_0}(x)$ are intersections of three types. Those of type (iii) are exactly the same as in $\mathcal{W}_{p,m,\mathcal{B}_0}(x)$. Those of type (i) and (ii) come from A 's in $(\alpha_{B(p',x)})_x$. By definition, $B(p', x)$ is either $B(p, m)$ or occurs before it in the enumeration of \mathcal{B} . Hence $\{B(p', x) : p' \in B(p, x)\}$ is finite. So there are only finitely many A 's going into types (i) and (ii), and hence $\mathcal{W}_{B(p,x) \times \{(m, \mathcal{B}_0)\}}(x)$ is finite, as claimed. ■

Further characterizations

THEOREM 2.8. *Let X be compact. Let P be a topological directed set which is separable metrizable, but not compact, down sets are compact, and every convergent sequence has an upper bound. Then the following are equivalent:*

- (0) X is P -Eberlein compact,
- (1) X has a P -point finite almost subbase,
- (2) X has a P -point additively Noetherian base,
- (3) X has a P -point finite expandable network,
- (4) X has a P -finite strong point network,
- (5) X has a P -finite point network.

THEOREM 2.9. *Let X be compact. Then the following are equivalent:*

- (0) X is Eberlein compact,
- (1) X has an \mathbb{N} -point finite almost subbase,
- (2) X has an \mathbb{N} -point finite expandable network,
- (3) X has an \mathbb{N} -finite point network.

As mentioned above, the equivalence of (0) and (2) is due to [DJP].

THEOREM 2.10. *Let X be compact. Then the following are equivalent:*

- (0) X is Talagrand compact,
- (1) X has an $\mathbb{N}^{\mathbb{N}}$ -point finite almost subbase,
- (2) X has an $\mathbb{N}^{\mathbb{N}}$ -point finite expandable network,
- (3) X has an $\mathbb{N}^{\mathbb{N}}$ -finite point network.

THEOREM 2.11. *Let X be compact. Then the following are equivalent:*

- (0) X is Gul'ko compact,
- (1) X has a $\mathcal{K}(M)$ -point finite almost subbase,

- (2) X has a $\mathcal{K}(M)$ -point finite expandable network,
- (3) X has a $\mathcal{K}(M)$ -finite point network,

where in (1)–(3), M is some separable metrizable space.

At first glance the equivalence of (0) and (2) in the preceding two results are related to results of García, Oncina and Orihuela [GOO]. However Orihuela [O], on behalf of the authors of [GOO], has informed the present authors that the definition of ‘index- M -point finite family’ (Definition 13) needs to be modified by adding the following sentence: ‘with $I_{K_1} \subset I_{K_2}$ and ξ be the inclusion map whenever $K_1 \subset K_2$ ’ to condition (ii). With this change their solution, presented in [GOO], to a problem of Gruenhage is correct. But it also means that the proof of Theorem 18 (which is used in their argument that every Gul’ko compact space has a Σ -point-finite expandable network) is incorrect. On the other hand, without this change Lemma 15 is false, and every space with network weight no more than the continuum has a Σ -point finite expandable network. Either way, no characterization of Gul’ko (or Talagrand) compacta in terms of a network is obtained, and the authors of [GOO] have withdrawn this claim.

Without compactness. Theorem 2.8 is stated for compact spaces. However the implications (1) \Rightarrow (2), (3) \Leftrightarrow (4), and (4) \Rightarrow (5) hold for general spaces. The authors do not know of an example when condition (5) does not give condition (4). The following example gives a strong (non-compact) counter-example to (3) \Rightarrow (2) with $P = \mathbb{N}$.

LEMMA 2.12. *There is a countable space with a point finite expandable network but no \mathbb{N} -point additively Noetherian base.*

Proof. Let F_ω be the Fréchet fan. So F_ω is obtained from $\omega \times (\omega + 1)$ by identifying the end points (m, ω) to a point $*$. Let $\mathcal{N} = \{(\{m, n\}, \{(m, n)\}) : m, n \in \omega\} \cup \{(\{*\}, F_\omega)\}$. This can easily be checked to be a point finite expandable network.

To complete the proof we show that $*$ has no \mathbb{N} -additively Noetherian base. If U is an open neighborhood of $*$ then define f_U in ω^ω by $f_U(m) = \min\{n : (m, n) \in U\}$. Conversely, if $g \in \omega^\omega$ then define $U_g = \{(m, n) : g(m) \leq n\} \cup \{*\}$ an open neighborhood of $*$. Suppose $\bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$ is a neighborhood base for $*$. We show there is an N such that \mathcal{B}_N is not additively Noetherian. Observe that $\bigcup\{\mathcal{F}_n : n \in \mathbb{N}\}$ where $\mathcal{F}_n = \{f_U : U \in \mathcal{B}_n\}$ is a dominating family in (ω^ω, \leq^*) (here \leq^* is the mod-finite order). So for some N we can see that \mathcal{F}_N is dominating in (ω^ω, \leq^*) . Let $\mathcal{B} = \mathcal{B}_N$ and $\mathcal{F} = \mathcal{F}_N$. Hence there is a subset $\mathcal{G} = \{g_\alpha : \alpha < \mathfrak{b}\}$ of \mathcal{F} such that \mathcal{G} is unbounded and $g_\alpha <^* g_\beta$ if $\alpha < \beta$, where \mathfrak{b} is the minimal size of an unbounded family in (ω^ω, \leq^*) . Define, by recursion, a decreasing sequence

$(S_n)_{n \in \omega}$ of cofinal subsets of \mathfrak{b} such that, for all $\alpha \in S_n$, $g_\alpha(n) = k_n$ for some fixed k_n . Define k in ω^ω by $k(n) = k_n$. Set $n_0 = 0$. As \mathcal{G} is unbounded, and S_0 is cofinal in \mathfrak{b} , there is an α_0 in S_0 such that $g_{\alpha_0} \not\leq^* k$. So, for infinitely many n , $k(n) < g_{\alpha_0}(n)$. Let n_1 be the first such n . Next choose $\alpha_1 \in S_{n_1}$, $\alpha_0 < \alpha_1$. Then $g_{\alpha_0} <^* g_{\alpha_1}$ and $g_{\alpha_1}(n_1) = k(n_1) < g_{\alpha_0}(n_1)$. And now choose $n_2 > n_1$ such that $k(n_2) < g_{\alpha_0}(n_2) < g_{\alpha_1}(n_2)$. Proceeding inductively, we find a strictly increasing sequence $(\alpha_i)_{i \in \omega}$ of ordinals and $(n_i)_i$ of integers such that, for every $p \in \omega$, $g_{\alpha_p}(n_p) = k(n_p) < \min\{g_{\alpha_i}(n_p) : 0 \leq i < p\}$ and $(g_{\alpha_i})_i$ is strictly increasing with respect to $<^*$. The g_{α_i} 's are in \mathcal{F} , and hence of the form $g_{\alpha_i} = f_{U_i}$ for some U_i in \mathcal{B} . So, by definition of f_{U_i} , for each p , $(n_p, g_{\alpha_p}(n_p)) \in U_p$, but, since $g_{\alpha_p}(n_p) < \min\{g_{\alpha_i}(n_p) : 0 \leq i < p\}$, for $i < p$, we have $(n_p, g_{\alpha_p}(n_p)) \notin U_i$. From the last line and Lemma 1.4(4), it follows that $\mathcal{B} = \mathcal{B}_N$ is not additively Noetherian, as desired. ■

Our final example shows that for non-compact spaces condition (2) in Theorem 2.8 does not imply (5), even with $P = 1$ in the hypothesis and ‘ P of calibre (ω_1, ω) ’ in the conclusion.

LEMMA 2.13. *There is a space with point additively Noetherian base but no countable point network, and hence no P -finite point network for any P with calibre (ω_1, ω) .*

Proof. Let $X = \{0, 1\}^{\omega_1}$. For any $\mathbf{x} \in X$, a basic open neighborhood of \mathbf{x} is $B_\alpha(\mathbf{x}) = \{\mathbf{y} : \mathbf{y}(\gamma) = \mathbf{x}(\gamma) \text{ for all } \gamma < \alpha\}$ for $\alpha < \omega_1$. Then $\mathcal{B} = \{B_\alpha(\mathbf{x}) : \mathbf{x} \in X \text{ and } \alpha < \omega_1\}$ is a base. Note that: (1) given any two elements of \mathcal{B} , either they have empty intersection or one is a subset of the other; (2) any countable intersection of \mathcal{B} is either empty or in \mathcal{B} ; and (3) any countable subset of X is closed and discrete. For any $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x} \in B_\alpha(\mathbf{y})$ implies that $B_\alpha(\mathbf{x}) = B_\alpha(\mathbf{y})$. Therefore, for any $\mathbf{x} \in X$, $(\mathcal{B})_{\mathbf{x}} = \{B_\alpha(\mathbf{x})\}$. We can see that $(\mathcal{B})_{\mathbf{x}}$ ordered by reverse inclusion is order-isomorphic to ω_1 . So \mathcal{B} is point additively Noetherian.

Suppose, for a contradiction, that X has a countable point network, $\mathcal{W} = \bigcup\{\mathcal{W}(\mathbf{x}) : \mathbf{x} \in X\}$. Also, for each $\mathbf{x} \in X$ and an open set U containing \mathbf{x} , we have the corresponding $V(\mathbf{x}, U)$. We can assume that $V(\mathbf{x}, U)$ is in \mathcal{B} .

Pick $\mathbf{x}_0 \in X$, and let $V_0 = V(\mathbf{x}_0, X)$. If α is a successor, pick $\mathbf{x}_\alpha \in V_{\alpha-}$ and $\mathbf{x}_\alpha \neq \mathbf{x}_\beta$ for all $\beta < \alpha$, and also let V_α be a basic open neighborhood of \mathbf{x}_α which is a subset of $V(x_\alpha, V_{\alpha-}) \cap B_\alpha(\mathbf{x}_\alpha) \setminus \{\mathbf{x}_\beta : \beta < \alpha\}$. If α is a limit ordinal, pick $\mathbf{x}_\alpha \in \bigcap\{V_\beta : \beta < \alpha\}$ with $\mathbf{x}_\alpha \neq \mathbf{x}_\beta$ for all $\beta < \alpha$, and let V_α be a basic open neighborhood of \mathbf{x}_α which is a subset of $V(\mathbf{x}_\alpha, \bigcap\{V_\beta : \beta < \alpha\}) \cap B_\alpha(\mathbf{x}_\alpha) \setminus \{\mathbf{x}_\beta : \beta < \alpha\}$. We get a transfinite sequence of points $\{\mathbf{x}_\alpha : \alpha < \omega_1\}$, together with a decreasing transfinite sequence of basic open sets $\{V_\alpha : \alpha < \omega_1\}$. Note that $\bigcap\{V_\alpha : \alpha < \omega_1\}$ is not empty. Then there

is a $\mathbf{y} \in \bigcap \{V_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$, there is a $W_\alpha \in \mathcal{W}(\mathbf{y})$ such that $\mathbf{x}_{\alpha+1} \in W_\alpha \subseteq V_\alpha$. Since $\mathcal{W}(\mathbf{y})$ is countable, there exists an uncountable subset A of ω_1 such that W_α is a fixed W for each $\alpha \in A$. Since $\mathbf{y} \in B_\alpha(\mathbf{x}_\alpha)$, we have $B_\alpha(\mathbf{x}_\alpha) = B_\alpha(\mathbf{y})$. Choose $\beta \in A$; then $\mathbf{x}_{\beta+1} \in W$. Since $\mathbf{x}_{\beta+1} \neq \mathbf{y}$ and A is uncountable, we can find $\gamma \in A$ such that $\mathbf{x}_{\beta+1} \notin B_\gamma(\mathbf{y})$. However, since $\mathbf{y} \in V(\mathbf{x}_{\gamma+1}, V_\gamma)$, we have $W \subseteq V_\gamma$, which is a subset of $B_\gamma(\mathbf{y})$. This contradicts the fact that $\mathbf{x}_{\beta+1} \notin B_\gamma(\mathbf{y})$. ■

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