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Operator Lipschitz functions on Banach spaces

by

Jan Rozendaal (Warszawa), Fedor Sukochev (Kensington, NSW), and Anna Tomskova (Kensington, NSW)

Abstract. Let X, Y be Banach spaces and let $\mathcal{L}(X,Y)$ be the space of bounded linear operators from X to Y. We develop the theory of double operator integrals on $\mathcal{L}(X,Y)$ and apply this theory to obtain commutator estimates of the form

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} \leq \text{const}\, \|BS - SA\|_{\mathcal{L}(X,Y)}$$

for a large class of functions f, where $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(Y)$ are scalar type operators and $S \in \mathcal{L}(X,Y)$. In particular, we establish this estimate for $f(t) := |t|$ and for diagonalizable operators on $X = \ell_p$ and $Y = \ell_q$ for $p < q$.

We also study the estimate above in the setting of Banach ideals in $\mathcal{L}(X,Y)$. The commutator estimates we derive hold for diagonalizable matrices with a constant independent of the size of the matrix.

1. Introduction. Let X be a Banach space and let $\mathcal{L}(X)$ be the space of all bounded linear operators on X. Let $A, B \in \mathcal{L}(X)$ be scalar type operators (see Definition 3.1 below) on X. Let $f : \mathrm{sp}(A) \cup \mathrm{sp}(B) \to \mathbb{C}$ be a bounded Borel function, where $\mathrm{sp}(A)$ (resp. $\mathrm{sp}(B)$) is the spectrum of the operator A (resp. B). We are interested in Lipschitz type estimates

$$\|f(B) - f(A)\|_{\mathcal{L}(X)} \leq \text{const}\, \|B - A\|_{\mathcal{L}(X)}, \tag{1.1}$$

where $\|\cdot\|_{\mathcal{L}(X)}$ is the uniform operator norm on the space $\mathcal{L}(X)$, and more generally in commutator estimates

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} \leq \text{const}\, \|BS - SA\|_{\mathcal{L}(X,Y)} \tag{1.2}$$

for Banach spaces X and Y, scalar type operators $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$, and $S \in \mathcal{L}(X,Y)$. This problem is well-known in the special case where $X = Y$ is a separable Hilbert space, such as ℓ_2, and A and B are normal

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operators on X. In this paper we study such estimates in the Banach space setting, and specifically for $X = \ell_p$ and $Y = \ell_q$ with $p, q \in [1, \infty]$.

In the special case where A, B are self-adjoint bounded operators on a Hilbert space H, the estimate

$$\|f(B) - f(A)\|_{\mathcal{L}(H)} \leq \text{const} \, \|B - A\|_{\mathcal{L}(H)} \tag{1.3}$$

was established by Peller [28, 26] (see also [12]) for $f : \mathbb{R} \to \mathbb{R}$ in the Besov class $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ (for the definition of $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ see Section 3.3). This result extended a long line of results from [7–9], in which the theory of double operator integration was developed to study the difference $f(B) - f(A)$ (see also [10]). This theory was revised and extended in various directions, including the Banach space setting, in [13]. However, until now the results in the general setting were much weaker than in the Hilbert space setting. In this paper we show that for scalar type operators on Banach spaces one can obtain results matching those on Hilbert spaces.

In Corollary 4.9 below we prove that (1.1) holds when $A, B \in \mathcal{L}(X)$ are scalar type operators with real spectrum and $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$. It is immediate from the definition of a scalar type operator that every normal operator on H is of scalar type. Therefore, Corollary 4.9 extends (1.3) to the Banach space setting. More generally, (1.2) holds for $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ and for all $S \in \mathcal{L}(X, Y)$ (see Corollary 4.8).

If f is the absolute value function then $f \notin \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ and the results mentioned above do not apply. Moreover, the techniques which we used to obtain (1.1) for $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ cannot be applied to the absolute value function (see Remark 8.3). However, the absolute value function is important in matrix analysis and perturbation theory (see [6, Sections VII.5 and X.2]). In the case where H is an infinite-dimensional Hilbert space, it was proved by Kato [19] that the function $t \mapsto |t|$, $t \in \mathbb{R}$, does not satisfy (1.3). An earlier example of McIntosh [24] showed the failure of the commutator estimate (1.2) for this function in the case $X = Y = H$. Later, it was proved by Davies [11] that for $1 \leq p \leq \infty$ and the Schatten–von Neumann ideal $\mathcal{S}_p$ with the norm $\|\cdot\|_{\mathcal{S}_p}$, the estimate

$$\big\||B| - |A|\big\|_{\mathcal{S}_p} \leq \text{const} \, \|B - A\|_{\mathcal{S}_p}$$

holds for all $A, B \in \mathcal{S}_p$ if and only if $1 < p < \infty$. Commutator estimates for the absolute value function and different Banach ideals in $\mathcal{L}(H)$ have also been studied in [16]. The proofs in [11, 13, 16] are based on Matsaev's celebrated theorem (see [18]) or on the UMD-property of the reflexive Schatten–von Neumann ideals. However, the spaces $\mathcal{L}(X, Y)$ are not UMD-spaces, and therefore the techniques used in [11, 13, 16] do not apply to them. To study (1.2) for $X = \ell_p$ and $Y = \ell_q$, we use completely different methods from those of [11, 13, 16], namely the theory of Schur multipliers on the space $\mathcal{L}(\ell_p, \ell_q)$ developed by Bennett [4, 5].

Let $p, q \in [1, \infty]$ with $p < q$. In Section 6 we show (see Theorem 6.8) that, for diagonalizable operators (for the definition see Section 5) $A \in \mathcal{L}(\ell_p)$ and $B \in \mathcal{L}(\ell_q)$ with real spectrum, and for the absolute value function f,

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell_p,\ell_q)} \leq \text{const}\,\|BS - SA\|_{\mathcal{L}(\ell_p,\ell_q)} \tag{1.4}$$

holds for all $S \in \mathcal{L}(\ell_p, \ell_q)$ (where ℓ_∞ should be replaced by c_0). Moreover, if $p, q \in [1, \infty]$ with $p = 1$ or $q = \infty$ then (1.4) holds for any Lipschitz function $f : \mathbb{C} \to \mathbb{C}$. In particular,

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell_1)} \leq \text{const}\,\|B - A\|_{\mathcal{L}(\ell_1)} \tag{1.5}$$

and

$$\|f(B) - f(A)\|_{\mathcal{L}(\mathrm{c}_0)} \leq \text{const}\,\|B - A\|_{\mathcal{L}(\mathrm{c}_0)} \tag{1.6}$$

for diagonalizable operators on ℓ_1 respectively c_0. Therefore we show that, even though (1.4) fails for $p = q = 2$ and f the absolute value function, and in particular (1.1) fails for $X = \ell_2$, (1.4) does hold for $p < q$ and f the absolute value function, and (1.1) holds for $X = \ell_1$ or $X = \mathrm{c}_0$ and each Lipschitz function f.

We also obtain results for $p \geq q$. In particular, for $p = q = 2$ we prove (see Corollary 6.15) that for each $\epsilon \in (0, 1]$ there exists a constant $C \geq 0$ such that the following holds. Let $A, B \in \mathcal{L}(\ell_2)$ be compact self-adjoint operators, and let $U, V \in \mathcal{L}(\ell_2)$ be unitaries such that

$$UAU^{-1} = \sum_{j=1}^{\infty} \lambda_j \mathcal{P}_j \quad \text{and} \quad VBV^{-1} = \sum_{j=1}^{\infty} \mu_j \mathcal{P}_j,$$

where $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\mu_j\}_{j=1}^{\infty}$ are sequences of real numbers and $\mathcal{P}_j \in \mathcal{L}(\ell_2)$, for $j \in \mathbb{N}$, is the jth standard basis projection. Then

$$\begin{aligned} &\big\||B| - |A|\big\|_{\mathcal{L}(\ell_2)} \\ &\quad \leq C \min(\|V(B - A)U^{-1}\|_{\mathcal{L}(\ell_2,\ell_{2-\epsilon})}, \|V(B - A)U^{-1}\|_{\mathcal{L}(\ell_{2+\epsilon},\ell_2)}) \end{aligned} \tag{1.7}$$

where the right-hand side equals infinity if $V(B - A)U^{-1} \notin \mathcal{L}(\ell_2, \ell_{2-\epsilon}) \cup \mathcal{L}(\ell_{2+\epsilon}, \ell_2)$.

The results stated here for the absolute value function in fact extend to a larger class of functions. This is briefly mentioned in Remark 6.16.

We note that the constants which appear in our results depend on the spectral constants of A and B from Section 3.1, and those in (1.4)–(1.6) on the diagonalizability constants of A and B from (5.5). These quantities are independent of the norms of A and B, and to obtain constants which do not depend on A and B in any way one merely has to restrict to operators with a sufficiently bounded spectral or diagonalizability constant. This is already done implicitly on Hilbert spaces by considering normal operators, for which these quantities are equal to 1. For example, in (1.7) the constant C does not depend on A or B in any way. Our results therefore truly extend the known estimates on Hilbert spaces, the main difference between Hilbert spaces and general Banach spaces being that on Hilbert spaces one has a large and easily identifiable class of operators that have spectral constant 1 and that are diagonalizable by an isometry.

We study the commutator estimate in (1.2) in the more general form

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq \text{const}\,\|BS - SA\|_{\mathcal{I}}, \tag{1.8}$$

where $\mathcal{I}$ is an operator ideal in $\mathcal{L}(X, Y)$. For example, we prove in Corollary 4.8 that (1.3) holds for a general Banach ideal $\mathcal{I}$ in $\mathcal{L}(X)$ with the strong convex compactness property (for definitions see Section 3.2), with respect to the norm $\|\cdot\|_{\mathcal{I}}$.

We also present (see Theorem 7.3) an example of a Banach ideal $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ in $\mathcal{L}(\ell_{p^*}, \ell_p)$, for $p \in [1, \infty)$ and $1/p + 1/p^* = 1$ (with ℓ_∞ replaced by c_0), namely the ideal of p-summing operators, such that any Lipschitz function f (in particular, the absolute value function) satisfies (1.8).

In the final section we apply our results to finite-dimensional spaces, and obtain commutator estimates for diagonalizable matrices. Any diagonalizable matrix is a scalar type operator, hence estimates (1.4)–(1.8) hold for diagonalizable matrices A and B with a constant independent of the size of the matrix.

2. Notation and terminology. The natural numbers are $\mathbb{N} = \{1, 2, \ldots\}$. All vector spaces are over the complex number field. Throughout, X and Y denote Banach spaces, the space of bounded linear operators from X to Y is $\mathcal{L}(X, Y)$, and $\mathcal{L}(X) := \mathcal{L}(X, X)$. We identify the algebraic tensor product $X^* \otimes Y$ with the space of finite rank operators in $\mathcal{L}(X, Y)$ via $(x^* \otimes y)(x) := \langle x^*, x\rangle y$ for $x \in X$, $x^* \in X^*$ and $y \in Y$. The spectrum of $A \in \mathcal{L}(X)$ is $\mathrm{sp}(A)$, and by $\mathrm{I}_X \in \mathcal{L}(X)$ we denote the identity operator on X. Throughout the text we use the abbreviations SOT and WOT for the strong and weak operator topology, respectively.

For $p \in [1, \infty]$, $\mathrm{L}^p(\mathbb{R})$ is the usual Lebesgue space of p-integrable functions on $\mathbb{R}$. We let ℓ_p, for $p \in [1, \infty]$, be the space of p-summable sequences $(x_k)_{k\in\mathbb{N}} \subseteq \mathbb{C}$, and c_0 consists of all sequences $(x_k)_{k\in\mathbb{N}} \subseteq \mathbb{C}$ which converge to zero.

The Borel σ-algebra on a Borel measurable subset $\sigma \subseteq \mathbb{C}$ will be denoted by $\mathfrak{B}_\sigma$, and $\mathfrak{B} := \mathfrak{B}_{\mathbb{C}}$. For measurable spaces (Ω_1, Σ_1) and (Ω_2, Σ_2) we denote by $\Sigma_1 \otimes \Sigma_2$ the σ-algebra on $\Omega_1 \times \Omega_2$ generated by all measurable rectangles $\sigma_1 \times \sigma_2$ with $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. If (Ω, Σ) is a measurable space then $\mathcal{B}(\Omega, \Sigma)$ is the space of all bounded Σ-measurable complex-valued functions on Ω, a Banach algebra with the supremum norm

$$\|f\|_{\mathcal{B}(\Omega,\Sigma)} := \sup_{\omega\in\Omega} |f(\omega)| \quad (f \in \mathcal{B}(\Omega, \Sigma)).$$

We simply write $\mathcal{B}(\Omega) := \mathcal{B}(\Omega, \Sigma)$ and $\|f\|_\infty := \|f\|_{\mathcal{B}(\Omega,\Sigma)}$ when no confusion can arise.

If μ is a complex Borel measure on a measurable space (Ω, Σ) and X is a Banach space, then a function $f : \Omega \to X$ is *μ-measurable* if there exists a sequence of X-valued simple functions converging to f μ-almost everywhere. For Banach spaces X and Y and a function $f : \Omega \to \mathcal{L}(X, Y)$, we say that f is *strongly measurable* if $\omega \mapsto f(\omega)x$ is a μ-measurable mapping $\Omega \to Y$ for each $x \in X$.

If μ is a positive measure on a measurable space (Ω, Σ) and $f : \Omega \to [0, \infty]$ is a function, we let

$$\overline{\int_\Omega} f(\omega)\,\mathrm{d}\mu(\omega) := \inf \int_\Omega g(\omega)\,\mathrm{d}\mu(\omega) \in [0, \infty],$$

where the infimum is taken over all measurable $g : \Omega \to [0, \infty]$ such that $g(\omega) \geq f(\omega)$ for $\omega \in \Omega$.

The Hölder conjugate of $p \in [1, \infty]$ is denoted by p^* and is defined by $1/p + 1/p^* = 1$. The indicator function of a subset σ of a set Ω is denoted by $\mathbf{1}_\sigma$. We will often identify functions defined on σ with their extensions to Ω by setting them equal to zero off σ.

3. Preliminaries

3.1. Scalar type operators. In this section we summarize some of the basics of scalar type operators, as taken from [17].

Let X be a Banach space. A *spectral measure* on X is a map $E : \mathfrak{B} \to \mathcal{L}(X)$ such that:

- $E(\emptyset) = 0$ and $E(\mathbb{C}) = \mathrm{I}_X$;
- $E(\sigma_1 \cap \sigma_2) = E(\sigma_1)E(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \mathfrak{B}$;
- $E(\sigma_1 \cup \sigma_2) = E(\sigma_1) + E(\sigma_2) - E(\sigma_1)E(\sigma_2)$ for all $\sigma_1, \sigma_2 \in \mathfrak{B}$;
- E is σ-additive in the strong operator topology.

Note that these conditions imply that E is projection-valued. Moreover, by [17, Corollary XV.2.4] there exists a constant K such that

$$\|E(\sigma)\|_{\mathcal{L}(X)} \leq K \quad (\sigma \in \mathfrak{B}). \tag{3.1}$$

An operator $A \in \mathcal{L}(X)$ is a *spectral operator* if there exists a spectral measure E on X such that $AE(\sigma) = E(\sigma)A$ and $\mathrm{sp}(A, E(\sigma)X) \subseteq \overline{\sigma}$ for all $\sigma \in \mathfrak{B}$, where $\mathrm{sp}(A, E(\sigma)X)$ denotes the spectrum of A in the space $E(\sigma)X$. For a spectral operator A, we let $\nu(A)$ denote the minimal constant K occurring in (3.1) and call $\nu(A)$ the *spectral constant of* A. This is well-defined since the spectral measure E associated with A is unique (cf. [17, Corollary XV.3.8]). Moreover, E is supported on $\mathrm{sp}(A)$ in the sense that $E(\mathrm{sp}(A)) = \mathrm{I}_X$ [17, Corollary XV.3.5]. Hence we can define an integral with respect to E of bounded Borel measurable functions on $\mathrm{sp}(A)$, as follows. For $f = \sum_{j=1}^{n} \alpha_j \mathbf{1}_{\sigma_j}$ a finite simple function with $\alpha_j \in \mathbb{C}$ and $\sigma_j \subseteq \mathrm{sp}(A)$ mutually disjoint Borel sets for $1 \leq j \leq n$, we let

$$\int\limits_{\mathrm{sp}(A)} f \,\mathrm{d}E := \sum_{j=1}^{n} \alpha_j E(\sigma_j). \tag{3.2}$$

This definition is independent of the representation of f, and

$$\begin{aligned}\Big\| \int\limits_{\mathrm{sp}(A)} f \,\mathrm{d}E \Big\|_{\mathcal{L}(X)} &= \sup_{\|x\|_X = \|x^*\|_{X^*}=1} \Big| \sum_{j=1}^{n} \alpha_j x^* E(\sigma_j) x \Big| \\ &\leq \sup_j |\alpha_j| \sup_{\|x\|_X = \|x^*\|_{X^*}=1} \|x^* E(\cdot) x\|_{\mathrm{var}} \\ &\leq 4\|f\|_{\mathcal{B}(\mathrm{sp}(A))} \sup_{\|x\|_X = \|x^*\|_{X^*}=1} \sup_{\sigma \subseteq \mathrm{sp}(A)} |x^* E(\sigma) x| \\ &\leq 4\nu(A) \|f\|_{\mathcal{B}(\mathrm{sp}(A))},\end{aligned}$$

where $\|x^* E(\cdot) x\|_{\mathrm{var}}$ is the variation norm of the measure $x^* E(\cdot) x$. Since the simple functions lie dense in $\mathcal{B}(\mathrm{sp}(A))$, for general $f \in \mathcal{B}(\mathrm{sp}(A))$ we can define

$$\int\limits_{\mathrm{sp}(A)} f \,\mathrm{d}E := \lim_{n\to\infty} \int\limits_{\mathrm{sp}(A)} f_n \,\mathrm{d}E \in \mathcal{L}(X)$$

for $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{B}(\mathrm{sp}(A))$ a sequence of simple functions with $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. This definition is independent of the choice of the approximating sequence and

$$\Big\| \int\limits_{\mathrm{sp}(A)} f \,\mathrm{d}E \Big\|_{\mathcal{L}(X)} \leq 4\nu(A) \|f\|_{\mathcal{B}(\mathrm{sp}(A))}. \tag{3.3}$$

It is straightforward to check that

$$\begin{aligned}\int\limits_{\mathrm{sp}(A)} (\alpha f + g) \,\mathrm{d}E &= \alpha \int\limits_{\mathrm{sp}(A)} f \,\mathrm{d}E + \int\limits_{\mathrm{sp}(A)} g \,\mathrm{d}E, \\ \int\limits_{\mathrm{sp}(A)} fg \,\mathrm{d}E &= \Big(\int\limits_{\mathrm{sp}(A)} f \,\mathrm{d}E \Big)\Big(\int\limits_{\mathrm{sp}(A)} g \,\mathrm{d}E \Big)\end{aligned}$$

for all $\alpha \in \mathbb{C}$ and simple $f, g \in \mathcal{B}(\mathrm{sp}(A))$, and approximation then extends these identities to general $f, g \in \mathcal{B}(\mathrm{sp}(A))$. Moreover, $\int_{\mathrm{sp}(A)} \mathbf{1} \,\mathrm{d}E = E(\mathrm{sp}(A)) = \mathrm{I}_X$. Hence the map $f \mapsto \int_{\mathrm{sp}(A)} f \,\mathrm{d}E$ is a continuous morphism $\mathcal{B}(\mathrm{sp}(A)) \to \mathcal{L}(X)$ of unital Banach algebras. Since the spectrum of A is compact, the identity function $\lambda \mapsto \lambda$ is bounded on $\mathrm{sp}(A)$ and $\int_{\mathrm{sp}(A)} \lambda \,\mathrm{d}E(\lambda) \in \mathcal{L}(X)$ is well-defined.

DEFINITION 3.1. A spectral operator $A \in \mathcal{L}(X)$ with spectral measure E is a *scalar type operator* if

$$A = \int_{\mathrm{sp}(A)} \lambda \,\mathrm{d}E(\lambda).$$

The class of scalar type operators on X is denoted by $\mathcal{L}_{\mathrm{s}}(X)$.

For $A \in \mathcal{L}_{\mathrm{s}}(X)$ with spectral measure E and $f \in \mathcal{B}(\mathrm{sp}(A))$ we define

$$f(A) := \int_{\mathrm{sp}(A)} f \,\mathrm{d}E. \tag{3.4}$$

As remarked above, $f \mapsto f(A)$ is a continuous morphism $\mathcal{B}(\mathrm{sp}(A)) \to \mathcal{L}(X)$ of unital Banach algebras with norm bounded by $4\nu(A)$. Note also that

$$\langle x^*, f(A)x\rangle = \int_{\mathrm{sp}(A)} f(\lambda)\,\mathrm{d}\langle x^*, E(\lambda)x\rangle \tag{3.5}$$

for all $f \in \mathcal{B}(\mathrm{sp}(A))$, $x \in X$ and $x^* \in X^*$. Indeed, for simple functions this follows from (3.2), and by taking limits one obtains (3.5) for general $f \in \mathcal{B}(\mathrm{sp}(A))$.

Finally, we note that a normal operator A on a Hilbert space H is a scalar type operator with $\nu(A) = 1$, and in this case (3.3) improves to

$$\Big\| \int_{\mathrm{sp}(A)} f \,\mathrm{d}E \Big\|_{\mathcal{L}(H)} \leq \|f\|_{\mathcal{B}(\mathrm{sp}(A))}, \tag{3.6}$$

as is known from the Borel functional calculus for normal operators.

3.2. Spaces of operators. In this section we discuss some properties of spaces of operators that we will need later on.

First we provide a lemma about approximation by finite rank operators. Recall that a Banach space X has the *bounded approximation property* if there exists $M \geq 1$ such that, for each $K \subseteq X$ compact and $\epsilon > 0$, there exists $S \in X^* \otimes X$ with $\|S\|_{\mathcal{L}(X)} \leq M$ and $\sup_{x\in K} \|Sx - x\|_X < \epsilon$.

LEMMA 3.2. *Let X and Y be Banach spaces such that X is separable and either X or Y has the bounded approximation property. Then each $T \in \mathcal{L}(X, Y)$ is the SOT-limit of a norm bounded sequence of finite rank operators.*

Proof. Fix $T \in \mathcal{L}(X, Y)$. By [23, Proposition 1.e.14] there exists a norm bounded net $\{T_j\}_{j\in J} \subseteq X^* \otimes Y$ having T as its SOT-limit. It is straightforward to see that the strong operator topology is metrizable on bounded subsets of $\mathcal{L}(X, Y)$ by

$$d(S_1, S_2) := \sum_{k=1}^{\infty} 2^{-k} \|S_1 x_k - S_2 x_k\|_Y \quad (S_1, S_2 \in \mathcal{L}(X, Y)),$$

where $\{x_k\}_{k\in\mathbb{N}} \subseteq X$ is a countable subset that is dense in the unit ball of X. Hence there exists a subsequence of $\{T_j\}_{j\in J}$ with SOT-limit T. ∎

Let X and Y be Banach spaces and let Z be a Banach space which is continuously embedded in $\mathcal{L}(X,Y)$. Following [38] (in the case where Z is a subspace of $\mathcal{L}(X,Y)$), we say that Z has the *strong convex compactness property* if the following holds. For any finite measure space (Ω,Σ,μ) and any strongly measurable bounded $f:\Omega\to Z$, the operator $T\in\mathcal{L}(X,Y)$ defined by

$$Tx := \int_{\Omega} f(\omega)x\,\mathrm{d}\mu(\omega) \qquad (x\in X), \tag{3.7}$$

belongs to Z with $\|T\|_Z \le \overline{\int}_{\Omega}\|f(\omega)\|_Z\,\mathrm{d}\mu(\omega)$. By the Pettis Measurability Theorem, any separable Z has this property. Indeed, if Z is separable then combining Propositions 1.9 and 1.10 in [37] shows that any strongly measurable $f:\Omega\to Z$ is μ-measurable as a map to Z. If f is bounded as well, then (3.7) defines an element of Z with

$$\|T\|_Z \le \int_{\Omega}\|f(\omega)\|_Z\,\mathrm{d}\mu(\omega).$$

It is shown in [38] and [33] that the subspaces of compact and weakly compact operators in $\mathcal{L}(X,Y)$ have the strong convex compactness property, but not all subspaces of $\mathcal{L}(X,Y)$ do. Moreover, if $\mathcal{N}$ is a semifinite von Neumann algebra on a separable Hilbert space H, with faithful normal semifinite trace τ, and $\mathcal{F}$ is a rearrangement invariant Banach function space with the Fatou property, then $\mathcal{E}=\mathcal{N}\cap\mathcal{F}(\mathcal{N},\tau)$ has the strong convex compactness property (see [3, Lemma 3.5]).

LEMMA 3.3. *Let X and Y be separable Banach spaces and Z a Banach space continuously embedded in $\mathcal{L}(X,Y)$. If $B_Z := \{z\in Z \mid \|z\|_Z\le 1\}$ is SOT-closed in $\mathcal{L}(X,Y)$, then Z has the strong convex compactness property.*

Proof. The proof follows that of [3, Lemma 3.5]. First we show that B_Z is a Polish space in the strong operator topology. As in the proof of Lemma 3.2, bounded subsets of $\mathcal{L}(X,Y)$ are SOT-metrizable. The finite rank operators are SOT-dense in $\mathcal{L}(X,Y)$, hence $\mathcal{L}(X,Y)$ is SOT-separable. Therefore B_Z is SOT-separable and metrizable. By assumption, B_Z is complete.

Now let (Ω,μ) be a finite measure space and let $f:\Omega\to Z$ be bounded and strongly measurable. Without loss of generality, we may assume that $f(\Omega)\subseteq B_Z$ and that μ is a probability measure. For each $y^*\in Y^*$ and $x\in X$, the mapping $B_Z\to[0,\infty)$, $T\mapsto|\langle y^*,Tx\rangle|$, is continuous. The collection of all these mappings, for $y^*\in Y^*$ and $x\in X$, separates the points of B_Z. Moreover, $\omega\mapsto|\langle y^*,f(\omega)x\rangle|$ is a measurable mapping $\Omega\to[0,\infty)$ for each $y^*\in Y^*$ and $x\in X$. By [37, Propositions 1.9 and 1.10], f is the

μ-almost everywhere SOT-limit of a sequence of B_Z-valued simple functions $\{f_k\}_{k=1}^{\infty}$. Let $T_k := \int_{\Omega} f_k \,\mathrm{d}\mu \in B_Z$ for $k \in \mathbb{N}$. By the dominated convergence theorem, $T_k(x) \to T(x) := \int_{\Omega} f(\omega)x \,\mathrm{d}\mu(\omega)$ as $k \to \infty$, for all $x \in X$. Since B_Z is SOT-closed by the assumption, we conclude that $T \in B_Z$.

Now let $g : \Omega \to [0, \infty)$ be measurable such that $1 \geq g(\omega) \geq \|f(\omega)\|_Z$ for $\omega \in \Omega$, and define

$$h(\omega) := \frac{f(\omega)}{g(\omega)} \quad \text{and} \quad \mathrm{d}\nu(\omega) := \frac{g(\omega)}{\int_{\Omega} g(\eta)\,\mathrm{d}\mu(\eta)}\mathrm{d}\mu(\omega)$$

for $\omega \in \Omega$. By what we have shown above, $x \mapsto \int_{\Omega} h(\omega)x \,\mathrm{d}\nu(\omega)$ defines an element of B_Z. Since

$$Tx = \int_{\Omega} f(\omega)x \,\mathrm{d}\mu(\omega) = \int_{\Omega} g(\omega)\,\mathrm{d}\mu(\omega) \int_{\Omega} h(\omega)x \,\mathrm{d}\nu(\omega),$$

we obtain $\|T\|_Z \leq \int_{\Omega} g(\omega)\,\mathrm{d}\mu(\omega)$, as remained to be shown. ∎

Remark 3.4. Note that the converse implication does not hold. Indeed, if X is a Hilbert space (or more generally, a Banach space with the metric approximation property) then the finite rank operators of norm less than or equal to 1 are SOT-dense in the unit ball of $\mathcal{L}(X)$. Therefore the compact operators of norm less than or equal to 1 are not SOT-closed in $\mathcal{L}(X)$ if X is infinite-dimensional. However, by [38, Theorem 1.3], the space of compact operators on X has the strong convex compactness property.

Let X and Y be Banach spaces and $\mathcal{I}$ a Banach space which is continuously embedded in $\mathcal{L}(X, Y)$. We say that $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a *Banach ideal* in $\mathcal{L}(X, Y)$ if

- for all $R \in \mathcal{L}(Y)$, $S \in \mathcal{I}$ and $T \in \mathcal{L}(X)$, $RST \in \mathcal{I}$ with $\|RST\|_{\mathcal{I}} \leq \|R\|_{\mathcal{L}(Y)}\|S\|_{\mathcal{I}}\|T\|_{\mathcal{L}(X)}$;
- $X^* \otimes Y \subseteq \mathcal{I}$ with $\|x^* \otimes y\|_{\mathcal{I}} = \|x^*\|_{X^*}\|y\|_Y$ for all $x^* \in X^*$ and $y \in Y$.

By Lemma 3.3 and [14, Proposition 17.21] (using the fact that the SOT and WOT closures of a convex set coincide), for separable X and Y, any *maximal* Banach ideal (for the definition see e.g. [29]) in $\mathcal{L}(X, Y)$ has the strong convex compactness property. This includes a large class of operator ideals, such as the ideal of absolutely p-summing operators, the ideal of integral operators, etc. (see [14, p. 203]).

3.3. Algebras of functions. In this section we discuss some algebras of functions that will be essential in later sections.

Let $\sigma_1, \sigma_2 \subseteq \mathbb{C}$ be Borel measurable subsets and let $\mathfrak{A}(\sigma_1 \times \sigma_2)$ be the class of Borel functions $\varphi : \sigma_1 \times \sigma_2 \to \mathbb{C}$ such that

$$\varphi(\lambda_1, \lambda_2) = \int_{\Omega} a_1(\lambda_1, \omega)a_2(\lambda_2, \omega)\,\mathrm{d}\mu(\omega) \tag{3.8}$$

for all $(\lambda_1,\lambda_2)\in\sigma_1\times\sigma_2$, where (Ω,Σ,μ) is a finite measure space (with μ positive) and $a_1\in\mathcal{B}(\sigma_1\times\Omega,\mathfrak{B}_{\sigma_1}\otimes\Sigma)$, $a_2\in\mathcal{B}(\sigma_2\times\Omega,\mathfrak{B}_{\sigma_2}\otimes\Sigma)$. For $\varphi\in\mathfrak{A}(\sigma_1\times\sigma_2)$ let

$$\|\varphi\|_{\mathfrak{A}(\sigma_1\times\sigma_2)}:=\inf\int_\Omega\|a_1(\cdot,\omega)\|_{\mathcal{B}(\sigma_1)}\|a_2(\cdot,\omega)\|_{\mathcal{B}(\sigma_2)}\,\mathrm{d}\mu(\omega),$$

where the infimum is taken over all possible representations ([1]) in (3.8) (it is straightforward to show that the map $\omega\mapsto\|a_1(\cdot,\omega)\|_{\mathcal{B}(\sigma_1)}\|a_2(\cdot,\omega)\|_{\mathcal{B}(\sigma_2)}$ is measurable).

Remark 3.5. The class $\mathfrak{A}(\sigma_1\times\sigma_2)$ is equal (see e.g. [30, 12]) to the class of functions $\varphi:\sigma_1\times\sigma_2\to\mathbb{C}$ admitting a representation

$$\varphi(\lambda_1,\lambda_2)=\int_\Omega b_1(\lambda_1,\omega)b_2(\lambda_2,\omega)\,\mathrm{d}\nu(\omega)\tag{3.9}$$

for all $(\lambda_1,\lambda_2)\in\sigma_1\times\sigma_2$, where (Ω,Σ,ν) is a measure space and $b_j:\sigma_j\times\Omega\to\mathbb{C}$, for $j=1,2$, are measurable functions such that $\omega\mapsto\|b_1(\cdot,\omega)\|_{\mathcal{B}(\sigma_1)}\|b_2(\cdot,\omega)\|_{\mathcal{B}(\sigma_2)}$ is measurable and

$$\int_\Omega\|b_1(\cdot,\omega)\|_{\mathcal{B}(\sigma_1)}\|b_2(\cdot,\omega)\|_{\mathcal{B}(\sigma_2)}\,\mathrm{d}\nu(\omega)<\infty.$$

Indeed, any $\varphi\in\mathfrak{A}(\sigma_1\times\sigma_2)$ has a representation as in (3.9), and conversely any $\varphi:\sigma_1\times\sigma_2\to\mathbb{C}$ satisfying (3.9) also satisfies (3.8) with $a_j\in\mathcal{B}(\sigma_j\times\Omega)$ defined by

$$a_j(\lambda_j,\omega):=\frac{b(\lambda_j,\omega)}{\|b(\cdot,\omega)\|_{\mathcal{B}(\sigma_j)}}$$

for $j=1,2$, $\lambda_j\in\sigma_j$ and $\omega\in\Omega$, and with the finite measure μ given by $\mathrm{d}\mu(\omega)=\|b_1(\cdot,\omega)\|_{\mathcal{B}(\sigma_1)}\|b_2(\cdot,\omega)\|_{\mathcal{B}(\sigma_2)}\mathrm{d}\nu(\omega)$.

Lemma 3.6. *For all $\sigma_1,\sigma_2\subseteq\mathbb{C}$ measurable, $\mathfrak{A}(\sigma_1\times\sigma_2)$ is a unital Banach algebra which is contractively included in $\mathcal{B}(\sigma_1\times\sigma_2)$.*

Proof. Showing that $\mathfrak{A}(\sigma_1\times\sigma_2)$ is a vector space is straightforward, and that it is normed algebra is proved in [30, Lemma 3] for $\sigma_1=\sigma_2=\mathbb{R}$ (the proof in our setting is identical). The completeness of $\mathfrak{A}(\sigma_1\times\sigma_2)$ follows by showing that an absolutely convergent series of elements in $\mathfrak{A}(\sigma_1\times\sigma_2)$ converges in $\mathfrak{A}(\sigma_1\times\sigma_2)$. This is done by considering a direct sum of the measure spaces involved. ∎

We now state sufficient conditions for a function to belong to $\mathfrak{A}$. The first will be used in the proof of Proposition 5.6. Let $\mathrm{W}^{1,2}(\mathbb{R})$ be the space

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([1]) This might seem problematic from a set-theoretic viewpoint. The problem can be fixed by choosing an equivalence class of such a representation for each real number which can occur in the infimum.

of all $g \in \mathrm{L}^2(\mathbb{R})$ with weak derivative $g' \in \mathrm{L}^2(\mathbb{R})$, endowed with the norm $\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})} := \|g\|_{\mathrm{L}^2(\mathbb{R})} + \|g'\|_{\mathrm{L}^2(\mathbb{R})}$ for $g \in \mathrm{W}^{1,2}(\mathbb{R})$.

LEMMA 3.7 ([30, Theorem 9]). *Let $g \in \mathrm{W}^{1,2}(\mathbb{R})$ and let*

$$\psi_g(\lambda_1, \lambda_2) := \begin{cases} g(\log(\lambda_1/\lambda_2)) & \text{if } \lambda_1, \lambda_2 > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Then $\psi_g \in \mathfrak{A}(\mathbb{R}^2)$ with $\|\psi_g\|_{\mathfrak{A}(\mathbb{R}^2)} \leq \sqrt{2}\, \|g\|_{\mathrm{W}^{1,2}(\mathbb{R})}$.

The second condition involves the Besov space $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$. Following [27], let $\{\psi_k\}_{k\in\mathbb{Z}}$ be a sequence of Schwartz functions on $\mathbb{R}$ such that, for each $k \in \mathbb{Z}$, the Fourier transform $\mathcal{F}\psi_k$ of ψ_k is supported on $[2^{k-1}, 2^{k+1}]$ and $\mathcal{F}\psi_{k+1}(x) = \mathcal{F}\psi_k(2x)$ for all $x > 0$, and such that $\sum_{k=-\infty}^{\infty} \mathcal{F}\psi_k(x) = 1$ for all $x > 0$. Let ψ_k^* be defined by $\mathcal{F}\psi_k^*(x) = \mathcal{F}\psi_k(-x)$ for $k \in \mathbb{Z}$ and $x \in \mathbb{R}$. If f is a distribution on $\mathbb{R}$ such that $\{2^k \|f * \psi_k\|_{\mathrm{L}^\infty(\mathbb{R})}\}_{k\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$ and $\{2^k \|f * \psi_k^*\|_{\mathrm{L}^\infty(\mathbb{R})}\}_{k\in\mathbb{Z}} \in \ell_1(\mathbb{Z})$, then f admits a representation

$$f = \sum_{k\in\mathbb{Z}} f * \psi_k + \sum_{k\in\mathbb{Z}} f * \psi_k^* + P, \tag{3.11}$$

where P is a polynomial.

We let the *homogeneous Besov space* $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ consist of all distributions as above for which $P = 0$. Then $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ is a Banach space when equipped with the norm

$$\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})} := \sum_{k=-\infty}^{\infty} 2^k \|f * \psi_k\|_{\mathrm{L}^\infty(\mathbb{R})} + \sum_{k=-\infty}^{\infty} 2^k \|f * \psi_k^*\|_{\mathrm{L}^\infty(\mathbb{R})} \quad (f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})).$$

For $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ define $\psi_f : \mathbb{R}^2 \to \mathbb{C}$ by

$$\psi_f(\lambda_1, \lambda_2) := \begin{cases} \dfrac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} & \text{if } (\lambda_1, \lambda_2) \in \mathbb{R}^2 \text{ and } \lambda_1 \neq \lambda_2, \\ f'(\lambda_1) & \text{if } \lambda_1 = \lambda_2 \in \mathbb{R}. \end{cases}$$

LEMMA 3.8. *There exists a constant $C \geq 0$ such that $\psi_f \in \mathfrak{A}(\mathbb{R}^2)$ for each $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, with $\|\psi_f\|_{\mathfrak{A}(\mathbb{R}^2)} \leq C\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}$.*

Proof. Let $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$. In [27, Theorem 2] (see also [28, p. 535]) it is shown that ψ_f has a representation

$$\psi_f(\lambda_1, \lambda_2) = \int_{\Omega} a_1(\lambda_1, \omega) a_2(\lambda_2, \omega)\, \mathrm{d}\mu(\omega)$$

for $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, where (Ω, μ) is a measure space and a_1 and a_2 are measurable functions on $\mathbb{R} \times \Omega$ such that

$$\int_{\Omega} \|a_1(\cdot, \omega)\|_\infty \|a_2(\cdot, \omega)\|_\infty \, \mathrm{d}|\mu|(\omega) \leq C\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}$$

for some constant $C \geq 0$ independent of f. Now apply Remark 3.5. ∎

In [27, Theorem 3] Peller also states a condition on a function f on $\mathbb{R}$ which is necessary for $\varphi_f \in \mathfrak{A}(\mathbb{R}^2)$ to hold, and this condition is only slightly weaker than $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$.

4. Double operator integrals and Lipschitz estimates

4.1. Double operator integrals. Fix Banach spaces X and Y, scalar type operators $A \in \mathcal{L}_{\mathrm{s}}(X)$ and $B \in \mathcal{L}_{\mathrm{s}}(Y)$ with spectral measures E respectively F, and $\varphi \in \mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$. Let a representation as in (3.8) for φ be given, with corresponding (Ω, μ) and $a_1 \in \mathcal{B}(\mathrm{sp}(A) \times \Omega)$, $a_2 \in \mathcal{B}(\mathrm{sp}(B) \times \Omega)$. For $\omega \in \Omega$, let $a_1(A, \omega) := a_1(\cdot, \omega)(A) \in \mathcal{L}(X)$ and $a_2(B, \omega) := a_2(\cdot, \omega)(B) \in \mathcal{L}(Y)$ be defined by the functional calculus for A respectively B from Section 3.1.

Lemma 4.1. *Let $S \in \mathcal{L}(X, Y)$ have separable range. Then, for each $x \in X$, $\omega \mapsto a_2(B, \omega) S a_1(A, \omega) x$ is a weakly measurable map $\Omega \to Y$.*

Proof. Fix $x \in X$. If $a_1 = \mathbf{1}_\sigma$ for some measurable $\sigma \subseteq \mathrm{sp}(A) \times \Omega$ then it is straightforward to show that $\langle x^*, a_1(A, \cdot) x\rangle$ is measurable for each $x^* \in X^*$. As S has separable range, $S a_1(A, \cdot) x$ is μ-measurable by the Pettis Measurability Theorem. If a_2 is an indicator function as well, the same argument shows that $a_2(B, \cdot) y$ is weakly measurable for each $y \in Y$. General arguments, approximating $S a_1(A, \cdot) x$ by simple functions, show that $a_2(B, \cdot) S a_1(A, \cdot) x$ is weakly measurable. By linearity this extends to simple a_1 and a_2, and for general a_1 and a_2 let $\{f_k\}_{k\in\mathbb{N}}$, $\{g_k\}_{k\in\mathbb{N}}$ be sequences of simple functions such that $a_1 = \lim_{k\to\infty} f_k$ and $a_2 = \lim_{k\to\infty} g_k$ uniformly. Then $a_1(A, \omega) = \lim_{k\to\infty} f_k(A)$ and $a_2(B, \omega) = \lim_{k\to\infty} g_k(B)$ in the operator norm, for each $\omega \in \Omega$. The desired measurability now follows. ∎

Now suppose that Y is separable and $\mathcal{I}$ is a Banach ideal in $\mathcal{L}(X, Y)$, and let $S \in \mathcal{L}(X, Y)$. By (3.3),

$$\|a_2(B, \omega) S a_1(A, \omega)\|_{\mathcal{I}} \leq 16 \nu(A) \nu(B) \|S\|_{\mathcal{I}} \|a_1(\cdot, \omega)\|_{\mathcal{B}(\mathrm{sp}(A))} \|a_2(\cdot, \omega)\|_{\mathcal{B}(\mathrm{sp}(B))} \tag{4.1}$$

for $w \in \Omega$. Since $\mathcal{I}$ is continuously embedded in $\mathcal{L}(X, Y)$, by the Pettis Measurability Theorem, Lemma 4.1 and (4.1) we can define the *double operator integral*

$$T^{A,B}_\varphi(S) x := \int_\Omega a_2(B, \omega) S a_1(A, \omega) x \,\mathrm{d}\mu(\omega) \in Y \qquad (x \in X). \tag{4.2}$$

Throughout, we will use T_φ for $T^{A,B}_\varphi$ when the operators A and B are clear from the context.

Proposition 4.2. *Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let $A \in \mathcal{L}_{\mathrm{s}}(X)$,*

$B \in \mathcal{L}_{\mathrm{s}}(Y)$, and $\varphi \in \mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$. Let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Then (4.2) *defines an operator $T_{\varphi}^{A,B} \in \mathcal{L}(\mathcal{I})$ which is independent of the choice of the representation of φ in* (3.8), *with*

$$\|T_{\varphi}^{A,B}\|_{\mathcal{L}(\mathcal{I})} \le 16\,\nu(A)\nu(B)\|\varphi\|_{\mathfrak{A}(\mathrm{sp}(A)\times\mathrm{sp}(B))}. \tag{4.3}$$

Proof. By (4.1) and the strong convex compactness property, we have $T_{\varphi}(S) \in \mathcal{L}(\mathcal{I})$ for all $S \in \mathcal{I}$, and

$$\|T_{\varphi}(S)\|_{\mathcal{I}} \le 16\nu(A)\nu(B)\|S\|_{\mathcal{I}} \int_{\Omega} \|a_1(\cdot,\omega)\|_{\mathcal{B}(\mathrm{sp}(A))}\|a_2(\cdot,\omega)\|_{\mathcal{B}(\mathrm{sp}(B))}\,\mathrm{d}\mu(\omega).$$

Clearly T_{φ} is linear, hence the result follows if we establish that T_{φ} is independent of the representation of φ. For this it suffices to let $\varphi \equiv 0$. Now, first consider $S = x^* \otimes y$ for $x^* \in X^*$ and $y \in Y$, and let $x \in X$, $y^* \in Y^*$ and $w \in \Omega$. Recall that E and F are the spectral measures of A and B, respectively. Then

$$\begin{aligned}
&\langle y^*, a_2(B,\omega)Sa_1(A,\omega)x\rangle \\
&\qquad= \int_{\mathrm{sp}(B)} a_2(\eta,\omega)\,\mathrm{d}\langle y^*, F(\eta)Sa_1(A,\omega)x\rangle \\
&\qquad= \int_{\mathrm{sp}(B)} a_2(\eta,\omega)\langle x^*, a_1(A,\omega)x\rangle\,\mathrm{d}\langle y^*, F(\eta)y\rangle \\
&\qquad= \int_{\mathrm{sp}(B)}\int_{\mathrm{sp}(A)} a_1(\lambda,\omega)a_2(\eta,\omega)\,\mathrm{d}\langle x^*, E(\lambda)x\rangle\,\mathrm{d}\langle y^*, F(\eta)y\rangle
\end{aligned}$$

by (3.5). Now Fubini's Theorem and the assumption on φ yield

$$\begin{aligned}
&\langle y^*, T_{\varphi}(S)x\rangle \\
&\quad= \int_{\Omega}\langle y^*, a_2(B,\omega)Sa_1(A,\omega)x\rangle\,\mathrm{d}\mu(\omega) \\
&\quad= \int_{\Omega}\int_{\mathrm{sp}(B)}\int_{\mathrm{sp}(A)} a_1(\lambda,\omega)a_2(\eta,\omega)\,\mathrm{d}\langle x^*, E(\lambda)x\rangle\,\mathrm{d}\langle y^*, F(\eta)y\rangle\,\mathrm{d}\mu(\omega) \\
&\quad= \int_{\mathrm{sp}(B)}\int_{\mathrm{sp}(A)}\int_{\Omega} a_1(\lambda,\omega)a_2(\eta,\omega)\,\mathrm{d}\mu(\omega)\,\mathrm{d}\langle x^*, E(\lambda)x\rangle\,\mathrm{d}\langle y^*, F(\eta)y\rangle \\
&\quad= \int_{\mathrm{sp}(B)}\int_{\mathrm{sp}(A)} \varphi(\lambda,\eta)\,\mathrm{d}\langle x^*, E(\lambda)x\rangle\,\mathrm{d}\langle y^*, F(\eta)y\rangle = 0.
\end{aligned}$$

By linearity, $T_{\varphi}(S) = 0$ for all $S \in X^* \otimes Y$. By Lemma 3.2, a general element $S \in \mathcal{I}$ is the SOT-limit of a bounded (in the operator norm on $\mathcal{L}(X,Y)$) sequence $\{S_n\}_{n\in\mathbb{N}} \subseteq X^* \otimes Y$. Hence for each $x \in X$ there exists a constant $C \geq 0$ such that, for all $n \in \mathbb{N}$,

$$\int_{\Omega} \|a_2(B,\omega)S_n a_1(A,\omega)x\|_Y \,\mathrm{d}\mu(\omega)$$
$$\leq 16\nu(A)\nu(B)\|S_n\|_{\mathcal{L}(X,Y)}\|x\| \int_{\Omega} \|a_1(\cdot,\omega)\|_{\mathcal{B}(\mathrm{sp}(A))}\|a_2(\cdot,\omega)\|_{\mathcal{B}(\mathrm{sp}(B))} \,\mathrm{d}\mu(\omega)$$
$$\leq C\int_{\Omega} \|a_1(\cdot,\omega)\|_{\mathcal{B}(\mathrm{sp}(A))}\|a_2(\cdot,\omega)\|_{\mathcal{B}(\mathrm{sp}(B))} \,\mathrm{d}\mu(\omega) < \infty,$$

where we have used (3.3). Now the dominated convergence theorem shows that $T_\varphi(S)x = \lim_{n\to\infty} T_\varphi(S_n)x = 0$ for all $x \in X$, which implies that T_φ is independent of the representation of φ and concludes the proof. ∎

If A and B are normal operators on separable Hilbert spaces X and Y, then (4.3) improves to

$$\|T_\varphi^{A,B}\|_{\mathcal{L}(\mathcal{I})} \leq \|\varphi\|_{\mathfrak{A}(\mathrm{sp}(A)\times\mathrm{sp}(B))} \tag{4.4}$$

by appealing to (3.6) instead of (3.3) in (4.1).

Remark 4.3. Let H be an infinite-dimensional separable Hilbert space and $\mathcal{S}_2$ the ideal of Hilbert–Schmidt operators on H. There is a natural definition (see [10]) of a double operator integral $\mathcal{T}_\varphi^{A,B} \in \mathcal{L}(\mathcal{S}_2)$ for all $\varphi \in \mathcal{B}(\mathbb{C}^2)$ and normal operators $A, B \in \mathcal{L}(H)$, such that $\mathcal{T}_\varphi^{A,B} = T_\varphi^{A,B}$ if $\varphi \in \mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$. One could wonder whether Proposition 4.2 can be extended to a larger class of functions than $\mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$ using an extension of the definition of $T_\varphi^{A,B}$ in (4.2) which coincides with $\mathcal{T}_\varphi^{A,B}$ on $\mathcal{S}_2$. But it follows from [26, Theorem 1] (see also Remark 3.5) that $\mathcal{T}_\varphi^{A,B}$ extends to a bounded operator on $\mathcal{I} = \mathcal{L}(H)$ if and only if $\varphi \in \mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$. Hence Proposition 4.2 cannot be extended to a larger function class than $\mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$ in general. However, for specific Banach ideals, e.g. ideals with the UMD-property, results have been obtained for larger classes of functions [13, 31].

Remark 4.4. The assumption in Proposition 4.2 that either X or Y has the bounded approximation property is only used, via Lemma 3.2, to ensure that each $S \in \mathcal{I}$ is the SOT-limit of a bounded (in $\mathcal{L}(X,Y)$) net of finite rank operators. Clearly this is true for general Banach spaces X and Y if $\mathcal{I}$ is the closure in $\mathcal{L}(X,Y)$ of $X^* \otimes Y$. In [22] the authors consider an assumption on X and $\mathcal{I}$, called the *local bounded approximation property for* $\mathcal{I}$, which guarantees that each $S \in \mathcal{I}$ is the SOT-limit of a bounded net of finite rank operators. It is shown in [22] that for certain non-trivial ideals this condition is strictly weaker than the bounded approximation property. In the results throughout the paper where we assume that X has the bounded approximation property, one may assume instead that X satisfies the local bounded approximation property for $\mathcal{I}$.

4.2. Commutator and Lipschitz estimates. Let $p_1, p_2 : \mathbb{C}^2 \to \mathbb{C}$ be the coordinate projections given by $p_1(\lambda_1, \lambda_2) := \lambda_1$, $p_2(\lambda_1, \lambda_2) := \lambda_2$ for $(\lambda_1, \lambda_2) \in \mathbb{C}^2$. Note that $f \circ p_1, f \circ p_2 \in \mathfrak{A}(\sigma_1 \times \sigma_2)$ for all $\sigma_1, \sigma_2 \subseteq \mathbb{C}$ Borel and $f \in \mathcal{B}(\sigma_1 \cup \sigma_2)$. For self-adjoint operators A and B on a Hilbert space and for a Schatten–von Neumann ideal $\mathcal{I}$, the following lemma is [30, Lemma 3].

Lemma 4.5. *Under the assumptions of Proposition 4.2, the following hold:*

(1) *The map $\varphi \mapsto T_\varphi^{A,B}$ is a morphism $\mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B)) \to \mathcal{L}(\mathcal{I})$ of unital Banach algebras.*
(2) *Let $f \in \mathcal{B}(\mathrm{sp}(A) \cup \mathrm{sp}(B))$ and $S \in \mathcal{L}(X, Y)$. Then $T_{f\circ p_1}(S) = Sf(A)$ and $T_{f\circ p_2}(S) = f(B)S$. In particular, $T_{p_1}(S) = SA$ and $T_{p_2}(S) = BS$.*

Proof. The structure of the proof is the same as that of [30, Lemma 3]. Linearity in (1) is straightforward. Fix $\varphi_1, \varphi_2 \in \mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$ with representations as in (3.8), with corresponding measure spaces (Ω_j, μ_j) and bounded Borel functions $a_{1,j} \in \mathcal{B}(\mathrm{sp}(A) \times \Omega_j)$ and $a_{2,j} \in \mathcal{B}(\mathrm{sp}(B) \times \Omega_j)$ for $j \in \{1, 2\}$. Then $\varphi := \varphi_1\varphi_2$ also has a representation as in (3.8), with $\Omega = \Omega_1 \times \Omega_2$, $\mu = \mu_1 \times \mu_2$ the product measure and $a_1 = a_{1,1}a_{1,2}$, $a_2 = a_{2,1}a_{2,2}$. By multiplicativity of the functional calculus for A,

$$a_1(A, (\omega_1, \omega_2)) = (a_{1,1}(\cdot, \omega_1)a_{1,2}(\cdot, \omega_2))(A) = a_{1,1}(A, \omega_1)a_{1,2}(A, \omega_2)$$

for all $(\omega_1, \omega_2) \in \Omega$, and similarly for $a_2(B, (\omega_1, \omega_2))$. Applying this to (4.2) yields

$$\begin{aligned} T_\varphi(S)x &= \int_\Omega a_2(B, \omega)Sa_1(A, \omega)x \,\mathrm{d}\mu(\omega) \\ &= \int_{\Omega_1} a_{2,1}(B, \omega_1)T_{\varphi_2}(S)a_{1,1}(A, \omega_1)x \,\mathrm{d}\mu_1(\omega_1) \\ &= T_{\varphi_1}(T_{\varphi_2}(S))x \end{aligned}$$

for all $S \in \mathcal{I}$ and $x \in X$, which proves (1). Part (2) follows from (4.2) and the fact that T_φ is independent of the representation of φ. ∎

For $f : \mathrm{sp}(A) \cup \mathrm{sp}(B) \to \mathbb{C}$ define

$$\varphi_f(\lambda_1, \lambda_2) := \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} \tag{4.5}$$

for $(\lambda_1, \lambda_2) \in \mathrm{sp}(A) \times \mathrm{sp}(B)$ with $\lambda_1 \neq \lambda_2$.

Theorem 4.6. *Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X, Y)$ with the strong convex compactness property. Let $A \in \mathcal{L}_{\mathrm{s}}(X)$ and*

$B \in \mathcal{L}_s(Y)$, and let $f \in \mathcal{B}(\mathrm{sp}(A) \cup \mathrm{sp}(B))$ be such that φ_f extends to an element of $\mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$. Then

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}(\mathrm{sp}(A)\times\mathrm{sp}(B))}\|BS - SA\|_{\mathcal{I}} \tag{4.6}$$

for all $S \in \mathcal{L}(X,Y)$ such that $BS - SA \in \mathcal{I}$.

In particular, if $X = Y$ and $B - A \in \mathcal{I}$,

$$\|f(B) - f(A)\|_{\mathcal{I}} \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}(\mathrm{sp}(A)\times\mathrm{sp}(B))}\|B - A\|_{\mathcal{I}}.$$

Proof. Note that $(p_2 - p_1)\varphi_f = f \circ p_2 - f \circ p_1$. By Lemma 4.5,

$$\begin{aligned} f(B)S - Sf(A) &= T_{f\circ p_2}(S) - T_{f\circ p_1}(S) = T_{(p_2-p_1)\varphi_f}(S) \\ &= T_{p_2\varphi_f}(S) - T_{p_1\varphi_f}(S) = T_{\varphi_f}(T_{p_2}(S) - T_{p_1}(S)) \\ &= T_{\varphi_f}(BS - SA) \end{aligned}$$

for each $S \in \mathcal{I}$. Proposition 4.2 now concludes the proof. ∎

Letting X and Y be Hilbert spaces and A and B normal operators, we generalize results from [10, 30] to all Banach ideals with the strong convex compactness property. As mentioned in Section 3.2, this includes all separable ideals and the so-called maximal operator ideals, which in turn is a large class of ideals containing the absolutely (p,q)-summing operators, the integral operators, and more [14, p. 203]. Note that, for normal operators, we can improve the estimate in (4.6) by appealing to (4.4) instead of (4.3).

Corollary 4.7. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ be normal operators on separable Hilbert spaces X and Y. Let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property, and let $f \in \mathcal{B}(\mathrm{sp}(A) \cup \mathrm{sp}(B))$ be such that φ_f extends to an element of $\mathfrak{A}(\mathrm{sp}(A) \times \mathrm{sp}(B))$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq \|\varphi_f\|_{\mathfrak{A}(\mathrm{sp}(A)\times\mathrm{sp}(B))}\|BS - SA\|_{\mathcal{I}}$$

for all $S \in \mathcal{L}(X,Y)$ such that $BS - SA \in \mathcal{I}$. In particular, if $X = Y$ and $B - A \in \mathcal{I}$,

$$\|f(B) - f(A)\|_{\mathcal{I}} \leq \|\varphi_f\|_{\mathfrak{A}(\mathrm{sp}(A)\times\mathrm{sp}(B))}\|B - A\|_{\mathcal{I}}.$$

Combining Theorem 4.6 with Lemma 3.8 yields the following generalization of [27, Theorem 4].

Corollary 4.8. *There exists a universal constant $C \geq 0$ such that the following holds. Let X and Y be separable Banach spaces such that X or Y has the bounded approximation property, and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, and let $A \in \mathcal{L}_s(X)$ and $B \in \mathcal{L}_s(Y)$ be such that $\mathrm{sp}(A) \cup \mathrm{sp}(B) \subseteq \mathbb{R}$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B)\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|BS - SA\|_{\mathcal{I}} \tag{4.7}$$

for all $S \in \mathcal{L}(X,Y)$ *such that* $BS - SA \in \mathcal{I}$. *In particular, if* $X = Y$ *and* $B - A \in \mathcal{I}$,

$$\|f(B) - f(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B)\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|B - A\|_{\mathcal{I}}.$$

In the case where the Banach ideal $\mathcal{I}$ is the space $\mathcal{L}(X,Y)$ of bounded operators from X to Y, we obtain the following corollary.

COROLLARY 4.9. *There exists a universal constant* $C \geq 0$ *such that the following holds. Let* X *and* Y *be separable Banach spaces such that either* X *or* Y *has the bounded approximation property. Let* $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, *and let* $A, B \in \mathcal{L}_{\mathrm{s}}(X)$ *be such that* $\mathrm{sp}(A) \cup \mathrm{sp}(B) \subseteq \mathbb{R}$. *Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} \leq C\nu(A)\nu(B)\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|BS - SA\|_{\mathcal{L}(X,Y)} \tag{4.8}$$

for all $S \in \mathcal{L}(X,Y)$. *In particular, if* $X = Y$ *then*

$$\|f(B) - f(A)\|_{\mathcal{L}(X)} \leq C\nu(A)\nu(B)\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|B - A\|_{\mathcal{L}(X)}.$$

REMARK 4.10. In [1] the requirement in [27, Theorem 4] that the operators in question are self-adjoint is removed by considering functions in the Besov class $\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R}^2)$ of two variables. More precisely, in [1, Theorem 7.2] it is shown that there exists a constant $C \geq 0$ such that if $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R}^2)$ then

$$\|f(B) - f(A)\|_{\mathcal{L}(H)} \leq C\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R}^2)}\|B - A\|_{\mathcal{L}(H)} \tag{4.9}$$

for all normal operators A and B on a separable Hilbert space H such that $A - B \in \mathcal{L}(H)$. It is quite conceivable that the assumption in Corollaries 4.8 and 4.9 that the operators A and B have real spectrum can be removed in a similar manner.

However, the results in the present section are mainly used in later sections to prove the operator Lipschitz estimate in (1.4). To this end, in Proposition 5.6 we relate operator Lipschitz estimates for the absolute value function to boundedness of triangular truncation operators. There the assumption that the operators in question have real spectrum is essential. Therefore, removing the spectral assumption in Corollaries 4.8 and 4.9 would not improve the main results of this article. On the other hand, considering operators with general spectrum would increase the length of this section considerably and change its scope (for comparison see [1, Sections 5 and 6]), since the double operator integral technique which was used in [27] does not suffice to yield (4.9). For this reason we choose not to pursue an analogue of (4.9) in the present article.

REMARK 4.11. Corollaries 4.8 and 4.9 yield global estimates, in the sense that (4.7) and (4.8) hold for all scalar type operators A and B with real spectrum, and the constant in the estimate depends on A and B only through their spectral constants $\nu(A)$ and $\nu(B)$. Local estimates follow if $f \in \mathcal{B}(\mathbb{R})$

is contained in the Besov class locally. More precisely, given scalar type operators $A \in \mathcal{L}_s(X)$ and $B \in \mathcal{L}_s(Y)$ with real spectrum, suppose there exists $g \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$ with $g(s) = f(s)$ for all $s \in \mathrm{sp}(A) \cup \mathrm{sp}(B)$. Then (with notation as in Corollary 4.8), Theorem 4.6 yields

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B)\|g\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|BS - SA\|_{\mathcal{I}} \tag{4.10}$$

for all $S \in \mathcal{L}(X, Y)$ such that $BS - SA \in \mathcal{I}$.

5. Spaces with an unconditional basis. In this section we prove some results for specific scalar type operators, namely operators which are diagonalizable with respect to an unconditional Schauder basis. These results will be used in later sections. In this section we assume that all spaces are infinite-dimensional, but the results and proofs carry over directly to finite-dimensional spaces. This will be used in Section 8.

5.1. Diagonalizable operators. Let X be a Banach space with an unconditional Schauder basis $\{e_j\}_{j=1}^\infty \subseteq X$. For $j \in \mathbb{N}$, let $\mathcal{P}_j \in \mathcal{L}(X)$ be the projection given by $\mathcal{P}_j(x) := x_j e_j$ for all $x = \sum_{k=1}^\infty x_k e_k \in X$.

Assumption 5.1. For simplicity, assume in this section that

$$\Big\| \sum_{j\in N} \mathcal{P}_j \Big\|_{\mathcal{L}(X)} = 1 \quad \text{for all non-empty} N \subseteq \mathbb{N}.$$

This condition is satisfied in the examples we consider in later sections, and simplifies the presentation. For general bases one merely gets additional constants in the results.

An operator $A \in \mathcal{L}(X)$ is *diagonalizable* (with respect to $\{e_j\}_{j=1}^\infty$) if there exists $U \in \mathcal{L}(X)$ invertible and a sequence $\{\lambda_j\}_{j=1}^\infty \in \ell_\infty$ of complex numbers such that

$$UAU^{-1}x = \sum_{j=1}^\infty \lambda_j \mathcal{P}_j x \quad (x \in X), \tag{5.1}$$

where the series converges since $\{e_k\}_{k=1}^\infty$ is unconditional (see [34, Lemma 16.1]). In this case A is a scalar type operator, with point spectrum equal to $\{\lambda_j\}_{j=1}^\infty$, $\mathrm{sp(A)} = \overline{\{\lambda_j\}_{j=1}^\infty}$ and spectral measure E given by

$$E(\sigma) = \sum_{\lambda_j \in \sigma} U^{-1}\mathcal{P}_j U \tag{5.2}$$

for $\sigma \subseteq \mathbb{C}$ Borel. The set of all diagonalizable operators on X is denoted by $\mathcal{L}_d(X)$. We do not explicitly mention the basis $\{e_j\}_{j=1}^\infty$ with respect to which an operator is diagonalizable, since this basis will be fixed throughout. Often we write $A \in \mathcal{L}_d(X, \lambda, U)$ to identify the operator U and the sequence $\lambda = \{\lambda_j\}_{j=1}^\infty$ above. For $A \in \mathcal{L}_d(X, \lambda, U)$ and $f \in \mathcal{B}(\mathbb{C})$, it follows from (3.4) that

$$f(A) = U^{-1}\Big(\sum_{j=1}^{\infty} f(\lambda_j)\mathcal{P}_j\Big)U. \tag{5.3}$$

Since any Banach space with a Schauder basis is separable and has the bounded approximation property, we can apply the results from the previous section to diagonalizable operators, and we obtain for instance the following.

Corollary 5.2. *There exists a universal constant $C \geq 0$ such that the following holds. Let X and Y be Banach spaces with unconditional Schauder bases, and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$, and let $A \in \mathcal{L}_{\mathrm{d}}(X)$ and $B \in \mathcal{L}_{\mathrm{d}}(Y)$ with $\mathrm{sp}(A) \cup \mathrm{sp}(B) \subseteq \mathbb{R}$. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B)\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|BS - SA\|_{\mathcal{I}}$$

for all $S \in \mathcal{L}(X,Y)$ such that $BS - SA \in \mathcal{I}$. In particular, if $X = Y$ and $B - A \in \mathcal{I}$,

$$\|f(B) - f(A)\|_{\mathcal{I}} \leq C\nu(A)\nu(B)\|f\|_{\dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})}\|B - A\|_{\mathcal{I}}.$$

Since this result does not apply to the absolute value function (and neither does the more general Theorem 4.6), and because of the importance of the absolute value function, we will now study Lipschitz estimates for more general functions.

Let Y be a Banach space with an unconditional Schauder basis $\{f_k\}_{k=1}^{\infty}$, and let the projections $\mathcal{Q}_k \in \mathcal{L}(Y)$ be given by $\mathcal{Q}_k(y) := y_k f_k$ for all $y = \sum_{l=1}^{\infty} y_l f_l \in Y$ and $k \in \mathbb{N}$. Let $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ and $\mu = \{\mu_k\}_{k=1}^{\infty}$ be sequences of complex numbers, and let $\varphi : \mathbb{C}^2 \to \mathbb{C}$. For $n \in \mathbb{N}$, define $T^{\lambda,\mu}_{\varphi,n} \in \mathcal{L}(\mathcal{L}(X,Y))$ by

$$T^{\lambda,\mu}_{\varphi,n}(S) := \sum_{j,k=1}^{n} \varphi(\lambda_j,\mu_k)\mathcal{Q}_k S\mathcal{P}_j \quad (S \in \mathcal{L}(X,Y)). \tag{5.4}$$

Note that $T^{\lambda,\mu}_{\varphi,n} \in \mathcal{L}(\mathcal{I})$ for each Banach ideal $\mathcal{I}$ in $\mathcal{L}(X,Y)$.

For $f \in \mathcal{B}(\mathbb{C})$ extend the divided difference from (4.5), $\varphi_f(\lambda_1,\lambda_2) := \frac{f(\lambda_2)-f(\lambda_1)}{\lambda_2-\lambda_1}$ for $(\lambda_1,\lambda_2) \in \mathbb{C}^2$ with $\lambda_1 \neq \lambda_2$, to a function $\varphi_f : \mathbb{C}^2 \to \mathbb{C}$.

Lemma 5.3. *Let X and Y be Banach spaces with unconditional Schauder bases, and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$. Let $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ and $\mu = \{\mu_k\}_{k=1}^{\infty}$ be sequences of complex numbers, and let $A \in \mathcal{L}_{\mathrm{d}}(X,\lambda,U)$, $B \in \mathcal{L}_{\mathrm{d}}(Y,\mu,V)$, $f \in \mathcal{B}(\mathbb{C})$ and $n \in \mathbb{N}$. Then*

$$\|f(B)S_n - S_n f(A)\|_{\mathcal{I}} \leq \|U\|_{\mathcal{L}(X)}\|V^{-1}\|_{\mathcal{L}(Y)}\|T^{\lambda,\mu}_{\varphi_f,n}(V(BS - SA)U^{-1})\|_{\mathcal{I}}$$

for all $S \in \mathcal{L}(X,Y)$ with $BS - SA \in \mathcal{I}$, where

$$S_n := \sum_{j,k=1}^{n} V^{-1}\mathcal{Q}_k VSU^{-1}\mathcal{P}_j U.$$

Proof. Let $S \in \mathcal{I}$ be such that $BS - SA \in \mathcal{I}$. For the duration of the proof write $P_j := U^{-1}\mathcal{P}_j U \in \mathcal{L}(X)$ and $Q_k := V^{-1}\mathcal{Q}_k V \in \mathcal{L}(Y)$ for $j,k \in \mathbb{N}$. By (5.3), and using $P_jP_k = 0$ and $Q_jQ_k = 0$ for $j \neq k$,

$$\begin{aligned} f(B)S_n - S_n f(A) &= \sum_{k=1}^{\infty} f(\mu_k)Q_k\Big(\sum_{i,l=1}^{n} Q_l S P_i\Big) - \sum_{j=1}^{\infty} f(\lambda_j)\Big(\sum_{i,l=1}^{n} Q_l S P_i\Big)P_j \\ &= \sum_{j,k=1}^{n} (f(\mu_k) - f(\lambda_j))Q_k S P_j \\ &= \sum_{j,k=1}^{n} \sum_{\mu_k \neq \lambda_j} \frac{f(\mu_k) - f(\lambda_j)}{\mu_k - \lambda_j}(\mu_k Q_k S P_j - \lambda_j Q_k S P_j) \\ &= \sum_{j,k=1}^{n} \varphi_f(\lambda_j,\mu_k)Q_k\Big(\Big(\sum_{l=1}^{\infty} \mu_l Q_l\Big)S - S\Big(\sum_{i=1}^{\infty} \lambda_i P_i\Big)\Big)P_j \\ &= \sum_{j,k=1}^{n} \varphi_f(\lambda_j,\mu_k)Q_k(BS - SA)P_j \\ &= V^{-1}T^{\lambda,\mu}_{\varphi_f,n}(V(BS - SA)U^{-1})U. \end{aligned}$$

Now use the ideal property of $\mathcal{I}$ to conclude the proof. ∎

For a sequence λ of complex numbers and $A \in \mathcal{L}_{\mathrm{d}}(X,\lambda,U)$, define

$$K_A := \inf\{\|U\|_{\mathcal{L}(X)}\|U^{-1}\|_{\mathcal{L}(X)} \mid A \in \mathcal{L}_{\mathrm{d}}(X,\lambda,U)\}. \tag{5.5}$$

We will call K_A the *diagonalizability constant* of A. Using the unconditionality of the Schauder basis of X and Assumption 5.1, one can show that K_A does not depend on the specific ordering of the sequence λ. Since the sequence λ is, up to ordering, uniquely determined by A (it is the point spectrum of A), K_A only depends on A. Moreover, by Assumption 5.1 and (5.2), $\|E(\sigma)\|_{\mathcal{L}(X)} \leq \|U^{-1}\|_{\mathcal{L}(X)}\|U\|_{\mathcal{L}(X)}$ for all $\sigma \subseteq \mathbb{C}$ Borel and $U \in \mathcal{L}(X)$ such that $A \in \mathcal{L}_{\mathrm{d}}(X,\lambda,U)$, where E is the spectral measure of A. Hence

$$\nu(A) \leq K_A, \tag{5.6}$$

where $\nu(A)$ is the spectral constant of A from Section 3.1.

We now derive commutator estimates for A and B in the operator norm, under a boundedness assumption which will be verified for specific X and Y in later sections.

PROPOSITION 5.4. *Let X and Y be Banach spaces with unconditional Schauder bases, $A \in \mathcal{L}_{\mathrm{d}}(X, \lambda, U)$, $B \in \mathcal{L}_{\mathrm{d}}(Y, \mu, V)$ and $f \in \mathcal{B}(\mathbb{C})$. Suppose that*

$$C := \sup_{n\in\mathbb{N}} \|T^{\lambda,\mu}_{\varphi_f,n}\|_{\mathcal{L}(\mathcal{L}(X,Y))} < \infty. \tag{5.7}$$

Then

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} \le CK_AK_B\|BS - SA\|_{\mathcal{L}(X,Y)}$$

for all $S \in \mathcal{L}(X, Y)$.

Proof. Let $S \in \mathcal{L}(X, Y)$ and for $n \in \mathbb{N}$ let $S_n \in \mathcal{L}(X, Y)$ be as in Lemma 5.3. It is straightforward to show that, for each $x \in X$, $S_nx \to Sx$ as $n \to \infty$. Hence $f(B)S_nx - S_nf(A)x \to f(B)Sx - Sf(A)x$ as $n \to \infty$, for each $x \in X$. Lemma 5.3 and (5.7) now yield

$$\begin{aligned}\|f(B)S - Sf(A)\|_{\mathcal{L}(X,Y)} &\le \limsup_{n\to\infty} \|f(B)S_n - S_nf(A)\|_{\mathcal{L}(X,Y)}\\ &\le C\|U\|\,\|V^{-1}\|\,\|V(BS - SA)U^{-1}\|_{\mathcal{L}(X,Y)}\\ &\le C\|U\|\,\|U^{-1}\|\,\|V\|\,\|V^{-1}\|\,\|BS - SA\|_{\mathcal{L}(X,Y)}.\end{aligned}$$

Taking the infimum over U and V concludes the proof. ∎

REMARK 5.5. Proposition 5.4 also holds for more general Banach ideals in $\mathcal{L}(X, Y)$. Indeed, let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X, Y)$ with the property that, if $\{S_m\}_{m=1}^{\infty} \subseteq \mathcal{I}$ is an $\mathcal{I}$-bounded sequence which SOT-converges to some $S \in \mathcal{L}(X, Y)$ as $m \to \infty$, then $S \in \mathcal{I}$ with $\|S\|_{\mathcal{I}} \le \limsup_{m\to\infty} \|S_m\|_{\mathcal{I}}$. If

$$C := \sup_{n\in\mathbb{N}} \|T^{\lambda,\mu}_{\varphi_f,n}\|_{\mathcal{L}(\mathcal{I})} < \infty$$

then the proof of Proposition 5.4 shows that

$$\|f(B)S - Sf(A)\|_{\mathcal{I}} \le CK_AK_B\|BS - SA\|_{\mathcal{I}}$$

for all $S \in \mathcal{L}(X, Y)$ such that $BS - SA \in \mathcal{I}$.

5.2. Estimates for the absolute value function. It is known that Lipschitz estimates for the absolute value function are related to estimates for so-called triangular truncation operators. For example, in [20] and [16] it was shown that the boundedness of the standard triangular truncation on many operator spaces is equivalent to Lipschitz estimates for the absolute value function. We prove that triangular truncation operators are related to Lipschitz estimates for the absolute value function in our setting as well. We do so by relating the assumption in (5.7) to triangular truncation operators associated to sequences. We will then bound the norms of these operators in later sections for specific X and Y.

Let $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ and $\mu = \{\mu_k\}_{k=1}^{\infty}$ be sequences of real numbers, and let X, Y, $\{\mathcal{P}_j\}_{j=1}^{\infty}$ and $\{\mathcal{Q}_k\}_{k=1}^{\infty}$ be as before. For $n \in \mathbb{N}$ define

$T^{\lambda,\mu}_{\triangle,n} \in \mathcal{L}(\mathcal{L}(X,Y))$ by

$$T^{\lambda,\mu}_{\triangle,n}(S) := \sum_{j,k=1}^{n} \sum_{\mu_k \le \lambda_j} \mathcal{Q}_k S \mathcal{P}_j \quad (S \in \mathcal{L}(X,Y)). \tag{5.8}$$

We call $T^{\lambda,\mu}_{\triangle,n}$ the *triangular truncation associated to* λ *and* μ.

For $f(t) := |t|$ for $t \in \mathbb{R}$, define $\varphi_f : \mathbb{C}^2 \to \mathbb{C}$ by

$$\varphi_f(\lambda_1,\lambda_2) := \begin{cases} \dfrac{|\lambda_1| - |\lambda_2|}{\lambda_1 - \lambda_2} & \text{if } \lambda_1 \neq \lambda_2, \\ 1 & \text{otherwise.} \end{cases}$$

The following result relates the norm of $T^{\lambda,\mu}_{\varphi_f,n}$ to that of $T^{\lambda,\mu}_{\triangle,n}$.

Proposition 5.6. *There exists a universal constant $C \ge 0$ such that the following holds. Let X and Y be Banach spaces with unconditional Schauder bases and let $\mathcal{I}$ be a Banach ideal in $\mathcal{L}(X,Y)$ with the strong convex compactness property. Let λ and μ be bounded sequences of real numbers. Let $f(t) := |t|$ for $t \in \mathbb{R}$. Then*

$$\|T^{\lambda,\mu}_{\varphi_f,n}(S)\|_{\mathcal{I}} \le C(\|S\|_{\mathcal{I}} + \|T^{\lambda,\mu}_{\triangle,n}(S)\|_{\mathcal{I}})$$

for all $n \in \mathbb{N}$ and $S \in \mathcal{I}$. In particular, if $\sup_{n\in\mathbb{N}} \|T^{\lambda,\mu}_{\triangle,n}\|_{\mathcal{L}(\mathcal{L}(X,Y))} < \infty$ *then* (5.7) *holds.*

Proof. Let $n \in \mathbb{N}$ and $S \in \mathcal{I}$, and write $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ and $\mu = \{\mu_k\}_{k=1}^{\infty}$. Throughout the proof we will only consider λ_j and μ_k for $1 \le j,k \le n$, but to simplify the presentation we will not mention this explicitly. We can decompose $T^{\lambda,\mu}_{\varphi_f,n}(S)$ as

$$\begin{aligned} T^{\lambda,\mu}_{\varphi_f,n}(S) = &\sum_{\lambda_k,\mu_k \ge 0} \mathcal{Q}_k S \mathcal{P}_j - \sum_{\mu_k < 0 < \lambda_j} \frac{\mu_k + \lambda_j}{\mu_k - \lambda_j} \mathcal{Q}_k S \mathcal{P}_j \\ &+ \sum_{\lambda_j < 0 < \mu_k} \frac{\mu_k + \lambda_j}{\mu_k - \lambda_j} \mathcal{Q}_k S \mathcal{P}_j - \sum_{\lambda_k,\mu_k \le 0} \mathcal{Q}_k S \mathcal{P}_j + \sum_{\lambda_k,\mu_k = 0} \mathcal{Q}_k S \mathcal{P}_j. \end{aligned}$$

Note that some of these terms may be zero. By the ideal property of $\mathcal{I}$ and Assumption 5.1,

$$\Big\| \sum_{\lambda_j,\mu_k \ge 0} \mathcal{Q}_k S \mathcal{P}_j \Big\|_{\mathcal{I}} \le \Big\| \sum_{\mu_k \ge 0} \mathcal{Q}_k \Big\|_{\mathcal{L}(Y)} \|S\|_{\mathcal{I}} \Big\| \sum_{\lambda_j \ge 0} \mathcal{P}_j \Big\|_{\mathcal{L}(X)} \le \|S\|_{\mathcal{I}}. \tag{5.9}$$

Similarly, $\| \sum_{\lambda_k,\mu_k \le 0} \mathcal{Q}_k S \mathcal{P}_j \|_{\mathcal{I}}$ and $\| \sum_{\lambda_k,\mu_k = 0} \mathcal{Q}_k S \mathcal{P}_j \|_{\mathcal{I}}$ are each bounded by $\|S\|_{\mathcal{I}}$. To bound the other terms it is sufficient to show that

$$\Big\| \sum_{\lambda_j,\mu_k > 0} \frac{\mu_k - \lambda_j}{\mu_k + \lambda_j} \mathcal{Q}_k S \mathcal{P}_j \Big\|_{\mathcal{I}} \le C'(\|S\|_{\mathcal{I}} + \|T^{\lambda,\mu}_{\triangle,n}(S)\|_{\mathcal{I}})$$

for some universal constant $C' \geq 0$. Indeed, replacing λ by $-\lambda$ and μ by $-\mu$ then yields the desired conclusion. Let

$$\Phi(S) := \sum_{\lambda_j,\mu_k>0} \frac{\mu_k - \lambda_j}{\mu_k + \lambda_j} \mathcal{Q}_k S \mathcal{P}_j,$$

and define $g \in \mathrm{W}^{1,2}(\mathbb{R})$ by $g(t) := \frac{2}{e^{|t|}+1}$ for $t \in \mathbb{R}$. Note that $\Phi(S)$ is equal to

$$\sum_{0<\mu_k\leq\lambda_j} \left(g\left(\log\frac{\lambda_j}{\mu_k}\right) - 1\right)\mathcal{Q}_k S\mathcal{P}_j + \sum_{0<\lambda_j<\mu_k} \left(1 - g\left(\log\frac{\lambda_j}{\mu_k}\right)\right)\mathcal{Q}_k S\mathcal{P}_j.$$

Now let $\psi_g : \mathbb{R}^2 \to \mathbb{C}$ be as in (3.10), and let $A := \sum_{j=1}^{\infty} \lambda_j \mathcal{P}_j \in \mathcal{L}(X)$ and $B := \sum_{k=1}^{\infty} \mu_k \mathcal{Q}_k \in \mathcal{L}(Y)$. Let $T_{\psi_g}^{A,B}$ be as in (4.2). One can check that

$$\begin{aligned}\Phi(S) = T_{\psi_g}^{A,B}(T_{\triangle,n}^{\lambda,\mu}(S)) &- \sum_{\lambda_j,\mu_k>0} \mathcal{Q}_k T_{\triangle,n}^{\lambda,\mu}(S)\mathcal{P}_j \\ &+ \sum_{\lambda_j,\mu_k>0} \mathcal{Q}_k(S - T_{\triangle,n}^{\lambda,\mu}(S))\mathcal{P}_j - T_{\psi_g}^{A,B}(S - T_{\triangle,n}^{\lambda,\mu}(S)).\end{aligned}$$

Since each Banach space with a Schauder basis is separable and has the bounded approximation property, Lemma 3.7 and Proposition 4.2 yield

$$\|T_{\psi_g}^{A,B}(T_{\triangle,n}^{\lambda,\mu}(S))\|_{\mathcal{I}} \leq 16\sqrt{2}\,\nu(A)\nu(B)\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})}\|T_{\triangle,n}^{\lambda,\mu}(S)\|_{\mathcal{I}}.$$

By (5.6), $\nu(A) = \nu(B) = 1$. Similarly,

$$\|T_{\psi_g}^{A,B}(S - T_{\triangle,n}^{\lambda,\mu}(S))\|_{\mathcal{I}} \leq 16\sqrt{2}\,\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})}(\|S\|_{\mathcal{I}} + \|T_{\triangle,n}^{\lambda,\mu}(S)\|_{\mathcal{I}}).$$

By the same arguments as in (5.9),

$$\begin{aligned}\Big\|\sum_{\lambda_j,\mu_k>0} \mathcal{Q}_k T_{\triangle,n}^{\lambda,\mu}(S)\mathcal{P}_j\Big\|_{\mathcal{I}} + \Big\|\sum_{\lambda_j,\mu_k>0} \mathcal{Q}_k(S - T_{\triangle,n}^{\lambda,\mu}(S))\mathcal{P}_j\Big\| \\ \leq 2\|S\|_{\mathcal{I}} + \|T_{\triangle,n}^{\lambda,\mu}(S)\|_{\mathcal{I}}.\end{aligned}$$

Combining all these estimates yields

$$\|\Phi(S)\|_{\mathcal{I}} \leq (2 + 32\sqrt{2}\,\|g\|_{\mathrm{W}^{1,2}(\mathbb{R})})(\|S\|_{\mathcal{I}} + \|T_{\triangle,n}^{\lambda,\mu}(S)\|_{\mathcal{I}}),$$

as desired. ∎

6. The absolute value function on $\mathcal{L}(\ell_p, \ell_q)$. In this section we study the absolute value function on $\mathcal{L}(\ell_p, \ell_q)$. We obtain the commutator estimate (1.2) for the absolute value function and $X = \ell_p$ and $Y = \ell_q$ with $p < q$, and we obtain (1.1) for each Lipschitz function and $X = \mathcal{L}(\ell_1)$ or $X = \mathcal{L}(\mathrm{c}_0)$. We also obtain results for $p \geq q$.

The key idea of the proof is entirely different from the techniques used in [11, 13, 16, 20], which are based on a special geometric property of the reflexive Schatten–von Neumann ideals (the UMD-property), a property which

$\mathcal{L}(\ell_p, \ell_q)$ does not have. Instead, we prove our results by relating estimates for the operators from (5.8) to the standard triangular truncation operator, defined in (6.1) below. For this we use the theory of Schur multipliers on $\mathcal{L}(\ell_p, \ell_q)$ developed in [5]. We then appeal to results from [4] about the boundedness of the standard triangular truncation on $\mathcal{L}(\ell_p, \ell_q)$.

6.1. Schur multipliers. For $p \in [1, \infty)$ let $\{e_j\}_{j=1}^{\infty} \subseteq \ell_p$ be the standard Schauder basis of ℓ_p, with the corresponding projections $\mathcal{P}_j(x) := x_j e_j$ for $x = \sum_{k=1}^{\infty} x_k e_k$ and $j \in \mathbb{N}$. We consider this basis and the corresponding projections on all ℓ_p-spaces simultaneously, for simplicity of notation. Note that Assumption 5.1 is satisfied for this basis. For $q \in [1, \infty]$, any operator $S \in \mathcal{L}(\ell_p, \ell_q)$ can be represented by an infinite matrix $\tilde{S} = \{s_{jk}\}_{j,k=1}^{\infty}$, where $s_{jk} := (S(e_k), e_j)$ for $j, k \in \mathbb{N}$. For an infinite matrix $M = \{m_{jk}\}_{j,k=1}^{\infty}$ the product $M * \tilde{S} := \{m_{jk} s_{jk}\}$ is the *Schur product* of the matrices M and $\tilde{S}$. The matrix M is a *Schur multiplier* if the mapping $\tilde{S} \mapsto M * \tilde{S}$ is a bounded operator on $\mathcal{L}(\ell_p, \ell_q)$. Throughout, we identify Schur multipliers with their corresponding operators.

The notion of a Schur multiplier is a discrete version of a double operator integral (for details see e.g. [32, 35]). Schur multipliers on the space $\mathcal{L}(\ell_p, \ell_q)$ are also called (p, q)-*multipliers*. We denote by $\mathcal{M}(p, q)$ the Banach space of (p, q)-multipliers with the norm

$$\|M\|_{(p,q)} := \sup\{\|M * \tilde{S}\|_{\mathcal{L}(\ell_p,\ell_q)} \mid \|S\|_{\mathcal{L}(\ell_p,\ell_q)} \leq 1\}.$$

Remark 6.1. We also consider (p, q)-multipliers M for $p = \infty$ and $q \in [1, \infty]$. Any operator $S \in \mathcal{L}(\mathrm{c}_0, \ell_q)$ corresponds to an infinite matrix $\tilde{S} = \{s_{jk}\}_{j,k=1}^{\infty}$, and M is said to be an (∞, q)-*multiplier* if the mapping $S \mapsto M * \tilde{S}$ is a bounded operator on $\mathcal{L}(\mathrm{c}_0, \ell_q)$. We define the Banach space $\mathcal{M}(\infty, q)$ in the obvious way. Often we do not explicitly distinguish the case $p = \infty$ from $1 \leq p < \infty$, but the reader should keep in mind that for $p = \infty$ the space ℓ_p should be replaced by c_0.

Remark 6.2. It is straightforward to see that $\|M\|_{(p,q)} \geq \sup_{j,k \in \mathbb{N}} |m_{j,k}|$ for all $p, q \in [1, \infty]$ and $M \in \mathcal{M}(p, q)$.

For $p, q \in [1, \infty]$ and $S \in \mathcal{L}(\ell_p, \ell_q)$, define

$$\mathcal{T}_{\triangle}(S) := \sum_{k \leq j} \mathcal{P}_k S \mathcal{P}_j, \tag{6.1}$$

which is a well-defined element of $\mathcal{L}(\ell_r, \ell_s)$ for suitable $r, s \in [1, \infty]$ by Proposition 6.3 below. The operator $\mathcal{T}_{\triangle}$ is the (*standard*) *triangular truncation* (see [21]). This operator can be identified with the following Schur multiplier. Let $T'_{\triangle} = \{t'_{jk}\}_{j,k=1}^{\infty}$ be a matrix given by $t'_{jk} = 1$ for $k \leq j$ and $t'_{jk} = 0$ otherwise. It is clear that $\mathcal{T}_{\triangle}$ extends to a bounded linear operator on $\mathcal{L}(\ell_p, \ell_q)$ if and only if $T'_{\triangle}$ is a (p, q)-multiplier. For $n \in \mathbb{N}$ and $r, s \in [1, \infty]$

we will consider the operators $\mathcal{T}_{\triangle,n} \in \mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\mathcal{L}(\ell_r,\ell_s))$ given by

$$\mathcal{T}_{\triangle,n}(S) := \sum_{1\leq k\leq j\leq n} \mathcal{P}_k S \mathcal{P}_j \quad (S \in \mathcal{L}(\ell_p,\ell_q)).$$

The dependence of the (p,q)-norm of $\mathcal{T}_\triangle$ on the indices p and q was determined in [4] and [21] (see also [36]), and is as follows.

PROPOSITION 6.3. *Let $p,q \in [1,\infty]$. Then the following statements hold.*

(i) [4, Theorem 5.1] *If either $p<q$, $1=p=q$ or $p=q=\infty$, then $\mathcal{T}_\triangle \in \mathcal{M}(p,q)$.*
(ii) [21, Proposition 1.2] *If $1\neq p\geq q\neq\infty$, then there is a constant $C>0$ such that*
$$\|\mathcal{T}_{\triangle,n}\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q))} \geq C\ln n$$
for all $n\in\mathbb{N}$.
(iii) [4, Theorem 5.2] *If $1\neq p\geq q\neq\infty$, then for each $s>q$ and $r<p$,*
$$\mathcal{T}_\triangle : \mathcal{L}(\ell_p,\ell_q)\to\mathcal{L}(\ell_p,\ell_s) \quad \text{and} \quad \mathcal{T}_\triangle : \mathcal{L}(\ell_p,\ell_q)\to\mathcal{L}(\ell_r,\ell_q)$$
are bounded.

REMARK 6.4. If $p=1$ or $q=\infty$, then, for a matrix $M=\{m_{jk}\}_{j,k=1}^\infty$, $M\in\mathcal{M}(p,q)$ if and only if $\sup_{j,k\in\mathbb{N}}|m_{jk}|<\infty$, in which case $\|M\|_{(p,q)} = \sup_{j,k\in\mathbb{N}}|m_{jk}|$. This follows immediately from the well-known identities (see [5, p. 605, (2) and (3)])

$$\|S\|_{\mathcal{L}(\ell_1,\ell_q)} = \sup_{k\in\mathbb{N}}\Big(\sum_{j=1}^\infty |s_{jk}|^q\Big)^{1/q}$$

for $q\in[1,\infty)$ and $S=\{s_{jk}\}_{j,k=1}^\infty\in\mathcal{L}(\ell_1,\ell_q)$, and

$$\|S\|_{\mathcal{L}(\ell_p,\ell_\infty)} = \sup_{j\in\mathbb{N}}\Big(\sum_{k=1}^\infty |s_{jk}|^{p^*}\Big)^{1/p^*}$$

for $p\in[1,\infty]$ and $S=\{s_{jk}\}_{j,k=1}^\infty\in\mathcal{L}(\ell_p,\ell_\infty)$ (with the obvious modification for $p=1$).

We will also need the following result, a generalization of [5, Theorem 4.1]. For a matrix $M=\{m_{jk}\}_{j,k=1}^\infty$, let $\widetilde{M}=\{\widetilde{m}_{jk}\}_{j,k=1}^\infty$ be obtained from M by repeating the first column, i.e. $\widetilde{m}_{j1}=m_{j1}$ and $\widetilde{m}_{jk}=m_{j(k-1)}$ for $j\in\mathbb{N}$ and $k\geq 2$.

PROPOSITION 6.5. *Let $p,q,r,s\in[1,\infty]$ with $r\leq p$. Let $M=\{m_{jk}\}_{j,k=1}^\infty$ be such that $S\mapsto M*S$ is a bounded mapping $\mathcal{L}(\ell_p,\ell_q)\to\mathcal{L}(\ell_r,\ell_s)$. Then $S\mapsto\widetilde{M}*S$ is also a bounded mapping $\mathcal{L}(\ell_p,\ell_q)\to\mathcal{L}(\ell_r,\ell_s)$, with*

$$\|M\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\mathcal{L}(\ell_r,\ell_s))} = \|\widetilde{M}\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\mathcal{L}(\ell_r,\ell_s))}.$$

In particular, if $M\in\mathcal{M}(p,q)$ then $\widetilde{M}\in\mathcal{M}(p,q)$ with $\|M\|_{(p,q)}=\|\widetilde{M}\|_{(p,q)}$.

Proof. The proof is almost identical to that of [5, Theorem 4.1], and the condition $r \leq p$ is used to ensure that $|x_1|^p + |x_2|^p \leq (|x_1|^r + |x_2|^r)^{p/r}$ for all $x_1, x_2 \in \mathbb{C}$ (with the obvious modification for $p = \infty$ or $r = \infty$). ∎

REMARK 6.6. By considering the transpose M' of a matrix M, and using the fact that $M' : \mathcal{L}(\ell_{q^*}, \ell_{p^*}) \to \mathcal{L}(\ell_{s^*}, \ell_{r^*})$ with

$$\|M'\|_{\mathcal{L}(\mathcal{L}(\ell_{q^*},\ell_{p^*}),\, \mathcal{L}(\ell_{s^*},\ell_{r^*}))} = \|M\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\, \mathcal{L}(\ell_r,\ell_s))},$$

Proposition 6.5 implies that the $\mathcal{L}(\mathcal{L}(\ell_p, \ell_q),\, \mathcal{L}(\ell_r, \ell_s))$-norm of a matrix is invariant under row repetitions if $s \leq q$. Moreover, since $\|S\|_{\mathcal{L}(\ell_p,\ell_q)}$ is invariant under permutations of the columns and rows of $S \in \mathcal{L}(\ell_p, \ell_q)$, rearrangements of the rows and columns of $M \in \mathcal{L}(\mathcal{L}(\ell_p, \ell_q),\, \mathcal{L}(\ell_r, \ell_s))$ leave $\|M\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\, \mathcal{L}(\ell_r,\ell_s))}$ invariant.

The following lemma is crucial to our main results.

LEMMA 6.7. *Let $p, q, r, s \in [1, \infty]$ with $r \leq p$ and $s \leq q$. Let $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ and $\mu = \{\mu_k\}_{k=1}^{\infty}$ be sequences of real numbers. Then*

$$\|T_{\triangle,n}^{\lambda,\mu}\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\, \mathcal{L}(\ell_r,\ell_s))} \leq \|\mathcal{T}_{\triangle,n}\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\, \mathcal{L}(\ell_r,\ell_s))}$$

for all $n \in \mathbb{N}$.

Proof. Note that $T_{\triangle,n}^{\lambda,\mu}(S) = M * S$ for all $S \in \mathcal{L}(\ell_p, \ell_q)$, where $M = \{m_{jk}\}_{j,k=1}^{\infty}$ is such that $m_{jk} = 1$ if $1 \leq j, k \leq n$ and $\mu_k \leq \lambda_j$, and $m_{jk} = 0$ otherwise. We show that

$$\|M\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\, \mathcal{L}(\ell_r,\ell_s))} \leq \|\mathcal{T}_{\triangle,n}\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\, \mathcal{L}(\ell_r,\ell_s))}.$$

Assume that M is non-zero, otherwise the statement is trivial. By Remark 6.6, rearrangement of the rows and columns of M does not change its norm. Hence we may assume that $\{\lambda_j\}_{j=1}^{n}$ and $\{\mu_k\}_{k=1}^{n}$ are decreasing. Now M has the property that if $m_{jk} = 1$ then $m_{il} = 1$ for all $i \leq j$ and $k \leq l \leq m_2$. By Proposition 6.5 and Remark 6.6, we may omit repeated rows and columns of M, and doing this repeatedly reduces M to $\mathcal{T}_{\triangle,N}$ for some $1 \leq N \leq n$. Noting that $\|\mathcal{T}_{\triangle,N}\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\, \mathcal{L}(\ell_r,\ell_s))} \leq \|\mathcal{T}_{\triangle,n}\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q),\, \mathcal{L}(\ell_r,\ell_s))}$ concludes the proof. ∎

6.2. The case $p < q$. We now combine the theory from the previous sections to deduce our main results.

THEOREM 6.8. *Let $p, q \in [1, \infty]$ with $p < q$, and let $f(t) := |t|$ for $t \in \mathbb{R}$. Then there exists a constant $C \geq 0$ such that the following holds (where ℓ_∞ should be replaced by c_0). Let $A \in \mathcal{L}_{\mathrm{d}}(\ell_p)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_q)$ have real spectrum. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell_p,\ell_q)} \leq CK_AK_B\|BS - SA\|_{\mathcal{L}(\ell_p,\ell_q)} \tag{6.2}$$

for all $S \in \mathcal{L}(\ell_p, \ell_q)$.

Proof. Simply combine Propositions 5.4 and 5.6, Lemma 6.7 and Proposition 6.3(i), using $\|\mathcal{T}_{\triangle,n}\|_{(p,q)} \le \|\mathcal{T}_{\triangle}\|_{(p,q)}$ for all $n \in \mathbb{N}$. ∎

We can deduce a stronger statement if $p = 1$ or $q = \infty$ in Theorem 6.8. For $f : \mathbb{C} \to \mathbb{C}$ a Lipschitz function, write

$$\|f\|_{\mathrm{Lip}} := \sup_{\substack{z_1,z_2\in\mathbb{C}\\ z_1\neq z_2}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|}.$$

Moreover, let $\varphi_f : \mathbb{C}^2 \to \mathbb{C}$ be given by

$$\varphi_f(\lambda_1, \lambda_2) := \begin{cases} \dfrac{f(\lambda_1) - f(\lambda_2)}{\lambda_1 - \lambda_2} & \text{if } \lambda_1 \neq \lambda_2, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

THEOREM 6.9. *Let $p, q \in [1, \infty]$ with $p = 1$ or $q = \infty$ (with ℓ_∞ replaced by c_0). Let $A \in \mathcal{L}_{\mathrm{d}}(\ell_p)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_q)$, and let $f : \mathbb{C} \to \mathbb{C}$ be Lipschitz. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell_p,\ell_q)} \le K_A K_B \|f\|_{\mathrm{Lip}} \|BS - SA\|_{\mathcal{L}(\ell_p,\ell_q)} \tag{6.4}$$

for all $S \in \mathcal{L}(\ell_p, \ell_q)$. In particular, for $p = q = 1$,

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell_1)} \le K_A K_B \|f\|_{\mathrm{Lip}} \|B - A\|_{\mathcal{L}(\ell_1)},$$

and for $p = q = \infty$,

$$\|f(B) - f(A)\|_{\mathcal{L}(c_0)} \le K_A K_B \|f\|_{\mathrm{Lip}} \|B - A\|_{\mathcal{L}(c_0)}.$$

Proof. Let $\lambda = \{\lambda_j\}_{j=1}^{\infty}$ and $\mu = \{\mu_k\}_{k=1}^{\infty}$ be sequences such that $A \in \mathcal{L}_{\mathrm{d}}(\ell_p, \lambda, U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_q, \mu, V)$ for certain $U \in \mathcal{L}(\ell_p)$ and $V \in \mathcal{L}(\ell_q)$. By Proposition 5.4, it suffices to prove that

$$\sup_{n\in\mathbb{N}} \|T^{\lambda,\mu}_{\varphi_f,n}\|_{\mathcal{L}(\mathcal{L}(\ell_p,\ell_q))} \le \|f\|_{\mathrm{Lip}}.$$

Fix $n \in \mathbb{N}$ and note that $T^{\lambda,\mu}_{\varphi_f,n}(S) = M * S$ for all $S \in \mathcal{L}(\ell_p, \ell_q)$, where $M = \{m_{jk}\}_{j,k=1}^{\infty}$ is the matrix given by $m_{jk} = \varphi_f(\lambda_j, \mu_k)$ for $1 \le j, k \le n$, and $m_{jk} = 0$ otherwise. Then

$$\sup_{j,k\in\mathbb{N}} |m_{jk}| \le \sup_{j,k\in\mathbb{N}} |\varphi_f(\lambda_j, \mu_k)| \le \|f\|_{\mathrm{Lip}}.$$

Remark 6.4 now concludes the proof. ∎

REMARK 6.10. Theorem 6.9 shows that each Lipschitz function f is operator Lipschitz on ℓ_1 and c_0, in the following sense. For fixed $M \ge 1$ and $f : \mathbb{C} \to \mathbb{C}$ Lipschitz, there exists a constant $C \ge 0$ such that

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell_1)} \le C\|B - A\|_{\mathcal{L}(\ell_1)}$$

for all $A, B \in \mathcal{L}_{\mathrm{d}}(\ell_1)$ such that $K_A, K_B \le M$, and C is independent of A and B. Similarly for c_0.

For $p < q$ an analogous statement holds. By considering $A, f(A) \in \mathcal{L}(\ell_p)$ and $B, f(B) \in \mathcal{L}(\ell_q)$ as operators from ℓ_p to ℓ_q, and by letting S be the inclusion mapping $\ell_p \hookrightarrow \ell_q$ in Theorems 6.8 and 6.9, one can suggestively write

$$\|f(B) - f(A)\|_{\mathcal{L}(\ell_p,\ell_q)} \leq C\|B - A\|_{\mathcal{L}(\ell_p,\ell_q)}$$

for all $A \in \mathcal{L}_{\mathrm{d}}(\ell_p)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_q)$ with $K_A, K_B \leq M$. Here f is the absolute value function for general $p < q$ in $[1,\infty]$ and any Lipschitz function if $p = 1$ or $q = \infty$.

This remark also applies to Corollaries 6.11 and 6.12 below.

In the case of Theorems 6.8 and 6.9 where $p = 2$ or $q = 2$, we can apply our results to compact normal operators. By the spectral theorem, any compact normal operator $A \in \mathcal{L}(\ell_2)$ has an orthonormal basis of eigenvectors, and therefore $A \in \mathcal{L}_{\mathrm{d}}(\ell_2, \lambda, U)$ for some sequence λ of real numbers and an isometry $U \in \mathcal{L}(\ell_2)$. Thus Theorems 6.8 and 6.9 yield the following corollaries.

Corollary 6.11. *Let $p \in (1,2)$. Then there exists a constant $C \geq 0$ such that the following holds. Let $A \in \mathcal{L}_{\mathrm{d}}(\ell_p)$ have real spectrum and let $B \in \mathcal{L}(\ell_2)$ be compact and self-adjoint. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell_p,\ell_2)} \leq CK_A\|BS - SA\|_{\mathcal{L}(\ell_p,\ell_2)}$$

for all $S \in \mathcal{L}(\ell_p, \ell_2)$, where $f(t) := |t|$ for $t \in \mathbb{R}$. Moreover,

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell_1,\ell_2)} \leq K_A\|f\|_{\mathrm{Lip}}\|BS - SA\|_{\mathcal{L}(\ell_1,\ell_2)}$$

for each $A \in \mathcal{L}_{\mathrm{d}}(\ell_1)$ and $S \in \mathcal{L}(\ell_1, \ell_2)$, each compact normal $B \in \mathcal{L}(\ell_2)$ and each Lipschitz function $f : \mathbb{C} \to \mathbb{C}$.

Corollary 6.12. *Let $q \in (2,\infty)$. Then there exists a constant $C \geq 0$ such that the following holds. Let $A \in \mathcal{L}(\ell_2)$ be compact and self-adjoint, and let $B \in \mathcal{L}_{\mathrm{d}}(\ell_q)$ have real spectrum. Then*

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell_2,\ell_q)} \leq CK_B\|BS - SA\|_{\mathcal{L}(\ell_2,\ell_q)}$$

for all $S \in \mathcal{L}(\ell_2, \ell_q)$, where $f(t) := |t|$ for $t \in \mathbb{R}$. Moreover,

$$\|f(B)S - Sf(A)\|_{\mathcal{L}(\ell_2,\mathrm{c}_0)} \leq K_B\|f\|_{\mathrm{Lip}}\|BS - SA\|_{\mathcal{L}(\ell_2,\mathrm{c}_0)}$$

for each compact normal $A \in \mathcal{L}_{\mathrm{d}}(\ell_2)$, each $B \in \mathcal{L}_{\mathrm{d}}(\mathrm{c}_0)$ and $S \in \mathcal{L}(\ell_2, \mathrm{c}_0)$, and each Lipschitz function $f : \mathbb{C} \to \mathbb{C}$.

6.3. The case $p \geq q$. We now examine the absolute value function f on $\mathcal{L}(\ell_p, \ell_q)$ for $p \geq q$, and obtain the following result.

Proposition 6.13. *Let $p, q \in (1,\infty]$ with $p \geq q$. Then for each $s < q$ there exists a constant $C \geq 0$ such that the following holds (where ℓ_∞ should be replaced by c_0). Let $A \in \mathcal{L}_{\mathrm{d}}(\ell_p, \lambda, U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_q, \mu, V)$ have real* *spectrum. Then*

$$\|f(B)S-Sf(A)\|_{\mathcal{L}(\ell_p,\ell_q)}\leq C\|U\|_{\mathcal{L}(\ell_p)}\|V^{-1}\|_{\mathcal{L}(\ell_q)}\|V(BS-SA)U^{-1}\|_{\mathcal{L}(\ell_p,\ell_s)}$$

for all $S\in\mathcal{L}(\ell_p,\ell_q)$ *such that* $V(BS-SA)U^{-1}\in\mathcal{L}(\ell_p,\ell_s)$.

In particular, if $p=q$ *and* $V(B-A)U^{-1}\in\mathcal{L}(\ell_p,\ell_s)$, *then*

$$\|f(B)-f(A)\|_{\mathcal{L}(\ell_p)}\leq C\|U\|_{\mathcal{L}(\ell_p)}\|V^{-1}\|_{\mathcal{L}(\ell_p)}\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell_p,\ell_s)}.$$

Proof. Let $R:=V(BS-SA)U^{-1}$. With notation as in Lemma 5.3,

$$\|f(B)S_n-S_nf(A)\|_{\mathcal{L}(\ell_p,\ell_q)}\leq\|U\|_{\mathcal{L}(\ell_p)}\|V^{-1}\|_{\mathcal{L}(\ell_q)}\|T^{\lambda,\mu}_{\varphi_f,n}(R)\|_{\mathcal{L}(\ell_p,\ell_q)}$$

for each $n\in\mathbb{N}$. Proposition 5.6, Lemma 6.7 (with $p=r$ and with q and s interchanged) and Proposition 6.3(iii) (with q and s interchanged) yield a constant $C'\geq 0$ such that

$$\|T^{\lambda,\mu}_{\varphi_f,n}(R)\|_{\mathcal{L}(\ell_p,\ell_q)}\leq C'(\|R\|_{\mathcal{L}(\ell_p,\ell_q)}+\|R\|_{\mathcal{L}(\ell_p,\ell_s)}).$$

Since $\mathcal{L}(\ell_p,\ell_s)\hookrightarrow\mathcal{L}(\ell_p,\ell_q)$ contractively,

$$\|f(B)S_n-S_nf(A)\|_{\mathcal{L}(\ell_p,\ell_q)}\leq C\|U\|_{\mathcal{L}(\ell_p)}\|V^{-1}\|_{\mathcal{L}(\ell_q)}\|V(BS-SA)U^{-1}\|_{\mathcal{L}(\ell_p,\ell_s)}$$

for all $n\in\mathbb{N}$, where $C=2C'$. Finally, as in the proof of Proposition 5.4, one lets n tend to infinity to conclude the proof. ∎

In the same way, appealing to the second part of Proposition 6.3(iii), one deduces the following result.

PROPOSITION 6.14. *Let* $p,q\in[1,\infty)$ *with* $p\geq q$. *Then for each* $r>p$ *there exists a constant* $C\geq 0$ *such that the following holds* (*where* ℓ_∞ *should be replaced by* c_0). *Let* $A\in\mathcal{L}_{\mathrm{d}}(\ell_p,\lambda,U)$ *and* $B\in\mathcal{L}_{\mathrm{d}}(\ell_q,\mu,V)$ *have real spectrum. Then*

$$\|f(B)S-Sf(A)\|_{\mathcal{L}(\ell_p,\ell_q)}\\ \leq C\|U\|_{\mathcal{L}(\ell_p)}\|V^{-1}\|_{\mathcal{L}(\ell_q)}\|V(BS-SA)U^{-1}\|_{\mathcal{L}(\ell_r,\ell_q)}$$

for all $S\in\mathcal{L}(\ell_p,\ell_q)$ *such that* $V(BS-SA)U^{-1}\in\mathcal{L}(\ell_r,\ell_q)$.

In particular, if $p=q$ *and* $V(B-A)U^{-1}\in\mathcal{L}(\ell_r,\ell_q)$, *then*

$$\|f(B)-f(A)\|_{\mathcal{L}(\ell_p)}\leq C\|U\|_{\mathcal{L}(\ell_p)}\|V^{-1}\|_{\mathcal{L}(\ell_p)}\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell_r,\ell_q)}.$$

We single out the case where $p=q=2$. Here we write $f(A)=|A|$ for a normal operator $A\in\mathcal{L}(\ell_2)$, since then $f(A)$ is equal to $|A|:=\sqrt{A^*A}$. Note also that the following result applies in particular to compact self-adjoint operators. For simplicity of presentation we only consider $\epsilon\in(0,1]$.

COROLLARY 6.15. *For each* $\epsilon\in(0,1]$ *there exists a constant* $C\geq 0$ *such that the following holds. Let* $A\in\mathcal{L}_{\mathrm{d}}(\ell_2,\lambda,U)$ *and* $B\in\mathcal{L}_{\mathrm{d}}(\ell_2,\mu,V)$ *be self-adjoint, with* U *and* V *unitaries, and let* $S\in\mathcal{L}(\ell_2)$. *If* $V(BS-SA)U^{-1}\in\mathcal{L}(\ell_2,\ell_{2-\epsilon})$, *then*

$$\big\||B|S-S|A|\big\|_{\mathcal{L}(\ell_2)}\leq C\|V(BS-SA)U^{-1}\|_{\mathcal{L}(\ell_2,\ell_{2-\epsilon})}$$

and if $V(BS-SA)U^{-1} \in \mathcal{L}(\ell_{2+\epsilon}, \ell_2)$ *then*

$$\big\||B|S - S|A|\big\|_{\mathcal{L}(\ell_2)} \le C\|V(BS-SA)U^{-1}\|_{\mathcal{L}(\ell_{2+\epsilon},\ell_2)}.$$

In particular, if $V(B-A)U^{-1} \in \mathcal{L}(\ell_2, \ell_{2-\epsilon})$, *then*

$$\big\||B| - |A|\big\|_{\mathcal{L}(\ell_2)} \le C\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell_2,\ell_{2-\epsilon})}$$

and if $V(B-A)U^{-1} \in \mathcal{L}(\ell_{2+\epsilon}, \ell_2)$, *then*

$$\big\||B| - |A|\big\|_{\mathcal{L}(\ell_2)} \le C\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell_{2+\epsilon},\ell_2)}.$$

Remark 6.16. Let $\mathcal{J}$ be the class of all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(t) = at + b + \int_{-\infty}^{t} (t-s)\,\mathrm{d}\mu(s) \tag{6.5}$$

for all $t \in \mathbb{R}$, where $a, b \in \mathbb{R}$ and μ is a signed measure of compact support. This class is introduced by Davies [11, p. 156], and he states that $f : \mathbb{R} \to \mathbb{R}$ satisfies (6.5) for a positive μ if and only if f is convex and linear for large $|t|$. The results in this section for f the absolute value function can be extended to all $f \in \mathcal{J}$, in the same way as in [11, Theorem 17]. We leave the details to the reader.

7. Lipschitz estimates on the ideal of p-summing operators. Let H be a separable infinite-dimensional Hilbert space. It was shown in [2] that a matrix $M = \{m_{jk}\}_{j,k=1}^{\infty}$ is a Schur multiplier on the Hilbert–Schmidt class $\mathcal{S}_2 \subset \mathcal{L}(H)$ if and only if $\sup_{j,k} |m_{jk}| < \infty$. By [25], $\mathcal{S}_2$ coincides with the Banach ideal $\Pi_p(H)$ of all p-summing operators (see the definition below) for all $p \in [1, \infty)$. Hence a matrix $M = \{m_{jk}\}_{j,k=1}^{\infty}$ is a Schur multiplier on $\Pi_p(H)$ if and only if $\sup_{j,k} |m_{jk}| < \infty$. In Corollary 7.2 below we show that the same statement is true for the Banach ideal $\Pi_p(\ell_{p^*}, \ell_p)$ in $\mathcal{L}(\ell_{p^*}, \ell_p)$, for $p \in [1, \infty)$. As a corollary we obtain operator Lipschitz estimates on $\Pi_p(\ell_{p^*}, \ell_p)$ for each Lipschitz function f on $\mathbb{C}$.

Let X and Y be Banach spaces and $1 \le p < \infty$. An operator $S : X \to Y$ is *p-absolutely summing* if there exists a constant C such that for each $n \in \mathbb{N}$ and each collection $\{x_j\}_{j=1}^{n} \subseteq X$,

$$\Big(\sum_{j=1}^{n} \|S(x_j)\|_Y^p\Big)^{1/p} \le C \sup_{\|x^*\|_{X^*} \le 1} \Big(\sum_{j=1}^{n} |\langle x^*, x_j\rangle|^p\Big)^{1/p}. \tag{7.1}$$

The smallest such constant is denoted by π_p, and $\Pi_p(X, Y)$ is the space of p-absolutely summing operators from X to Y. We let $\Pi_p(X) := \Pi_p(X, X)$. By [15, Propositions 2.3, 2.4 and 2.6], $(\Pi_p(X, Y), \pi_p(\cdot))$ is a Banach ideal in $\mathcal{L}(X, Y)$.

Below we consider p-absolutely summing operators from ℓ_{p^*} to ℓ_p. We first present the following result.

LEMMA 7.1. *Let* $p \in [1,\infty)$ *and* $S = \{s_{jk}\}_{j,k=1}^{\infty}$. *Then* $S \in \Pi_p(\ell_{p^*}, \ell_p)$ (*with* ℓ_∞ *replaced by* c_0) *if and only if*

$$c_p := \Big(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|s_{jk}|^p\Big)^{1/p} < \infty.$$

In this case, $\pi_p(S) = c_p$.

Proof. It follows from [15, Example 2.11] that, if $c_p < \infty$ for $p \in (1,\infty)$, then $S \in \Pi_p(\ell_{p^*}, \ell_p)$ with $\pi_p(S) \le c_p$. An inspection of the proof of [15, Example 2.11] shows that this statement in fact also holds for $p = 1$. For the converse, let $n \in \mathbb{N}$ and let $x_j := e_j \in \ell_{p^*}$ for $1 \le j \le n$. By (7.1) (with $X = \ell_{p^*}$ and $Y = \ell_p$),

$$\Big(\sum_{k=1}^{n}\sum_{j=1}^{\infty}|s_{jk}|^p\Big)^{1/p} \le \pi_p(S).$$

Letting n tend to infinity concludes the proof. ∎

For the following corollary of Lemma 7.1, recall that a matrix M is said to be a *Schur multiplier* on a subspace $\mathcal{I} \subseteq \mathcal{L}(\ell_p, \ell_q)$ if $S \mapsto M * S$ is a bounded map on $\mathcal{I}$. Recall also the definition of the standard triangular truncation $\mathcal{T}_\triangle$ from (6.1).

COROLLARY 7.2. *Let* $p \in [1,\infty)$ *and let* $M = \{m_{jk}\}_{j,k=1}^{\infty}$ *be a matrix. Then* M *is a Schur multiplier on* $\Pi_p(\ell_{p^*}, \ell_p)$ (*with* ℓ_∞ *replaced by* c_0) *if and only if* $\sup_{j,k\in\mathbb{N}} |m_{jk}| < \infty$. *In this case,*

$$\|M\|_{\mathcal{L}(\Pi_p(\ell_{p^*},\ell_p))} = \sup_{j,k\in\mathbb{N}} |m_{jk}|.$$

In particular, $\mathcal{T}_\triangle \in \mathcal{L}(\Pi_p(\ell_{p^*}, \ell_p))$ *with* $\|\mathcal{T}_\triangle\|_{\mathcal{L}(\Pi_p(\ell_{p^*},\ell_p))} = 1$.

Observe that $\mathcal{T}_\triangle \notin \mathcal{L}(\mathcal{L}(\ell_{p^*}, \ell_p))$ if $p^* \ge p$, by Proposition 6.3(ii). Nevertheless, $\mathcal{T}_\triangle$ is bounded on the ideal $\Pi_p(\ell_{p^*}, \ell_p) \subset \mathcal{L}(\ell_{p^*}, \ell_p)$ for all $p \in [1,\infty)$.

We now prove our main result concerning commutator estimates on $\Pi_p(\ell_{p^*}, \ell_p)$.

THEOREM 7.3. *Let* $p \in [1,\infty)$, $A \in \mathcal{L}_{\mathrm{d}}(\ell_{p^*})$ (*with* ℓ_∞ *replaced by* c_0) *and* $B \in \mathcal{L}_{\mathrm{d}}(\ell_p)$. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be Lipschitz. Then*

$$\pi_p(f(B)S - Sf(A)) \le K_A K_B \|f\|_{\mathrm{Lip}} \pi_p(BS - SA) \tag{7.2}$$

for all $S \in \mathcal{L}(\ell_{p^*}, \ell_p)$ *such that* $BS - SA \in \Pi_p(\ell_{p^*}, \ell_p)$.

Proof. Let $A \in \mathcal{L}_{\mathrm{d}}(\ell_{p^*}, \lambda, U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_p, \mu, V)$ for certain $\lambda = \{\lambda_j\}_{j=1}^{\infty}$, $\mu = \{\mu_k\}_{k=1}^{\infty}$, $U \in \mathcal{L}(\ell_{p^*})$ and $V \in \mathcal{L}(\ell_p)$. If $\{S_m\}_{m=1}^{\infty} \subseteq \Pi_p(\ell_{p^*}, \ell_p)$ is a π_p-bounded sequence which SOT-converges to $S \in \mathcal{L}(X,Y)$, then $S \in \Pi_p(\ell_{p^*}, \ell_p)$ with $\pi_p(S) \le \limsup_{m\to\infty} \pi_p(S_m)$, by (7.1). Hence, by Remark 5.5, it suffices to prove that $\sup_{n\in\mathbb{N}} \|T^{\lambda,\mu}_{\varphi_f,n}\|_{\mathcal{L}(\Pi_p(\ell_{p^*},\ell_p))} \le \|f\|_{\mathrm{Lip}}$, where

φ_f is as in (6.3). This is done as in the proof of Theorem 6.9, using Corollary 7.2 instead of Remark 6.4. ■

PROBLEM 7.4. Let $p, q, r \in [1, \infty]$ be such that $q \geq 2$, $1/q - 1/p < 1/2$ and $1/r = 1/p - 1/q + 1/2$. Then the Schatten class $\mathcal{S}_r$ coincides with the Banach ideal $\Pi_{p,q}(\ell_2)$ of (p, q)-summing operators on ℓ_2 (for the definition of (p, q)-summing operators see [15]). Hence, by [16], the standard triangular truncation is bounded on $\Pi_{p,q}(\ell_2)$. For which $r_1, r_2 \in [1, \infty]$ is the standard triangular truncation bounded on $\Pi_{p,q}(\ell_{r_1}, \ell_{r_2})$? For which ideals $\mathcal{I}$ in $\mathcal{L}(\ell_p, \ell_q)$ is the standard triangular truncation bounded on $\mathcal{I}$? As shown in Theorem 7.3, answers to these questions are linked to commutator estimates for diagonalizable operators.

8. Matrix estimates. In this section we apply the theory developed in Sections 4–6 to finite-dimensional spaces. We leave the derivation of the dimension-independent estimates that follow from the results in Section 7 to the reader.

8.1. Finite-dimensional spaces. For $n \in \mathbb{N}$ let X be an n-dimensional Banach space with basis $\{e_1, \ldots, e_n\} \subset X$ and the corresponding basis projections $\mathcal{P}_k \in \mathcal{L}(X)$ for $1 \leq k \leq n$. Recall that an operator $A \in \mathcal{L}(X)$ is *diagonalizable* if there exists $U \in \mathcal{L}(X)$ invertible such that

$$UAU^{-1} = \sum_{k=1}^{n} \lambda_k \mathcal{P}_k$$

for some $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. We then write $A \in \mathcal{L}_{\mathrm{d}}(X, \lambda, U)$. Recall also the definition of spectral and scalar type operators from Section 3.1.

LEMMA 8.1. *Let $A \in \mathcal{L}(X)$. Then A is a spectral operator, and A is a scalar type operator if and only if A is diagonalizable. If $A \in \mathcal{L}_{\mathrm{d}}(X, \lambda, U)$ then the spectral measure E of A is given by $E(\sigma) = 0$ if $\sigma \cap \mathrm{sp}(A) = \emptyset$, $\sigma \in \mathfrak{B}$ and $E(\{\lambda\}) = \sum_{\lambda_j = \lambda} U^{-1} \mathcal{P}_j U$ for $\lambda \in \mathrm{sp}(A)$.*

Proof. It was already remarked in Section 5 that any diagonalizable operator is a scalar type operator, with spectral measure as specified. By [17, Theorem XV.4.5], an operator $T \in \mathcal{L}(Y)$ on an arbitrary Banach space Y is a spectral operator if and only if $T = S + N$ for a commuting scalar type operator $S \in \mathcal{L}(Y)$ and a generalized nilpotent operator $N \in \mathcal{L}(Y)$, and this decomposition is unique. The Jordan decomposition for matrices yields a commuting diagonalizable S and a nilpotent N such that $A = S + N$, hence A is a spectral operator. If A is a scalar type operator, then the Jordan decomposition yields a commuting diagonalizable S and a nilpotent N such that $A = S + N$. By the uniqueness of such a decomposition [17, Theorem XV.4.5], $N = 0$ and $A = S$ is diagonalizable. ■

Let Y be a finite-dimensional Banach space. As in [6], a norm $\|\cdot\|$ on $\mathcal{L}(X,Y)$ is said to be *symmetric* if

- $\|RST\| \leq \|R\|_{\mathcal{L}(Y)}\|S\|\|T\|_{\mathcal{L}(X)}$ for all $R \in \mathcal{L}(Y)$, $S \in \mathcal{L}(X,Y)$ and $T \in \mathcal{L}(X)$;
- $\|x^* \otimes y\| = \|x^*\|_{X^*}\|y\|_Y$ for all $x^* \in X^*$ and $y \in Y$.

Clearly $(\mathcal{L}(X,Y),\|\cdot\|)$ is a Banach ideal in $\mathcal{L}(X,Y)$ in the sense of Section 3.2 if and only if $\|\cdot\|$ is symmetric. Note that, for $A \in \mathcal{L}_{\mathrm{d}}(X,\lambda,U)$,

$$f(A) = U^{-1}\Big(\sum_{k=1}^{n} f(\lambda_k)\mathcal{P}_k\Big)U,$$

as in (5.3). Let $\mathfrak{A} := \mathfrak{A}(\mathbb{C}\times\mathbb{C})$ be as in Section 3.3, and for $f \in \mathcal{B}(\mathbb{C})$ and $(\lambda_1,\lambda_2) \in \mathbb{C}^2$ with $\lambda_1 \neq \lambda_2$ let $\varphi_f(\lambda_1,\lambda_2) := \frac{f(\lambda_2)-f(\lambda_1)}{\lambda_2-\lambda_1}$, as in (4.5). The following corollary of Theorem 4.6 extends results for self-adjoint operators and unitarily invariant norms (see e.g. [20] and [6, Chapter X]) to diagonalizable operators and symmetric norms. Note that a symmetric norm on $\mathcal{L}(X,Y)$ need not be unitarily invariant.

Corollary 8.2. *Let $f \in \mathcal{B}(\mathbb{C})$ be such that φ_f extends to an element of $\mathfrak{A}$. Let X and Y be finite-dimensional Banach spaces, let $\|\cdot\|$ be a symmetric norm on $\mathcal{L}(X,Y)$, and let $A \in \mathcal{L}_{\mathrm{d}}(X)$ and $B \in \mathcal{L}_{\mathrm{d}}(Y)$. Then*

$$\|f(B)S - Sf(A)\| \leq 16\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}}\|BS - SA\|$$

for all $S \in \mathcal{L}(X,Y)$. In particular, if $X = Y$,

$$\|f(B) - f(A)\| \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}}\|B - A\|. \tag{8.1}$$

Corollaries 4.8 and 4.9 yield dimension-independent estimates for $f \in \dot{\mathrm{B}}^1_{\infty,1}(\mathbb{R})$.

Remark 8.3. Let $\sigma_1,\sigma_2 \subset \mathbb{C}$ be finite sets. Then any $\varphi : \sigma_1\times\sigma_2 \to \mathbb{C}$ belongs to $\mathfrak{A}(\sigma_1\times\sigma_2)$. Indeed, one can find a representation as in (3.8) by letting Ω be finite and solving a system of linear equations. Therefore Theorem 4.6 yields an estimate

$$\|f(B)S - Sf(A)\| \leq 16\,\nu(A)\nu(B)\|\varphi_f\|_{\mathfrak{A}(\mathrm{sp}(A)\times\mathrm{sp}(B))}\|BS - SA\|$$

as in (4.6) for all $f \in \mathcal{B}(\mathbb{C})$. This might lead one to think that the assumption in Theorem 8.2 that φ_f extends to an element of $\mathfrak{A}$ is not really necessary. However, for general $f \in \mathcal{B}(\mathbb{C})$ the norm $\|\varphi_f\|_{\mathfrak{A}(\mathrm{sp}(A)\times\mathrm{sp}(B))}$ may blow up as the number of points in $\mathrm{sp}(A)$ and $\mathrm{sp}(B)$ grows to infinity. Indeed, for $f \in \mathcal{B}(\mathbb{C})$ the absolute value function and $\|\cdot\|$ the operator norm, a dimension-independent estimate as in (8.1) does not hold for all self-adjoint operators on all finite-dimensional Hilbert spaces [6, (X.25)]. Hence φ_f does not extend to an element of $\mathfrak{A}$, and one cannot expect to obtain Theorem 8.2 for all bounded Borel functions on $\mathbb{C}$.

8.2. The absolute value function. We now apply our results for the absolute value function to finite-dimensional spaces. First note that Lemma 5.3 and Proposition 5.6 relate commutator estimates for general symmetric norms to triangular truncation operators.

For $n \in \mathbb{N}$ and $p \in [1,\infty]$ let ℓ_p^n denote $\mathbb{C}^n$ with the p-norm

$$\|(x_1,\ldots,x_n)\|_p := \Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p} \qquad ((x_1,\ldots,x_n)\in\mathbb{C}^n),$$

with the obvious modification for $p=\infty$. Applying Theorem 6.8 with $S : \mathbb{C}^n \to \mathbb{C}^n$ the identity operator yields the following. This result is false for $p=q=2$.

Corollary 8.4. *Let $p,q \in [1,\infty]$ with $p<q$ and let $f(t) := |t|$ for $t \in \mathbb{R}$. Then there exists a constant $C \geq 0$ such that the following holds. Let $n \in \mathbb{N}$ and let $A \in \mathcal{L}_{\mathrm{d}}(\ell_p^n)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_q^n)$ have real spectrum. Then*

$$\|f(B)-f(A)\|_{\mathcal{L}(\ell_p^n,\ell_q^n)} \leq CK_AK_B\|B-A\|_{\mathcal{L}(\ell_p^n,\ell_q^n)}.$$

Theorem 6.9 shows that, for $p=1$ or $q=\infty$, Corollary 8.4 extends to all Lipschitz functions $f:\mathbb{C}\to\mathbb{C}$, with $C=\|f\|_{\mathrm{Lip}}$. Corollaries 6.11 and 6.12 imply that for $p=2$ or $q=2$ and A or B self-adjoint, the estimate in Corollary 8.4 simplifies.

From the results for $p \geq q$ we obtain for instance the following.

Corollary 8.5. *For each $\epsilon \in (0,1]$ there exists a constant $C\geq 0$ such that the following holds. Let $n\in\mathbb{N}$ and let $A \in \mathcal{L}_{\mathrm{d}}(\ell_2^n,\lambda,U)$ and $B \in \mathcal{L}_{\mathrm{d}}(\ell_2^n,\mu,V)$ be self-adjoint operators, with U and V unitaries. Then*

$$\begin{aligned}&\big\||B|-|A|\big\|_{\mathcal{L}(\ell_2^n)}\\ &\qquad\leq C\min(\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell_2^n,\ell_{2-\epsilon}^n)},\|V(B-A)U^{-1}\|_{\mathcal{L}(\ell_{2+\epsilon}^n,\ell_2^n)}).\end{aligned}$$

Finally note that, under the assumptions of Corollary 8.5,

$$\big\||B|-|A|\big\|_{\mathcal{L}(\ell_2^n)} \leq C\|B-A\|_{\mathcal{L}(\ell_2^n)}\min(\|V\|_{\mathcal{L}(\ell_2^n,\ell_{2-\epsilon}^n)},\|U^{-1}\|_{\mathcal{L}(\ell_{2+\epsilon}^n,\ell_2^n)}).$$

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Jan Rozendaal
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
E-mail: janrozendaalmath@gmail.com

Fedor Sukochev, Anna Tomskova
School of Mathematics & Statistics
University of NSW
Kensington, NSW 2052, Australia
E-mail: f.sukochev@unsw.edu.au
a.tomskova@unsw.edu.au