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HALF-LIBERATED REAL SPHERES AND THEIR SUBSPACES

BY

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Abstract. We describe the quantum subspaces of Banica–Goswami's half-liberated real spheres, showing in particular that there is a bijection between the symmetric ones and the conjugation stable closed subspaces of the complex projective spaces.

1. Introduction. Let $n \geq 1$. The *half-liberated real sphere* $S^{n-1}_{\mathbb{R},*}$ was defined by Banica and Goswami [4] as the quantum space corresponding to the C^*-algebra

$$C(S^{n-1}_{\mathbb{R},*}) = C^*\Big(v_1, \dots, v_n \;\Big|\; \sum_{i=1}^n v_i^2 = 1,\, v_i^* = v_i,\\ v_iv_jv_k = v_kv_jv_i,\, 1 \leq i,j,k \leq n\Big).$$

It corresponds to a natural quantum homogeneous space over the half-liberated orthogonal quantum group O_n^* introduced by Banica and Speicher [5]. These quantum spaces and groups, although defined by very simple means, namely the intriguing half-commutativity relations $abc = cba$ arising from representation-theoretic considerations via Woronowicz' Tannaka–Krein duality [11], turned out to be new in the field, and definitely of interest. See [1, 2] for recent developments and general discussions on non-commutative spheres.

The aim of this paper is to describe the quantum subspaces of $S^{n-1}_{\mathbb{R},*}$. We will show in particular that there is a natural bijection between the symmetric ones (see Section 3 for the definition) and the conjugation stable closed subspaces of the complex projective space $P^{n-1}_{\mathbb{C}}$. The description of all subspaces is also given, but it is more technical, and uses representation theory methods, inspired by those employed by Podleś [9] in the determination of the quantum subgroups of $\mathrm{SU}_{-1}(2) \simeq O_2^*$. It follows from our analysis that the quantum subspaces of $S^{n-1}_{\mathbb{R},*}$ can be completely described by means of classical spaces, a fact already noted in [7] in the description of the quantum subgroups of O_n^*. As in [7], a crossed product model pro-

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vides the bridge linking the quantum subspaces and the subspaces of an appropriate classical space.

In fact, for the description of all the quantum subspaces, we will work in a more general context, where a quantum space $Z_{\mathbb{R},*}$ is associated to any compact space Z endowed with an appropriate $\mathbb{T} \rtimes \mathbb{Z}_2 \simeq O_2$ continuous action, and where $S^{n-1}_{\mathbb{R},*}$ is obtained from the complex sphere $S^{n-1}_{\mathbb{C}}$. We do not know if our general framework really furnishes new examples of interest, however we think that, as often with abstract settings, it has the merit to clean up arguments, can ultimately simplify the theory, and could be well-suited to other developments, such as K-theory computations.

The paper is organized as follows. Section 2 consists of preliminaries. In Section 3, we construct a faithful crossed product representation of $C(S^{n-1}_{\mathbb{R},*})$ and use the sign automorphism to construct a $\mathbb{Z}_2$-grading on $C(S^{n-1}_{\mathbb{R},*})$. We then show that there is an explicit bijective correspondence between symmetric quantum subspaces of $S^{n-1}_{\mathbb{R},*}$ (corresponding to $\mathbb{Z}_2$-graded ideals of $C(S^{n-1}_{\mathbb{R},*})$) and conjugation stable closed subspaces of the complex projective space $P^{n-1}_{\mathbb{C}}$. Section 4, in which we work in a slightly more general framework, is devoted to the description of all the quantum subspaces of $S^{n-1}_{\mathbb{R},*}$, in terms of pairs of certain subspaces of the complex sphere $S^{n-1}_{\mathbb{C}}$.

2. Notation and preliminaries

2.1. Conventions. All C^*-algebras are assumed to be unital, as well as all C^*-algebra maps. By a compact space we mean a compact Hausdorff space, unless otherwise specified. The categories of compact spaces and of commutative C^*-algebras are anti-equivalent by the classical Gelfand duality, and we use, in a standard way, the language of quantum spaces: a C^*-algebra A is viewed as the algebra of continuous functions on a (uniquely determined) quantum space X, and we write $A = C(X)$. Quantum subspaces $Y \subset X$ correspond to surjective $*$-algebra maps $C(X) \to C(Y)$, and hence as well to ideals of $C(X)$. A quantum space X is said to be *classical* (resp. *non-classical*) if the C^*-algebra $C(X)$ is commutative (resp. non-commutative).

2.2. Classical spaces. The real and complex spheres are denoted respectively by $S^{n-1}_{\mathbb{C}}$ and $S^{n-1}_{\mathbb{R}}$, and their C^*-algebras of continuous functions always are endowed with their usual presentations:

$$C(S^{n-1}_{\mathbb{C}}) = C^*\Big(z_1, \dots, z_n \,\Big|\, \sum_{i=1}^n z_i z_i^* = 1,\, z_i z_j = z_j z_i,\\ z_i z_j^* = z_j^* z_i,\, 1 \le i, j, \le n\Big)$$

where $z_1, \dots, z_n$ are the standard coordinate functions, and

$$C(S_{\mathbb{R}}^{n-1}) = C^*\Big(x_1, \dots, x_n \;\Big|\; \sum_{i=1}^{n} x_i^2 = 1,\, x_i^* = x_i,$$
$$x_i x_j = x_j x_i,\, 1 \le i, j \le n\Big)$$

where $x_1, \dots, x_n$ are the standard coordinate functions. The sphere $S_{\mathbb{C}}^0$ is, as usual, denoted $\mathbb{T}$.

The complex projective space $P_{\mathbb{C}}^{n-1}$ is the orbit space $S_{\mathbb{C}}^{n-1}/\mathbb{T}$ for the usual $\mathbb{T}$-action by multiplication: $C(P_{\mathbb{C}}^{n-1})$ is isomorphic to a C^*-subalgebra of $C(S_{\mathbb{C}}^{n-1})$, via the natural identification $C(S_{\mathbb{C}}^{n-1}/\mathbb{T}) \simeq C(S_{\mathbb{C}}^{n-1})^{\mathbb{T}}$, the latter C^*-algebra being the C^*-subalgebra of $C(S_{\mathbb{C}}^{n-1})$ generated by the elements $z_i z_j^*$ (by the Stone–Weierstrass theorem). Moreover, the C^*-algebra $C(P_{\mathbb{C}}^{n-1})$ has the following presentation, communicated to me by T. Banica.

Lemma 2.1. *The C^*-algebra $C(P_{\mathbb{C}}^{n-1})$ has the presentation*

$$C(P_{\mathbb{C}}^{n-1}) \simeq C^*(p_{ij},\, 1 \le i, j \le n \mid p = p^* = p^2,\, \mathrm{Tr}(p) = 1,$$
$$p_{ij} p_{kl} = p_{kl} p_{ij},\, 1 \le i, j \le n)$$

where p denotes the matrix (p_{ij}), and where the element p_{ij} corresponds to the element $z_i z_j^*$.

Proof. Denote by A the C^*-algebra on the right. It is straightforward to check that there exists a $*$-algebra map

$$A \to C(P_{\mathbb{C}}^{n-1}), \qquad p_{ij} \mapsto z_i z_j^*,$$

which is surjective. To show the injectivity, it is enough, by Gelfand duality, to show that the corresponding continuous map

$$P_{\mathbb{C}}^{n-1} = S_{\mathbb{C}}^{n-1}/\mathbb{T} \to \{p \in M_n(\mathbb{C}) \mid p = p^* = p^2,\, \mathrm{Tr}(p) = 1\},$$
$$(z_1, \dots, z_n) \mapsto (z_i z_j^*),$$

is surjective, which follows from the structure of rank 1 projections. ∎

To conclude this section, we introduce a last piece of notation. The complex conjugation induces an order 2 automorphism of $C(S_{\mathbb{C}}^{n-1})$, which we denote τ, with $\tau(z_i) = z_i^*$. This enables us to form the crossed product $C(S_{\mathbb{C}}^{n-1}) \rtimes \mathbb{Z}_2$, which we use intensively in the rest of the paper.

3. Half-liberated spheres and their symmetric subspaces. Recall from the introduction that the half-liberated real sphere $S_{\mathbb{R},*}^{n-1}$ [4] is the quantum space corresponding to the C^*-algebra

$$C(S_{\mathbb{R},*}^{n-1}) = C^*\Big(v_1, \dots, v_n \;\Big|\; \sum_{i=1}^{n} v_i^2 = 1,\, v_i^* = v_i,$$
$$v_i v_j v_k = v_k v_j v_i,\, 1 \le i, j, k \le n\Big).$$

3.1. $\mathbb{Z}_2$-grading on $C(S^{n-1}_{\mathbb{R},*})$

DEFINITION 3.1. The *sign automorphism* of $C(S^{n-1}_{\mathbb{R},*})$, denoted ν, is the automorphism defined by $\nu(v_i) = -v_i$ for any i.

The sign automorphism defines a $\mathbb{Z}_2$-grading on the algebra $C(S^{n-1}_{\mathbb{R},*})$:

$$C(S^{n-1}_{\mathbb{R},*}) = C(S^{n-1}_{\mathbb{R},*})_0 \oplus C(S^{n-1}_{\mathbb{R},*})_1$$

where $C(S^{n-1}_{\mathbb{R},*})_0 = \{a \in C(S^{n-1}_{\mathbb{R},*}) \mid \nu(a) = a\}$ and $C(S^{n-1}_{\mathbb{R},*})_1 = \{a \in C(S^{n-1}_{\mathbb{R},*}) \mid \nu(a) = -a\}$. Here of course $C(S^{n-1}_{\mathbb{R},*})_0$ is the fixed point algebra for the $\mathbb{Z}_2$-action on $C(S^{n-1}_{\mathbb{R},*})$ defined by ν. It has the following description.

LEMMA 3.2. *The C^*-subalgebra generated by the elements v_iv_j, $1 \leq i,j \leq n$, is commutative, coincides with $C(S^{n-1}_{\mathbb{R},*})_0$, and we have a $*$-algebra isomorphism*

$$\Phi : C(P^{n-1}_{\mathbb{C}}) \to C(S^{n-1}_{\mathbb{R},*})_0, \qquad z_iz_j^* \mapsto v_iv_j.$$

Proof. The commutativity of $C^*(v_iv_j)$, a fundamental and direct observation, is known [6, 4]. It is clear that $v_iv_j \in C(S^{n-1}_{\mathbb{R},*})_0$ for any i,j, hence $C^*(v_iv_j) \subset C(S^{n-1}_{\mathbb{R},*})_0$. Denote by $\mathcal{O}(S^{n-1}_{\mathbb{R},*})$ the dense $*$-subalgebra of $C(S^{n-1}_{\mathbb{R},*})$ generated by the elements v_i. It is clear that $\mathcal{O}(S^{n-1}_{\mathbb{R},*})_0$ is the linear span of monomials of even length, hence is generated as an algebra by the elements v_iv_j. Therefore we get the announced result, since $\mathcal{O}(S^{n-1}_{\mathbb{R},*})_0$ is dense in $C(S^{n-1}_{\mathbb{R},*})_0$ (if A is a C^*-algebra acted on by a finite group and $\mathcal{A} \subset A$ is a dense $*$-subalgebra, then $\mathcal{A}^G$ is dense in A^G).

The existence of Φ follows from Lemma 2.1, and Φ is surjective by the previous discussion. The injectivity is [4, Theorem 3.3]. For the sake of completeness, we will present, during the proof of Theorem 3.4 below, another elementary proof of the injectivity of Φ (the reasoning in [4] relied on much more sophisticated diagrammatic quantum group techniques). ∎

DEFINITION 3.3. A quantum subspace $X \subset S^{n-1}_{\mathbb{R},*}$ is said to be *symmetric* if the corresponding ideal I is $\mathbb{Z}_2$-graded, i.e.

$$I = I_0 \oplus I_1, \quad \text{with} \quad I_0 = I \cap C(S^{n-1}_{\mathbb{R},*})_0 \quad \text{and} \quad I_1 = I \cap C(S^{n-1}_{\mathbb{R},*})_1,$$

or in other words, if ν induces an automorphism of $C(S^{n-1}_{\mathbb{R},*})/I$.

3.2. Faithful crossed product representation of $C(S^{n-1}_{\mathbb{R},*})$. We now describe a crossed product model for $C(S^{n-1}_{\mathbb{R},*})$, using the crossed product $C(S^{n-1}_{\mathbb{C}}) \rtimes \mathbb{Z}_2$ associated to the conjugation action on $S^{n-1}_{\mathbb{C}}$ (see the previous section). A related construction was already considered in [7] in the quantum group setting.

THEOREM 3.4. *There exists an injective $*$-algebra map*

$$\pi : C(S^{n-1}_{\mathbb{R},*}) \to C(S^{n-1}_{\mathbb{C}}) \rtimes \mathbb{Z}_2, \qquad v_i \mapsto z_i \otimes \tau.$$

Proof. It is straightforward to construct π, and this is left to the reader. We have $\pi(v_iv_j) = (z_i \otimes \tau)(z_j \otimes \tau) = z_iz_j^* \otimes 1$, hence $\pi\Phi = \mathrm{id} \otimes 1$, and this indeed gives another proof for the injectivity of Φ in Lemma 3.2.

We have to show that π is injective. First recall the following general fact: if A and B are $\mathbb{Z}_2$-graded C^*-algebras and $\pi : A \to B$ is a $*$-algebra map preserving the $\mathbb{Z}_2$-grading, then π is injective if and only if the restriction of π to A_0 is injective. Indeed, assume that $\pi_{|A_0}$ is injective. To show the injectivity of π, we just have to show that $\pi_{|A_1}$ is injective (since π preserves the $\mathbb{Z}_2$-grading). So let $a \in A_1$ with $\pi(a) = 0$. We have $a^*a \in A_0$ and $\pi(a)^*\pi(a) = \pi(a^*a) = 0$, hence $a^*a = 0$ and $a = 0$; thus π is injective.

Now note that $C(S^{n-1}_{\mathbb{C}}) \rtimes \mathbb{Z}_2$ is $\mathbb{Z}_2$-graded as well, with grading defined by

$$(C(S^{n-1}_{\mathbb{C}}) \rtimes \mathbb{Z}_2)_0 = C(S^{n-1}_{\mathbb{C}}) \otimes 1, \qquad (C(S^{n-1}_{\mathbb{C}}) \rtimes \mathbb{Z}_2)_1 = C(S^{n-1}_{\mathbb{C}}) \otimes \tau,$$

and that π preserves the respective $\mathbb{Z}_2$-gradings. The previous discussion shows that it is enough to show that the restriction of π to $C(S^{n-1}_{\mathbb{R},*})_0$ is injective, which is immediate since the restriction of π to $C(S^{n-1}_{\mathbb{R},*})_0$ is the injective map $\Phi^{-1} \otimes 1$. ■

3.3. Symmetric subspaces of half-liberated spheres. Before describing the symmetric subspaces of $S^{n-1}_{\mathbb{R},*}$, we need another ingredient.

DEFINITION 3.5. We denote by γ the linear endomorphism of $C(S^{n-1}_{\mathbb{R},*})$ defined by

$$\gamma(a) = \sum_{i=1}^{n} v_i a v_i.$$

The main properties of γ are summarized in the following lemma.

LEMMA 3.6. *The endomorphism γ preserves the $\mathbb{Z}_2$-grading of $C(S^{n-1}_{\mathbb{R},*})$, and induces a $*$-algebra automorphism of $C(S^{n-1}_{\mathbb{R},*})_0$. Moreover, the following diagram commutes:*

$$\begin{array}{ccc} C(P^{n-1}_{\mathbb{C}}) & \xrightarrow{\Phi} & C(S^{n-1}_{\mathbb{R},*})_0 \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\gamma} \\ C(P^{n-1}_{\mathbb{C}}) & \xrightarrow{\Phi} & C(S^{n-1}_{\mathbb{R},*})_0 \end{array}$$

where τ is the automorphism induced by complex conjugation, i.e. $\tau(z_iz_j^*) = z_jz_i^*$. Hence there is a bijective correspondence between γ-stable ideals of $C(S^{n-1}_{\mathbb{R},*})_0$ and conjugation stable closed subsets of $P^{n-1}_{\mathbb{C}}$.

Proof. It is clear that γ preserves the $\mathbb{Z}_2$-grading of $C(S^{n-1}_{\mathbb{R},*})$, that $\gamma(1) = 1$ and that γ commutes with the involution. We have

$$\begin{aligned}\gamma(v_{i_1}v_{j_1}\cdots v_{i_m}v_{j_m}) &= \sum_k v_k v_{i_1}v_{j_1}\cdots v_{i_m}v_{j_m}v_k \\ &= \sum_k v_{j_1}v_{i_1}\cdots v_{j_m}v_{i_m}v_k v_k = v_{j_1}v_{i_1}\cdots v_{j_m}v_{i_m},\end{aligned}$$

which is shown by induction on m, using the half-commutation relations. This proves that the diagram commutes, and at the same time γ preserves multiplication, and is an automorphism. The last assertion of the lemma then follows immediately from the correspondence between conjugation closed subspaces of $P^{n-1}_{\mathbb{C}}$ and τ-stable ideals of $C(P^{n-1}_{\mathbb{C}})$. ∎

The description of the $\mathbb{Z}_2$-graded ideals of $C(S^{n-1}_{\mathbb{R},*})$ is then as follows.

THEOREM 3.7. *We have a bijective correspondence*

$$\begin{aligned}\{\mathbb{Z}_2\text{-}graded\ ideals\ of\ C(S^{n-1}_{\mathbb{R},*})\} &\leftrightarrow \{\gamma\text{-}stable\ ideals\ of\ C(S^{n-1}_{\mathbb{R},*})_0\},\\ I &\mapsto I_0 = I\cap C(S^{n-1}_{\mathbb{R},*})_0,\\ \langle J\rangle = J + C(S^{n-1}_{\mathbb{R},*})_1 J &\mapsfrom J.\end{aligned}$$

Proof. For notational simplicity, we write $A = C(S^{n-1}_{\mathbb{R},*})$. Let $I = I_0 + I_1$ be a $\mathbb{Z}_2$-graded ideal of A. It is clear that I_0 is γ-stable since it is an ideal in A, thus the first map is well-defined; we call it $\mathcal{F}$.

Let $J \subset A_0$ be a γ-stable ideal. It is clear that $J + A_1J$ is a left ideal, and, in order to show that it is a right ideal as well, it is enough to see that $A_1J = JA_1$. This will follow from the following claim: for $x \in A_0$, we have $v_ix = \gamma(x)v_i$ for any i. This is shown

(1) for elements of type v_jv_k by using half-commutation,
(2) for polynomials in v_jv_k by induction, and
(3) by density for any x.

From this, we have $v_iJ = \gamma(J)v_i = Jv_i$, and then $A_1J = \sum_i v_iA_0J = \sum_i v_iJ = \sum_i Jv_i = \sum_i JA_0v_i = JA_1$, as required. Thus $J + A_1J$ is an ideal, and is $\mathbb{Z}_2$-graded by construction. The second map is well-defined, we call it $\mathcal{G}$.

Let $I = I_0 + I_1$ be a $\mathbb{Z}_2$-graded ideal of A. Then for $x \in I_1$ we have

$$x = \sum_i v_iv_ix \in A_1I_0, \qquad x = \sum_i xv_iv_i \in I_0A_1$$

hence $I_1 = A_1I_0 = I_0A_1$, and so $\mathcal{GF}(I) = I_0 + A_1I_0 = I$.

Finally it is clear that $\mathcal{FG}(J) = J$ for any γ-stable ideal $J \subset A_0$. ∎

In terms of subspaces, we have the following immediate translation.

COROLLARY 3.8. *We have a bijection*

$$\{\text{conjugation stable closed subspaces } Y \subset P^{n-1}_{\mathbb{C}}\}$$
$$\updownarrow$$
$$\{\text{symmetric quantum subspaces } X \subset S^{n-1}_{\mathbb{R},*}\}.$$

The bijection is as follows. Let $Y \subset P^{n-1}_{\mathbb{C}}$ be a conjugation stable closed subspace, and let $\mathcal{J}_Y \subset C(P^{n-1}_{\mathbb{C}})$ be the ideal of functions vanishing on Y. Then the corresponding symmetric closed subspace X of $S^{n-1}_{\mathbb{R},*}$ is defined by $C(X) = C(S^{n-1}_{\mathbb{R},*})/\langle \Phi(\mathcal{J}_Y)\rangle$.

The correspondence is easy to use in practice, because Φ and Φ^{-1} are completely explicit, and Φ transforms the monomial $z_{i_1}\cdots z_{i_m} z^*_{j_1}\cdots z^*_{j_m}$ into the monomial $v_{i_1}v_{j_1}\cdots v_{i_m}v_{j_m}$. For example if $f_1,\dots,f_r$ are polynomials in $z_iz_j^*$ that generate the ideal $\mathcal{J}_Y$, then the corresponding quotient of $C(S^{n-1}_{\mathbb{R},*})$ is $C(S^{n-1}_{\mathbb{R},*})/(\Phi(f_1),\dots,\Phi(f_r))$.

4. General setup. We now describe all the quantum subspaces of $S^{n-1}_{\mathbb{R},*}$. For this, it will be convenient to work in a more general framework.

We denote by $\mathbb{T}\rtimes\mathbb{Z}_2$ the semidirect product associated to the conjugation action of $\mathbb{Z}_2$ on $\mathbb{T}$. This is a compact group, isomorphic to the orthogonal group O_2, but the above description will be more convenient.

General setup. Let Z be a compact space endowed with a continuous action of $\mathbb{T}\rtimes\mathbb{Z}_2$, such that the action is $\mathbb{T}$-free. The $\mathbb{Z}_2$-action on Z corresponds, unless otherwise specified, to the $\mathbb{Z}_2$-action of the second factor of $\mathbb{T}\rtimes\mathbb{Z}_2$, and the corresponding automorphism is denoted τ.

We set $Z_{\mathbb{R}} = Z^{\mathbb{Z}_2} = \{z \in Z \mid \tau(z) = z\}$, $Z_{\mathrm{reg}} = Z \setminus \mathbb{T}Z_{\mathbb{R}}$, and

$$C(Z)^{\mathbb{T}} = \{f \in C(Z) \mid f(\omega z) = f(z),\ \forall \omega\in\mathbb{T},\ \forall z \in Z\},$$
$$C_{\mathbb{T}}(Z) = \{f \in C(Z) \mid f(\omega z) = \omega f(z),\ \forall \omega\in\mathbb{T},\ \forall z \in Z\}.$$

The $\mathbb{Z}_2$-action on Z enables us to form the crossed product $C(Z)\rtimes\mathbb{Z}_2$.

DEFINITION 4.1. For a compact space Z endowed with a continuous $\mathbb{T}\rtimes\mathbb{Z}_2$-action that is $\mathbb{T}$-free, we set

$$C(Z_{\mathbb{R},*}) = \{f_0\otimes 1 + f_1\otimes\tau \mid f_0 \in C(Z)^{\mathbb{T}},\ f_1 \in C_{\mathbb{T}}(Z)\} \subset C(Z)\rtimes\mathbb{Z}_2.$$

It is a direct verification that $C(Z_{\mathbb{R},*})$ is a C^*-subalgebra of $C(Z)\rtimes\mathbb{Z}_2$.

The following lemma links the present construction to the half-liberated spheres.

Lemma 4.2. *Assume that there exist $f_1, \dots, f_n \in C_{\mathbb{T}}(Z)$ such that*

(1) $C(Z)^{\mathbb{T}} = C^*(f_i f_j^*,\ 1 \le i, j \le n)$,

(2) $C_{\mathbb{T}}(Z) = f_1 C(Z)^{\mathbb{T}} + \cdots + f_n C(Z)^{\mathbb{T}}$.

Then $C(Z_{\mathbb{R},*}) = C^*(f_1 \otimes \tau, \dots, f_n \otimes \tau)$, and the elements $f_1 \otimes \tau, \dots, f_n \otimes \tau$ half-commute.

Proof. Set $A = C^*(f_1 \otimes \tau, \dots, f_n \otimes \tau)$. Then

$$(f_i \otimes \tau)(f_j \otimes \tau)^* = (f_i \otimes \tau)(\tau(f_j^*) \otimes \tau) = f_i f_j^* \otimes 1 \in A,$$

hence by (1) we have $C(Z)^{\mathbb{T}} \otimes 1 \subset A$. For $f \in C_{\mathbb{T}}(Z)$, we have $f = \sum_i g_i f_i$ for some $g_1, \dots, g_n \in C(Z)^{\mathbb{T}}$ by (2), so

$$f \otimes \tau = \sum_i g_i f_i \otimes \tau = \sum_i (g_i \otimes 1)(f_i \otimes \tau) \in A,$$

and this shows that $C(Z_{\mathbb{R},*}) = A$. Moreover,

$$(f_i \otimes \tau)(f_j \otimes \tau)(f_k \otimes \tau) = f_i \tau(f_j) f_k \otimes \tau = (f_k \otimes \tau)(f_j \otimes \tau)(f_i \otimes \tau). \blacksquare$$

Example 4.3. Consider the natural $\mathbb{T} \rtimes \mathbb{Z}_2$-action on $S_{\mathbb{C}}^{n-1}$, where the $\mathbb{T}$-action is by multiplication and the $\mathbb{Z}_2$-action is by conjugation. The $\mathbb{T}$-action is indeed free, we have $(S_{\mathbb{C}}^{n-1})_{\mathbb{R}} = S_{\mathbb{R}}^{n-1}$, and

$$\begin{aligned}(S_{\mathbb{C}}^{n-1})_{\mathrm{reg}} =: S_{\mathbb{C},\mathrm{reg}}^{n-1} &= S_{\mathbb{C}}^{n-1} \setminus \mathbb{T} S_{\mathbb{R}}^{n-1} \\ &= \{g = (g_1, \dots, g_n) \in S_{\mathbb{C}}^{n-1} \mid \exists i, j \text{ with } g_i \overline{g_j} \ne g_j \overline{g_j}\}.\end{aligned}$$

The coordinate functions $z_1, \dots, z_n \in C_{\mathbb{T}}(S_{\mathbb{C}}^{n-1})$ satisfy the conditions of the previous lemma, because $1 = \sum_i z_i z_i^*$, and hence the image of the injective morphism π of Theorem 3.4 is precisely $C((S_{\mathbb{C}}^{n-1})_{\mathbb{R},*})$. Therefore we identify the two algebras.

The description of the quantum subspaces of $Z_{\mathbb{R},*}$, which has $S_{\mathbb{R},*}^{n-1}$ as a particular case, is as follows.

Theorem 4.4. *There exists a bijection between the set of quantum subspaces $X \subset Z_{\mathbb{R},*}$ and the set of pairs (E, F) where*

(1) *$E \subset Z_{\mathrm{reg}}$ is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable and $E = \overline{E} \cap Z_{\mathrm{reg}}$ (i.e. E is closed in Z_{reg}),*

(2) *$F \subset Z_{\mathbb{R}}$ is closed and satisfies $\overline{E} \cap Z_{\mathbb{R}} \subset F$.*

Moreover, the quantum subspace X is non-classical if and only if in the corresponding pair (E, F) we have $E \ne \emptyset$.

The proof is given in the next subsections.

4.1. Representation theory of $C(Z_{\mathbb{R},*})$. We now provide a description of the irreducible representations of the C^*-algebra $C(Z_{\mathbb{R},*})$, the main step towards a description of the subspaces of $Z_{\mathbb{R},*}$. It is certainly possible to use general results on crossed products [10] to provide this description, but since everything can be done in a quite direct and elementary manner, we will proceed directly.

The C^*-algebra $C(Z_{\mathbb{R},*})$ is, by definition, a C^*-subalgebra of the crossed product C^*-algebra $C(Z) \rtimes \mathbb{Z}_2$. In particular, it can be seen as a C^*-subalgebra of $M_2(C(Z))$, and hence all its irreducible representations have dimension ≤ 2, and it is a 2-subhomogeneous C^*-algebra. A precise description of the irreducible representations of $C(Z_{\mathbb{R},*})$ is given in the following result.

PROPOSITION 4.5.

(1) *Any $z \in Z$ defines a representation*

$$\theta_z : C(Z_{\mathbb{R},*}) \to M_2(\mathbb{C}),$$
$$f_0 \otimes 1 + f_1 \otimes \tau \mapsto \begin{pmatrix} f_0(z) & f_1(z) \\ f_1(\tau(z)) & f_0(\tau(z)) \end{pmatrix},$$

and any irreducible representation of $C(Z_{\mathbb{R},*})$ is isomorphic to a sub-representation of some θ_z for some $z \in Z$. The representation θ_z is irreducible if and only if $z \in Z_{\mathrm{reg}}$. Moreover, for $z, x \in Z$, the representations θ_z and θ_x are isomorphic if and only if $(\mathbb{T} \rtimes \mathbb{Z}_2)z = (\mathbb{T} \rtimes \mathbb{Z}_2)x$.

(2) *Any $z \in Z_{\mathbb{R}}$ defines a one-dimensional representation*

$$\phi_z : C(Z_{\mathbb{R},*}) \to \mathbb{C}, \qquad f_0 \otimes 1 + f_1 \otimes \tau \mapsto f_0(z) + f_1(z),$$

and any one-dimensional representation arises in this way. Moreover, for $z, y \in Z_{\mathbb{R}}$ we have $\phi_z = \phi_y \Leftrightarrow z = y$.

(3) *If π is an irreducible representation of $C(Z_{\mathbb{R},*})$, then either $\pi \simeq \theta_z$ for some $z \in Z_{\mathrm{reg}}$, or $\pi = \phi_z$ for some $z \in Z_{\mathbb{R}}$.*

Proof. That θ_z defines a representation of $C(Z_{\mathbb{R},*})$ can be checked directly, or by using the standard embedding of the crossed product $C(Z) \rtimes \mathbb{Z}_2$ into $M_2(C(Z))$,

$$C(Z_{\mathbb{R},*}) \to M_2(C(Z)), \qquad f_0 \otimes 1 + f_1 \otimes \tau \mapsto \begin{pmatrix} f_0 & f_1 \\ f_1\tau & f_0\tau \end{pmatrix},$$

composed with evaluation at $z \in Z$. Since any irreducible representation of $M_2(C(Z))$ is obtained by evaluation at an element $z \in Z$, we deduce that any irreducible representation of $C(Z_{\mathbb{R},*})$ is isomorphic to a subrepresentation of θ_z for some $z \in Z$ (see e.g. [8]).

Now assume that $z \in \mathbb{T}Z_{\mathbb{R}}$: $z = \lambda y$ for some $y \in Z_{\mathbb{R}}$. Then $\tau(z) = \tau(\lambda y) = \overline{\lambda}\tau(y) = \overline{\lambda} y = \overline{\lambda}^2 z$, and for $f_0 \otimes 1 + f_1 \otimes \tau \in C(Z_{\mathbb{R},*})$, we have

$$\theta_z(f_0 \otimes 1 + f_1 \otimes \tau) = \begin{pmatrix} f_0(z) & f_1(z) \\ f_1(\tau(z)) & f_0(\tau(z)) \end{pmatrix} = \begin{pmatrix} f_0(z) & f_1(z) \\ \overline{\lambda}^2 f_1(z) & f_0(z) \end{pmatrix}.$$

This implies that $\theta_z(C(Z_{\mathbb{R},*}))$ is abelian, and hence θ_z is not irreducible.

Assume that $z \in Z_{\mathrm{reg}}$. To prove that θ_z is irreducible, it is enough to show that there exist $f_0 \in C(Z)^{\mathbb{T}}$ and $f_1 \in C_{\mathbb{T}}(Z)$ such that

$$f_0(z) \neq f_0(\tau(z)), \qquad f_1(z) \neq 0.$$

Indeed, then $\theta_z(f_0 \otimes 1)$ and $\theta_z(f_1 \otimes \tau)$ do not commute, and hence $\theta_z(C(Z_{\mathbb{R},*}))$ is a non-commutative C^*-subalgebra of $M_2(\mathbb{C})$, so the two algebras are equal and θ_z is irreducible.

Since $z \in Z_{\mathrm{reg}}$, we have $\mathbb{T}z \neq \mathbb{T}\tau(z)$. Otherwise $z = \lambda\tau(z)$ for some $\lambda \in \mathbb{T}$, and for $\mu \in \mathbb{T}$ such that $\mu^2 = \overline{\lambda}$ we have $z = \overline{\mu}\mu z$ with $\mu z \in Z_{\mathbb{R}}$, a contradiction. Hence by Urysohn's lemma and the fact that $X/\mathbb{T}$ is Hausdorff, there exists $f_0 \in C(Z)^{\mathbb{T}} \simeq C(Z/\mathbb{T})$ such that $f_0(z) \neq f_0(\tau(z))$, as needed. Finally since the $\mathbb{T}$-action is free, there exists a (continuous) map $f : \mathbb{T}z \to \mathbb{C}$ such that $f(\lambda z) = \lambda$ for any $\lambda \in \mathbb{T}$, that we extend to a continuous function f on Z (Tietze's extension theorem). Now let $f_1 \in C(Z)$ be defined by

$$f_1(y) = \int_{\mathbb{T}} \lambda^{-1} f(\lambda y)\, d\lambda.$$

We have $f_1 \in C_{\mathbb{T}}(Z)$ and $f_1(z) = 1$, as needed, and we conclude that θ_z is irreducible.

A finite-dimensional representation is determined by its character, and the character of θ_z is given by $\chi_z(f_0 \otimes 1 + f_1 \otimes \tau) = f_0(z) + f_0(\tau(z))$.

Let $z, x \in Z$ be such that $(\mathbb{T} \rtimes \mathbb{Z}_2)z \neq (\mathbb{T} \rtimes \mathbb{Z}_2)x$. Then there exists $f_0 \in C(Z)^{\mathbb{T} \rtimes \mathbb{Z}_2}$ such that $f_0(z) = 1 = f_0(\tau(z))$ and $f_0(x) = 0 = f_0(\tau(x))$. We have $f_0 \in C(Z)^{\mathbb{T}}$ and $\chi_z(f) = 2$, $\chi_x(f) = 0$, hence the representations θ_z and θ_x are not isomorphic.

If $(\mathbb{T} \rtimes \mathbb{Z}_2)z = (\mathbb{T} \rtimes \mathbb{Z}_2)x$, then either $z = \lambda x$ or $z = \lambda\tau(x)$ for some $\lambda \in \mathbb{T}$. From this we see that $f(z) + f(\tau(z)) = f(x) + f(\tau(x))$ for any $f \in C(Z)^{\mathbb{T}}$, and hence $\chi_z = \chi_x$, which shows that θ_z and θ_x are isomorphic, and concludes the proof of (1).

For $z \in Z_{\mathbb{R}}$, it is a direct verification to check that ϕ_z above defines a $*$-algebra map $C(Z_{\mathbb{R},*}) \to \mathbb{C}$. Now let $\psi : C(Z_{\mathbb{R},*}) \to \mathbb{C}$ be a $*$-algebra map. The representation defined by ψ is isomorphic to a subrepresentation of θ_z for some $z \in Z$, and with $z \notin Z_{\mathrm{reg}}$ because θ_z is not irreducible. Hence $z = \lambda y$ for $y \in Z_{\mathbb{R}}$. We then have, for $f_0 \otimes 1 + f_1 \otimes \tau \in C(Z_{\mathbb{R},*})$,

$$\theta_z(f_0 \otimes 1 + f_1 \otimes \tau) = \begin{pmatrix} f_0(y) & \lambda f_1(y) \\ \overline{\lambda} f_1(y) & f_0(y) \end{pmatrix},$$

and from this we see that the lines generated by $(1, \overline{\lambda})$ and $(1, -\overline{\lambda})$ are both stable under $C(Z_{\mathbb{R},*})$, so that $\theta_z \simeq \phi_y \oplus \phi_{-y}$ (this can also be seen using characters), and finally $\psi = \phi_y$ or $\psi = \phi_{-y}$.

Let $y, z \in Z_{\mathbb{R}}$ be such that $\phi_y = \phi_z$. Then for any $f_0 \in C(Z)^{\mathbb{T}}$, we have $f_0(y) = f_0(z)$, hence $\mathbb{T}y = \mathbb{T}z$, and so $y = \pm z$. As before, there exists $f_1 \in C_{\mathbb{T}}(Z)$ such that $f_1(\lambda z) = \lambda$ for any $\lambda \in \mathbb{T}$. Therefore $\phi_z(f_1 \otimes \tau) = f_1(\tau(z)) = f_1(z) = 1 = \pm f_1(y) = \pm\phi_y(f_1 \otimes \tau)$, hence $z = y$.

This proves (2), and (3) follows by combining (1) and (2). ∎

COROLLARY 4.6. *The C^*-algebra $C(Z_{\mathbb{R},*})$ is non-commutative if and only if $Z_{\mathrm{reg}} \neq \emptyset$.*

Proof. This follows from the previous proposition, because a C^*-algebra is non-commutative if and only if it has an irreducible representation of dimension > 1. ∎

4.2. Closed subspaces of $\widehat{C(Z_{\mathbb{R},*})}$. We now discuss $\widehat{C(Z_{\mathbb{R},*})}$, the spectrum of $C(Z_{\mathbb{R},*})$, endowed with its usual topology (since all the irreducible representations of A are finite-dimensional, we know [8] that the topological spaces $\widehat{A}$ and $\mathrm{Prim}(A)$ are canonically homeomorphic).

For $E \subset Z_{\mathrm{reg}}$ and $F \subset Z_{\mathbb{R}}$, we set

$$M(E,F) = (\{\theta_z \mid z \in E\} \cup \{\phi_z \mid z \in F\})/\!\sim\, \subset \widehat{C(Z_{\mathbb{R},*})}$$

where of course $\sim$ means that we identify isomorphic representations. Proposition 4.5 ensures that any subset of $\widehat{C(Z_{\mathbb{R},*})}$ is of the form $M(E,F)$ for $E \subset Z_{\mathrm{reg}}$ a $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subspace and $F \subset Z_{\mathbb{R}}$. The next result describes the closed subsets of $\widehat{C(Z_{\mathbb{R},*})}$.

PROPOSITION 4.7. *Let $E \subset Z_{\mathrm{reg}}$ be a $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subspace, and let $F \subset Z_{\mathbb{R}}$. Then*

$$\overline{M(E,F)} = M(\overline{E} \cap Z_{\mathrm{reg}}, \overline{F} \cup (\overline{E} \cap Z_{\mathbb{R}})).$$

In particular there exists a bijection between closed subsets of $\widehat{C(Z_{\mathbb{R},*})}$ and pairs (E,F) such that

(1) *$E \subset Z_{\mathrm{reg}}$ is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable and $E = \overline{E} \cap Z_{\mathrm{reg}}$,*
(2) *$F \subset Z_{\mathbb{R}}$ is closed and satisfies $\overline{E} \cap Z_{\mathbb{R}} \subset F$.*

Proof. Set $A = C(Z_{\mathbb{R},*})$. For $S \subset \widehat{A}$, we have

$$\overline{S} = \Big\{\pi \in \widehat{A} \,\Big|\, \bigcap_{\rho \in S} \mathrm{Ker}(\rho) \subset \mathrm{Ker}(\pi)\Big\}.$$

For E, F as in the statement of the proposition, we have

$$\overline{M(E,F)} = \overline{M(E,\emptyset) \cup M(\emptyset,F)} = \overline{M(E,\emptyset)} \cup \overline{M(\emptyset,F)}.$$

Hence we can study the two pieces separately. The bijective map

$$Z_{\mathbb{R}} \to \widehat{A}_1, \quad z \mapsto \phi_z,$$

where $\widehat{A}_1$ is the set of 1-dimensional representations, is clearly continuous, and since $Z_{\mathbb{R}}$ and $\widehat{A}_1$ are both compact, this is a homeomorphism. Hence it sends the closure of a subset in $Z_{\mathbb{R}}$ to the closure of the image in $\widehat{A}_1$, and hence to the closure of the image in $\widehat{A}$, since $\widehat{A}_1$ is closed in $\widehat{A}$. Thus $\overline{M(\emptyset, F)} = M(\emptyset, \overline{F})$. Consider now the following two claims:

$$(*) \qquad \text{For } y \in Z_{\mathbb{R}},\ \bigcap_{z\in E} \mathrm{Ker}(\theta_z) \subset \mathrm{Ker}(\phi_y) \Leftrightarrow y \in \overline{E} \cap Z_{\mathbb{R}}.$$

$$(**) \qquad \text{For } y \in Z_{\mathrm{reg}},\ \bigcap_{z\in E} \mathrm{Ker}(\theta_z) \subset \mathrm{Ker}(\theta_y) \Leftrightarrow y \in \overline{E} \cap Z_{\mathrm{reg}}.$$

Once these claims are proved, we will indeed have

$$\overline{M(E,\emptyset)} = M(\overline{E} \cap Z_{\mathrm{reg}}, \overline{E} \cap Z_{\mathbb{R}}),$$

as required.

We begin with $(*)$. Let $y \in Z_{\mathbb{R}}$. Assume first that $y \in \overline{E} \cap Z_{\mathbb{R}}$. Let $f_0 \otimes 1 + f_1 \otimes \tau \in \cap_{z\in E}\mathrm{Ker}(\theta_z)$. Then f_0 and f_1 vanish on E, and hence on $\overline{E}$, and $f_0 \otimes 1 + f_1 \otimes \tau \in \mathrm{Ker}(\phi_y)$. Thus $\bigcap_{z\in E} \mathrm{Ker}(\theta_z) \subset \mathrm{Ker}(\phi_y)$. Conversely, assume that $y \notin \overline{E} \cap Z_{\mathbb{R}}$. Then $\mathbb{T}y \cap \overline{E} = \emptyset$ since E is $\mathbb{T}$-stable and there exists $f_0 \in C(Z)^{\mathbb{T}}$ such that $f_0(y) = 1$ and $f_0(\overline{E}) = 0$. We then have $f_0 \otimes 1 \in \bigcap_{z\in E} \mathrm{Ker}(\theta_z)$ while $f_0 \otimes 1 \notin \mathrm{Ker}(\phi_y)$, and $(*)$ is proved.

Now let $y \in Z_{\mathrm{reg}}$. Assume first $y \in \overline{E} \cap Z_{\mathrm{reg}}$. Let $f_0 \otimes 1 + f_1 \otimes \tau \in \bigcap_{z\in E} \mathrm{Ker}(\theta_z)$. Then f_0 and f_1 vanish on E, and hence on $\overline{E}$, and $f_0 \otimes 1 + f_1 \otimes \tau \in \mathrm{Ker}(\theta_y)$. Thus $\bigcap_{z\in E} \mathrm{Ker}(\theta_z) \subset \mathrm{Ker}(\theta_y)$. Conversely, assume that $y \notin \overline{E} \cap Z_{\mathrm{reg}}$. Then $(\mathbb{T} \rtimes \mathbb{Z}_2)y \cap \overline{E} = \emptyset$ since E is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable and there exists $f_0 \in C(Z)^{\mathbb{T}\rtimes\mathbb{Z}_2}$ such that $f_0(y) = 1$ and $f_0(\overline{E}) = 0$. Therefore $f_0 \otimes 1 \in \bigcap_{z\in E} \mathrm{Ker}(\theta_z)$ while $f_0 \otimes 1 \notin \mathrm{Ker}(\theta_y)$, and $(**)$ is proved.

If $E, E' \subset Z_{\mathrm{reg}}$ and $F, F' \subset Z_{\mathbb{R}}$, then by Proposition 4.5 we have $\mathcal{M}(E, F) = \mathcal{M}(E', F') \Rightarrow F = F'$ and $E = E'$ if E and E' are $\mathbb{T} \rtimes \mathbb{Z}_2$-stable, hence the last assertion of Proposition 4.7 follows from the first one. ∎

Proof of Theorem 4.4. Recall that we have to show that there is a bijection between the set of quantum subspaces $X \subset Z_{\mathbb{R},*}$ and the set of pairs (E, F) where

(1) $E \subset Z_{\mathrm{reg}}$ is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable and $E = \overline{E} \cap Z_{\mathrm{reg}}$ (i.e. E is closed in Z_{reg}),
(2) $F \subset Z_{\mathbb{R}}$ is closed and satisfies $\overline{E} \cap Z_{\mathbb{R}} \subset F$.

By Proposition 4.7 the set of pairs (E, F) as in the statement is in bijection with the set of closed subsets of $\widehat{C(Z_{\mathbb{R},*})}$, hence the result follows from the standard correspondence between quotients of a C^*-algebra and closed subsets of its spectrum [8]. More precisely, to such a pair (E, F) is associated the C^*-algebra $C(Z_{\mathbb{R},*})/\bigcap_{\pi\in M(E,F)} \mathrm{Ker}(\pi)$, and conversely, to a quotient

C^*-algebra map $q : C(Z_{\mathbb{R},*}) \to B$ is associated the pair (E, F) such that $\{\pi \in \widehat{C(Z_{\mathbb{R},*})} \mid \mathrm{Ker}(q) \subset \mathrm{Ker}(\pi)\} = M(E, F)$, i.e.

$$E = \{z \in Z_{\mathrm{reg}} \mid \mathrm{Ker}(q) \subset \mathrm{Ker}(\theta_z)\}, \qquad F = \{z \in Z_{\mathbb{R}} \mid \mathrm{Ker}(q) \subset \mathrm{Ker}(\phi_z)\}.$$

The C^*-algebra B is non-commutative if and only if it has an irreducible representation of dimension > 1, if and only if the corresponding E is non-empty. ■

REMARK 4.8. It follows from the above considerations that plenty of intermediate quantum subspaces $S^{n-1}_{\mathbb{R}} \subset X \subset S^{n-1}_{\mathbb{R},*}$ exist for $n \geq 2$. Indeed, for any $m \geq 1$, there exists $S^{n-1}_{\mathbb{R}} \subset X \subset S^{n-1}_{\mathbb{R},*}$ such that $C(X)$ has precisely m isomorphism classes of irreducible representations of dimension 2. This follows from the fact that the orbit space $S^{n-1}_{\mathbb{C},\mathrm{reg}}/(\mathbb{T} \rtimes \mathbb{Z}_2)$ is infinite, and hence if we pick $E \subset S^{n-1}_{\mathbb{C},\mathrm{reg}}$ (closed and $\mathbb{T} \rtimes \mathbb{Z}_2$-stable) such that $E/(\mathbb{T} \rtimes \mathbb{Z}_2)$ has m elements, the quantum space corresponding to the pair $(E, S^{n-1}_{\mathbb{R}})$ has the above property.

This contrasts with the situation for quantum groups, where there are no intermediate quantum subgroups $O_n \subset G \subset O^*_n$ [3]. The absence of such intermediate quantum subgroups can probably be roughly explained by some strong rigidity arising from the associated algebraic structure (Lie and Hopf algebras) used in [3]. It would certainly be interesting and useful to find axioms on half-liberated spaces that would ensure that the "no intermediate subobjects" situation in the quantum group case still holds.

4.3. Symmetric subspaces. It is now natural to wonder about the link between the description of the symmetric subspaces of $S^{n-1}_{\mathbb{R},*}$ in Section 3 and the one that should arise from Theorem 4.4. We first note that there also exists a notion of symmetric subspace in the general framework. Indeed, define an automorphism σ of $C(Z_{\mathbb{R},*})$ by $\sigma(f_0 \otimes 1 + f_1 \otimes \tau) = f_0 \otimes 1 - f_1 \otimes \tau$. We say that a subspace $X \subset Z_{\mathbb{R},*}$ is *symmetric* if $\sigma(X) = X$, or in other words, if σ induces an automorphism on the corresponding quotient of $C(Z_{\mathbb{R},*})$. When $Z = S^{n-1}_{\mathbb{C}}$, the automorphism σ is the sign automorphism ν of Section 3.

LEMMA 4.9. *Let $X \subset Z_{\mathbb{R},*}$ be a quantum subspace. Then X is symmetric if and only if in the corresponding pair (E, F) of Theorem 4.4 we have $F = -F$.*

Proof. We retain the notation of Proposition 4.5. It is straightforward to check that for $z \in Z_{\mathrm{reg}}$ we have $\theta_z \sigma = \theta_{-z} \simeq \theta_z$, and that for $z \in Z_{\mathbb{R}}$ we have $\phi_z \sigma = \phi_{-z}$.

Now if $X \subset Z_{\mathbb{R},*}$ is a quantum subspace, then saying that X is symmetric precisely means that for the corresponding pair (E, F) the corresponding ideal $\bigcap_{\pi\in M(E,F)} \mathrm{Ker}(\pi)$ is σ-stable. We have

$$\begin{aligned}\sigma\Big(\bigcap_{\pi\in M(E,F)} \mathrm{Ker}(\pi)\Big) &= \sigma\Big(\bigcap_{z\in E} \mathrm{Ker}(\theta_z)\Big) \cap \sigma\Big(\bigcap_{z\in F} \mathrm{Ker}(\phi_z)\Big)\\ &= \Big(\bigcap_{z\in E} \mathrm{Ker}(\theta_z\sigma^{-1})\Big) \cap \Big(\bigcap_{z\in E} \mathrm{Ker}(\phi_z\sigma^{-1})\Big)\\ &= \Big(\bigcap_{z\in E} \mathrm{Ker}(\theta_z)\Big) \cap \Big(\bigcap_{z\in F} \mathrm{Ker}(\phi_{-z})\Big) = \bigcap_{\pi\in M(E,-F)} \mathrm{Ker}(\pi).\end{aligned}$$

Hence X is symmetric if and only if $F = -F$. ∎

THEOREM 4.10. *There exists a bijection*

$$\{\textit{symmetric quantum subspaces } X \subset Z_{\mathbb{R},*}\}$$
$$\updownarrow$$
$$\{\mathbb{T} \rtimes \mathbb{Z}_2\textit{-stable closed subspaces } Y \subset Z\}.$$

Moreover, the subspace $X \subset Z_{\mathbb{R},*}$ is non-classical if and only if, for the corresponding $Y \subset Z$, we have $Y_{\mathrm{reg}} = Y \cap Z_{\mathrm{reg}} \neq \emptyset$.

Proof. To a pair (E, F) as in Theorem 4.4 and satisfying $F = -F$ we associate the closed $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subset $Y = E \cup \mathbb{T}F$. Conversely, if Y is $\mathbb{T} \rtimes \mathbb{Z}_2$-stable, then $(Y_{\mathrm{reg}}, Y_{\mathbb{R}})$ is a pair as above. The two maps are inverse bijections (in particular because $(E \cup \mathbb{T}F)_{\mathbb{R}} = \pm F$). ∎

Of course the above theorem reproves the first part of Corollary 3.8, since closed $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subspaces of $S_{\mathbb{C}}^{n-1}$ correspond to conjugation stable closed subspaces of $P_{\mathbb{C}}^{n-1}$.

To finish, we provide another explicit description of the bijection in the proof of Theorem 4.10. If $Y \subset Z$ is a closed $\mathbb{T} \rtimes \mathbb{Z}_2$-stable subspace, then we may form the C^*-algebra $C(Y_{\mathbb{R},*})$ as before, and restriction of functions yields a $*$-algebra map $C(Z_{\mathbb{R},*}) \to C(Y_{\mathbb{R},*})$, which is easily seen to be surjective thanks to the standard extension theorems. We thus get a quantum subspace $Y_{\mathbb{R},*} \subset Z_{\mathbb{R},*}$.

THEOREM 4.11. *The bijection*

$$\{\textit{symmetric quantum subspaces } X \subset Z_{\mathbb{R},*}\}$$
$$\updownarrow$$
$$\{\mathbb{T} \rtimes \mathbb{Z}_2\textit{-stable closed subspaces } Y \subset Z\}$$

is induced by the map $Y \mapsto Y_{\mathbb{R},*}$.

Proof. Let $q : C(Z_{\mathbb{R},*}) \to C(Y_{\mathbb{R},*})$ be the surjective $*$-algebra map associated to restriction of functions. It is a direct verification to check that

$$\mathrm{Ker}(q) = \Big(\bigcap_{y\in Y_{\mathrm{reg}}} \mathrm{Ker}(\theta_y)\Big) \cap \Big(\bigcap_{y\in Y_{\mathbb{R}}} \mathrm{Ker}(\phi_y)\Big) = \bigcap_{\pi\in M(Y_{\mathrm{reg}},Y_{\mathbb{R}})} \mathrm{Ker}(\pi).$$

Hence, in terms of pairs (E, F) corresponding to closed subsets of $\widehat{C(Z_{\mathbb{R},*})}$, our map associates the pair $(Y_{\mathrm{reg}}, Y_{\mathbb{R}})$ to Y. As already discussed, this map is a bijection. ∎

To close the paper, we mention that it is possible to get the description of the quantum subgroups of O_n^* from [7] using Theorem 4.4, with some more work, i.e. studying when the ideal corresponding to a pair (E, F) is a Hopf ideal. As pointed out in the introduction, that was the original approach of Podleś [9] in the determination of the quantum subgroups of $\mathrm{SU}_{-1}(2) \simeq O_2^*$.

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