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DYNAMICS OF A MODIFIED DAVEY–STEWARTSON SYSTEM IN $\mathbb{R}^3$

BY

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Abstract. We study the Cauchy problem in $\mathbb{R}^3$ for the modified Davey–Stewartson system

$$i\partial_t u + \Delta u = \lambda_1 |u|^4 u + \lambda_2 b_1 u v_{x_1}, \quad -\Delta v = b_2(|u|^2)_{x_1}.$$

Under certain conditions on λ_1 and λ_2, we provide a complete picture of the local and global well-posedness, scattering and blow-up of the solutions in the energy space. Methods used in the paper are based upon the perturbation theory from [Tao et al., Comm. Partial Differential Equations 32 (2007), 1281–1343] and the convexity method from [Glassey, J. Math. Phys. 18 (1977), 1794–1797].

1. Introduction. The Davey–Stewartson system of partial differential equations has its origin in fluid mechanics. These are model equations in the theory of shallow-water waves [6] for the functions u and v, related to the amplitude and the mean velocity potential of the water wave, which satisfy the equations

$$\begin{cases} i\partial_t u + u_{xx} + \mu u_{yy} = -a|u|^2 u + b_1 u v_{x_1}, \\ \nu v_{xx} + v_{yy} = b_2(|u|^2)_{x_1}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2. \end{cases} \tag{1.1}$$

Here $u = u(t,x)$ is a complex-valued function, $v = v(t,x)$ is a real-valued function, and μ, ν, a, b_1, b_2 are real constants. This system provides a canonical description of the amplitude dynamics of a weakly nonlinear two-dimensional wave packet when a mean field is driven by a modulation (see [6]). Electrostatic ion wave packets propagating in an arbitrary direction in a magnetized plasma is an example of physical application of such equations. The Davey–Stewartson system is classified as elliptic-elliptic $(+,+)$, elliptic-hyperbolic $(-,+)$, hyperbolic-elliptic $(+,-)$, hyperbolic-hyperbolic $(-,-)$ according to the signs of μ, ν.

A large amount of work has been devoted to the study of the Davey–Stewartson system (1.1). Ghidaglia and Saut [9] studied the Cauchy problem for (1.1) and (except for the case $\mu, \nu < 0$) proved its solvability in $H^1(\mathbb{R}^2)$.

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In the elliptic-hyperbolic case, Tsutsumi [29] obtained the $L^p(\mathbb{R}^2)$ decay estimates of solutions of system (1.1) for $2 < p < \infty$. Ozawa [24] gave exact blow-up solutions of the Cauchy problem for (1.1). Ohta [22, 23] discussed the existence and nonexistence of stable standing waves under certain conditions. Guo and Wang [11] studied the Cauchy problem for (1.1) in the case $\mu, \nu > 0$. Gan and Zhang [8] used the cross-constrained variational method to study the sharp threshold for the global existence and instability of standing waves for (1.1). The extension of the Davey–Stewartson system to high dimensions was considered by Zakharov and Schulman [30] and Nishinari, Abe and Satsuma [21] (see also [15, 25, 26] and the references therein).

In the present paper, we consider the following modified three-dimensional Davey–Stewartson system:

$$\begin{cases} i\partial_t u + \Delta u = \lambda_1 |u|^4 u + \lambda_2 u v_{x_1}, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ -\Delta v = (|u|^2)_{x_1}, \end{cases} \tag{1.2}$$

where $u(t,x)$ and $v(t,x)$ are complex- and real-valued functions, respectively. This system is a three-dimensional extension of equations (1.1) in the elliptic-elliptic case $\mu = \nu = 1$. Notice first that system (1.2) can be reduced to a single equation by introducing the pseudo-differential operator defined by

$$\widehat{E_1 f}(\xi) = \frac{\xi_1^2}{|\xi|^2} \hat{f}(\xi).$$

Indeed, solving the second equation in (1.2) with respect to v and substituting it into the first one, we obtain the Cauchy problem

$$\begin{cases} iu_t + \Delta u = \lambda_1 |u|^4 u + \lambda_2 E_1(|u|^2) u, \\ u_0 = u(0,x) \in H^1(\mathbb{R}^3). \end{cases} \tag{1.3}$$

If the nonlinearity $N(u) = \lambda_1 |u|^4 u + \lambda_2 E_1(|u|^2) u$ is replaced with $N(u) = \lambda_1 |u|^4 u + \lambda_2 |u|^2 u$ in (1.3), Tao et al. [28] and Miao et al. [**?**] have systematically studied this type of combined nonlinear Schrödinger equations. The nonlinearity in our paper contains a nonlocal form $E_1(|u|^2)u$, which causes much trouble because it does not obey the relation

$$\mathrm{Re}(N(u)\nabla \bar{u}) = \nabla(\mathcal{N}(u)) \quad \text{for some real-valued function } \mathcal{N}(u) \tag{1.4}$$

(i.e. it is not Hamiltonian). Hence, a delicate analysis is needed to deal with such a nonlinearity. To get over several difficulties, we take into account some almost local estimates of E_1 by making use of singular integral operators and Fourier analysis. Here, let us mention that related considerations concerning the nonlocal nonlinearity $f(u) = (|x|^{-\gamma} * |u|^2)u$, $0 < \gamma < N$, can be found in [7, 16–19].

Notice that, by a standard reasoning, every H^1-solution of the Cauchy problem (1.3) conserves the following physical quantities:

$$\text{Mass:} \qquad M(u) = \int_{\mathbb{R}^3} |u(x,t)|^2\,dx = M(u_0),$$

$$\text{Energy:} \qquad E(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(x,t)|^2\,dx + \frac{\lambda_1}{6}\int_{\mathbb{R}^3} |u(x,t)|^6\,dx + \frac{\lambda_2}{4}\int_{\mathbb{R}^3} E_1(|u(x,t)|^2)|u(x,t)|^2\,dx = E(u_0),$$

$$\text{Momentum:} \qquad P(u) = \operatorname{Im}\int_{\mathbb{R}^3} \bar{u}(x,t)\nabla u(x,t)\,dx = P(u_0).$$

In this paper, we will systematically study the local and global well-posedness, scattering and blow-up results for the Cauchy problem (1.3) under certain assumptions on the parameters λ_1, λ_2 and initial data u_0. The local theory for problem (1.3) is considered in Section 3. Here, standard techniques involving the Banach fixed point theorem can be used to construct local-in-time solutions. The term $|u|^4 u$ is energy-critical, thus the maximal time of existence for these local solutions depends on the profile of the initial data, rather than on its H^1_x-norm.

Now we state the main results.

THEOREM 1.1 (Global well-posedness). *For every $u_0 \in H^1_x$ and $\lambda_1 > 0$, there exists a unique global-in-time solution $u(t,x)$ to problem* (1.3). *Moreover, for every compact interval I, the solution $u(t,x)$ satisfies the space-time bound*

$$\|u\|_{S^1(I\times\mathbb{R}^3)} \le C(|I|, \|u_0\|_{H^1_x}),$$

where $|I|$ is the length of the interval and the space $S^1(I\times\mathbb{R}^3)$ is defined in (2.1) *below.*

To prove this theorem, we combine an a priori estimate of the kinetic energy $\|u(t,x)\|_{\dot{H}^1_x}$ together with a "good" local well-posedness result, where the time of existence of a solution to problem (1.3) depends on the H^1_x-norm of the initial datum only.

Next, we study long time behavior of solutions.

THEOREM 1.2 (Energy-space scattering). *Let $u_0 \in H^1_x$, $\lambda_1 > 0$ and $u(t,x)$ be the unique solution to problem* (1.3). *There exist unique $u_\pm \in H^1_x$ such that*

$$\|u(t) - e^{it\Delta}u_\pm\|_{H^1_x} \to 0 \qquad \text{as } t \to \pm\infty$$

under the small mass condition $M \le c(\|\nabla u_0\|_2)$ for a small number $c > 0$ depending only on $\|\nabla u_0\|_2$.

In the proof of this theorem, we first obtain a bound for a finite global Strichartz norm of a solution to problem (1.3) using the small mass assumption and the stability result from Lemma 2.5 for the energy-critical NLS. Then, by a standard argument, the finite global Strichartz norm implies scattering for problem (1.3).

Finally, we will prove blow-up in a finite time using the convexity method from [10].

THEOREM 1.3 (Blow-up of solutions). *Let* $\lambda_1 < 0$, $u_0 \in H^1_x(\mathbb{R}^3)$, $xu_0 \in L^2(\mathbb{R}^3)$, $\Im \int_{\mathbb{R}^3} \bar{u}_0 x \cdot \nabla u_0 \, dx > 0$, *and* $E(u_0) < 0$. *Then the solution* $u(t,x)$ *of problem* (1.3) *blows up in finite time; more precisely, there exists* $T_* > 0$ *such that* $\lim_{t\to T_*} \|\nabla u(t,x)\|_{L^2_x} = \infty$.

The remainder of the paper is organized as follows. We introduce notation and some well-known results in Section 2. In Section 3, we prove the local well-posedness result and some linear estimates. Section 4 is devoted to global well-posedness. In Section 5, we use perturbation theory and the small mass assumption to obtain the global scattering result. Finally, we consider the finite time blow-up in Section 6.

2. Preliminaries. First, we introduce the notation and several fundamental lemmas needed in this paper. The notation $A \lesssim B$ means that $A \le CB$ for some constant C. Likewise, if $A \lesssim B \lesssim A$, we say that $A \sim B$. We use $L^r_x(\mathbb{R}^N)$ to denote the Lebesgue space of functions $f : \mathbb{R}^N \to \mathbb{C}$ with

$$\|f\|_{L^r} := \Big(\int_{\mathbb{R}^N} |f(x)|^r \, dx \Big)^{1/r} < \infty,$$

with the usual modification when $r = \infty$. We also use the space-time Lebesgue spaces $L^q_t L^r_x$ which are equipped with the norm

$$\|f\|_{L^q_t L^r_x} := \Big(\int_I \|f\|^q_{L^r_x} \, dt \Big)^{1/q}$$

for any space-time slab $I \times \mathbb{R}^N$. When $q = r$, we abbreviate $L^q_t L^r_x$ by $L^q_{t,x}$.

A pair (q,r) is called *Schrödinger-admissible* if

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2} \quad \text{for } 2 \le q, r \le \infty.$$

For a spacetime slab $I \times \mathbb{R}^3$, we define

$$\|u\|_{\dot{S}^0(I\times\mathbb{R}^3)} := \sup \|u\|_{L^q_t L^r_x(I\times\mathbb{R}^3)},$$

where the sup is taken over all admissible pairs (q,r). We also use the norm

$$\|u\|_{\dot{S}^1(I\times\mathbb{R}^3)} := \|\nabla u\|_{\dot{S}^0(I\times\mathbb{R}^3)},$$

and we introduce the space

$$S^1(I \times \mathbb{R}^3) = \dot{S}^0(I \times \mathbb{R}^3) \cap \dot{S}^1(I \times \mathbb{R}^3) \tag{2.1}$$

with the usual norm.

Denote by $\dot{N}^0(I\times\mathbb{R}^3)$ the dual space of $\dot{S}^0(I\times\mathbb{R}^3)$. Moreover, we denote

$$\dot{N}^1(I \times \mathbb{R}^3) = \{u : \nabla u \in \dot{N}^0(I \times \mathbb{R}^3)\},$$
$$N^1(I \times \mathbb{R}^3) = \dot{N}^0(I \times \mathbb{R}^3) \cap \dot{N}^1(I \times \mathbb{R}^3).$$

Finally, we deal with the norms

$$\|u\|_{V(I)} = \|u\|_{L^{10/3}_{t,x}(I\times\mathbb{R}^3)}, \quad \|u\|_{W(I)} = \|u\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)},$$

and we introduce the spaces

$$\dot{X}^0(I) = L^8_t L^{12/5}_x(I \times \mathbb{R}^3) \cap V(I) \cap L^{10}_t L^{30/13}_x(I \times \mathbb{R}^3),$$
$$\dot{X}^1(I) = \{u : \nabla u \in \dot{X}^0(I)\}, \qquad X^1(I) = \dot{X}^0(I) \cap \dot{X}^1(I).$$

LEMMA 2.1 (Strichartz estimates [1, 14, 27]). *Let I be a compact time interval, $k \in \{0,1\}$, and $u : I\times\mathbb{R}^3 \to \mathbb{C}$ be an $\dot{S}^k$-solution to the Schrödinger equation*

$$iu_t + \Delta u = F$$

for a given function F. Then

$$\|u\|_{\dot{S}^k(I\times\mathbb{R}^3)} \lesssim \|u(t_0)\|_{\dot{H}^k_x(\mathbb{R}^3)} + \|F\|_{\dot{N}^k(I\times\mathbb{R}^3)}$$

for every $t_0 \in I$.

Next, we recall some known facts from [4, 5].

LEMMA 2.2. *Let E_1 be the singular integral operator defined in Fourier variables by*

$$\widehat{E_1 f}(\xi) = \frac{\xi_1^2}{|\xi|^2}\hat{f}(\xi).$$

For $1 < p < \infty$, the operator E_1 has the following properties:

(i) *$E_1 \in \mathcal{L}(L^p, L^p)$, where $\mathcal{L}(L^p, L^p)$ denotes the space of bounded liner operators from L^p to L^p.*
(ii) *If $\psi \in H^s$, then $E_1(\psi) \in H^s$, $s \in \mathbb{R}$.*
(iii) *If $\psi \in W^{m,p}$, then $E_1(\psi) \in W^{m,p}$ and*

$$\partial_k E_1(\psi) = E_1(\partial_k \psi), \qquad k = 1, \dots, N.$$

(iv) *E_1 preserves the following operations:*
Translation: $E_1(\psi(\cdot + y))(x) = E_1(\psi)(x + y)$, $y \in \mathbb{R}^N$,
Dilation: $E_1(\psi(\lambda\cdot))(x) = E_1(\psi)(\lambda x)$, $\lambda > 0$,
Conjugation: $\overline{E_1(\psi)} = E_1(\overline{\psi})$, where $\overline{\psi}$ is the complex conjugate of ψ.

Remark 2.3. Notice that, from the definition of E_1 and from the Parseval identity, we immediately obtain the following relations:

$$\int_{\mathbb{R}^N} |\psi|^2 E_1(|\psi|^2)\,dx \le \int_{\mathbb{R}^N} |\psi|^4\,dx,$$

$$\int_{\mathbb{R}^N} |\psi|^2 E_1(|\psi|^2)\,dx = \int_{\mathbb{R}^N} \frac{\xi_1^2}{|\xi|^2}\big|\widehat{(|\psi|^2)}\big|^2\,d\xi > 0.$$

Lemma 2.4. *For all Schwartz functions ϕ,*

$$\int_{\mathbb{R}^N} \phi x\cdot\nabla\phi\,dx = -\frac{N}{2}\int_{\mathbb{R}^N} |\phi|^2\,dx,$$

$$\int_{\mathbb{R}^N} |\phi|^{p-1}\phi x\cdot\nabla\phi\,dx = -\frac{N}{p+1}\int_{\mathbb{R}^N} |\phi|^{p+1}\,dx,$$

$$\int_{\mathbb{R}^N} E_1(|\phi|^2)\phi x\cdot\nabla\phi\,dx = -\frac{N}{4}\int_{\mathbb{R}^N} |\phi|^2 E_1(|\phi|^2)\,dx.$$

Since the energy-critical NLS is well-understood, we treat equation (1.3) as its perturbation. Thus, to conclude this section, we show the following stability result, which will be frequently used in this paper and the proof of which can be found in [**?**].

Lemma 2.5 ($\dot H^1_x$ critical stability result, Tao et al. [**?**]). *Let I be a compact time interval and let $\tilde w$ be an approximate solution of the equation*

$$(i\partial_t+\Delta)w = |w|^4 w \tag{2.2}$$

on $I\times\mathbb{R}^3$ in the sense that

$$(i\partial_t+\Delta)\tilde w = |\tilde w|^4\tilde w + e \tag{2.3}$$

for some function e. Assume that

$$\|\tilde w\|_{W(I)} \le L, \tag{2.4}$$

$$\|\tilde w\|_{L^\infty_t \dot H^1_x} \le E_0 \tag{2.5}$$

for some constants $L, E_0>0$. Let $t_0\in I$ and let $w(t_0)$ be close to $\tilde w(t_0)$ in the sense that

$$\|w(t_0)-\tilde w(t_0)\|_{\dot H^1_x} \le E' \tag{2.6}$$

for some $E'>0$. Assume also the smallness conditions

$$\Big(\sum_N \|P_N\nabla e^{i(t-t_0)\Delta}(w(t_0)-\tilde w(t_0))\|_{L^{10}_t L^{30/13}_x}\Big)^{1/2} \le \varepsilon, \tag{2.7}$$

$$\|\nabla e\|_{\dot N^0(I\times\mathbb{R}^3)} \le \varepsilon \tag{2.8}$$

for some $0<\varepsilon<\varepsilon_2$, where $\varepsilon_2=\varepsilon_2(E_0,E',L)$ is a small constant. Then there exists a solution w to equation (2.2) on $I\times\mathbb{R}^3$ with the initial data

$w(t_0)$ at time $t = t_0$ satisfying

$$\|\nabla(w-\tilde{w})\|_{L_t^{10}L_x^{30/13}} \le c(E_0,E',L)(\varepsilon+\varepsilon^7), \tag{2.9}$$

$$\|w-\tilde{w}\|_{\dot{S}^1(I\times\mathbb{R}^3)} \le c(E_0,E',L)(E'+\varepsilon+\varepsilon^7), \tag{2.10}$$

$$\|w\|_{\dot{S}^1(I\times\mathbb{R}^3)} \le c(E_0,E',L). \tag{2.11}$$

3. Local theory. Before we construct local-in-time solutions, we prove two linear estimates.

LEMMA 3.1. *Let I be a compact time interval, λ_1,λ_2 be nonzero real numbers, and $k\in\{0,1\}$. Then*

$$\begin{aligned}\|\lambda_1|u|^4u+\lambda_2E_1(|u|^2)u\|_{\dot{N}^k(I\times\mathbb{R}^3)} &\lesssim |I|^{1/2}\|u\|^2_{\dot{X}^1(I\times\mathbb{R}^3)}\|u\|_{\dot{X}^k(I\times\mathbb{R}^3)}\\ &\quad+\|u\|^4_{\dot{X}^1(I\times\mathbb{R}^3)}\|u\|_{\dot{X}^k(I\times\mathbb{R}^3)}\end{aligned}$$

and

$$\begin{aligned}&\|(\lambda_1|u|^4u+\lambda_2E_1(|u|^2)u)-(\lambda_1|v|^4v+\lambda_2E_1(|v|^2)v)\|_{\dot{N}^0(I\times\mathbb{R}^3)}\\ &\quad\lesssim |I|^{1/2}(\|u\|^2_{\dot{X}^1(I\times\mathbb{R}^3)}+\|v\|^2_{\dot{X}^1(I\times\mathbb{R}^3)})\|u-v\|_{\dot{X}^0(I\times\mathbb{R}^3)}\\ &\qquad+(\|u\|^4_{\dot{X}^1(I\times\mathbb{R}^3)}+\|v\|^4_{\dot{X}^1(I\times\mathbb{R}^3)})\|u-v\|_{\dot{X}^0(I\times\mathbb{R}^3)}.\end{aligned}$$

Proof. We only estimate the quantity $\lambda_2E_1(|u|^2)u$, because the reasoning in the case of $|u^4|u$ is similar. Using the Hölder and Sobolev inequalities and the boundedness of E_1 on L^p, we have

$$\begin{aligned}\|E_1(|u|^2)u\|_{\dot{N}^k(I\times\mathbb{R}^3)} &\lesssim \big\||\nabla|^k(E_1(|u|^2)u)\big\|_{L_t^{8/7}L_x^{12/7}(I\times\mathbb{R}^3)}\\ &\lesssim |I|^{1/2}\big\||\nabla|^k(E_1(|u|^2)u)\big\|_{L_t^{8/3}L_x^{12/7}(I\times\mathbb{R}^3)}\\ &\lesssim |I|^{1/2}\big\|\|E_1(|u|^2)\|_{L_x^6}\big\||\nabla|^ku\big\|_{L_x^{12/5}}+\big\||\nabla|^kE_1(|u|^2)\big\|_{L_x^2}\|u\|_{L_x^{12}}\big\|_{L_t^{8/3}(I)}\\ &\lesssim |I|^{1/2}\big\||u|^2\big\|_{L_t^4L_x^6(I\times\mathbb{R}^3)}\big\||\nabla|^ku\big\|_{L_t^8L_x^{12/5}(I\times\mathbb{R}^3)}\\ &\quad+|I|^{1/2}\big\|u|\nabla|^ku\big\|_{L_t^4L_x^2(I\times\mathbb{R}^3)}\|u\|_{L_t^8L_x^{12}(I\times\mathbb{R}^3)}\\ &\lesssim |I|^{1/2}\|u\|^2_{L_t^8L_x^{12}(I\times\mathbb{R}^3)}\big\||\nabla|^ku\big\|_{L_t^8L_x^{12/5}(I\times\mathbb{R}^3)}\\ &\lesssim |I|^{1/2}\|\nabla u\|^2_{L_t^8L_x^{12/5}(I\times\mathbb{R}^3)}\big\||\nabla|^ku\big\|_{L_t^8L_x^{12/5}(I\times\mathbb{R}^3)}\\ &\lesssim |I|^{1/2}\|u\|^2_{\dot{X}^1(I\times\mathbb{R}^3)}\|u\|_{\dot{X}^k(I\times\mathbb{R}^3)}.\end{aligned}$$

By a completely analogous reasoning, we obtain

$$\begin{aligned}
&\|\lambda_2 E_1(|u|^2)u - \lambda_2 E_1(|v|^2)v\|_{\dot N^0(I\times\mathbb{R}^3)}\\
&\quad\lesssim \|E_1(|u|^2)u - E_1(|v|^2)v\|_{L_t^{8/7}L_x^{12/7}(I\times\mathbb{R}^3)}\\
&\quad\lesssim |I|^{1/2}\|E_1(|u|^2)u - E_1(|v|^2)v\|_{L_t^{8/3}L_x^{12/7}(I\times\mathbb{R}^3)}\\
&\quad\lesssim |I|^{1/2}\|E_1(|u|^2)(u-v) + E_1(|u|^2-|v|^2)v\|_{L_t^{8/3}L_x^{12/7}(I\times\mathbb{R}^3)}\\
&\quad\lesssim |I|^{1/2}\|u\|^2_{\dot X^1(I\times\mathbb{R}^3)}\|u-v\|_{\dot X^0(I\times\mathbb{R}^3)}\\
&\qquad + |I|^{1/2}(\|u\|_{\dot X^1(I\times\mathbb{R}^3)} + \|v\|_{\dot X^1(I\times\mathbb{R}^3)})\|v\|_{\dot X^1(I\times\mathbb{R}^3)}\|u-v\|_{\dot X^0(I\times\mathbb{R}^3)}\\
&\quad\lesssim |I|^{1/2}(\|u\|^2_{\dot X^1(I\times\mathbb{R}^3)} + \|v\|^2_{\dot X^1(I\times\mathbb{R}^3)})\|u-v\|_{\dot X^0(I\times\mathbb{R}^3)}. \;\blacksquare
\end{aligned}$$

LEMMA 3.2. *Let $I\times\mathbb{R}^3$ be an arbitrary spacetime slab and $k\in\{0,1\}$. Then*

$$\|E_1(|u|^2)u\|_{\dot N^k(I\times\mathbb{R}^3)} \lesssim \|u\|_{V(I)}\|u\|_{W(I)}\big\||\nabla|^k u\big\|_{V(I)},$$

$$\big\||u|^4u\big\|_{\dot N^k(I\times\mathbb{R}^3)} \lesssim \|u\|^4_{W(I)}\big\||\nabla|^k u\big\|_{V(I)}.$$

Proof. By the boundedness of E_1 on $L^p(\mathbb{R}^3)$ for every $1<p<\infty$, and by the Hölder and interpolation inequalities, we have

$$\begin{aligned}
\|E_1(|u|^2)u\|_{\dot N^k(I\times\mathbb{R}^3)} &\lesssim \big\||\nabla|^k(E_1(|u|^2)u)\big\|_{L_{t,x}^{10/7}(I\times\mathbb{R}^3)}\\
&\lesssim \|E_1(|u|^2)\|_{L_{t,x}^{5/2}(I\times\mathbb{R}^3)}\big\||\nabla|^k u\big\|_{L_{t,x}^{10/3}(I\times\mathbb{R}^3)}\\
&\quad + \big\||\nabla|^k E_1(|u|^2)\big\|_{L_{t,x}^{5/3}(I\times\mathbb{R}^3)}\|u\|_{L_{t,x}^{10}(I\times\mathbb{R}^3)}\\
&\lesssim \|u\|^2_{L_{t,x}^{5}(I\times\mathbb{R}^3)}\big\||\nabla|^k u\big\|_{L_{t,x}^{10/3}(I\times\mathbb{R}^3)}\\
&\quad + \big\||\nabla|^k u\big\|_{L_{t,x}^{10/3}(I\times\mathbb{R}^3)}\|u\|_{L_{t,x}^{10/3}(I\times\mathbb{R}^3)}\|u\|_{L_{t,x}^{10}(I\times\mathbb{R}^3)}\\
&\lesssim \|u\|_{L_{t,x}^{10}(I\times\mathbb{R}^3)}\|u\|_{L_{t,x}^{10/3}(I\times\mathbb{R}^3)}\big\||\nabla|^k u\big\|_{L_{t,x}^{10/3}(I\times\mathbb{R}^3)}\\
&\lesssim \|u\|_{V(I)}\|u\|_{W(I)}\big\||\nabla|^k u\big\|_{V(I)}.
\end{aligned}$$

The estimate of $\big\||u|^4u\big\|_{\dot N^k(I\times\mathbb{R}^3)}$ can be obtained similarly. ■

Based on the linear estimates from Lemmas 3.1 and 3.2, we may use the standard argument from [2, 3] to achieve the following proposition (see also [12–14] for more details).

PROPOSITION 3.3 (Local well-posedness). *Let $u_0\in H_x^1(\mathbb{R}^3)$ and λ_1, λ_2 be nonzero real constants. Then for every $T>0$, there exists $\eta=\eta(T)$ such that if*

$$\|e^{it\Delta}u_0\|_{X^1([-T,T])} \le \eta,$$

then problem (1.3) *admits a unique strong* H^1_x*-solution* u *defined on* $[-T,T]$. *Let* $(-T_{\min}, T_{\max})$ *be the maximal time interval on which* u *is defined. Then* $u \in S^1(I \times \mathbb{R}^3)$ *for every compact time interval* $I \subset (-T_{\min}, T_{\max})$. *Furthermore,*

- *if* $T_{\max} < \infty$, *then*

$$\text{either} \quad \lim_{t\to T_{\max}} \|u\|_{\dot H^1_x} = \infty, \quad \text{or} \quad \|u\|_{\dot S^1((0,T_{\max})\times\mathbb{R}^3)} = \infty;$$

  similarly, if $T_{\min} < \infty$, *then*

$$\text{either} \quad \lim_{t\to -T_{\min}} \|u\|_{\dot H^1_x} = \infty, \quad \text{or} \quad \|u\|_{\dot S^1((-T_{\min},0)\times\mathbb{R}^3)} = \infty.$$

- *The solution* u *depends continuously on the initial datum* u_0 *in the following sense: if* $u_0^{(m)} \to u_0$ *in* H^1_x *and if* $u^{(m)}$ *is the maximal solution to problem* (1.3) *with initial datum* $u_0^{(m)}$, *then* $u^{(m)} \to u$ *in* $L^q_t H^1_x([-S,T]\times\mathbb{R}^3)$ *for every* $q < \infty$ *and every interval* $[-S,T] \subset (-T_{\min}, T_{\max})$.

LEMMA 3.4 (Blow-up criterion). *Let* $u_0 \in H^1_x$ *and let* u *be the unique strong solution to problem* (1.3) *on the spacetime slab* $[0,T_0]\times\mathbb{R}^3$ *such that* $\|u\|_{\dot X^1([0,T_0])} < \infty$. *Then there exists* $\delta = \delta_{u_0}$ *such that the solution* u *can be extended to a strong* H^1_x*-solution on the slab* $[0, T_0+\delta]\times\mathbb{R}^3$.

The proof of the blow-up criterion is based on a standard contradiction argument: if the time existence interval of the solution cannot be extended beyond a time T_0, then the $\dot X^1$-norm must blow-up at T_0 (see e.g. [1] for more details).

LEMMA 3.5. *Let* $k \in \{0,1\}$, I *be a compact time interval, and* u *be a unique solution to problem* (2.2) *on* $I\times\mathbb{R}^3$ *obeying the bound* $\|u\|_{W(I)} \le L$, *where* $L > 0$. *If* $t_0 \in I$ *and* $u(t_0) \in H^k_x$, *then* $\|u\|_{\dot S^k(I\times\mathbb{R}^3)} \le c(L)\|w(t_0)\|_{\dot H^k_x}$.

Proof. Divide the interval I into $N \sim (1+M/\eta)^6$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{W(I_j)} \le \eta,$$

where η is a small constant to be chosen later. By the Strichartz estimate (see Lemma 2.1), in each I_j, we obtain

$$\begin{aligned}\|u\|_{\dot S^k(I_j\times\mathbb{R}^3)} &\lesssim \|u(t_j)\|_{\dot H^k_x} + \|u\|_{\dot S^k(I_j\times\mathbb{R}^3)}\|u\|^4_{W(I_j)} + \|u\|_{W(I_j)}\|u\|^2_{\dot S^k(I_j\times\mathbb{R}^3)}\\ &\le \|u(t_j)\|_{\dot H^k_x} + \|u\|_{\dot S^k(I_j\times\mathbb{R}^3)}\eta^4 + \eta\|u\|^2_{\dot S^k(I_j\times\mathbb{R}^3)}.\end{aligned}$$

Hence, choosing η sufficiently small, we get

$$\|u\|_{\dot S^k(I_j\times\mathbb{R}^3)} \lesssim \|u(t_j)\|_{\dot H^k_x} \quad \text{for all } j \in \{0,1,2,\dots\}.$$

Indeed, in the interval I_0, we have

$$\|u(t_1)\|_{\dot{H}^k_x} \le \|u\|_{\dot{S}^k(I_0\times\mathbb{R}^3)} \le C\|u(t_0)\|_{\dot{H}^k_x}.$$

Moreover, in I_1, we get

$$\|u(t_2)\|_{\dot{H}^k_x} \le \|u\|_{\dot{S}^k(I_1\times\mathbb{R}^3)} \le C\|u(t_1)\|_{\dot{H}^k_x} \le C^2\|u(t_0)\|_{\dot{H}^k_x}.$$

Similarly, for each interval I_j, we obtain

$$\|u(t_j)\|_{\dot{H}^k_x} \le C^j\|u(t_0)\|_{\dot{H}^k_x}.$$

Summing up the above estimates over all subintervals I_j, we complete the proof. ∎

4. Global well-posedness. In order to obtain a global well-posedness result, we first get an a priori bound on the kinetic energy of a solution. Then we establish a "good" local well-posedness result, which shows that the existence time of a H^1_x-solution depends only on the H^1_x-norm of the initial datum. The above two steps combined with the conservation of mass lead to the global-in-time solution by a standard iterative method.

PROPOSITION 4.1 (Kinetic energy control). *Let $u_0 \in H^1_x$ and $\lambda_1 > 0$. There exists a unique global-in-time solution $u(t,x)$ to problem* (1.3)*. Moreover, there exists a number $C(E,M) > 0$ such that*

$$\|u(t,x)\|_{\dot{H}^1_x} \le C(E,M) \quad \textit{for } t \in \mathbb{R}.$$

Proof. Recall that the energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(x,t)|^2\,dx + \frac{\lambda_1}{6}\int_{\mathbb{R}^3} |u(x,t)|^6\,dx + \frac{\lambda_2}{4}\int_{\mathbb{R}^3} E_1(|u(x,t)|^2)|u(x,t)|^2\,dx$$

is conserved in time. We consider the following two cases:

(1) Let $\lambda_1, \lambda_2 > 0$. Then by the inequality $\frac{1}{4}\int_{\mathbb{R}^3} E_1(|u(x,t)|^2)|u(x,t)|^2\,dx \ge 0$ and the conservation of energy, we have

$$\|u(t,x)\|_{\dot{H}^1_x} \le E(u(t,x)) = E(u_0(x)).$$

(2) Let $\lambda_1 > 0$, $\lambda_2 < 0$. Hence,

$$\frac{\lambda_1}{6}|u|^6 + \frac{-|\lambda_2|}{4}|u|^4 \ge -C(\lambda_1,\lambda_2)|u|^2 \tag{4.1}$$

for a constant $C(\lambda_1,\lambda_2) > 3\lambda_2^2/(32\lambda_1)$. Indeed, this is an easy property of the quadratic function $f(x) = (\lambda_1/6)x^2 - (|\lambda_2|/4)x + C(\lambda_1,\lambda_2)$ with the discriminant $(-|\lambda_2|/4)^2 - 4(\lambda_1/6)C(\lambda_1,\lambda_2) < 0$ for $C(\lambda_1,\lambda_2) > 3\lambda_2^2/(32\lambda_1)$.

Combining (4.1) with the estimate

$$\int_{\mathbb{R}^3} E_1(|u(x,t)|^2)|u(x,t)|^2\,dx \le \int_{\mathbb{R}^3} |u(x,t)|^4\,dx$$

we obtain

$$\begin{aligned}\frac{1}{2}&\int_{\mathbb{R}^3} |\nabla u(x,t)|^2\,dx\\ &= E(u) - \left[\frac{\lambda_1}{6}\int_{\mathbb{R}^3}|u(x,t)|^6\,dx + \frac{\lambda_2}{4}\int_{\mathbb{R}^3} E_1(|u(x,t)|^2)|u(x,t)|^2\,dx\right]\\ &\le E(u) - \left[\frac{\lambda_1}{6}\int_{\mathbb{R}^3}|u(x,t)|^6\,dx - \frac{|\lambda_2|}{4}\int_{\mathbb{R}^3}|u(x,t)|^4\,dx\right]\\ &\le E(u) + C(\lambda_1,\lambda_2)|u|^2 \le C(E,M).\end{aligned}$$

Thus, we have proved $\|u(t,x)\|_{\dot H^1_x} \le C(E,M)$. ∎

From now on, we will treat the quantity $\lambda_2 E_1(|u|^2)u$ as a perturbation in the energy-critical NLS.

PROPOSITION 4.2 ("Good" local well-posedness result). *Let* $u_0 \in H^1_x$ *and* $\lambda_1 > 0$. *There exists* $T = T(\|u_0\|_{H^1_x}) > 0$ *such that problem* (1.3) *admits a unique strong solution* $u \in S^1(I\times\mathbb{R}^3)$ *satisfying*

$$\|u\|_{S^1(I\times\mathbb{R}^3)} \le C(E,M), \quad I = [-T,T].$$

Proof. By the local result from Proposition 3.3, it suffices to prove an a priori $\dot X^1$-bound for u, namely $\|u\|_{\dot X^1(I)} \le C(\|u_0\|_{H^1_x})$. In fact, if we assume the existence of a strong solution u to problem (1.3), we should prove that the norm $\|u\|_{\dot X^1(I)}$ is finite as long as $T = T(\|u_0\|_{H^1_x})$ is sufficiently small.

Let w be a unique strong global-in-time solution to the NLS equation (2.2) with the initial datum $w_0 = u_0$ at time $t_0 = 0$. By a known result, the function w satisfies

$$\|w\|_{\dot S^1(\mathbb{R}\times\mathbb{R}^3)} \le C(\|u_0\|_{H^1_x}).$$

By Lemma 3.5, we also have

$$\|w\|_{\dot S^0(I\times\mathbb{R}^3)} \le C(\|u_0\|_{\dot H^1_x})\|u_0\|_{L^2_x} \le C(E,M).$$

By time reversal symmetry, it suffices to prove the required result forward in time.

First, we prove

$$\|u\|_{\dot S^1([0,T]\times\mathbb{R}^3)} \le C(E,M). \tag{4.2}$$

Now we partition $\mathbb{R}^+$ into $J = J(E,\eta)$ disjoint subintervals I_j such that

$$\|w\|_{\dot X^1(I_j)} \sim \eta,$$

where $I_j = [t_j, t_{j+1}]$ $(j = 0, 1, \dots, J-1)$, $I_J = [t_J, \infty)$ and η will be chosen later. We may assume that there exists $J' < J$ such that for all $0 \le j < j'-1$, $[0,T] \cap I_j \neq \emptyset$. Thus, $[0,T] = \bigcup_{j=0}^{J'-1}([0,T] \cap I_j)$.

Then by the Strichartz estimate, we have

$$\begin{aligned}\|e^{i(t-t_j)\Delta} w(t_j)\|_{\dot{X}^1(I_j)} &\le \|w\|_{\dot{X}^1(I_j)} + \|\nabla(|w|^4 w)\|_{L^{10/7}_{t,x}(I_j\times\mathbb{R}^3)} \\ &\le \|w\|_{\dot{X}^1(I_j)} + C\|\nabla w\|_{L^{10/3}_{t,x}(I_j\times\mathbb{R}^3)}\|w\|^4_{L^{10}_{t,x}(I_j\times\mathbb{R}^3)} \\ &\le \|w\|_{\dot{X}^1(I_j)} + C\|\nabla w\|_{L^{10/3}_{t,x}(I_j\times\mathbb{R}^3)}\|\nabla w\|^4_{L^{10}_t L^{30/13}_x(I_j\times\mathbb{R}^3)} \\ &\le \eta + C\|w\|^5_{\dot{X}^1(I_j)} \le \eta + C\eta^5,\end{aligned}$$

where C depends only on the Strichartz constant.

Now we use the Stability Lemma 2.5 in the time interval I_0, with the perturbation term $e = \lambda_2 E_1(|u|^2)u$. Note that $u_0 = w_0$. Hence by the Strichartz estimate, we have

$$\begin{aligned}\|u\|_{\dot{X}^1(I_0)} &\le \|e^{it\Delta}u_0)\|_{\dot{X}^1(I_0)} + C|I_0|^{1/2}\|u\|^3_{\dot{X}^1} + C\|u\|^5_{\dot{X}^1(I_0)} \\ &\le \eta + C\eta^5 + CT^{1/2}\|u\|^3_{\dot{X}^1} + C\|u\|^5_{\dot{X}^1(I_0)}.\end{aligned}$$

Assuming η, T are sufficiently small, a standard continuity method yields

$$\|u\|_{\dot{X}^1(I_0)} \le 2\eta.$$

Thus,

$$\|u\|_{W(I_0)} = \|u\|_{L^{10}_{t,x}(I_0\times\mathbb{R}^3)} \le \|\nabla u\|_{L^{10}_t L^{30/13}_x(I_0\times\mathbb{R}^3)} \le \|u\|_{\dot{X}^1(I_0)} \le 2\eta.$$

By Proposition 4.1, we have

$$\|u\|_{L^\infty_t \dot{H}^1_x(I_0\times\mathbb{R}^3)} \le E_0 = C(E, M).$$

By the Hölder inequality, we obtain

$$\|\nabla e\|_{\dot{N}^0(I_0\times\mathbb{R}^3)} \lesssim T^{1/2}\|u\|^3_{\dot{X}^1(I_0)} \lesssim T^{1/2}\eta^3.$$

Choosing T sufficiently small depending only on E, M, we get

$$\|\nabla e\|_{\dot{N}^0(I_0\times\mathbb{R}^3)} < \varepsilon, \tag{4.3}$$

where $\varepsilon = \varepsilon(E, M)$ will be chosen later. Hence, using the stability theory from Lemma 2.5, we obtain the estimate

$$\|u - w\|_{\dot{S}^1(I_0\times\mathbb{R}^3)} \le C(E, M)\varepsilon^7,$$

which implies

$$\begin{aligned}&\|u(t_1) - w(t_1)\|_{\dot{H}^1_x} \le C(E, M)\varepsilon^7, \\ &\|e^{i(t-t_1)\Delta}(u(t_1) - w(t_1))\|_{\dot{X}^1(I_1)} \le C(E, M)\varepsilon^7.\end{aligned}$$

Thus, by the above two inequalities,

$$\begin{aligned}\|u\|_{\dot X^1(I_1)} &\le \|e^{i(t-t_1)\Delta}u(t_1)\|_{\dot X^1(I_1)} + |I_1|^{1/2}\|u\|^3_{\dot X^1(I_1)} + C\|u\|^5_{\dot X^1(I_1)}\\ &\le \|e^{i(t-t_1)\Delta}w(t_1)\|_{\dot X^1(I_1)} + \|e^{i(t-t_1)\Delta}(u(t_1)-w(t_1))\|_{\dot X^1(I_1)}\\ &\quad + |T|^{1/2}\|u\|^3_{\dot X^1(I_1)} + C\|u\|^5_{\dot X^1(I_1)}\\ &\le \eta + C\eta^5 + C(E,M)\varepsilon^7 + CT^{1/2}\|u\|^3_{\dot X^1(I_0)} + C\|u\|^5_{\dot X^1(I_1)}.\end{aligned}$$

Choosing a sufficiently small $\varepsilon > 0$ (depending on E, M), by a standard continuity method, we get

$$\|u\|_{\dot X^1(I_1)} \le 2\eta.$$

Furthermore, inequality (4.3) holds true when I_0 is replaced by I_1.

Applying Lemma 2.5 on I_1 again, we obtain

$$\|u-w\|_{\dot S^1(I_1\times\mathbb{R}^3)} \le C(E,M)\varepsilon^{7^2}.$$

By induction, for every interval I_j with $0 \le j \le J(E,\eta)-1$, we have

$$\|u\|_{\dot X^1(I_j)} \le 2\eta.$$

Combining these estimates for all intervals I_j, we obtain

$$\|u\|_{\dot X^1([0,T])} \le 2\eta J \le C(E). \tag{4.4}$$

By estimate (4.4), Proposition 4.1, and the Strichartz estimate, we obtain

$$\|u\|_{\dot S^1([0,T]\times\mathbb{R}^3)} \le \|u_0\|_{H^1_x} + T^{1/2}\|u\|^3_{\dot X^1(I)} + \|u\|^5_{\dot X^1(I)} \le C(E,M). \tag{4.5}$$

Next, we will prove

$$\|u\|_{\dot S^0([0,T]\times\mathbb{R}^3)} \le C(E,M). \tag{4.6}$$

By Proposition 4.1 and the Strichartz estimate, we get

$$\begin{aligned}\|u\|_{\dot S^0([0,T]\times\mathbb{R}^3)} &\le \|u_0\|_{L^2_x} + T^{1/2}\|u\|^2_{\dot X^1(I)}\|u\|_{\dot X^0(I)} + \|u\|^4_{\dot X^1(I)}\|u\|_{\dot X^0(I)}\\ &\le M^{1/2} + C(E,M)\|u\|^2_{\dot X^1(I)}\|u\|_{\dot S^0(I)} + \|u\|^4_{\dot X^1(I)}\|u\|_{\dot S^0(I)}.\end{aligned}$$

Hence, we decompose $[0,T]$ into $N = N(E,M,\delta)$ subintervals J_k such that $\|u\|_{\dot X^1(J_k)} \sim \delta$ for some small constant $\delta > 0$ to be chosen later. Thus

$$\|u\|_{\dot S^0(J_k\times\mathbb{R}^3)} \lesssim M^{1/2} + C(E,M)\delta^2\|u\|_{\dot S^0(J_k)} + \delta^4\|u\|_{\dot S^0(J_k)}.$$

Choosing δ sufficiently small depending on E, M, a standard continuity method yields

$$\|u\|_{\dot S^0(J_k\times\mathbb{R}^3)} \le C(E,M).$$

Summing up these bounds over all subintervals J_k, we get

$$\|u\|_{\dot S^0([0,T]\times\mathbb{R}^3)} \le C(E,M). \tag{4.7}$$

Finally, combining (4.5) and (4.7), we obtain

$$\|u\|_{S^1([0,T]\times\mathbb{R}^3)} \le C(E,M), \tag{4.8}$$

where T only depends on energy and mass.

If we divide the interval I into subintervals of length T, and sum up the corresponding $\dot{S}^1$-bounds in these subintervals, we complete the proof of Proposition 4.2. ∎

5. Scattering theory. In the case $\lambda_1 > 0$, in order to obtain the scattering result, we need a small mass condition.

The first step is to show that the S^1-norm of the solution to problem (1.3) is bounded on the whole line, namely

$$\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^3)} \le C(E,M).$$

Let w be a unique strong global-in-time solution to the NLS equation (2.2) with the initial datum $w_0 = u_0$ at time $t_0 = 0$. By a known result, the function w satisfies

$$\|w\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^3)} \le C(E).$$

By Lemma 3.5, we obtain

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^3)} \le C(E)M^{1/2}.$$

Now, we define the following spaces:

$$\dot{Y}^0(I) = V(I) \cap L_t^{10} L_x^{30/13}(I\times\mathbb{R}^3),$$
$$\dot{Y}^1(I) = \{u : \nabla u \in \dot{Y}^0(I)\}, \qquad Y^1(I) = \dot{Y}^0(I) \cap \dot{Y}^1(I).$$

Thus, by Lemma 3.2 we have, for $k \in \{0,1\}$,

$$\big\||\nabla|^k (E_1(|u|^2)u)\big\|_{\dot{N}^0(I\times\mathbb{R}^3)} \lesssim \|u\|_{\dot{Y}^0(I)} \|u\|_{\dot{Y}^1(I)} \|u\|_{\dot{Y}^k(I)}, \tag{5.1}$$

$$\big\||\nabla|^k (|u|^4 u)\big\|_{\dot{N}^0(I\times\mathbb{R}^3)} \lesssim \|u\|^4_{\dot{Y}^1(I)} \|u\|_{\dot{Y}^k(I)}. \tag{5.2}$$

For sake of simplicity, we only consider the domain $\mathbb{R}^+\times\mathbb{R}^3$. We divide the half-line $\mathbb{R}^+$ into $J = J(E,\eta)$ subintervals I_j such that

$$\|w\|_{\dot{Y}^1(I_j)} \sim \eta,$$

where $I_j = [t_j, t_{j+1}]$ $(j = 0,1,\dots,J-1)$, $I_J = [t_J,\infty)$. Assuming that $M = M(E,\eta)$ is sufficiently small, we have

$$\|w\|_{\dot{S}^0(\mathbb{R}\times\mathbb{R}^3)} \le C(E)M^{1/2} \le \eta.$$

Thus,

$$\|w\|_{Y^1(I_j)} \sim \eta. \tag{5.3}$$

By relation (5.3) and the Strichartz estimate, we obtain

$$\|e^{i(t-t_j)\Delta} w(t_j)\|_{Y^1(I_j)} \le \|w\|_{Y^1(I_j)} + \|w\|^5_{Y^1(I_j)} \le \eta + C\eta^5 \le 2\eta,$$

provided η is sufficiently small.

First, we consider the time interval $I_0 = [t_0, t_1]$. By (5.1), (5.5) and the Strichartz estimate,

$$\|u\|_{Y^1(I_0)} \le 2\eta + C\|u\|^3_{Y^1(I_0)} + C\|u\|^5_{Y^1(I_0)}. \tag{5.4}$$

By a standard continuity method, for sufficiently small η,

$$\|u\|_{Y^1(I_0)} \le 4\eta.$$

Similarly, by (5.1), (5.5) and the Strichartz estimate,

$$\begin{aligned}\|u\|_{\dot Y^0(I_0)} &\le \|u_0\|_{L^2_x} + C\|u\|^2_{\dot Y^0(I_0)}\|u\|_{\dot Y^1(I_0)} + \|u\|_{\dot Y^0(I_0)}\|u\|^4_{\dot Y^1(I_0)} \\ &\le M^{1/2} + \eta\|u\|^2_{\dot Y^0(I_0)} + \eta^4\|u\|_{\dot Y^0(I_0)},\end{aligned}$$

which implies that

$$\|u\|_{\dot Y^0(I_0)} \lesssim M^{1/2},$$

provided that η and M are sufficiently small.

For the perturbation term $E_1(|u|^2)u$, by (5.1) we have

$$\|E_1(|u|^2)u\|_{\dot N^1(I_0\times\mathbb{R}^3)} \lesssim \|u\|_{\dot Y^0(I_0)}\|u\|^2_{\dot Y^1(I_0)} \lesssim M^{1/2}\eta \le M^{\delta_0},$$

where δ_0 is a small constant.

Applying Lemma 2.5, we have

$$\|u-w\|_{\dot S^1(I_0\times\mathbb{R}^3)} \le M^{c\delta_0},$$

which implies that

$$\|e^{i(t-t_1)\Delta}(u(t_1)-w(t_1))\|_{\dot S^1(I_0\times\mathbb{R}^3)} \le M^{c\delta_0}.$$

Thus,

$$\begin{aligned}\|u\|_{Y^1(I_1)} \le{}& \|e^{i(t-t_1)\Delta}u(t_1)\|_{\dot Y^0(I_1)} + \|e^{i(t-t_1)\Delta}w(t_1)\|_{\dot Y^1(I_1)} \\ &+ \|e^{i(t-t_1)\Delta}(u(t_1)-w(t_1))\|_{\dot Y^1(I_1)} + c\|u\|^3_{Y^1(I_1)} + C\|u\|^5_{Y^1(I_1)} \\ \le{}& M^{1/2} + M^{c\delta_0} + \eta + c\|u\|^3_{Y^1(I_1)} + C\|u\|^5_{Y^1(I_1)}.\end{aligned}$$

Hence, the standard continuity method yields

$$\|u\|_{Y^1(I_1)} \le 4\eta, \quad \|u\|_{\dot Y^0(I_1)} \le M^{1/2}.$$

Using Lemma 3.5, we further obtain

$$\|u-w\|_{\dot S^1(I_1\times\mathbb{R}^3)} \le M^{c\delta_1}$$

for some δ_1 satisfying $0 < \delta_1 < \delta_0$.

By induction, for any time interval I_j we obtain

$$\|u\|_{Y^1(I_j)} \le 4\eta.$$

Adding all these intervals, we have

$$\|u\|_{Y^1(\mathbb{R}^+)} \lesssim J\eta \le C(E). \tag{5.5}$$

Hence, by the Strichartz estimate,

$$\begin{aligned}\|u\|_{S^1(\mathbb{R}^+\times\mathbb{R}^3)} &\lesssim \|u_0\|_{H_x^1} + \|u\|^3_{Y^1(\mathbb{R}^+)} + \|u\|^5_{Y^1(\mathbb{R}^+)}\\ &\lesssim M + E + C(E) \le C(E,M).\end{aligned}\tag{5.6}$$

Next, we will prove that boundedness of global Strichartz norms implies scattering.

For $0 < t < \infty$, define

$$u_+(t) = u_0 - i\int_0^t e^{-is\Delta}\big(\lambda_1|u|^4u + \lambda_2 E_1(|u|^2)u\big)(s)\,ds.$$

Since $u \in S^1(\mathbb{R}\times\mathbb{R}^3)$, by the Strichartz estimate, we have $u_+(t) \in H_x^1$. Now, we will show that $u_+(t)$ converges in H_x^1 as $t \to \infty$. For $0 < \tau < t$, we obtain

$$\begin{aligned}\|u_+(t) - u_+(\tau)\|_{H_x^1} &= \Big\|\int_\tau^t e^{-is\Delta}\big(\lambda_1|u|^4u + \lambda_2E_1(|u|^2)u\big)(s)\,ds\Big\|_{H_x^1}\\
&\le \Big\|\int_\tau^t e^{-is\Delta}\big(\lambda_1|u|^4u + \lambda_2E_1(|u|^2)u\big)(s)\,ds\Big\|_{L_t^\infty H_x^1}\\
&\lesssim C(\lambda_1,\lambda_2)\|u\|_{V([\tau,t])}\|u\|_{W([\tau,t])}\big\||\nabla|^k u\big\|_{V([\tau,t])} + \|u\|^4_{W([\tau,t])}\big\||\nabla|^k u\big\|_{V([\tau,t])}\\
&\le \|u\|_{V([\tau,t])}\|u\|_{W([\tau,t])}\|(1+|\nabla|)u\|_{V([\tau,t])} + \|u\|^4_{W([\tau,t])}\|(1+|\nabla|)u\|_{V([\tau,t])}.\end{aligned}$$

Since $\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^3)} < \infty$, for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$\|u_+(t) - u_+(\tau)\|_{H_x^1} \le \varepsilon \quad \text{for all } \tau, t > T_\varepsilon.$$

Next, we will show that $u(t)$ converges to $e^{it\Delta}u_+$ in the norm of H_x^1 as $t \to \infty$. Indeed,

$$\begin{aligned}\|e^{-it\Delta}u(t) - u_+\|_{H_x^1} &= \Big\|\int_t^\infty e^{-is\Delta}\big(\lambda_1|u|^4u + \lambda_2E_1(|u|^2)u\big)(s)\,ds\Big\|_{H_x^1}\\
&= \Big\|\int_t^\infty e^{i(t-s)\Delta}\big(\lambda_1|u|^4u + \lambda_2E_1(|u|^2)u\big)(s)\,ds\Big\|_{H_x^1}\\
&\lesssim \|u\|_{V([t,\infty])}\|u\|_{W([t,\infty])}\|(1+|\nabla|)u\|_{V([t,\infty])}\\
&\quad + \|u\|^4_{W([t,\infty])}\|(1+|\nabla|)u\|_{V([t,\infty])}.\end{aligned}$$

Hence, using the boundedness of $\|u\|_{S^1(\mathbb{R}\times\mathbb{R}^3)}$, we obtain

$$\|e^{-it\Delta}u(t) - u_+\|_{H_x^1} \to 0 \quad \text{as } t \to \infty.$$

This completes the proof of scattering for problem (1.3).

6. Blow-up.

6. Blow-up. In this section, following the convexity method of Glassey [10], we will prove the blow-up result stated in Theorem 1.3.

Consider the strong H^1_x-solution $u(t,x)$ of problem (1.3) with an initial datum $u_0 \in H^1_x(\mathbb{R}^3)$ such that $xu_0 \in L^2(\mathbb{R}^3)$. By the Hardy inequality and the conservation of mass, we have

$$\|u_0\|_2^2 = \int_{\mathbb{R}^3} |x|\,|u|\frac{|u|}{|x|}\,dx \lesssim \|xu\|_2\|\nabla u\|_2.$$

Hence, in order to prove the blow-up result, we only need to show the existence of $T>0$ such that

$$\lim_{t\to T} \int_{\mathbb{R}^3} |x|^2|u(t,x)|^2\,dx = 0. \tag{6.1}$$

Let

$$V(t) = \int_{\mathbb{R}^3} |x|^2|u(t,x)|^2\,dx.$$

Then a direct computation leads to the equalities

$$V'(t) = 4\,\mathrm{Im}\int_{\mathbb{R}^3} \bar{u}x\cdot\nabla u\,dx = -4y(t) \quad \text{and} \quad V''(t) = -4y'(t),$$

where

$$y'(t) = -2\int_{\mathbb{R}^3}|\nabla u|^2\,dx - 2\lambda_1\int_{\mathbb{R}^3}|u(t,x)|^6\,dx - \frac{3}{2}\lambda_2\int_{\mathbb{R}^3}E_1(|u|^2)|u|^2\,dx.$$

We only need to show that

$$y'(t) \geq C\|\nabla u\|_2^2 > 0 \quad \text{for some constant } C>0. \tag{6.2}$$

Indeed, if $\lambda_2 > 0$ and $E<0$, then

$$\begin{aligned} y'(t) &= -2\int_{\mathbb{R}^3}|\nabla u|^2\,dx + 12\left\{\frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda_2}{4}\int_{\mathbb{R}^3}E_1(|u|^2)|u|^2\,dx - E\right\} \\ &\quad - \frac{3\lambda_2}{2}\int_{\mathbb{R}^3}E_1(|u|^2)|u|^2\,dx \\ &= 4\|\nabla u\|_2^2 + \frac{3\lambda_2}{2}\int_{\mathbb{R}^3}E_1(|u|^2)|u|^2\,dx - 12E > 4\|\nabla u\|_2^2 > 0. \end{aligned}$$

Otherwise, if $\lambda_2, E<0$, then

$$\begin{aligned} y'(t) &= -2\int_{\mathbb{R}^3}|\nabla u|^2\,dx + 6\left\{\frac{1}{2}\|\nabla u\|_2^2 + \frac{\lambda_1}{6}\int_{\mathbb{R}^3}|u|^6\,dx - E\right\} - 2\lambda_1\int_{\mathbb{R}^3}|u|^6\,dx \\ &= \|\nabla u\|_2^2 - \lambda_1\|u\|_6^6 - 6E > \|\nabla u\|_2^2 > 0. \end{aligned}$$

By the assumption, $y(t) > y_0 > 0$ for all $t\in[0,T]$. Combining this fact with the differential inequality (6.2), we see that $V(t)$ is decreasing and concave, which implies that relation (6.1) is satisfied, which is impossible. Thus, we have completed the proof.

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