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On consecutive integers divisible by the number of their divisors

by

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1. Introduction. Let $d(n)$ be the number of divisors of the positive integer n. Positive integers n such that $d(n)\mid n$ have been studied by Spiro [7]. She showed that for large x, the number of such $n\le x$ is

$$(x/\sqrt{\log x})(\log\log x)^{-1+o(1)}$$

as $x\to\infty$. In [8], she studied the positive integers n such that $d(n)$ is a divisor of $n+1$, showing that for large x, the number of such $n\le x$ is asymptotically equal to $cx/\sqrt{\log x}$ for some positive constant c.

Here, we look at positive integers n such that $d(n+k)\mid n+k$ for $k=0,1,\ldots,s$ for some $s\ge 1$. The first result we prove is the following.

Theorem 1.1. *If $d(n)\mid n$ and $d(n+1)\mid n+1$ and $n>1$, then n is even.*

In particular, it follows from Theorem 1.1 that there is no n such that $d(n+k)\mid n+k$ for $k=0,1,2$. So, from now on, we set

$$\mathcal{N}=\{n: d(n)\mid n \text{ and } d(n+1)\mid n+1\}.$$

We have

$$\begin{aligned}\mathcal{N}=\{&1,8,1520,50624,62000,103040,199808,221840,269360,463760,\\&690560,848240,986048,1252160,1418480,2169728,2692880,2792240,\\&3448448,3721040,3932288,5574320,5716880,6066368,6890624,\\&6922160,8485568,\ldots\}.\end{aligned}$$

We study the cardinality of the set $\mathcal{N}(x)=\mathcal{N}\cap[1,x]$ for large real x. We have the following result.

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Theorem 1.2. *We have*

$$\#\mathcal{N}(x) = \frac{x^{1/2}(\log\log x)^{O(1)}}{(\log x)^c}$$

as $x \to \infty$, *where* $c = 2 - 1/\sqrt{3} = 1.42265\ldots$.

Throughout this paper, we use the Landau symbols O and o and the Vinogradov symbols $\ll$ and $\gg$ with their regular meanings. Recall that $A = O(B)$, $A \ll B$ and $B \gg A$ all mean that $|A| < \kappa B$ for some constant κ, while $A = o(B)$ means that $A/B \to 0$. Further, $A \sim B$ means that $A/B \to 1$. We use $\log_k x$ for the kth fold composition of the natural logarithm function with itself, and assume that the input is large enough so that its value is positive real.

2. The proof of Theorem 1.1. Suppose that $n > 1$ is odd and $d(n) \mid n$. Then $n = m^2$ for some odd m. Thus, $n + 1 = m^2 + 1 \equiv 2 \pmod 4$. Since $d(n+1) \mid n+1$, it follows that there is exactly one prime p such that the exponent of p in the factorization of $n+1$ is odd, and all remaining prime factors of $n+1$ appear at even exponents. Since already $2 \parallel n+1$, we get $n + 1 = 2u^2$ for some odd positive integer u. Hence, $m^2 - 2u^2 = -1$. We now look at the factorization of u. Assume that there exists a prime p such that $p^{2\alpha+1} \parallel u$. Then $4\alpha + 3 \mid d(n+1) \mid n+1$, so there exists a prime $q \equiv 3 \pmod 4$ with $q \mid n+1$. Reducing the equation $m^2 - 2u^2 = -1$ modulo q, we get $m^2 \equiv -1 \pmod q$, contradicting the fact that $q \equiv 3 \pmod 4$. This shows that the exponent of every prime appearing in the factorization of u is even, so $u = v^2$. We get the equation

$$m^2 - 2v^4 = -1,$$

and its only solution in positive integers (m, v) with $m > 1$ is $(m, v) = (239, 13)$ by a result of Ljunggren [5]. Thus, $n + 1 = 2 \times 13^4$, which is not convenient because $d(n+1) = 10$ does not divide $n+1$.

3. The proof of Theorem 1.2

3.1. The upper bound. Let x be large. We cover $\mathcal{N}(x)$ with a bounded number of sets labeled $\mathcal{N}_i(x)$ for $i = 1, 2, 3$ each of whose cardinalities has the order of magnitude indicated by the conclusion of the theorem. By using Theorem 1.1, we infer that if $n \in \mathcal{N}$ and $n > 1$, then n is even, so $n+1$ is odd. Hence, $d(n+1)$ is odd, showing that $n + 1 = m^2$ for some odd m. Hence, if additionally $n \in \mathcal{N}(x)$, then $m \le \sqrt{x+1}$.

Recall that a positive integer s is *square-full* if $p^2 \mid s$ whenever p is a prime factor of s. By using a result of Erdős and Szekeres [2] to the effect that

$$\#\{s \le x : s \text{ square-full}\} \sim \kappa\sqrt{x} \quad (x \to \infty)$$

for some positive constant κ, and the Abel summation formula, we get

$$\sum_{\substack{s\ge t\\ s\,\text{square-full}}} \frac{1}{s} \ll \frac{1}{\sqrt{t}} \tag{3.1}$$

for all $t \ge 1$. We then set $z := (\log x)^4$ and

$\mathcal{N}_1(x) := \{n \in \mathcal{N}(x) : s \mid m{+}i \text{ for } i \in \{-1,0,1\} \text{ and some square-full } s > z\}$.

For a fixed $i \in \{-1,0,1\}$ and $s > z$, we have

$$m+i \le \sqrt{x+1}+i \le \sqrt{x+1}+1 \quad \text{ and } m+i \text{ is a multiple of } s > z,$$

and the number of such m is at most $\lfloor(\sqrt{x+1}+1)/s\rfloor \le 2\sqrt{x}/s$ for x large. Thus,

$$\#\mathcal{N}_1(x) \le \sum_{i\in\{-1,0,1\}} \sum_{\substack{s>z\\ s\,\text{square-full}}} \frac{2\sqrt{x}}{s} \ll \frac{\sqrt{x}}{(\log x)^2}, \tag{3.2}$$

where for the last estimate we used (3.1) with $t := z$.

Next we set $K := \lfloor 15 \log\log x\rfloor$ and

$$\mathcal{N}_2(x) := \{n \in \mathcal{N} : \Omega(m(m-1)(m+1)) > K\}.$$

If $n \in \mathcal{N}_2(x)$, then there exists $i \in \{-1,0,1\}$ such that

$$\Omega(m+i) \ge K/3 \ge \lfloor 5\log\log x\rfloor.$$

It then follows that

$$\#\mathcal{N}_2(x) \le 3 \sum_{\substack{m\le\sqrt{x+1}+1\\ \Omega(m)\ge K/3}} 1 \ll \frac{K\sqrt{x}\log x}{2^{K/3}} \ll \frac{\sqrt{x}}{(\log x)^2}, \tag{3.3}$$

because $5\log 2 - 1 > 2$, where the middle inequality follows from [6, Lemma 13]. We now set $U := (\log x)^4$, $V := x^{1/(\log\log x)^2}$ and

$$P := \prod_{U<p\le V} p.$$

Pick $n \in \mathcal{N}_3(x) := \mathcal{N}(x) \setminus (\mathcal{N}_1(x) \cup \mathcal{N}_2(x))$. Let

$$M := \gcd(m(m-1)(m+1), P).$$

Fix M. Since primes $q \in (U, V]$ have the property that $q^2 \nmid m(m-1)(m+1)$ (because $n \notin \mathcal{N}_1(x)$), it follows that M is square-free and the quotient $m(m-1)(m+1)/M$ is coprime to P. Thus, we can write

$$m - i = M_i N_i \quad \text{ for } i \in \{-1,0,1\}, \tag{3.4}$$

where $M_i \mid M$ and $\gcd(N_i, M) = 1$ for $i \in \{-1,0,1\}$. Furthermore, let $k := \omega(M)$. Write $k = r + s$, where $r := \omega(M_0)$ and $s := \omega(M_{-1}M_1)$. In what follows, we assume that $s \ge 2$, and the argument can be easily adapted to $s \in \{0,1\}$.

Since $n \in \mathcal{N}$, we deduce that $2^s \mid d(n) = d(m^2-1) \mid M_1M_{-1}N_1N_{-1}$, and $3^r \mid m^2 \mid (M_0N_0)^2$. In particular, either $2^{s-1} \mid N_{-1}$ and $2 \mid N_1$, or $2^{s-1} \mid N_1$ and $2 \mid N_{-1}$. Furthermore, $3^{\lfloor r/2 \rfloor} \mid N_0$. The above relations put m into two progressions modulo $2^{s-1}3^{\lfloor r/2\rfloor}M$ of the form

$$m = 3^{\lfloor r/2\rfloor}M_0(2^{s-1}M_{-1}M_1\lambda + A_0),$$

with some nonnegative integer λ, and either

$$3^{\lfloor r/2\rfloor}M_0A_0 - 1 = 2^{s-1}M_{-1}A_{-1} \quad \text{and} \quad 3^{\lfloor r/2\rfloor}M_0A_0 + 1 = 2M_1A_1, \quad \text{or}$$
$$3^{\lfloor r/2\rfloor}M_0A_0 - 1 = 2M_{-1}A_{-1} \qquad \text{and} \quad 3^{\lfloor r/2\rfloor}M_0A_0 + 1 = 2^{s-1}M_1A_1$$

for some positive integers A_i with $i \in \{-1,0,1\}$. Here, we assume that $A_0 < 2^{s-1}M_{-1}M_1$. We only treat the first case above since the second one is similar. Hence,

$$\begin{aligned} m(m-1)(m+1) = 2^s3^{\lfloor r/2\rfloor}M(2^{s-1}M_{-1}M_1\lambda + A_0)(3^{\lfloor r/2\rfloor}M_0M_1\lambda + A_{-1}) \\ \times (2^{s-2}3^{\lfloor r/2\rfloor}M_{-1}M_0\lambda + A_1). \end{aligned}$$

Consider the polynomial

$$\begin{aligned} f(X) := (2^{s-1}M_{-1}M_1X + A_0)(3^{\lfloor r/2\rfloor}M_0M_1X + A_{-1}) \\ \times (2^{s-1}3^{\lfloor r/2\rfloor}M_{-1}M_0X + A_{-1}). \end{aligned}$$

Then λ is a nonnegative integer such that

$$2^{s-1}M_{-1}M_1\lambda + A_0 \le \frac{\sqrt{x+1}}{3^{\lfloor r/2\rfloor}M_0}$$

and $f(\lambda)$ is coprime to P. If $q \in (U,V]$, the polynomial $f(X)$ has three roots modulo q except when $q \mid M$, in which case it has only one root. Note also that

$$\frac{\sqrt{x}}{2^s3^{r/2}M} > \frac{x^{1/2-K/(\log\log x)^2}}{(\log x)^{K\log 3}} > V$$

for all $x > x_0$ sufficiently large. Thus, by the sieve, the number of such λ is

$$\begin{aligned} &\ll \frac{\sqrt{x}}{2^s3^{r/2}M}\Bigg(\prod_{U<p\le V}\Big(1-\frac{3}{p}\Big)\Bigg)\Big(\frac{M}{\phi(M)}\Big)^2 \\ &\ll \frac{\sqrt{x}(\log\log x)^2}{2^s3^{r/2}M}\Big(\frac{\log U}{\log V}\Big)^3 = \frac{\sqrt{x}(\log\log x)^{O(1)}}{2^s3^{r/2}M(\log x)^3}. \end{aligned}$$

Here, we have used, aside from the sieve, the minimal order $\phi(M)/M \gg 1/\log\log x$ of the Euler function ϕ when $M \le x$, as well as the estimate

$$\prod_{p\le t}\Big(1-\frac{3}{p}\Big) = \frac{c_0}{(\log t)^3}(1+o(1))$$

as $t \to \infty$ with a suitable positive constant c_0, with $t = U$ and $t = V$, respectively. This was when M_0, M_{-1} and M_1 were fixed. By keeping only

M, r and s fixed, the divisor M_0 of M can be chosen in $\binom{r+s}{r}$ ways. Choosing M_0 determines $M_{-1}M_1$ uniquely, but then M_{-1} (so also M_1) can be chosen in 2^s additional ways. By accounting for all such possibilities, we find that the number of possibilities when M, r and s are all fixed is

$$\ll \binom{r+s}{r} \frac{1}{3^{r/2}} \frac{\sqrt{x}(\log\log x)^{O(1)}}{M(\log x)^3}.$$

We now vary r and s so that $r+s=k$ and sum up the above bounds, getting a bound of

$$\frac{c_1^k \sqrt{x}(\log\log x)^{O(1)}}{M(\log x)^3},$$

where $c_1 := 1 + 1/\sqrt{3}$, and where we have used the binomial formula

$$\sum_{r+s=k} \binom{r+s}{r} \frac{1}{3^{r/2}} = c_1^k.$$

We now sum up over all the possibilities for M, a square-free number with k prime factors all in $(U,V]$, and use the fact that

$$\sum_{\substack{\mu^2(M)=1\\ \omega(M)=k\\ p\,|\,M \Rightarrow p\in(U,V]}} \frac{1}{M} \le \frac{1}{k!}\bigg(\sum_{U<p\le V} \frac{1}{p}\bigg)^k \le \frac{1}{k!}\big(\log\log x + O(\log_3 x)\big)^k,$$

where we have also used Mertens' estimate

$$\sum_{p\le t} \frac{1}{p} = \log\log t + O(1) \tag{3.5}$$

with $t \in \{U, V\}$. Using the inequality $k! \ge (k/e)^k$ shows that the above bound is

$$\begin{aligned} &\le \left(\frac{e\log\log x + O(\log_3 x)}{k}\right)^k \\ &\le \left(\frac{e\log\log x}{k}\right)^k \left(1 + O\left(\frac{\log_3 x}{k}\right)\right)^k \\ &\ll \left(\frac{e\log\log x}{k}\right)^k (\log\log x)^{O(1)}. \end{aligned} \tag{3.6}$$

For the last inequality above, we have used the fact that $(1+z)^u < e^{uz}$ for all positive real numbers u and z with $u := k$ and $z := c_2(\log_3 x)/k$, where c_2 is the constant implied by the O in (3.6) above. Thus, we get a bound of

$$\frac{\sqrt{x}(\log\log x)^{O(1)}}{(\log x)^3}\left(\frac{c_1 e\log\log x}{k}\right)^k.$$

By finally summing this up over $k \le K = O(\log\log x)$, we get

$$\#\mathcal{N}_3(x) \ll \frac{\sqrt{x}(\log\log x)^{O(1)}}{(\log x)^3} \max\left\{\left(\frac{c_1 e \log\log x}{k}\right)^k : k \le K\right\}.$$

For a fixed positive real number A, the maximum of $t \mapsto (eA/t)^t$ is obtained when $t = A$, and equals e^A. By applying this with $A = c_1 \log\log x$, we see that the maximum of the expression on the right-hand side above is

$$\le e^{c_1 \log\log x} = (\log x)^{c_1}.$$

We thus get

$$\#\mathcal{N}_3(x) \ll \frac{\sqrt{x}(\log\log x)^{O(1)}}{(\log x)^{3-c_1}}. \tag{3.7}$$

The desired upper bound from Theorem 1.2 follows from (3.2), (3.3) and (3.7) because

$$\#\mathcal{N}(x) = \#\Big(\bigcup_{j=1}^{3} \mathcal{N}_j(x)\Big) \le \#\mathcal{N}_1(x) + \#\mathcal{N}_2(x) + \#\mathcal{N}_3(x).$$

3.2. The lower bound. We let x be large. Set $K := \log_2 x$, $h := \lfloor \log_3 x \rfloor$, $s := \lfloor K \rfloor$ and $r := \lfloor (1/\sqrt{3})K \rfloor$. We define

$$s_1 := s + A, \quad \text{where} \quad A \in [h, 2h]$$

is such that the minimal prime factor of s_1 exceeds h. This is possible by an extension of Bertrand's postulate due to Sylvester. Next, we take

$$r_1 := r + B, \quad \text{where} \quad B \in [h, 2h],$$

such that:

(i) the smallest prime factor of r_1 exceeds $\log h$;
(ii) $\gcd(r_1, s_1) = 1$.

To justify that it is possible to choose r_1 satisfying both (i) and (ii) above for large x, we argue as follows. Let $\mathcal{I} := [r + h, r + 2h]$. This is an interval of length h, so by the Erathostenes sieve, the number of integers fulfilling (i) above in $\mathcal{I}$ is $\gg h/\log\log h$. For large x, the implied constant can be chosen to be $c_3 := e^{-0.6} = .548812\ldots$. Now let us give an upper bound for the number of numbers in $\mathcal{I}$ failing (ii). Observe that since s_1 has no prime factor smaller than h, it follows that

$$\Omega(s_1) \le \frac{\log s_1}{\log h} \ll \frac{h}{\log h}.$$

The implied constant above can be chosen to be $c_4 := 2$ for large enough x. Let $h < p_1 < \cdots < p_t$ with $t \ll h/\log h$ be all the prime factors of s_1. For each $i \in \{1, \ldots, t\}$, the interval $\mathcal{I}$ contains at most one multiple of p_i. Thus, the number of numbers in $\mathcal{I}$ which are not coprime to s_1 is at most $t \ll h/\log h$.

Hence, the number of numbers r_1 passing both (i) and (ii) above is

$$\geq c_3 \frac{h}{\log\log h} - c_4 \frac{h}{\log h}$$

for large x, and in particular it is positive.

Now let

$$S_1 := \min\{S : s_1 \mid S \text{ and } d(S) \text{ is a power of } 2\}.$$

To estimate S_1, observe that if we write

$$s_1 = \prod_{q \in \Pi(s_1)} q^{\alpha_q},$$

where $\Pi(s_1)$ is the set of prime factors of s_1, then letting 2^{β_q} be the unique power of 2 in the interval $[\alpha_q + 1, 2\alpha_q]$, we get

$$S_1 = \prod_{q \in \Pi(s_1)} q^{2^{\beta_q} - 1}.$$

Since $2^{\beta_q} - 1 \leq 2\alpha_q - 1 < 2\alpha_q$, we infer that

$$S_1 \leq s_1^2 = O((\log_2 x)^2) \quad \text{and} \quad \Omega(S_1) \leq 2\Omega(s_1). \tag{3.8}$$

Similarly, we let

$$R_1 := \min\{R : r_1 \mid R \text{ and } 2\nu_q(R) + 1 \text{ is a power of 3 for all primes } q \mid r_1\}.$$

Here, as is customary, we write, for a prime p and an integer n, $\nu_p(n)$ for the exact exponent at which p appears in the factorization of n. If we write this time

$$r_1 = \prod_{q \in \Pi(r_1)} q^{\gamma_q},$$

then the interval $[2\gamma_q + 1, 6\gamma_q + 1]$ contains a power of 3, say 3^{δ_q}. Then

$$R_1 = \prod_{q \in \Pi(r_1)} q^{(3^{\delta_q} - 1)/2}.$$

Since $(3^{\delta_q} - 1)/2 \leq 3\gamma_q$, it follows that

$$R_1 \leq r_1^3 = O((\log_2 x)^3) \quad \text{and} \quad \Omega(R_1) \leq 3\Omega(r_1). \tag{3.9}$$

Note that S_1 and s_1 have the same prime factors and so do R_1 and r_1, so in particular $\gcd(R_1, S_1) = 1$ because of condition (ii).

We once again let M be a square-free positive integer with exactly $L := r + s$ prime factors all in $(U, V]$ as in the proof of the upper bound. We write $M = M_0 M_{-1} M_1$, where $\omega(M_0) = r$ and $\omega(M_{-1} M_1) = s$. We let m be a solution of the Chinese Remainder System

$$\begin{aligned} m &\equiv 1 \pmod{2^{s_1 - 2} S_1 M_1}, \\ m &\equiv 0 \pmod{3^{(r_1 - 1)/2} R_1 M_0}, \\ m &\equiv -1 \pmod{2M_{-1}}. \end{aligned}$$

Note that this system is similar to (3.4), except that (r, s) have been replaced by (r_1, s_1) and there are the additional parameters R_1, S_1. The above system is solvable because $\gcd(R_1, S_1) = 1$, R_1S_1 is coprime to 6 for large x (since its smallest prime factor exceeds $\log h$), and R_1S_1 is coprime to M, since the largest prime factor of R_1S_1 is $\le 2\log\log x < U$ for large x. The above congruences put m into an arithmetic progression modulo $2^{s_1-2}3^{(r_1-1)/2}R_1S_1M$. Writing

$$m = 3^{(r_1-1)/2}R_1M_0(2^{s_1-2}S_1M_{-1}M_1\lambda + A_0)$$

for some integer $A_0 < 2^{s_1-2}S_1M_{-1}M_1$ shows that

$$3^{(r_1-1)/2}R_1M_0A_0 - 1 = 2^{s_1-2}S_1M_1A_1 \text{ and } 3^{(r_1-1)/2}R_1M_0A_0 + 1 = 2M_{-1}A_{-1}$$

for some positive integers A_{-1} and A_1. Then

$$m(m-1)(m+1) = 2^{s_1-1}3^{(r_1-1)/2}R_1S_1Mf(\lambda),$$

where $f(X) \in \mathbb{Z}[X]$ is

$$\begin{aligned} f(X) = {} & (2^{s_1-2}S_1M_{-1}M_1X + A_0)(3^{(r_1-1)/2}R_1M_{-1}M_0X + A_1) \\ & \times (3^{(r_1-1)/2}2^{s_1-3}R_1S_1M_0M_1X + A_{-1}). \end{aligned}$$

We take

$$\lambda < \frac{\sqrt{x}}{2 \times 2^{s_1-2}3^{(r_1-1)/2}R_1S_1M}.$$

Note that the above upper bound on λ is of the form

$$\frac{\sqrt{x}}{(\log x)^{O(1)}M} > \frac{\sqrt{x}}{(\log x)^{O(1)}x^{(r+s)/(\log\log x)^2}} > x^{1/2-3/\log\log x}$$

for large values of x. We let $\rho(p)$ be the number of solutions of $f(X) \equiv 0 \pmod{p}$. Note that $\rho(p) = 3$ if $p \nmid 6R_1S_1M$, and $\rho(p) = 1$ otherwise. By the Fundamental Lemma of Brun's combinatorial sieve (see [3, Theorem 2.5, p. 82]), there exists an absolute constant g such that the set of $\lambda < \sqrt{x}/(2^{s_1-1}3^{(r_1-1)/2}R_1S_1M)$ such that the smallest prime factor of $f(\lambda)$ exceeds $x^{1/g}$ is of cardinality

$$\gg \frac{\sqrt{x}}{2^{s_1}3^{r_1/2}R_1S_1M}\prod_{p<x^{1/g}}\left(1 - \frac{\rho(p)}{p}\right) \gg \frac{\sqrt{x}(\log\log x)^{O(1)}}{2^s3^{r/2}M(\log x)^3}.$$

Above, we have used the fact that

$$\prod_{p<x^{1/g}}\left(1 - \frac{\rho(p)}{p}\right) \ge \prod_{p\le x}\left(1 - \frac{3}{p}\right) \gg \frac{1}{(\log x)^3},$$

as well as the fact that

$$2^{s_1} = 2^{s+A} = 2^s(\log\log x)^{O(1)} \quad \text{and} \quad S_1 = O(\log\log x),$$

and similarly

$$3^{(r_1-1)/2} = 3^{r/2+(B-1)/2} = 3^{r/2}(\log\log x)^{O(1)} \quad \text{and} \quad R_1 = O((\log\log x)^2).$$

We now argue that for most of our λ, $m(m-1)(m+1)$ is not divisible by the square of any prime $p > x^{1/g}$. Indeed, let $p > x^{1/g}$ be a prime such that $p^2 \mid m(m+1)(m-1)$ for some $m \le \sqrt{x+1}$. This puts m into 1 of 3 arithmetic progressions modulo p^2, and the number of such m is $\ll \sqrt{x}/p^2$. By summing this up over p, we find that the number of such m's is bounded above by

$$\ll \sqrt{x} \sum_{p > x^{1/g}} \frac{1}{p^2} \ll x^{1/2-1/g}.$$

Thus, the number of λ's yielding such n's is also bounded by $O(x^{1/2-1/g})$. Since we have just said that the number of available λ's is

$$\gg \frac{\sqrt{x}(\log\log x)^{O(1)}}{2^s 3^{r/2} M} > x^{1/2-3/\log\log x},$$

it follows that for most of our λ's, the number $f(\lambda)$ is not a multiple of p^2 for any prime $p > x^{1/g}$. Hence, $f(\lambda)$ is square-free and has at most g prime factors.

Let us now show that $n := m^2 - 1 \in \mathcal{N}(x)$.

Note that $m^2 - 1 = 2^{s_1-1} M_1 M_{-1} S_1 f_1(\lambda)$ and $m^2 = 3^{r_1-1} R_1^2 M_0 f_2(\lambda)$, where $f_1(X)$ and $f_2(X)$ are factors of degree 2 and 1 of $f(X)$, respectively. Since 2^{s_1-1}, $M_1 M_{-1}$, S_1 and $f_1(\lambda)$ are pairwise coprime, we have

$$d(m^2-1) = d(2^{s_1-1}) d(M_1 M_{-1}) d(S_1) d(f_1(\lambda)).$$

Clearly, $d(2^{s_1-1}) = s_1 \mid S_1 \mid n$ and s_1 is odd. Further, $d(M_1 M_{-1}) = 2^s$, $d(f_1(\lambda)) = 2^w$ for some $w \le g$, and $d(S_1)$ is a power of 2. Note also that

$$d(S_1) \le 2^{\Omega(S_1)} \le 2^{2\Omega(s_1)}$$

and $\Omega(s_1) \le \log s_1/\log h \ll h/\log h$. Thus,

$$\nu_2(d(n)) \le s + g + O(h/\log h)$$

and in particular $\nu_2(d(n)) < s_1 - 1 = s + A - 1$ for large values of x. Hence, $d(n) \mid n$. Similarly,

$$d(n+1) = d(m^2) = d(3^{r_1-1}) d(M_0^2) d(R_1^2) d(f_2(\lambda)^2).$$

Note that $d(3^{r_1-1}) = r_1 \mid R_1 \mid m \mid n+1$ and r_1 is coprime to 3. Further, $d(M_0^2) = 3^r$, $d(f_2(\lambda)^2) = 3^u$ for some $u \le g$, and $d(R_1^2)$ is a power of 3. As in the previous case,

$$d(R_1^2) \le 3^{\Omega(R_1^2)} = 3^{2\Omega(R_1)} \le 3^{6\Omega(r_1)}$$

and $\Omega(r_1) \ll \log r_1/\log\log h \ll h/\log\log h$. Hence,

$$\nu_3(d(n+1)) \le r + g + O(h/\log\log h),$$

so in particular $\nu_3(d(n+1)) < r_1 - 1 = r + B - 1$ for large values of x. This shows that $d(n+1) \mid n+1$.

So, we have created several suitable values of n from a fixed M and for a fixed choice of the divisors $M_0,\ M_1,\ M_1$ of M such that $\omega(M_0) = r$. Keeping M fixed and looping over all the $\binom{r+s}{s}2^s$ possibilities of choosing such triples of divisors $\{M_{-1}, M_0, M_1\}$ of M, we get a lower bound of

$$\binom{r+s}{s}\frac{1}{3^{r/2}}\,\frac{\sqrt{x}(\log\log x)^{O(1)}}{(\log x)^3 M}$$

on the number of possibilities when M is fixed. Let us notice that

$$\binom{r+s}{s}\frac{1}{3^{r/2}} \gg \frac{c_1^L}{\sqrt{\log\log x}}. \tag{3.10}$$

Indeed, let us go through the details. Using the Stirling formula $k! = (k/e)^k\sqrt{2\pi k}e^{o(1)}$ and our choices for r and s, we get

$$\begin{aligned}\binom{r+s}{s}\frac{1}{3^{r/2}} &\gg \frac{1}{\sqrt{K}}\,\frac{(r+s)^{r+s}}{(\sqrt{3}\,r)^r s^s} \\ &= \frac{1}{\sqrt{K}}\exp\biggl(-(r+s)\biggl(-\frac{s}{r+s}\log\Bigl(\frac{s}{r+s}\Bigr) - \frac{r}{r+s}\log\Bigl(\frac{\sqrt{3}r}{r+s}\Bigr)\biggr)\biggr).\end{aligned} \tag{3.11}$$

Since

$$\frac{s}{r+s} = \frac{1}{c_1} + O\Bigl(\frac{1}{K}\Bigr), \qquad \frac{\sqrt{3}\,r}{r+s} = \frac{1}{c_1} + O\Bigl(\frac{1}{K}\Bigr),$$

it follows that the expression inside the exponential is

$$\begin{aligned}(r+s)\biggl(-\frac{s}{r+s}\log\Bigl(\frac{s}{r+s}\Bigr) - \frac{r}{r+s}\log\Bigl(\frac{\sqrt{3}\,r}{r+s}\Bigr)\biggr)& \\ = (r+s)(\log c_1 + O(1/K)) &= (r+s)\log c_1 + O(1),\end{aligned}$$

which together with (3.11) gives us (3.10). Recalling that $L = r+s$, we deduce that the number of possibilities for $n \in \mathcal{N}(x)$ created by our construction for a fixed M is

$$\gg \frac{\sqrt{x}\,c_1^L(\log\log x)^{O(1)}}{(\log x)^3 M}.$$

We now sum up over all the square-free M's with L prime factors all in $(U,V]$, getting a bound of

$$\frac{\sqrt{x}\,c_1^L(\log\log x)^{O(1)}}{(\log x)^3}\sum_{\substack{\mu^2(M)=1\\ \omega(M)=L\\ p\,\mid\,M\Rightarrow U<p\le V}}\frac{1}{M}.$$

We need to find a lower bound on the last sum. A simple combinatorial argument shows that

$$\sum_{\substack{\mu^2(M)=1\\ \omega(M)=L\\ p\,|\,M\Rightarrow U<p\le V}} \frac{1}{M} \ge \frac{1}{L!}\Bigg(\sum_{U<p\le V}\frac{1}{p}\Bigg)^{L} - \frac{1}{(L-1)!}\Bigg(\sum_{U<p\le V}\frac{1}{p^2}\Bigg)\Bigg(\sum_{\substack{U<p\le V\\ \alpha\ge 1}}\frac{1}{p^\alpha}\Bigg)^{L-1}.$$

Set

$$T_1 := \sum_{U<p\le V}\frac{1}{p}, \qquad T_2 := \sum_{\substack{U<p\le V\\ \alpha\ge 1}}\frac{1}{p^\alpha}, \qquad T_3 := \sum_{U<p\le V}\frac{1}{p^2}.$$

Clearly, using estimate (3.5), we have

$$T_1 = \log\log V - \log\log U + O(1) = \log\log x + O(\log_3 x), \tag{3.12}$$

while

$$T_2 = T_1 + \sum_{U<p\le V}\sum_{\alpha\ge 2}\frac{1}{p^\alpha} = T_1 + O(T_3) = T_1 + O\Big(\frac{1}{U}\Big).$$

Thus,

$$\frac{1}{L!}T_1^L - \frac{T_3}{(L-1)!}T_2^{L-1} = \frac{T_1^L}{L!}\Bigg(1 - T_3\Big(\frac{L}{T_1}\Big)\Big(\frac{T_2}{T_1}\Big)^{L-1}\Bigg).$$

Now $T_3 \ll 1/U$, $L/T \ll 1$ and

$$\Big(\frac{T_2}{T_1}\Big)^{L-1} = \Bigg(1 + O\Big(\frac{1}{T_1U}\Big)\Bigg)^{L-1} = \exp\Big(\frac{L}{UT_1}\Big) = \exp(o(1)) = 1 + o(1)$$

as $x \to \infty$. Thus,

$$1 - T_3\Big(\frac{L}{T_1}\Big)\Big(\frac{T_2}{T_1}\Big)^{L-1} \ge \frac{1}{2}$$

for all large enough x. So, we arrived at the conclusion that the number of $n \in \mathcal{N}(x)$ is of order at least

$$\frac{\sqrt{x}\, c_1^L (\log\log x)^{O(1)}}{(\log x)^3}\,\frac{T_1^L}{L!}.$$

Using the Stirling formula and the estimate (3.12) for T_1 implies that this last expression is

$$\gg \frac{\sqrt{x}\, c_1^L (\log\log x)^{O(1)}}{(\log x)^3}\Bigg(\frac{e\log\log x + O(\log_3 x)}{L}\Bigg)^{L}. \tag{3.13}$$

The last factor is

$$\left(\frac{e\log\log x+O(\log_3 x)}{L}\right)^L=\left(\frac{e}{c_1}+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)^L$$
$$=\frac{e^L}{c_1^L}\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right)^{O(\log_2 x)}=\frac{(\log x)^{c_1}(\log\log x)^{O(1)}}{c_1^L},$$

which inserted in (3.13) gives the desired bound

$$\#\mathcal{N}(x)\gg\frac{\sqrt{x}(\log\log x)^{O(1)}}{(\log x)^c}.$$

4. Comments. As pointed out in the introduction, Spiro proved that the number of $n\le x$ such that $d(n)\,|\,n$ is $x(\log\log x)^{-1+o(1)}/(\log x)^{1/2}$ as $x\to\infty$. Our result is a bit weaker in that we have not given an explicit limiting exponent for the $\log\log x$ in our version of the problem—we only showed that it is bounded. Spiro also looked at positive integers n such that $d(n)\,|\,n+1$. Following Spiro, we can also ask about the counting function of the positive integers n such that $d(n)\,|\,n+1$ and $d(n+1)\,|\,n$, or both $d(n)$ and $d(n+1)$ dividing n.

By weakening the condition that $d(n+k)$ divides $n+k$ for $k=0,1,\dots,s$ to $d(n+k)\,|\,m+k$ for some positive integer m and all $k=0,1,\dots,s$, we found several examples with $s=2$. Here are a few such (n,m):

$\{(3,2)$, $(15,4)$, $(35,8)$, $(63,6)$, $(195,8)$, $(255,8)$, $(399,104)$, $(1023,208)$, $(1088,350)$, $(1295,24)$, $(1599,104)$, $(2915,104)$, $(3135,272)$, $(4355,80)$, $(6399,350)$, $(7055,224)$, $(8099,224)$, $(8835,80)$, $(12099,80)$, $(15375,224)$, $(15875,224)$, $(16899,80)$, $(17955,224)$, $(22499,44)$, $(24335,224)$, $(25599,32)$, $(32399,224)$, $(33123,728)$, $(33855,272)$, $(41615,224)$, $(44099,80)$, $(52899,80)$, $(55695,224)$, $(57599,80)$, $(62499,104)$, $(65535,16)$, $(69695,440)$, $(72899,188)$, $(80655,224)$, $(89999,224)$, $(93635,224)$, $(115599,224)$, $(122499,224)$, $(147455,224)$, $(156815,224)$, $(159999,224)$, $(164835,80)$, $(176399,404)$, $(184899,80)$, $(190095,224)$, $(197135,224)$, $(215295,80)$, $(217155,80)$, $(220899,80)$, $(240099,224)$, $(249999,664)$, $(287295,272)$, $(295935,32)$, $(309135,224)$, $(324899,80)$, $(331775,324)$, $(352835,944)$, $(401955,80)$, $(414735,224)$, $(417315,80)$, $(427715,80)$, $(462399,944)$, $(470595,272)$, $(476099,80)$, $(484415,944)$, $(489999,224)$, $(495615,896)$, $(512655,224)$, $(547599,224)$, $(562499,440)$, $(577599,944)$, $(608399,944)$, $(614655,224)$, $(665855,80)$, $(739599,224)$, $(746495,384)$, $(792099,80)$, $(820835,80)$, $(846399,440)$, $(876095,944)$, $(894915,512)$, $(902499,224)$, $(933155,80)$, $(972195,512)$, $(1008015,224)$, $(1020099,80)$, $(1110915,80)$, $(1123599,224)$, $(1136355,512)$, $(1196835,80)$, $(1201215,272)$, $(1223235,512)$, $(1299599,944)$, $(1313315,80)$, $(1322499,224)$, $(1464099,224)$, $(1547535,224)$, $(1552515,512)$, $(1664099,80),\dots\}$.

We do not know how to prove that the set of such n is infinite. However, consider the following examples:

(i) $m-1,\ m,\ m+1$ are all square-free.
(ii) m^2+1 is prime.

Then taking $n := m^2 - 1$, the number n has the desired property for $s = 3$ since $d(m^2 - 1)$ is a power of 2, $d(m^2)$ is a power of 3 and $d(m^2 + 1) = 2$. That there should be infinitely many such examples is predicted by [4, Conjecture 5], which generalizes the Bateman–Horn conjectures, and which predicts that the number of such $n \leq x$ should be asymptotically $C\sqrt{x}/\log x$ for some positive constant C. We also observe that many of examples have the form $n = 4k^2 - 1$.

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References

[1] E. R. Canfield, P. Erdős and C. Pomerance, *On a problem of Oppenheim concerning 'Factorisatio Numerorum'*, J. Number Theory 17 (1983), 1–28.

[2] P. Erdős und G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem*, Acta Sci. Math. (Szeged) 7 (1934), 95–102.

[3] H. Halberstam and H.-E. Rickert, *Sieve Methods*, Academic Press, 1974.

[4] J. Jiménez Urroz, F. Luca and I. E. Shparlinski, *On the number of isogeny classes of pairing-friendly elliptic curves and statistics of MNT curves*, Math. Comp. 81 (2012), 1093–1110.

[5] W. Ljunggren, *Zur Theorie der Gleichung* $x^2 + 1 = Dy^4$, Avh. Norske Vid. Akad. Oslo 1 (1942), 27 pp.

[6] F. Luca and C. Pomerance, *Irreducible radical extensions and Euler-function chains*, in: Combinatorial Number Theory, B. Landman et al. (eds.), de Gruyter, 2007, 351–362.

[7] C. Spiro, *How often is the number of divisors of* n *a divisor of* n*?*, J. Number Theory 21 (1985), 81–100.

[8] C. Spiro, *How often does the number of divisors of an integer divide its successor?*, J. London Math. Soc. 31 (1985), 30–40.

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