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# Reproducing kernels, Engliš algebras and some applications

by

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**Abstract.** We introduce the notion of Engliš algebras, defined in terms of reproducing kernels and Berezin symbols. Such algebras were apparently first investigated by Engliš (1995). Here we give some new results on Engliš $C^*$-algebras on abstract reproducing kernel Hilbert spaces and some applications to various questions of operator theory. In particular, we give applications to Riccati operator equations, zero Toeplitz products, and the existence of invariant subspaces for some operators.

**1. Introduction and preliminaries.** In this paper we study some $C^*$-operator algebras defined in terms of reproducing kernels. Such algebras were investigated by Engliš in his seminal paper [E2], and for this reason we call them Engliš algebras. We give new applications of reproducing kernels and Engliš algebras to some problems in operator theory. Since we will mainly deal with an abstract Reproducing Kernel Hilbert Space (briefly, RKHS) $\mathcal{H} = \mathcal{H}(\Omega)$, and also more concrete RKHS $H^2 = H^2(\mathbb{D})$ (Hardy space) and $L_a^2 = L_a^2(\mathbb{D})$ (Bergman space), let us start with necessary definitions, notation and preliminaries about these spaces.

By a *reproducing kernel Hilbert space* we mean a Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set $\Omega$ such that evaluation at any point of $\Omega$ is a continuous linear functional on $\mathcal{H}$. The classical Riesz representation theorem ensures that the Hilbert function space $\mathcal{H}$ has a reproducing kernel, that is, a function $k_{\mathcal{H}} : \Omega \times \Omega \to \mathbb{C}$ with $\langle f, k_{\mathcal{H},\lambda}\rangle = f(\lambda)$ for all $f \in \mathcal{H}$ and $\lambda \in \Omega$, where $k_{\mathcal{H},\lambda} = k_{\mathcal{H}}(\cdot, \lambda) \in \mathcal{H}$. Let $\widehat{k}_{\mathcal{H},\lambda} := k_{\mathcal{H},\lambda}/\|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}$ be the normalized reproducing kernel of $\mathcal{H}$. For any bounded linear operator $T$ on $\mathcal{H}$, its *Berezin symbol* $\widetilde{T}$ is defined by (see [NR])

$$\widetilde{T}(\lambda) := \langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle, \qquad \lambda \in \Omega.$$

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The Berezin symbol of an operator provides important information about the operator. Namely, it is well known that on the most familiar RKHS, including the Hardy, Bergman and Fock Hilbert spaces, the Berezin symbol uniquely determines the operator (i.e., $\widetilde{T_1}(\lambda) = \widetilde{T_2}(\lambda)$ for all $\lambda \in \Omega$ implies $T_1 = T_2$); see, for instance, [E2] and [Zhu].

The RKHS is said to be *standard* (see Nordgren and Rosenthal [NR]) if the underlying set $\Omega$ is a subset of a topological space and its boundary $\partial\Omega$ is nonempty and has the property that $\{\widehat{k}_{\mathcal{H},\lambda_n}\}$ converges weakly to 0 whenever $\{\lambda_n\}$ is a sequence in $\Omega$ that converges to a point in $\partial\Omega$. The common RKHSs of analytic functions, including the Hardy, Bergman and Fock Hilbert spaces, are standard in this sense.

For a compact operator $K$ on the standard RKHS $\mathcal{H}$, it is clear that

$$\lim_{n\to\infty} \widetilde{K}(\lambda_n) = 0$$

whenever $\{\lambda_n\}$ converges to a point of $\partial\Omega$ (since compact operators send weakly convergent sequences to strongly convergent ones). In this sense, the Berezin symbol of a compact operator on a standard RKHS vanishes on the boundary. In [NR], Nordgren and Rosenthal characterized compact operators $K$ on RKHS in terms of the Berezin symbols of $U^{-1}KU$, where $U : \mathcal{H} \to \mathcal{H}$ is unitary:

Lemma 1.1. *An operator $K$ on a standard RKHS is compact if and only if all the Berezin symbols $\widetilde{U^{-1}KU}$ for all unitary $U$ vanish on the boundary.*

Let $\mathbb{T} = \partial\mathbb{D} = \{e^{it} : 0 \le t < 2\pi\}$ be the unit circle, $dt$ be the arc-length measure on $\mathbb{T}$, the Lebesgue measure on $\mathbb{T}$, and $m(e^{it}) = dt/(2\pi)$ the normalized Lebesgue measure.

Recall that the *Hardy space* $H^2 = H^2(\mathbb{D})$ is the Hilbert space of analytic functions $f(z) = \sum_{n\ge 0} a_n z^n$ defined in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $\sum_{n\ge 0} |a_n|^2 < \infty$. It is convenient to establish a natural embedding of $H^2$ in $L^2 = L^2(\mathbb{T}, m)$, by associating to each $f \in H^2$ its boundary value $(bf)(\zeta) := \lim_{r\to 1^-} f(r\zeta)$, which exists by the Fatou theorem (see Hoffman [Hof]) for $m$-almost all $\zeta \in \mathbb{T}$. Then we have $H^2 = \{f \in L^2 : \widehat{f}(n) = 0,\ n < 0\}$, where $\widehat{f}(n) := \int_{\mathbb{T}} \overline{\zeta}^n f(\zeta)\, dm(\zeta)$ is the $n$th Fourier coefficient of $f$. In what follows, we will not distinguish the functions $f$ and $bf$.

For $\varphi \in L^1 = L^1(\mathbb{T})$, we will denote by $\widetilde{\varphi}$ its harmonic extension into $\mathbb{D}$ defined by

$$\widetilde{\varphi}(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\tau}) \frac{1 - r^2}{1 + r^2 - 2r\cos(t - \tau)}\, d\tau, \qquad re^{it} \in \mathbb{D}.$$

For $\varphi \in L^2(\mathbb{T})$, this harmonic function is analytic if and only if $\varphi \in H^2$.

Let $H^\infty = H^\infty(\mathbb{D})$ denote the Banach algebra of bounded analytic functions on $\mathbb{D}$, normed by $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. Let $\overline{H^\infty} := \{\overline{g} : g \in H^\infty\}$ be the space of bounded co-analytic functions on $\mathbb{D}$, and $L^\infty = L^\infty(\mathbb{T})$ the space of all (equivalence classes of) essentially bounded functions on $\mathbb{T}$, normed by the essential supremum norm (relative to Lebesgue measure $m = m(\zeta)$ on $\mathbb{T}$). For $g \in L^\infty(\mathbb{T})$, $\|g\|_\infty$ stands for the essential supremum of $|g|$. It is well known (see Hoffman [Hof]) that to every $f \in H^\infty$ there corresponds a function $bf \in L^\infty(\mathbb{T})$, defined almost everywhere by $(bf)(e^{it}) = \lim_{r\to 1^-} f(re^{it})$. The equality $\|f\|_\infty = \|bf\|_\infty$ holds.

For any *inner function* $\theta \in H^\infty$ (i.e., $|\theta(z)| \le 1$ on $\mathbb{D}$ and $|\theta(\xi)| = 1$ for almost all $\xi \in \mathbb{T}$) the corresponding model space $K_\theta$ is defined as $K_\theta := H^2 \ominus \theta H^2$.

If $\varphi \in L^\infty(\mathbb{T})$, then the *Toeplitz operator* $T_\varphi$ acting on $H^2$ is defined by $T_\varphi f = P_+(\varphi f)$, where $P_+ : L^2 \to H^2$ is the Riesz orthoprojector. It is clear from the definition of that $T_\varphi^* = T_{\overline{\varphi}}$. It is also well known (see Halmos [Hal]) that for $\psi$ and $\varphi$ in $L^\infty(\mathbb{T})$, $T_\psi T_\varphi$ is a Toeplitz operator on $H^2$ if and only if either $\psi \in \overline{H^\infty}$ or $\varphi \in H^\infty$. In both cases, $T_\psi T_\varphi = T_{\psi\varphi}$. Therefore $T_f T_g T_h = T_{fgh}$ for any $f \in \overline{H^\infty}$, $g \in L^\infty$, $h \in H^\infty$.

The following is well-known (see Engliš [E2]).

LEMMA 1.2. *If $\varphi \in L^\infty$, then $\widetilde{T_\varphi} = \widetilde{\varphi}$, i.e., the Berezin symbol of the Toeplitz operator $T_\varphi$ is equal to the harmonic extension $\widetilde{\varphi}$ of its symbol $\varphi$.*

The present paper is organized as follows. In Section 2, we introduce Engliš algebras and give some of their properties. We give a criterion for a truncated Toeplitz operator to belong to the Engliš algebra $\mathcal{A}^0_{K_\theta}$.

In Section 3, we investigate the solvability of the Riccati operator equation

$$XAX + XB - CX - D = 0 \tag{R}$$

on the Engliš algebra $\mathcal{A}_\mathcal{H}$ of operators on a RKHS $\mathcal{H}(\Omega)$ over some set $\Omega$. More exactly, we prove a necessary condition for the solvability of (R) in terms of Berezin symbols. It turns out that if (R) is solvable on an appropriate subset of $\mathcal{A}_\mathcal{H}$, then the solution of (R) is unique and it is represented in terms of Berezin symbols of the coefficient operators $A$ and $D$.

Section 4 studies zero operator products, in particular, zero Toeplitz products. It is proved that if $\varphi_1, \dots, \varphi_n \in L^\infty(\mathbb{T})$ and $H_1, \dots, H_n$ are operators from the Engliš algebra $\mathcal{F}_{H^2}$ such that $(T_{\varphi_1} + H_1) \dots (T_{\varphi_n} + H_n) = 0$, then $\varphi_1 \dots \varphi_n = 0$. This generalizes the known Douglas lemma. In particular, we obtain a new proof of the Brown–Halmos theorem. Similar results are proved for zero Toeplitz products on the Bergman space $L^2_a$. In particular, we partially solve a conjecture due to Čučković [C] (see Corollary 4.7).

In Section 5, we study the maximal Berezin set $\widetilde{W}_0(T_\varphi)$ of the Toeplitz operator $T_\varphi$ on $H^2$, and prove that $\widetilde{W}_0(T_\varphi) = \{\|\varphi\|_\infty\}$. In general, we show that for any $T \in \mathcal{A}_{\mathcal{H}}$ (Engliš's algebra), we have $\widetilde{W}_0(T) \subset \mathbb{T}_{\|T\|}$, where $\mathbb{T}_{\|T\|}$ denotes the circle with center 0 and radius $\|T\|$.

In Section 6, we are mainly interested in the following question: if $A, B, C$ belong to the Engliš algebra $\mathcal{A}^0_{\mathcal{H}}$, then under which conditions is $AB - C$ a compact operator on $\mathcal{H}$? Here we characterize the compactness of $AB - C$ in terms of Berezin symbols.

Section 7 proves an Axler–Chang–Sarason–Volberg type theorem for the semi-commutator $[T_u, T_v) := T_{uv} - T_u T_v$ of the Toeplitz operators $T_u$ and $T_v$ on the Bergman space $L^2_a$.

In Section 8, we discuss the extended eigenvalues and extended eigenvectors of operators on some Engliš algebras. In Section 9, we give some sufficient conditions for the existence of a nontrivial invariant subspace in $H^2$ in terms of reproducing kernels and Duhamel operators.

**2. Engliš algebras and some of their properties.** In this section, we will discuss some operator algebras defined in terms of reproducing kernels and study their properties. These algebras were introduced and investigated mainly by Engliš [E2].

Following [E2], let $\mathcal{T}$ be the $C^*$-algebra generated by the set $\{T_\Phi : \Phi \in L^\infty(\mathbb{T})\}$ of Toeplitz operators on $H^2$. The following famous result is due to Douglas [Dou].

THEOREM 2.1 (Douglas). *There is a $C^*$-homomorphism $\sigma : \mathcal{T} \to L^\infty(\mathbb{T})$ which satisfies $\sigma(T_\Phi) = \Phi$ for all $\Phi \in L^\infty(\mathbb{T})$. The kernel of $\sigma$ coincides with the commutator ideal of $\mathcal{T}$, i.e. the ideal in $\mathcal{T}$ generated by all commutators*

$$[R, S] = RS - SR, \qquad R, S \in \mathcal{T}.$$

*$\sigma$ is sometimes called the symbol map.*

Engliš [E2] gives an alternative method for proving results akin to the Douglas theorem. The symbol of an operator $T$ is then obtained as the non-tangential boundary value of the Berezin symbol $\widetilde{T}$ of $T$. As is shown in [E2], this method also works for operator algebras larger than the Toeplitz algebra. The same technique is also applicable to the Bergman space $L^2_a = L^2_a(\mathbb{D})$, which is the closed subspace of $L^2(\mathbb{D})$ consisting of analytic functions on $\mathbb{D}$.

Here we give other applications of reproducing kernels and Berezin symbol techniques; in particular, we will investigate other properties and applications of Engliš algebras. Before stating our results, let us give some necessary definitions and notation.

Recall that if $f$ is a bounded continuous function on $\mathbb{D}$, then we say that $f \to 0$ *radially* if

$$\lim_{r\to 1^-} f(re^{it}) = 0$$

for all $t \in [0, 2\pi)$, except possibly for a set of measure zero.

We define the following *Engliš algebras* on the RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ (in the case of $\mathcal{H} = H^2(\mathbb{D})$ and $\mathcal{H} = L_a^2(\mathbb{D})$, these algebras were defined in terms of radial, nontangential and uniform limits and investigated by Engliš [E2]):

$$\mathcal{F}_{\mathcal{H}} := \{T \in \mathcal{B}(\mathcal{H}) : \|T\widehat{k}_{\mathcal{H},\lambda}\|, \|T^*\widehat{k}_{\mathcal{H},\lambda}\| \to 0 \text{ as } \lambda \to \partial\Omega\};$$
$$\mathcal{A}_1 := \{T_\Phi + T : \Phi \in L^\infty(\mathbb{T}),\ T \in \mathcal{F}_{H^2}\};$$
$$\mathcal{A}_{\mathcal{H}} := \{T \in \mathcal{B}(\mathcal{H}) : \|T\widehat{k}_{\mathcal{H},\lambda}\|^2 - |\widetilde{T}(\lambda)|^2, \|T^*\widehat{k}_{\mathcal{H},\lambda}\|^2 - |\widetilde{T}(\lambda)|^2 \to 0 \text{ as } \lambda \to \partial\Omega\},$$

where $\widehat{k}_{\mathcal{H},\lambda} := k_{\mathcal{H},\lambda}/\|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}$ is the normalized reproducing kernel of the space $\mathcal{H}(\Omega)$; and

$$\mathcal{A}_{\mathcal{H}}^0 := \{T \in \mathcal{B}(\mathcal{H}) : \|T\widehat{k}_{\mathcal{H},\lambda}\|^2 - |\widetilde{T}(\lambda)|^2 \to 0 \text{ as } \lambda \to \partial\Omega\}.$$

Engliš [E2] proved that $\mathcal{A}_{H^2}$ is a $C^*$-algebra. Also, it follows from the following result of Engliš [E2] (see also Karaev [K2]) that $T_\Phi \in \mathcal{A}_{H^2}$ for any $\Phi \in L^\infty(\mathbb{T})$.

LEMMA 2.2. *Let $\Phi \in L^\infty(\mathbb{T})$, and denote, as before, by $\widetilde{\Phi}$ its harmonic extension* (*by the Poisson formula*) *into $\mathbb{D}$. Then $T_\Phi\widehat{k}_{H^2,\lambda} - \widetilde{\Phi}(\lambda)\widehat{k}_{H^2,\lambda} \to 0$ radially, i.e.,*

$$\lim_{r\to 1} \|T_\Phi\widehat{k}_{H^2,re^{it}} - \widetilde{\Phi}(re^{it})\widehat{k}_{H^2,re^{it}}\| = 0$$

*for almost all $t \in [0, 2\pi)$.*

Let $\theta$ be an inner function and $\varphi \in L^\infty(\mathbb{T})$. The truncated Toeplitz operator $A_\varphi^\theta$ is defined on the model space

$$K_\theta := H^2 \ominus \theta H^2$$

by the formula $A_\varphi^\theta f := P_\theta T_\varphi | K_\theta$, where $P_\theta : H^2 \to K_\theta$ is the orthogonal projection and $T_\varphi : H^2 \to H^2$ is the Toeplitz operator of symbol $\varphi$.

Note that in the case of $\mathcal{H} = H^2(\mathbb{D})$, $\mathcal{H} = L_a^2(\mathbb{D})$ and $\mathcal{H} = K_\theta$, the limits in the definition of the corresponding Engliš algebras are assumed, as mentioned above, to be radial, or nontangential.

So, the next result proves membership of truncated Toeplitz operators in the Engliš algebra

$$\mathcal{A}_{K_\theta}^0 = \{T \in \mathcal{B}(K_\theta) : \|T\widehat{k}_\lambda^\theta\|^2 - |\widetilde{T}(\lambda)|^2 \to 0 \text{ radially}\}.$$

THEOREM 2.3. *Let $A_\varphi^\theta$ be a truncated Toeplitz operator on the space $K_\theta$. Then $A_\varphi^\theta \in \mathcal{A}_{K_\theta}^0$ if and only if*

$$\lim_{\substack{\lambda\to\partial\mathbb{D}\\ \textit{radially}}} \frac{1}{1-|\theta(\lambda)|^2}\big[\|T_{\varphi(1-\overline{\theta(\lambda)}\theta)}\widehat{k}_\lambda\|^2 - \|T_{\overline{\varphi\theta-\theta(\lambda)}}\widehat{k}_\lambda\|^2$$

$$- 2\operatorname{Re}\big(\widetilde{A_\varphi^\theta}(\lambda)\big)\big(\varphi|1-\overline{\theta(\lambda)}\theta|^2\big)^{\sim}(\lambda) + (1-|\theta(\lambda)|^2)|\widetilde{A_\varphi^\theta}(\lambda)|^2\big] = 0.$$

*Proof.* First, note that the normalized reproducing kernel of the RKHS $K_\theta$ is

$$\widehat{k}_\lambda^\theta(z) := \left(\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\right)^{1/2}\frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z}, \qquad \lambda\in\mathbb{D},$$

and hence the Berezin symbol of the operator $A_\varphi^\theta$ is

$$\widetilde{A_\varphi^\theta}(\lambda) := \langle A_\varphi^\theta \widehat{k}_\lambda^\theta, \widehat{k}_\lambda^\theta\rangle, \qquad \lambda\in\mathbb{D}.$$

Since $A_\varphi^\theta\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}\widehat{k}_\lambda^\theta \perp \widetilde{A_\varphi^\theta}\widehat{k}_\lambda^\theta$, we have

$$\|A_\varphi^\theta\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta\|^2 = \|A_\varphi^\theta\widehat{k}_\lambda^\theta\|^2 - |\widetilde{A_\varphi^\theta}(\lambda)|^2, \qquad \lambda\in\mathbb{D}.$$

Hence $A_\varphi^\theta\in\mathcal{A}^0_{K_\theta}$ if and only if $\|A_\varphi^\theta\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta\|\to 0$ as $\lambda\to\partial\mathbb{D}$ radially. Therefore we will examine the boundary behavior of $\|A_\varphi^\theta\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta\|$.

Indeed, since $P_\theta = I - T_\theta T_{\overline{\theta}}$, we have

$$\begin{aligned}
&\|A_\varphi^\theta\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta\|^2\\
&\quad= \|P_\theta T_\varphi\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta\|^2 = \|(I-T_\theta T_{\overline{\theta}})T_\varphi\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta\|^2\\
&\quad= \langle T_\varphi\widehat{k}_\lambda^\theta - T_\theta T_{\overline{\theta}}T_\varphi\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta, T_\varphi\widehat{k}_\lambda^\theta - T_\theta T_{\overline{\theta}}T_\varphi\widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta\rangle\\
&\quad= \|T_\varphi\widehat{k}_\lambda^\theta\|^2 - \langle T_\varphi\widehat{k}_\lambda^\theta, T_\theta T_{\overline{\theta}}T_\varphi\widehat{k}_\lambda^\theta\rangle - \widetilde{A_\varphi^\theta}(\lambda)\langle T_\varphi\widehat{k}_\lambda^\theta,\widehat{k}_\lambda^\theta\rangle\\
&\qquad - \langle T_\theta T_{\overline{\theta}}T_\varphi\widehat{k}_\lambda^\theta, T_\varphi\widehat{k}_\lambda^\theta\rangle + \|T_\theta T_{\overline{\theta}}T_\varphi\widehat{k}_\lambda^\theta\|^2\\
&\qquad - \widetilde{A_\varphi^\theta}(\lambda)\langle\widehat{k}_\lambda^\theta, T_\varphi\widehat{k}_\lambda^\theta\rangle + |\widetilde{A_\varphi^\theta}(\lambda)|^2\\
&\quad= \|T_\varphi\widehat{k}_\lambda^\theta\|^2 - 2\operatorname{Re}[\widetilde{A_\varphi^\theta}(\lambda)\langle T_\varphi\widehat{k}_\lambda^\theta,\widehat{k}_\lambda^\theta\rangle] - 2\|T_{\overline{\theta}\varphi}\widehat{k}_\lambda^\theta\|^2\\
&\qquad + \|T_\theta T_{\overline{\theta}\varphi}\widehat{k}_\lambda^\theta\|^2 + |\widetilde{A_\varphi^\theta}(\lambda)|^2\\
&\quad= \|T_\varphi\widehat{k}_\lambda^\theta\|^2 - 2\operatorname{Re}[\widetilde{A_\varphi^\theta}(\lambda)\langle T_\varphi\widehat{k}_\lambda^\theta,\widehat{k}_\lambda^\theta\rangle] - \|T_{\overline{\theta}\varphi}\widehat{k}_\lambda^\theta\|^2 + |\widetilde{A_\varphi^\theta}(\lambda)|^2\\
&\quad= \frac{1}{1-|\theta(\lambda)|^2}\|T_{\varphi(1-\overline{\theta(\lambda)}\theta)}\widehat{k}_\lambda\|^2\\
&\qquad - 2\operatorname{Re}[\widetilde{A_\varphi^\theta}(\lambda)\langle T_{\varphi|1-\overline{\theta(\lambda)}\theta|^2}\widehat{k}_\lambda,\widehat{k}_\lambda\rangle]\frac{1}{1-|\theta(\lambda)|^2}\\
&\qquad + |\widetilde{A_\varphi^\theta}(\lambda)|^2 - \|T_{\overline{\varphi}(1-\overline{\theta(\lambda)}\theta)}\widehat{k}_\lambda\|^2\frac{1}{1-|\theta(\lambda)|^2},
\end{aligned}$$

and hence

$$\|A_\varphi^\theta \widehat{k}_\lambda^\theta - \widetilde{A_\varphi^\theta}(\lambda)\widehat{k}_\lambda^\theta\|^2 = \frac{1}{1-|\theta(\lambda)|^2}\big[\|T_{\varphi(1-\overline{\theta(\lambda)}\theta)}\widehat{k}_\lambda\|^2 - \|T_{\varphi\overline{\theta-\theta(\lambda)}}\widehat{k}_\lambda\|^2 - 2\,\mathrm{Re}\big(\widetilde{A_\varphi^\theta}(\lambda)(\varphi|1-\overline{\theta(\lambda)}\theta|^2)^\sim(\lambda)\big) + (1-|\theta(\lambda)|^2)|\widetilde{A_\varphi^\theta}(\lambda)|^2\big]$$

for all $\lambda \in \mathbb{D}$, which shows that $A_\varphi^\theta$ is in the Engliš algebra $\mathcal{A}_{K_\theta}^0$ if and only if

$$\lim_{\substack{\lambda\to\partial\mathbb{D}\\ \text{radially}}} \frac{1}{1-|\theta(\lambda)|^2}\big[\|T_{\varphi(1-\overline{\theta(\lambda)}\theta)}\widehat{k}_\lambda\|^2 - \|T_{\varphi\overline{\theta-\theta(\lambda)}}\widehat{k}_\lambda\|^2 - 2\,\mathrm{Re}(\widetilde{A_\varphi^\theta}(\lambda))\big(\varphi|1-\overline{\theta(\lambda)}\theta|^2\big)^\sim(\lambda) + (1-|\theta(\lambda)|^2)|\widetilde{A_\varphi^\theta}(\lambda)|^2\big] = 0,$$

as desired. ∎

Let $\mathcal{A}_{L_a^2}$ denote the $C^*$-algebra of operators in $\mathcal{B}(L_a^2)$ (see [E2]) defined by

$$\mathcal{A}_{L_a^2} := \{T \in \mathcal{B}(L_a^2) : \|T\widehat{k}_{L_a^2,\lambda}\|^2 - |\widetilde{T}(\lambda)|^2 \to 0 \text{ radially, and similarly for } T^*\}.$$

The main properties of the Engliš algebras are collected in the following proposition, which can be found in [E2].

Proposition 2.4. *We have:*

(i) $\mathcal{F}_\mathcal{H}$ *is a* $C^*$*-algebra.*
(ii) $T_\Phi \in \mathcal{F}_\mathcal{H} \Rightarrow \Phi = 0$.
(iii) $\mathcal{A}_1$ *is a* $C^*$*-algebra.*
(iv) *For any* $T \in \mathcal{A}_1$, *there exists* $\sigma(T) \in L^\infty(\mathbb{T})$ *such that*

$$\widetilde{T}(\lambda) \to \sigma(T) \quad \text{radially as } \lambda \to \partial\mathbb{D}.$$

(v) *For* $\Phi \in L^\infty(\mathbb{T})$, $T_\Phi \in \mathcal{A}_{H^2}$.
(vi) $\mathcal{A}_\mathcal{H}$ *is a* $C^*$*-algebra.*
(vii) $\mathcal{A}_\mathcal{H}^0$ *is an algebra.*

Proposition 2.5. *Let* $T \in \mathcal{B}(L_a^2)$. *Then* $T \in \mathcal{A}_{L_a^2}$ *if and only if*

$$\|(T-\widetilde{T}(\lambda)I)\widehat{k}_{L_a^2,\lambda}\|_2,\ \|(T^*-\widetilde{T^*}(\lambda)I)\widehat{k}_{L_a^2,\lambda}\|_2 \to 0 \text{ radially as } \lambda \to \partial\mathbb{D}.$$

*Proof.* Since

$$\|T\widehat{k}_{L_a^2,\lambda}\|_2^2 = \|T\widehat{k}_{L_a^2,\lambda} - \widetilde{T}(\lambda)\widehat{k}_{L_a^2,\lambda}\|_2^2 + |\widetilde{T}(\lambda)|^2,$$
$$\|T^*\widehat{k}_{L_a^2,\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{L_a^2,\lambda}\|_2^2 = \|T^*\widehat{k}_{L_a^2,\lambda}\|_2^2 - |\widetilde{T}(\lambda)|^2,$$

the desired result follows. ∎

**3. On the solvability of a Riccati operator equation on Engliš algebras.** In this section, we investigate the solvability of the Riccati operator equation

$$XAX + XB - CX - D = 0 \tag{1}$$

on the Engliš algebra $\mathcal{A}_{\mathcal{H}}$ of operators on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ over some set $\Omega$. More exactly, we give a necessary condition for the solvability of (1) in terms of Berezin symbols. In turns out that if (1) is solvable on an appropriate subset of $\mathcal{A}_{\mathcal{H}}$, then the solution of (1) is unique and it is represented in terms of Berezin symbols of the coefficient operators $A$ and $D$. To state our results, denote, for any $A \in \mathcal{B}(\mathcal{H})$,

$$\widetilde{A}^{\mathrm{bv}}(\xi) := \lim_{\lambda \to \xi \in \partial\Omega} \widetilde{A}(\lambda)$$

if it exists and if $\widetilde{A}^{\mathrm{bv}} \in L^\infty(\partial\Omega)$. We also set

$$\mathcal{A}_{\mathcal{H}}^u := \{A \in \mathcal{A}_{\mathcal{H}} : \widetilde{A}^{\mathrm{bv}} \text{ is in } L^\infty(\partial\Omega) \text{ and uniquely determines } A\}.$$

The main result of this section essentially improves and generalizes [K2, Theorem 2.1]:

THEOREM 3.1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS over a set $\Omega$, and let $A, B, C, D \in \mathcal{B}(\mathcal{H})$ with $\widetilde{A}^{\mathrm{bv}}, \widetilde{B}^{\mathrm{bv}}, \widetilde{C}^{\mathrm{bv}}, \widetilde{D}^{\mathrm{bv}} \in L^\infty(\partial\Omega)$, $B \in \mathcal{A}_{\mathcal{H}}^0$ and $\widetilde{B}^{\mathrm{bv}} \neq \widetilde{C}^{\mathrm{bv}}$ almost everywhere on $\partial\Omega$.*

*If equation* (1) *is solvable on $\mathcal{A}_{\mathcal{H}}^u$, i.e., if there exists $T \in \mathcal{A}_{\mathcal{H}}^u$ satisfying* (1), *then:*

(a) $\widetilde{D}^{\mathrm{bv}}/(B-C)^{\sim\mathrm{bv}} \in L^\infty(\partial\Omega)$ *if* $\widetilde{A}^{\mathrm{bv}} = 0$;
(b) $((B-C)^{\sim\mathrm{bv}})^2 + 4\widetilde{A}^{\mathrm{bv}}\widetilde{D}^{\mathrm{bv}} = 0$ *if* $\widetilde{A}^{\mathrm{bv}} \neq 0$ *almost everywhere on* $\partial\Omega$.

*Proof.* Suppose that there exists $T \in \mathcal{A}_{\mathcal{H}}^u$ such that

$$TAT + TB - CT - D = 0,$$

where $A, B, C, D$ satisfy the hypotheses of the theorem. Then, for all $\lambda \in \Omega$,

$$\begin{aligned}
0 &= \langle (TAT + TB - CT - D)\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle \\
&= \langle TAT\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle + \langle TB\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle - \langle CT\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle - \langle D\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle \\
&= \langle AT\widehat{k}_{\mathcal{H},\lambda}, T^*\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\rangle + \widetilde{T}(\lambda)\langle AT\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle \\
&\quad + \langle T(B\widehat{k}_{\mathcal{H},\lambda} - \widetilde{B}(\lambda)\widehat{k}_{\mathcal{H},\lambda}), \widehat{k}_{\mathcal{H},\lambda}\rangle + \widetilde{B}(\lambda)\widetilde{T}(\lambda) \\
&\quad - \langle C(T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}), \widehat{k}_{\mathcal{H},\lambda}\rangle - \widetilde{C}(\lambda)\widetilde{T}(\lambda) - \widetilde{D}(\lambda) \\
&= \langle AT\widehat{k}_{\mathcal{H},\lambda}, T^*\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\rangle \\
&\quad + \widetilde{T}(\lambda)[\langle A(T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}), \widehat{k}_{\mathcal{H},\lambda}\rangle + \widetilde{T}(\lambda)\widetilde{A}(\lambda)]
\end{aligned}$$

$$
\begin{aligned}
&+ \langle B\widehat{k}_{\mathcal{H},\lambda} - \widetilde{B}(\lambda)\widehat{k}_{\mathcal{H},\lambda}, T^*\widehat{k}_{\mathcal{H},\lambda}\rangle + \widetilde{B}(\lambda)\widetilde{T}(\lambda)\\
&- \langle T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}, C^*\widehat{k}_{\mathcal{H},\lambda}\rangle - \widetilde{C}(\lambda)\widetilde{T}(\lambda) - \widetilde{D}(\lambda)\\
= {}& [\widetilde{A}(\lambda)(\widetilde{T}(\lambda))^2 + (\widetilde{B}(\lambda) - \widetilde{C}(\lambda))\widetilde{T}(\lambda) - \widetilde{D}(\lambda)]\\
&+ \langle AT\widehat{k}_{\mathcal{H},\lambda}, T^*\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\rangle + \widetilde{T}(\lambda)\langle T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}, A^*\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&+ \langle B\widehat{k}_{\mathcal{H},\lambda} - \widetilde{B}(\lambda)\widehat{k}_{\mathcal{H},\lambda}, T^*\widehat{k}_{\mathcal{H},\lambda}\rangle,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&|\widetilde{A}(\lambda)(\widetilde{T}(\lambda))^2 + (B-C)^{\sim}(\lambda)\widetilde{T}(\lambda) - \widetilde{D}(\lambda)|\\
&\qquad \le \|AT\|\,\|T^*\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}} + \|T\|\,\|A\|\,\|T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}}\\
&\qquad\qquad + \|T\|\,\|B\widehat{k}_{\mathcal{H},\lambda} - \widetilde{B}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}} \to 0 \quad \text{as } \lambda \to \partial\Omega,
\end{aligned}
$$

because $T \in \mathcal{A}_{\mathcal{H}}$ and $B \in \mathcal{A}^0_{\mathcal{H}}$. This implies that

$$
\widetilde{A}^{\mathrm{bv}}(\xi)(\widetilde{T}^{\mathrm{bv}}(\xi))^2 + (B-C)^{\sim\mathrm{bv}}(\xi)\widetilde{T}^{\mathrm{bv}}(\xi) - \widetilde{D}^{\mathrm{bv}}(\xi) = 0 \tag{2}
$$

for a.a. $\xi \in \partial\Omega$.

(a) If $\widetilde{A}^{\mathrm{bv}}(\xi) = 0$ almost everywhere on $\partial\Omega$ then obviously equation (2) has a unique solution

$$
\widetilde{T}^{\mathrm{bv}} = \widetilde{D}^{\mathrm{bv}}/(B-C)^{\sim\mathrm{bv}},
$$

and since $\widetilde{T}^{\mathrm{bv}} \in L^\infty(\partial\Omega)$, we have $\widetilde{D}^{\mathrm{bv}}/(B-C)^{\sim\mathrm{bv}} \in L^\infty(\partial\Omega)$.

(b) If $\widetilde{A}^{\mathrm{bv}}(\xi) \neq 0$ almost everywhere on $\partial\Omega$ then since $\widetilde{T}^{\mathrm{bv}}$ uniquely determines the operator $T$, we deduce that the quadratic equation (2) has only one solution. This means that

$$
((B-C)^{\sim\mathrm{bv}})^2 + 4\widetilde{A}^{\mathrm{bv}}\widetilde{D}^{\mathrm{bv}} = 0,
$$

as desired. ∎

It is clear from the proof that, in fact, Theorem 3.1 proves that if the Riccati equation (1) is solvable on $\mathcal{A}^u_{\mathcal{H}}$, and $T \in \mathcal{A}^u_{\mathcal{H}}$ is a solution, then the solution $T$ is unique and

$$
\widetilde{T}^{\mathrm{bv}} = \begin{cases} \dfrac{\widetilde{D}^{\mathrm{bv}}}{(B-C)^{\sim\mathrm{bv}}} & \text{if } \widetilde{A}^{\mathrm{bv}} = 0,\\[2ex] \dfrac{(C-B)^{\sim\mathrm{bv}}}{2\widetilde{A}^{\mathrm{bv}}} & \text{if } \widetilde{A}^{\mathrm{bv}} \neq 0 \text{ almost everywhere on } \partial\Omega. \end{cases}
$$

Note that the solvability of the Riccati operator equation in concrete operator classes is an important problem of operator theory and its applications. For instance, the existence of a nontrivial solution of (1) for fixed $A \in \mathcal{B}(H)$, $B = D = 0$ and $C = A$ on the set $\mathcal{P}_H$ of all orthogonal projectors $P \in \mathcal{B}(H)$ is equivalent to the solution of the famous Invariant Subspace Problem in the infinite-dimensional separable Hilbert space $H$; here nontriviality means that $0 \neq X \neq I_H$, where $I_H$ is the identity operator on $H$.

## 4. Zero operator products

**4.1. Zero operator products on $H^2$.** In the following proposition we investigate the so-called zero products for $\mathcal{A}_1$-class operators which implies, in particular, the well-known Douglas lemma [Dou] for zero Toeplitz products on the Hardy space $H^2$.

PROPOSITION 4.1. *Let $\varphi_1,\dots,\varphi_n \in L^\infty(\mathbb{T})$ and $H_1,\dots,H_n \in \mathcal{F}_{H^2}$, where $n$ is an integer, be such that*

$$(T_{\varphi_1}+H_1)\dots(T_{\varphi_n}+H_n)=0.$$

*Then $\varphi_1\dots\varphi_n=0$.*

*Proof.* Since $(T_{\varphi_1}+H_1)\dots(T_{\varphi_n}+H_n)=0$, we have

$$\begin{aligned}0&=[(T_{\varphi_1}+H_1)\dots(T_{\varphi_n}+H_n)]^{\sim}(\lambda)\\&=\langle (T_{\varphi_1}+H_1)\dots(T_{\varphi_n}+H_n)\widehat{k}_{H^2,\lambda},\widehat{k}_{H^2,\lambda}\rangle\end{aligned}\tag{3}$$

for all $\lambda\in\mathbb{D}$. It is easy to see that

$$(T_{\varphi_1}+H_1)\dots(T_{\varphi_n}+H_n)=T_{\varphi_1}\dots T_{\varphi_n}+X_{\substack{\varphi_1,\dots,\varphi_n\\ H_1,\dots H_n}}+H_1\dots H_n\tag{4}$$

where $X_{\substack{\varphi_1,\dots,\varphi_n\\ H_1,\dots H_n}}$ is a sum of products of operators

$$T_{\varphi_1},\dots,T_{\varphi_n},H_1,\dots,H_n.$$

Since every such product has a factor equal to at least one of $H_1,\dots,H_n$, and $H_i\in\mathcal{F}_{H^2}$ for any $i\in\{1,\dots,n\}$, by Lemma 2.2 it is not difficult to deduce that

$$|\widetilde{X}_{\substack{\varphi_1,\dots,\varphi_n\\ H_1,\dots H_n}}(\lambda)|\to 0\quad\text{radially,}$$

that is,

$$\lim_{r\to 1^-}|\widetilde{X}_{\substack{\varphi_1,\dots,\varphi_n\\ H_1,\dots H_n}}(re^{it})|=0\tag{5}$$

for almost all $t\in[0,2\pi)$. From (3) and (4) we obtain

$$0=(T_{\varphi_1}\dots T_{\varphi_n})^{\sim}(\lambda)+\widetilde{X}_{\substack{\varphi_1,\dots,\varphi_n\\ H_1,\dots H_n}}(\lambda)+(H_1\dots H_n)^{\sim}(\lambda)\tag{6}$$

for all $\lambda\in\mathbb{D}$. On the other hand,

$$\begin{aligned}&(T_{\varphi_1}\dots T_{\varphi_n})^{\sim}(\lambda)\\&\quad=\langle T_{\varphi_1}\dots T_{\varphi_n}\widehat{k}_{H^2,\lambda},\widehat{k}_{H^2,\lambda}\rangle=\langle T_{\varphi_2}\dots T_{\varphi_n}\widehat{k}_{H^2,\lambda},T_{\overline{\varphi}_1}\widehat{k}_{H^2,\lambda}\rangle\\&\quad=\langle T_{\varphi_2}\dots T_{\varphi_n}\widehat{k}_{H^2,\lambda},(T_{\overline{\varphi}_1}\widehat{k}_{H^2,\lambda}-\widetilde{\overline{\varphi}}_1(\lambda)\widehat{k}_{H^2,\lambda})+\widetilde{\overline{\varphi}}_1(\lambda)\widehat{k}_{H^2,\lambda}\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle T_{\varphi_2} \dots T_{\varphi_n} \widehat{k}_{H^2,\lambda}, T_{\overline{\varphi}_1} \widehat{k}_{H^2,\lambda} - \widetilde{\overline{\varphi}}_1(\lambda) \widehat{k}_{H^2,\lambda} \rangle \\
&\quad + \widetilde{\overline{\varphi}}_1 \langle T_{\varphi_2} \dots T_{\varphi_n} \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \rangle \\
&= A_{1,\lambda} + \widetilde{\varphi}_1(\lambda) [\langle T_{\varphi_3} \dots T_{\varphi_n} \widehat{k}_{H^2,\lambda}, T_{\overline{\varphi}_2} \widehat{k}_{H^2,\lambda} - \widetilde{\overline{\varphi}}_2(\lambda) \widehat{k}_{H^2,\lambda} \rangle \\
&\qquad\qquad\qquad\qquad\qquad + \widetilde{\overline{\varphi}}_2(\lambda) \langle T_{\varphi_3} \dots T_{\varphi_n} \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \rangle] \\
&= A_{1,\lambda} + \widetilde{\varphi_1}(\lambda) A_{2,\lambda} \\
&\quad + \widetilde{\varphi_1}(\lambda) \widetilde{\varphi_2}(\lambda) [A_{3,\lambda} + \widetilde{\varphi_3}(\lambda) \langle T_{\varphi_4} \dots T_{\varphi_n} \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \rangle] \\
&= A_{1,\lambda} + \widetilde{\varphi_1}(\lambda) A_{2,\lambda} + \widetilde{\varphi_1}(\lambda) \widetilde{\varphi_2}(\lambda) A_{3,\lambda} \\
&\quad + \widetilde{\varphi_1}(\lambda) \widetilde{\varphi_2}(\lambda) \widetilde{\varphi_3}(\lambda) \langle T_{\varphi_4} \dots T_{\varphi_n} \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \rangle \\
&= \cdots \\
&= A_{1,\lambda} + \widetilde{\varphi_1}(\lambda) A_{2,\lambda} + \widetilde{\varphi_1}(\lambda) \widetilde{\varphi_2}(\lambda) A_{3,\lambda} + \widetilde{\varphi_1}(\lambda) \widetilde{\varphi_2}(\lambda) \widetilde{\varphi_3}(\lambda) A_{4,\lambda} \\
&\quad + \cdots + \widetilde{\varphi_1}(\lambda) \widetilde{\varphi_2}(\lambda) \widetilde{\varphi_3}(\lambda) \dots \widetilde{\varphi_n}(\lambda)
\end{aligned}
$$

for every $\lambda \in \mathbb{D}$, where

$$
A_{k,\lambda} := \langle T_{\varphi_{k+1} \dots \varphi_n} \widehat{k}_{H^2,\lambda}, T_{\overline{\varphi}_k} \widehat{k}_{H^2,\lambda} - \widetilde{\overline{\varphi_k}}(\lambda) \widehat{k}_{H^2,\lambda} \rangle, \qquad k = 1, \dots, n-1.
$$

Hence, for every $\lambda \in \mathbb{D}$, we infer from (6) that

$$
\begin{aligned}
&|\widetilde{\varphi_1}(\lambda) \dots \widetilde{\varphi_n}(\lambda)| \\
&\quad = \big| A_{1,\lambda} + \widetilde{\varphi_1}(\lambda) A_{2,\lambda} + \widetilde{\varphi_1}(\lambda) \widetilde{\varphi_2}(\lambda) A_{3,\lambda} + \cdots + \widetilde{\varphi_1}(\lambda) \dots \widetilde{\varphi_{n-2}}(\lambda) A_{n-1,\lambda} \\
&\qquad\qquad\qquad\qquad\qquad + \widetilde{X}_{\substack{\varphi_1,\dots,\varphi_n \\ H_1,\dots H_n}}(\lambda) + (H_1 \dots H_n)^{\sim}(\lambda) \big|.
\end{aligned}
$$

Since $(H_1 \dots H_n)^{\sim}(\lambda) \to 0$ radially, by (5) and the Cauchy–Schwarz inequality, we conclude that

$$
\lim_{r \to 1^-} |\widetilde{\varphi_1}(re^{it}) \dots \widetilde{\varphi_n}(re^{it})| = 0,
$$

and hence $\varphi_1(e^{it}) \dots \varphi_n(e^{it}) = 0$ for almost all $t \in [0, 2\pi)$, as claimed. ■

As a corollary of Proposition 4.1 (for $H_i = 0$, $i = 1, \dots, n$), we obtain the following well-known Douglas lemma.

Corollary 4.2. *Let $\varphi_1, \dots, \varphi_n \in L^\infty(\mathbb{T})$. If $T_{\varphi_1} \dots T_{\varphi_n} = 0$, then $\varphi_1 \dots \varphi_n = 0$.*

Recall that in the Hardy space $H^2$ it is routine that if $\overline{u}$ or $v$ is holomorphic then $T_u T_v = T_{uv}$. In [BH], it was shown by A. Brown and P. Halmos that the converse is also true. That is, if $T_u T_v = T_w$ then one of the two symbols $\overline{u}$ or $v$ must be holomorphic and in this case $w = uv$. From this they easily deduce that if $T_u T_v = 0$, then one of the two symbols $u$ or $v$ must be identically zero. There are many other interesting applications of their result. (For the history of the Brown–Halmos theorem and also new Brown–Halmos type theorems, see, for instance, Ahern and Čučković [AC]).

A special case of Corollary 4.2 is the following, which gives a new simpler proof of the Brown–Halmos result.

Corollary 4.3. *Let $\varphi, \psi \in L^\infty(\mathbb{T})$ with $\psi(\xi) \neq 0$ almost everywhere on $\mathbb{T}$. If $T_\varphi T_\psi = 0$, then $T_\varphi = 0$.*

**4.2. Zero operator products on $L_a^2$.** Following Axler and Zheng [AZ1], let $\mathcal{U}$ denote the $C^*$-subalgebra of $L^\infty(\mathbb{D})$ generated by $H^\infty$. It is well known (see [AZ1, Proposition 4.5]) that $\mathcal{U}$ equals the closed subalgebra of $L^\infty$ generated by the set of bounded harmonic functions on $\mathbb{D}$.

It is also well known (see Ahern, Flores and Rudin [AFR] and Engliš [E1]) that a function in $L^\infty(\mathbb{D})$ equals its Berezin symbol (transform) if and only if it is harmonic.

Recall that for $f \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator $T_f$ with symbol $f$ is the operator on $L_a^2 = L_a^2(\mathbb{D})$ defined by $T_f g = P(fg)$, where $P$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L_a^2$.

The *Berezin symbol* (*transform*) $\widetilde{f}$ of a function $f \in L^\infty(\mathbb{D}, dA)$ is defined to be the Berezin symbol of the Toeplitz operator $T_f$ on $L_a^2$. In other words, $\widetilde{f} := \widetilde{T_f}$. Because $\langle T_f \widehat{k}_\lambda, \widehat{k}_\lambda\rangle = \langle P(f\widehat{k}_\lambda), \widehat{k}_\lambda\rangle = \langle f\widehat{k}_\lambda, \widehat{k}_\lambda\rangle$, we obtain the formula

$$\widetilde{f}(\lambda) = \int_{\mathbb{D}} f(z)|\widehat{k}_\lambda(z)|^2\, dA(z).$$

The Berezin transform of a function in $L^\infty(\mathbb{D}, dA)$ often plays the same important role in the theory of Bergman spaces as the harmonic extension of a function in $L^\infty(\partial\mathbb{D}, dm)$ plays in the theory of Hardy spaces (for more details see, for instance, Axler and Zheng [AZ1] and Zhu [Zhu]).

The following two results are due to Axler and Zheng [AZ1, Corollary 3.4 and 3.7].

Lemma 4.4. *If $u \in \mathcal{U}$, then $\widetilde{u} - u$ has nontangential limit $0$ at almost every point of $\mathbb{T}$.*

Lemma 4.5. *If $u \in \mathcal{U}$, then the function*

$$\lambda \mapsto \|(T_u - u(\lambda)I)\widehat{k}_{L_a^2,\lambda}\|_{L_a^2}$$

*has nontangential limit $0$ at almost every point of $\mathbb{T}$.*

Now, by the same arguments used in the proof of Proposition 4.1 and also by applying Lemmas 4.4 and 4.5, one can prove the following (the proof is omitted).

Proposition 4.6. *Let $u_1, \dots u_n \in \mathcal{U}$ and $T_{u_i}$, $i = 1, \dots, n$, be the Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$. If $T_{u_1} \dots T_{u_n} = 0$, then $u_1(\xi) \dots u_n(\xi) = 0$ for almost all $\mathbb{T}$; here $u_i(\xi)$ denotes the nontangential boundary value of $u$ at $\xi \in \mathbb{T}$, which exists by Lemma 4.4.*

COROLLARY 4.7. *Let $f, g \in \mathcal{U}$ with $g$ harmonic. Then $T_f T_g = 0$ on $L^2_a(\mathbb{D})$ has only a trivial solution.*

This corollary, in particular, generalizes [AC, Corollary 2], and also partially solves the following known conjecture (see [AC]).

CONJECTURE 4.8. Let $f, g \in L^\infty(\mathbb{D})$ with $g$ harmonic. Then $T_f T_g = 0$ on $L^2_a(\mathbb{D})$ has only a trivial solution.

For more information about zero Toeplitz products on the Hardy and Bergman spaces, see, for instance, Ahern and Čučković [AC], Aleman and Vukotić [AV], Čučković [C], and their references.

**5. Maximal Berezin set.** Recall that for any operator $A \in \mathcal{B}(\mathcal{H}(\Omega))$ the *maximal Berezin set* $\widetilde{W}_0(A)$ is defined by (see [GGSU])

$$\widetilde{W}_0(A) := \Big\{\lambda \in \mathbb{C} : \exists\{\lambda_n\} \subset \Omega \text{ such that } \lambda = \lim_{\lambda_n \to \partial\Omega} \widetilde{A}(\lambda_n) \text{ and } \lim_{\lambda_n \to \partial\Omega} \|A\widehat{k}_{\mathcal{H},\lambda_n}\| = \|A\|\Big\}.$$

Our next result improves [GGSU, Theorem 1] (which actually means that $\{\|T_\varphi\|\} \subset \widetilde{W}_0(T_\varphi)$) for any $\varphi \in H^{\infty,d} := \{f \in H^\infty : |f| = \|f\|_\infty$ on a subset of $\mathbb{T}$ of positive measure$\}$.

PROPOSITION 5.1. *For any Toeplitz operator $T_\varphi$ on $H^2$, we have*

$$\widetilde{W}_0(T_\varphi) = \{\|\varphi\|_\infty\}.$$

*Proof.* By Lemma 1.2, it is elementary that for any $\varphi \in L^\infty(\mathbb{T})$,

$$\begin{aligned}\|T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{T_\varphi}(\lambda)\widehat{k}_{H^2,\lambda}\|^2 &= \|T_\varphi \widehat{k}_{H^2,\lambda}\|^2 - \|\widetilde{T_\varphi}(\lambda)\|^2 \\ &= \|T_\varphi \widehat{k}_{H^2,\lambda}\|^2 - |\widetilde{\varphi}(\lambda)|^2 \quad (\forall \lambda \in \mathbb{D}).\end{aligned}$$

Since $\|T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{T_\varphi}(\lambda)\widehat{k}_{H^2,\lambda}\| \to 0$ radially (see Lemma 2.2), we deduce that

$$\lim_{|\lambda| \to 1^-} \|T_\varphi \widehat{k}_{H^2,\lambda}\| = \lim_{|\lambda| \to 1^-} |\widetilde{\varphi}(\lambda)|,$$

which implies that $\widetilde{W}_0(T_\varphi) = \{\|T_\varphi\|\} = \{\|\varphi\|_\infty\}$. ∎

By the same argument as in the proof of Proposition 5.1, one can prove the following for the Engliš algebra $\mathcal{A}_\mathcal{H}$.

PROPOSITION 5.2. *For any $T \in \mathcal{A}_\mathcal{H}$, we have $\widetilde{W}_0(T) \subset \mathbb{T}_{\|T\|}$ and $\widetilde{W}_0(T^*) \subset \mathbb{T}_{\|T\|}$, where $\mathbb{T}_{\|T\|}$ denotes the circle with center $0$ and radius $\|T\|$.*

**6. Compactness and related problems.** Recall (see Hoffman [Hof]) that the maximal ideal space of the Banach algebra $H^\infty(\mathbb{D})$, denoted by $\mathcal{M}$, is defined to be the set of multiplicative linear functionals from $H^\infty$ to the field of complex numbers. With the weak-star topology, $\mathcal{M}$ is a compact

Hausdorff space. If $z \in \mathbb{D}$, then the point evaluation $f \mapsto f(z)$ at $z$ is a multiplicative linear functional on $\mathcal{M}$. Thus we can think of $z$ as an element of $\mathcal{M}$, and of $\mathbb{D}$ as a subset of $\mathcal{M}$. The celebrated Carleson Corona Theorem (see [Hof]) states that $\mathbb{D}$ is dense in $\mathcal{M}$.

For $\lambda \in \mathbb{D}$, let $\varphi_\lambda$ be the Möbius map on $\mathbb{D}$ defined by

$$\varphi_\lambda(z) := \frac{\lambda - z}{1 - \overline{\lambda} z}.$$

Following Axler and Zheng [AZ1], we define HOP (which stands for "harmonic on parts") to be the set of functions $u \in \mathcal{U}$ such that $u \circ \varphi_m$ is harmonic on $\mathbb{D}$ for every $m \in \mathcal{M} \setminus \mathbb{D}$.

Every bounded harmonic function on $\mathbb{D}$ is in HOP; also every function in $C(\overline{\mathbb{D}})$, the space of continuous functions on $\overline{\mathbb{D}}$, is in HOP (see [AZ1]).

Proposition 6.1. *Suppose $S$ is a finite sum of operators of the form $T_{u_1} \ldots T_{u_n}$, where $u_j \in \mathcal{U}$ and $T_{u_j} : L_a^2 \to L_a^2$ for $j = 1, \ldots, n$.*

(a) *If $\widetilde{S} \in \mathrm{HOP}$ and $A \in \mathcal{B}(L_a^2)$ with $A(S - T_{\widetilde{S}}) = (S - T_{\widetilde{S}})A$, then $A$ has a nontrivial hyperinvariant subspace on $L_a^2$.*

(b) *If $\widetilde{S} \in \mathrm{HOP}$ and $\widetilde{S}$ has limit $0$ on $\partial\mathbb{D}$ then $S$ has a representation*

$$S = \textit{compact Toeplitz operator} + \textit{compact operator}.$$

*Proof.* The proofs are immediate from [AZ1, Corollary 3.12] and the famous Lomonosov theorem [Lom]. Indeed, if $\widetilde{S} \in \mathrm{HOP}$, then by [AZ1, Corollary 3.12], $S - T_{\widetilde{S}}$ is a compact operator, and Lomonosov's result applies. This proves (a).

(b) Since $\widetilde{S}$ vanishes on $\partial\mathbb{D}$, we know by Axler–Zheng's theorem [AZ1] that $T_{\widetilde{S}}$ is a compact operator on $L_a^2$. Also, since $\widetilde{S} \in \mathrm{HOP}$, we see that $S - T_{\widetilde{S}}$ is compact, $S - T_{\widetilde{S}} = K$, which implies the desired representation $S = T_{\widetilde{S}} + K$, where $T_{\widetilde{S}}$ is a compact Toeplitz operator on $L_a^2$. ∎

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a standard RKHS over a set $\Omega$, and consider the Engliš algebra

$$\mathcal{A}_{\mathcal{H}}^0 := \Big\{ T \in \mathcal{B}(\mathcal{H}) : \lim_{\lambda \to \partial\Omega} (\|T\widehat{k}_{\mathcal{H},\lambda}\|^2 - |\widetilde{T}(\lambda)|^2) = 0 \Big\}.$$

It can be proved as in [E2, proof of (A1), p. 186] that $\mathcal{A}_{\mathcal{H}}^0$ is a norm-closed algebra (the proof is omitted). Also it is clear that $\mathcal{A}_{\mathcal{H}} \subset \mathcal{A}_{\mathcal{H}}^0$, where $\mathcal{A}_{\mathcal{H}}$ is the Engliš algebra defined in Section 2. Further, $\mathcal{A}_{\mathcal{H}}^0$ contains all compact operators on $\mathcal{H}$.

In the next theorem we will study compactness of the operator $AB - C$, where $A, B, C$ belong to $\mathcal{A}_{\mathcal{H}}^0$ and are noncompact operators on $\mathcal{H}$.

Theorem 6.2. *Let $A, B, C \in \mathcal{A}_{\mathcal{H}}^0$. Then $AB - C$ is a compact operator on $\mathcal{H}$ if and only if*

$$\lim_{\lambda\to\partial\Omega}\big[\langle U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda},(AB-C)^*U\widehat{k}_{\mathcal{H},\lambda}\rangle \\ +|\widetilde{U}(\lambda)|^2(\widetilde{A}(\lambda)\widetilde{B}(\lambda)-\widetilde{C}(\lambda))\big]=0$$

*for all unitary operators $U$ on $\mathcal{H}$.*

*Proof.* We will apply Lemma 1.1. Let $U:\mathcal{H}\to\mathcal{H}$ be a unitary operator. Let us calculate the Berezin symbol of $U^{-1}(AB-C)U$:

$$\begin{aligned}
&(U^{-1}(AB-C)U)^{\sim}(\lambda)=\langle ABU\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle-\langle CU\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\quad=\langle AB(U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda})\rangle+\widetilde{U}(\lambda)\langle AB\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad-\langle C(U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle-\widetilde{U}(\lambda)\langle C\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\quad=\langle AB(U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle+\widetilde{U}(\lambda)\langle A(B\widehat{k}_{\mathcal{H},\lambda}-\widetilde{B}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad+\widetilde{U}(\lambda)\widetilde{B}(\lambda)\langle A\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle-\langle C(U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad-\widetilde{U}(\lambda)\langle C\widehat{k}_{\mathcal{H},\lambda}-\widetilde{C}(\lambda)\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle-\widetilde{U}(\lambda)\widetilde{C}(\lambda)\langle\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\quad=\langle AB(U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad+\widetilde{U}(\lambda)\langle A(B\widehat{k}_{\mathcal{H},\lambda}-\widetilde{B}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad+\widetilde{U}(\lambda)\widetilde{B}(\lambda)\langle A\widehat{k}_{\mathcal{H},\lambda}-\widetilde{A}(\lambda)\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad+\widetilde{U}(\lambda)\widetilde{B}(\lambda)\widetilde{A}(\lambda)\langle\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle-\langle C(U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad-\widetilde{U}(\lambda)\langle C\widehat{k}_{\mathcal{H},\lambda}-\widetilde{C}(\lambda)\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle-\widetilde{U}(\lambda)\widetilde{C}(\lambda)\langle\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\quad=\langle AB(U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad+\widetilde{U}(\lambda)\langle A(B\widehat{k}_{\mathcal{H},\lambda}-\widetilde{B}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad+\widetilde{U}(\lambda)\widetilde{B}(\lambda)\langle A\widehat{k}_{\mathcal{H},\lambda}-\widetilde{A}(\lambda)\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad+|\widetilde{U}(\lambda)|^2\widetilde{A}(\lambda)\widetilde{B}(\lambda)-\langle C(U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\qquad-\widetilde{U}(\lambda)\langle C\widehat{k}_{\mathcal{H},\lambda}-\widetilde{C}(\lambda)\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle-|\widetilde{U}(\lambda)|^2\widetilde{C}(\lambda).
\end{aligned}$$

Thus,

$$\begin{aligned}
(U^{-1}(AB-C)U)^{\sim}(\lambda)&=\langle U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda},(AB-C)^*U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\quad+|\widetilde{U}(\lambda)|^2[\widetilde{A}(\lambda)\widetilde{B}(\lambda)-\widetilde{C}(\lambda)]\\
&\quad+\widetilde{U}(\lambda)\langle A(B\widehat{k}_{\mathcal{H},\lambda}-\widetilde{B}(\lambda)\widehat{k}_{\mathcal{H},\lambda}),U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\quad+\widetilde{U}(\lambda)\widetilde{B}(\lambda)\langle A\widehat{k}_{\mathcal{H},\lambda}-\widetilde{A}(\lambda)\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle\\
&\quad-\widetilde{U}(\lambda)\langle C\widehat{k}_{\mathcal{H},\lambda}-\widetilde{C}(\lambda)\widehat{k}_{\mathcal{H},\lambda},U\widehat{k}_{\mathcal{H},\lambda}\rangle
\end{aligned}\tag{7}$$

for any unitary operator $U$ on $\mathcal{H}$ and any $\lambda\in\Omega$.

By the hypotheses of theorem and the obvious inequality

$$\|U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\|\le 2,$$

we infer from (7) by the Cauchy–Schwarz inequality that

$$\lim_{\lambda\to\partial\Omega}(U^{-1}(AB-C)U)^{\sim}(\lambda)=0$$

if and only if

$$\lim_{\lambda\to\partial\Omega}\big[\langle U\widehat{k}_{\mathcal{H},\lambda}-\widetilde{U}(\lambda)\widehat{k}_{\mathcal{H},\lambda},(AB-C)^{*}U\widehat{k}_{\mathcal{H},\lambda}\rangle \\ +|\widetilde{U}(\lambda)|^{2}(\widetilde{A}(\lambda)\widetilde{B}(\lambda)-\widetilde{C}(\lambda))\big]=0.$$

Since $U$ is arbitrary, by Lemma 1.1 this proves the theorem. ∎

COROLLARY 6.3. *If $AB-C$ is compact, then*

$$\lim_{\lambda\to\partial\Omega}\big(\widetilde{A}(\lambda)\widetilde{B}(\lambda)-\widetilde{C}(\lambda)\big)=0.$$

**7. Asymptotic multiplicativity and compactness of semi-commutators on Bergman space.** In what follows, nt-$\lim_{\lambda\to\partial\mathbb{D}} a(\lambda)$ will denote the nontangential limit of $a(\lambda)$. It is well known that the Berezin transform is not multiplicative even over the space of harmonic functions. However, $\widetilde{uv}(\lambda)-\widetilde{u}(\lambda)\widetilde{v}(\lambda)\to 0$ as $\lambda\to\partial\mathbb{D}$ for some pairs of functions $u,v$. In [AZ1], the authors describe the bounded harmonic functions for which this happens. Their proof is based on Hankel operators (see [AZ1, Lemma 4.1 and Theorem 4.5]).

In this section, we consider more general situations for $u$ and $v$, and give necessary and sufficient conditions for the asymptotic multiplicativity of the Berezin transform for functions. We prove, in particular, the following Axler–Chang–Sarason–Volberg type theorem (see [ACS] and [V]) for Bergman space Toeplitz operators with symbols in $\mathcal{U}$ (see ($b_1$) and ($b_2$) below).

THEOREM 7.1. *Let $u,v\in L^{\infty}(\mathbb{D},dA)$, and let $T_u,T_v,T_{uv}$ be the corresponding Toeplitz operators on $L^2_a$. Then*

(a) $\lim_{\lambda\to\partial\mathbb{D}}(\widetilde{uv}(\lambda)-\widetilde{u}(\lambda)\widetilde{v}(\lambda))=0$ *if and only if*

$$\lim_{\lambda\to\partial\mathbb{D}}[(T_{uv}-T_uT_v)^{\sim}(\lambda)+\langle T_{v-\widetilde{v}(\lambda)}\widehat{k}_{L^2_a,\lambda},T_{\overline{u}}\widehat{k}_{L^2_a,\lambda}\rangle]=0.$$

(b) *If $u,v\in\mathcal{U}$, then:*

($b_1$) nt-$\lim_{\lambda\to\partial\mathbb{D}}(\widetilde{uv}(\lambda)-\widetilde{u}(\lambda)\widetilde{v}(\lambda))=0$ *if and only if $T_{uv}-T_uT_v$ is compact on $L^2_a$;*

($b_2$) nt-$\lim_{\lambda\to\partial\mathbb{D}}(\widetilde{uv}(\lambda)-\widetilde{u}(\lambda)\widetilde{v}(\lambda))=0$ *if and only if $T_{uv}-T_vT_u$ is compact on $L^2_a$.*

*Proof.* For $u,v\in L^{\infty}(\mathbb{D},dA)$ we have

$$\begin{aligned}(T_{uv}-T_uT_v)^{\sim}(\lambda)&=\widetilde{T_{uv}}(\lambda)-\widetilde{T_uT_v}(\lambda)\\&=\langle T_{uv}\widehat{k}_{L^2_a,\lambda},\widehat{k}_{L^2_a,\lambda}\rangle-\langle T_uT_v\widehat{k}_{L^2_a,\lambda},\widehat{k}_{L^2_a,\lambda}\rangle\end{aligned}$$

$$= \langle T_{uv}\widehat{k}_{L^2_a,\lambda} - \widetilde{T_{uv}}(\lambda)\widehat{k}_{L^2_a,\lambda}, \widehat{k}_{L^2_a,\lambda}\rangle + \langle \widetilde{T_{uv}}(\lambda)\widehat{k}_{L^2_a,\lambda}, \widehat{k}_{L^2_a,\lambda}\rangle$$
$$- \langle T_v\widehat{k}_{L^2_a,\lambda} - \widetilde{T_v}(\lambda)\widehat{k}_{L^2_a,\lambda}, T_{\overline{u}}\widehat{k}_{L^2_a,\lambda}\rangle - \langle \widetilde{T_v}(\lambda)\widehat{k}_{L^2_a,\lambda}, T_{\overline{u}}\widehat{k}_{L^2_a,\lambda}\rangle$$
$$= \widetilde{uv}(\lambda) - \widetilde{u}(\lambda)\widetilde{v}(\lambda) - \langle T_{v-\widetilde{v}(\lambda)}\widehat{k}_{L^2_a,\lambda}, T_{\overline{u}}\widehat{k}_{L^2_a,\lambda}\rangle,$$

and hence

$$\widetilde{uv}(\lambda) - \widetilde{u}(\lambda)\widetilde{v}(\lambda) = (T_{uv} - T_uT_v)^{\sim}(\lambda) + \langle T_{v-\widetilde{v}(\lambda)}\widehat{k}_{L^2_a,\lambda}, T_{\overline{u}}\widehat{k}_{L^2_a,\lambda}\rangle \tag{8}$$

for all $\lambda \in \mathbb{D}$, which implies (a).

($b_1$) Since $u, v \in \mathcal{U}$ and $\mathcal{U}$ is an algebra, we have $uv \in \mathcal{U}$ and $\widetilde{u}, \widetilde{v} \in \mathcal{U}$. On the other hand, it is easy to see from [AZ1, proof of Corollary 3.7] that by the same arguments it can be shown that the function $\lambda \mapsto \|T_{v-\widetilde{v}(\lambda)}\widehat{k}_{L^2_a,\lambda}\|_2$ has nontangential limit 0 at almost every point of $\partial\mathbb{D}$. For this, it is enough to note that

$$\begin{aligned}\|(v - \widetilde{v}(\lambda))\widehat{k}_{L^2_a,\lambda}\|_2^2 &= \int_{\mathbb{D}} |v(z) - \widetilde{v}(\lambda)|^2 |\widehat{k}_{L^2_a,\lambda}(z)|^2 dA(z)\\ &= \int_{\mathbb{D}} \big(|v(z)|^2 - 2\operatorname{Re}\overline{\widetilde{v}(\lambda)}v(z) + |\widetilde{v}(\lambda)|^2\big)|\widehat{k}_{L^2_a,\lambda}(z)|^2\, dA(z)\\ &= \widetilde{|v|^2}(\lambda) - 2\operatorname{Re}\overline{\widetilde{v}(\lambda)}\widetilde{v}(\lambda) + |\widetilde{v}(\lambda)|^2\end{aligned}$$

for all $\lambda \in \mathbb{D}$. Hence, by the same argument as in [AZ1, proof of Corollary 3.7], the function $\lambda \mapsto \|(v - \widetilde{v}(\lambda))\widehat{k}_{L^2_a,\lambda}\|_2$ belongs to $\mathcal{U}$ and it has nontangential limit 0 almost everywhere on $\partial\mathbb{D}$. Consequently,

$$|\langle T_{v-\widetilde{v}(\lambda)}\widehat{k}_{L^2_a,\lambda}, T_{\overline{u}}\widehat{k}_{L^2_a,\lambda}\rangle| \le \|T_u\|\, \|T_{v-\widetilde{v}(\lambda)}\widehat{k}_{L^2_a,\lambda}\|_2 \to 0$$

nontangentially almost everywhere on $\partial\mathbb{D}$. Hence, by (8),

$$\operatorname*{nt-lim}_{\lambda\to\partial\mathbb{D}}(\widetilde{uv}(\lambda) - \widetilde{u}(\lambda)\widetilde{v}(\lambda)) = 0$$

almost everywhere on $\partial\mathbb{D}$ if and only if $\text{nt-lim}_{\lambda\to\partial\mathbb{D}}(T_{uv} - T_uT_v)^{\sim}(\lambda) = 0$ almost everywhere on $\partial\mathbb{D}$. To complete the proof of ($b_1$), it suffices to note that the last condition is equivalent to the compactness of $T_{uv} - T_uT_v$ on $L^2_a$ (see Axler and Zheng [AZ2, Theorem 2.2]). The proof of ($b_2$) is the same. ∎

By similar arguments one can prove the following result, which improves [AC, Corollary 3].

PROPOSITION 7.2. *If $f, g, h \in \mathcal{U}$ are such that $T_fT_g = T_fT_h$ and $f$ is not identically* 0*, then $g = h$.*

*Proof.* Since $T_fT_g = T_fT_h$, we have $(T_fT_g - T_fT_h)^{\sim}(\lambda) = 0$ for all $\lambda \in \mathbb{D}$. Hence, as in the previous proofs,

$$(g - h)^{\sim}(\lambda)\widetilde{f}(\lambda) = \langle T_{h-g-(h-g)^{\sim}(\lambda)}\widehat{k}_{L^2_a,\lambda}, T_{\overline{f}}\widehat{k}_{L^2_a,\lambda}\rangle, \qquad \lambda \in \mathbb{D}.$$

As in the proof of Theorem 7.1(b), since

$$\langle T_{h-g-(h-g)^{\sim}(\lambda)}\widehat{k}_{L^2_a,\lambda}, T_{\overline{f}}\widehat{k}_{L^2_a,\lambda}\rangle \to 0$$

nontangentially almost everywhere on $\partial\mathbb{D}$, we conclude that

$$(g-h)(\xi)f(\xi) = (g-h)^{\sim}(\xi)\widetilde{f}(\xi) = 0$$

almost everywhere on $\partial\mathbb{D}$ (because if $u \in \mathcal{U}$, then $\widetilde{u}(\xi) = u(\xi)$ for almost all $\xi \in \partial\mathbb{D}$, see [AZ1]). By assumption, $f$ is not identically 0, and therefore $f(\xi) \neq 0$ almost everywhere on $\partial\mathbb{D}$. Then $(g-h)(\xi) = 0$ for almost all $\xi \in \partial\mathbb{D}$. As $g-h \in \mathcal{U}$ and $\mathcal{U}$ is the $C^*$-algebra in $L^\infty(\mathbb{D}, dA)$ generated by $H^\infty$, the Riesz brothers' theorem shows that $(g-h)(z) = 0$ for all $z \in \mathbb{D}$. ■

Theorem 7.1 can also be reformulated as follows, which essentially improves a result due to Axler and Zheng [AZ1, Theorem 4.5], because every bounded harmonic function is in the $C^*$-algebra $\mathcal{U}$.

Theorem 7.3. *Let $u, v \in \mathcal{U}$. Then* $\text{nt-lim}_{\lambda\to\partial\mathbb{D}}(\widetilde{uv}(\lambda) - \widetilde{u}(\lambda)\widetilde{v}(\lambda)) = 0$ *if and only if $2T_{uv} - T_uT_v - T_vT_u$ is a compact operator on the Bergman space $L_a^2(\mathbb{D})$.*

*Proof.* Since

$$(2T_{uv} - T_uT_v - T_vT_u)^{\sim}(\lambda) = (T_{uv} - T_uT_v)^{\sim}(\lambda) + (T_{uv} - T_vT_u)^{\sim}(\lambda),$$

it follows from (8) that

$$\begin{aligned}\widetilde{uv}(\lambda) - \widetilde{u}(\lambda)\widetilde{v}(\lambda) &= \tfrac{1}{2}(2T_{uv} - T_uT_v - T_vT_u)^{\sim}(\lambda) \\ &\quad + \tfrac{1}{2}[\langle T_{v-\widetilde{v}(\lambda)}\widehat{k}_{L_a^2,\lambda}, T_{\overline{u}}\widehat{k}_{L_a^2,\lambda}\rangle + \langle T_{u-\widetilde{u}(\lambda)}\widehat{k}_{L_a^2,\lambda}, T_{\overline{v}}\widehat{k}_{L_a^2,\lambda}\rangle]\end{aligned}$$

for all $\lambda \in \mathbb{D}$. From this, as in the proof of Theorem 7.1(b), by applying Axler and Zheng's arguments (see [AZ2, proof of Corollary 3.7]) we deduce that $\widetilde{uv}(\lambda) - \widetilde{u}(\lambda)\widetilde{v}(\lambda) \to 0$ nontangentially almost everywhere on $\mathbb{T}$ if and only if $(2T_{uv} - T_uT_v - T_vT_u)^{\sim}(\lambda) \to 0$ nontangentially almost everywhere on $\mathbb{T}$. It remains to note that by [AZ2, Theorem 2.2], the last assertion is equivalent to the compactness of $2T_{uv} - T_uT_v - T_vT_u$. ■

Let $V$ denote the subalgebra of $L^\infty(\mathbb{D}, dA)$ consisting of all functions $v \in L^\infty(\mathbb{D}, dA)$ such that the function $\lambda \mapsto \|T_{v-\widetilde{v}(\lambda)}\widehat{k}_{L^2_{\alpha,\lambda}}\|_2$ has nontangential limit 0 almost everywhere on $\partial\mathbb{D}$.

It is not difficult to see from the proofs of Theorems 7.1 and 7.3 that they can be proved in more general cases (we omit the proofs).

Theorem 7.4. *If $u, v \in V$, then* $\text{nt-lim}_{\lambda\to\partial\mathbb{D}}(\widetilde{uv}(\lambda) - \widetilde{u}(\lambda)\widetilde{v}(\lambda)) = 0$ *if and only if $T_{uv} - T_uT_v$ is compact on $L_a^2$.*

Theorem 7.5. *Let $u, v \in V$. Then* $\text{nt-lim}_{\lambda\to\partial\mathbb{D}}(\widetilde{uv}(\lambda) - \widetilde{u}(\lambda)\widetilde{v}(\lambda)) = 0$ *if and only if $2T_{uv} - T_uT_v - T_vT_u$ is a compact operator on $L_a^2$.*

**8. Some remarks on extended eigenvalues and extended eigenvectors of operators.** Let $X$ be a Banach space and $T \in \mathcal{B}(X)$. A complex number $\lambda$ is said to be an *extended eigenvalue* for $T$ provided that there exists a nonzero operator $A \in \mathcal{B}(X)$ such that

$$TA = \lambda AT.$$

Such an operator is called an *extended eigenvector* or an *eigenoperator* for $T$ corresponding to the extended eigenvalue $\lambda$. We denote by

$$\{T\}'_\lambda := \{A \in \mathcal{B}(X) : TA = \lambda AT\}$$

the set of extended eigenvectors for $T$ corresponding to the extended eigenvalue $\lambda$. We also denote by $\operatorname{ext}(T)$ the set of all extended eigenvalues of $T$. The set of extended eigenvectors will be denoted by $\operatorname{Ext}(T)$.

The notions of extended eigenvalue and extended eigenvector became popular in the 1970's when searching for invariant subspaces, especially in the work of Lomonosov [Lom], S. Brown [B] and Kim, Moore and Pearcy [KMP]. Beginning from Malamud's papers [Mal1, Mal2], these concepts have received a considerable amount of attention, both in the context of invariant subspaces (see Lacruz [Lac] and Lambert [Lam]) and in the study of extended eigenvalues and extended eigenvectors for some special classes of operators in the work of Biswas, Lambert and Petrovic [BLP], Lacruz et al. [LLPZ], Lacruz [Lac], Domanov and Malamud [DM], Karaev [K3], Bourdan and Shapiro [BS], Lauric [Lau], Cassier and Alkanjo [CA]. Note that since $IT = TI$, where $I$ is the identity operator on $X$, we always have $1 \in \operatorname{ext}(T)$. Also recall that the extended eigenvalues of analytic Toeplitz operators $T_\varphi$, $\varphi \in H^\infty$, on the Hardy space $H^2$ were investigated by Deddens [D], Bourdon and Shapiro [BS] and Gürdal [G] (see also Alkanjo [Alk]).

Note that if $\varphi$ is a constant function whose value is nonzero, then $\operatorname{ext}(T_\varphi) = \mathbb{C}$. So, we will assume that $\varphi$ is nonconstant. Hence, $T_\varphi$ is one-to-one so that 0 is never an extended eigenvalue. In fact,

$$\operatorname{ext}(T_\varphi) \subset \{z : |z| \geq 1\}$$

for every (nonconstant) $\varphi \in H^\infty$ (see, for instance, Bourdon and Shapiro [BS] and Deddens [D]).

In this section, we consider the operators from Engliš algebras, in particular, the Toeplitz operators $T_\varphi$ on the Hardy space $H^2$, and prove some results on the commutant, extended eigenvalues and extended eigenvectors.

Proposition 8.1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS on the unit disc $\Omega$, and $\mathcal{M}ult(\mathcal{H})$ be the set of all multipliers of $\mathcal{H}$. Let $\varphi \in \mathcal{M}ult(\mathcal{H})$ be a nonconstant function and $M_\varphi$ be the associated multiplication operator on $\mathcal{H}$,*

$M_\varphi f := \varphi f$. *Then*

$$\{M_\varphi\}' = \{A \in \mathcal{B}(\mathcal{H}) : \langle \varphi - \varphi(\lambda)\widehat{k}_{\mathcal{H},\lambda}, A^*\widehat{k}_{\mathcal{H},\lambda}\rangle = 0 \text{ for all } \lambda \in \Omega\},$$

*where* $\{M_\varphi\}'$ *denotes the commutant of* $M_\varphi$.

*Proof.* By the closed graph theorem, $M_\varphi$ is a bounded operator on $\mathcal{H}$. Let $AM_\varphi = M_\varphi A$. Then obviously $\widetilde{AM_\varphi}(\lambda) = \widetilde{M_\varphi A}(\lambda)$ for all $\lambda \in \Omega$, and hence in view of $M_\varphi^* k_{\mathcal{H},\lambda} = \overline{\varphi(\lambda)} k_{\mathcal{H},\lambda}$, we have

$$\begin{aligned}&\langle A(M_\varphi \widehat{k}_{\mathcal{H},\lambda} - \varphi(\lambda)\widehat{k}_{\mathcal{H},\lambda}), \widehat{k}_{\mathcal{H},\lambda}\rangle + \varphi(\lambda)\widetilde{A}(\lambda)\\ &\quad = \langle M_\varphi A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle = \langle A\widehat{k}_{\mathcal{H},\lambda}, M_\varphi^*\widehat{k}_{\mathcal{H},\lambda}\rangle = \langle A\widehat{k}_{\mathcal{H},\lambda}, \overline{\varphi(\lambda)}\,\widehat{k}_{\mathcal{H},\lambda}\rangle = \varphi(\lambda)\widetilde{A}(\lambda).\end{aligned}$$

Thus,

$$\langle A(M_\varphi \widehat{k}_{\mathcal{H},\lambda} - \varphi(\lambda)\widehat{k}_{\mathcal{H},\lambda}), \widehat{k}_{\mathcal{H},\lambda}\rangle = 0,$$

or equivalently $\langle (\varphi - \varphi(\lambda))\widehat{k}_{\mathcal{H},\lambda}, A^*\widehat{k}_{\mathcal{H},\lambda}\rangle = 0$ for all $\lambda \in \Omega$, as desired. ∎

Apparently, the following is known, but we will give a new proof.

PROPOSITION 8.2. *Let* $\varphi \in L^\infty(\mathbb{T})$, *let* $T_\varphi$ *be a Toeplitz operator on* $H^2$, *and let* $h \in H^\infty$. *Then* $T_h \in \{T_\varphi\}'$ *if and only if* $\varphi \in H^\infty$.

*Proof.* As in the proof of the previous proposition, by passing to Berezin symbols, we have

$$T_\varphi T_h = T_h T_\varphi$$

if and only if

$$\langle T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{\varphi}(\lambda)\widehat{k}_{H^2,\lambda}, \overline{h(\lambda)}\,\widehat{k}_{H^2,\lambda}\rangle = \langle T_h\widehat{k}_{H^2,\lambda}, T_{\overline{\varphi}}\widehat{k}_{H^2,\lambda} - \overline{\widetilde{\varphi}(\lambda)}\,\widehat{k}_{H^2,\lambda}\rangle,$$

hence

$$h(\lambda)\langle T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{\varphi}(\lambda)\widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda}\rangle = \langle T_h\widehat{k}_{H^2,\lambda}, T_{\overline{\varphi}}\widehat{k}_{H^2,\lambda} - \overline{\widetilde{\varphi}(\lambda)}\,\widehat{k}_{H^2,\lambda}\rangle,$$

for all $\lambda \in \mathbb{D}$. Since $\langle T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{\varphi}(\lambda)\widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda}\rangle = 0$ for all $\lambda \in \mathbb{D}$, we deduce that $T_\varphi T_h = T_h T_\varphi$ if and only if $\langle T_h\widehat{k}_{H^2,\lambda}, T_{\overline{\varphi}}\widehat{k}_{H^2,\lambda} - \overline{\widetilde{\varphi}(\lambda)}\,\widehat{k}_{H^2,\lambda}\rangle = 0$ for all $\lambda \in \mathbb{D}$, or equivalently

$$0 = \langle \widehat{k}_{H^2,\lambda}, T_{\overline{h\varphi}}\widehat{k}_{H^2,\lambda} - \overline{\widetilde{\varphi}(\lambda)h(\lambda)}\,\widehat{k}_{H^2,\lambda}\rangle = \langle \widehat{k}_{H^2,\lambda}, T_{\overline{h\varphi}}\widehat{k}_{H^2,\lambda} - \overline{(\widetilde{\varphi}h)(\lambda)}\,\widehat{k}_{H^2,\lambda}\rangle$$

for all $\lambda \in \mathbb{D}$. Thus, $T_\varphi T_h = T_h T_\varphi$ if and only if

$$\langle \widehat{k}_{H^2,\lambda}, T_{\overline{h\varphi}}\widehat{k}_{H^2,\lambda} - \overline{(\widetilde{\varphi}h)(\lambda)}\,\widehat{k}_{H^2,\lambda}\rangle = 0 \tag{9}$$

for all $\lambda \in \mathbb{D}$. On the other hand, always

$$\langle \widehat{k}_{H^2,\lambda}, T_{\overline{h\varphi}}\widehat{k}_{H^2,\lambda} - \widetilde{\overline{h\varphi}}(\lambda)\widehat{k}_{H^2,\lambda}\rangle = 0,$$

or equivalently

$$\langle \widehat{k}_{H^2,\lambda}, T_{\overline{h\varphi}}\widehat{k}_{H^2,\lambda} - \overline{\widetilde{h\varphi}(\lambda)}\widehat{k}_{H^2,\lambda}\rangle = 0 \tag{10}$$

for all $\lambda \in \mathbb{D}$. So, by combining (9) and (10), we find that $T_\varphi T_h = T_h T_\varphi$ if and only if $\widetilde{\varphi h} = \widetilde{\varphi} h$.

Since $\widetilde{\varphi}$, $h$ and $\widetilde{\varphi h}$ are harmonic functions on $\mathbb{D}$, from the equality $\widetilde{\varphi h} = \widetilde{\varphi} h$ we deduce by [AZ1, Lemma 4.2] that $\varphi$ is analytic on $\mathbb{D}$. This shows that $\varphi \in H^\infty$, as desired. ∎

Remark 8.3. The assertion that $T_\varphi T_h = T_h T_\varphi$ if and only if $\widetilde{\varphi h} = \widetilde{\varphi} h$ can also be proved in the following simpler way:

$$T_\varphi T_h = T_h T_\varphi \;\Leftrightarrow\; \widetilde{T_h T_\varphi}(\lambda) = \widetilde{T_\varphi T_h}(\lambda),\ \forall\lambda \;\Leftrightarrow\; h(\lambda)\widetilde{\varphi}(\lambda) = \widetilde{T_{\varphi h}}(\lambda),\ \forall\lambda$$
$$\Leftrightarrow\; h(\lambda)\widetilde{\varphi}(\lambda) = \widetilde{\varphi h}(\lambda),\ \forall\lambda.$$

Remark 8.4. In Remark 8.3, we have used the unicity theorem for the Berezin symbols of operators, which says that $A = B$ if and only if $\widetilde{A} = \widetilde{B}$ (see, for example, Zhu [Zhu]). Now, by using this unicity theorem we give, in terms of Berezin symbols, a new characterization of Toeplitz operators on the Hardy space $H^2$, which is quite different from the classical characterization that an operator $A$ on $H^2$ is a Toeplitz operator if and only if $S^*AS = A$, where $S$ is the unilateral shift operator defined on $H^2$ by $Sf(z) = zf(z)$ (see, for example, Halmos [Hal]).

Proposition 8.5. *An operator $A$ in $\mathcal{B}(H^2)$ is a Toeplitz operator if and only if its Berezin symbol $\widetilde{A}$ is a harmonic function on $\mathbb{D}$.*

*Proof.* If $A = T_\varphi$ for some $\varphi$, then by Lemma 1.2, $\widetilde{A} = \widetilde{\varphi}$ is harmonic on $\mathbb{D}$. Conversely, if $\widetilde{A}$ is harmonic on $\mathbb{D}$, then the Toeplitz operator $T_{\widetilde{A}}$ with harmonic symbol $\widetilde{A}$ satisfies

$$\widetilde{T_{\widetilde{A}}}(\lambda) = \widetilde{A}(\lambda)$$

for all $\lambda \in \mathbb{D}$, which implies in view of the above unicity theorem for Berezin symbols that $A = T_{\widetilde{A}}$, as desired. ∎

Of course, a similar result for the Bergman space is only true in the following form: *if an operator $A$ on $L_2^a$ has a harmonic Berezin symbol $\widetilde{A}$ then $A = T_{\widetilde{A}}$.*

For any operator $T \in \mathcal{B}(\mathcal{H}(\mathbb{D}))$ on the RKHS $\mathcal{H} = \mathcal{H}(\mathbb{D})$, we will denote $\widetilde{T}^{\mathrm{rad}}(e^{it}) := \lim_{r\to 1^-} \widetilde{T}(re^{it})$ if these radial limits exist almost everywhere on the unit circle $\mathbb{T}$, and if $\widetilde{T}^{\mathrm{rad}} \in L^\infty(\mathbb{T})$.

For any Engliš algebra $\mathcal{A}_\mathcal{H}$ on the RKHS $\mathcal{H} = \mathcal{H}(\mathbb{D})$, we set

$$\widetilde{\mathcal{A}}_{\mathcal{H}}^{\mathrm{rad}} := \{T \in \mathcal{A}_\mathcal{H} : \widetilde{T}^{\mathrm{rad}} \neq 0 \text{ almost everywhere on } \partial\mathbb{D}\}.$$

Proposition 8.6. *Let $T \in \widetilde{\mathcal{A}}_{\mathcal{H}}^{\mathrm{rad}}$, and let $\mu \neq 1$ be a complex number. If $A \in \mathcal{B}(\mathcal{H}(\mathbb{D}))$ and $AT = \mu TA$, then $\widetilde{A}^{\mathrm{rad}} = 0$.*

*Proof.* As in the proof of the preceding proposition, we obtain

$$\langle AT\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle = \mu\langle TA\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\rangle,$$

and hence

$$\begin{aligned}(\mu-1)\widetilde{T}(\lambda)\widetilde{A}(\lambda) &= \langle T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}, A^*\widehat{k}_{\mathcal{H},\lambda}\rangle \\ &\quad - \mu\langle A\widehat{k}_{\mathcal{H},\lambda}, T^*\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\rangle\end{aligned}$$

for all $\lambda \in \mathbb{D}$. From this, as $T \in \mathcal{A}_{\mathcal{H}}$, we find that

$$\begin{aligned}&|\mu-1|\,|\widetilde{T}(\lambda)|\,|\widetilde{A}(\lambda)| \\ &\quad \le \|A\|\,\|T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}} + |\mu|\,\|A\|\,\|T^*\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\|_{\mathcal{H}} \to 0\end{aligned}$$

radially, which implies in view of $\mu \neq 1$ and $\widetilde{T}^{\mathrm{rad}} \neq 0$ that $\widetilde{A}^{\mathrm{rad}} = 0$, as desired. ∎

COROLLARY 8.7. *If $\lambda \neq 1$ is an extended eigenvalue of an analytic Toeplitz operator $T_\varphi$ on $H^2$, then $\{T_\varphi\}'_\lambda$ does not contain any (nonzero) Toeplitz operator $T_\psi$ with $\psi \in L^\infty(\mathbb{T})$.*

Another similar result is formulated in terms of the extended spectrum of an operator $T$, and gives some structure for the extended eigenvectors of $T$.

COROLLARY 8.8. *Let $T \in \widetilde{\mathcal{A}}^{\mathrm{rad}}_{\mathcal{H}}$ on the RKHS $\mathcal{H} = \mathcal{H}(\mathbb{D})$ be such that $\operatorname{ext}(T) \subseteq \mathbb{D}$. Then*

$$\operatorname{Ext}(T) \subset \mathcal{B}(\mathcal{H}) \setminus \widetilde{\mathcal{A}}^{\mathrm{rad}}_{\mathcal{H}}.$$

*Proof.* Let $A \in \operatorname{Ext}(T)$. If $\widetilde{A}^{\mathrm{rad}}$ does not exist almost everywhere on $\partial\mathbb{D}$, then obviously $A \in \mathcal{B}(\mathcal{H}) \setminus \widetilde{\mathcal{A}}^{\mathrm{rad}}_{\mathcal{H}}$. So, we will assume that $\widetilde{A}^{\mathrm{rad}}$ exists for any $A \in \operatorname{Ext}(T)$.

Suppose on the contrary that there exists $B \in \operatorname{Ext}(T)$ such that $B \notin \mathcal{B}(\mathcal{H}) \setminus \widetilde{\mathcal{A}}^{\mathrm{rad}}_{\mathcal{H}}$, that is, $BT = \beta TB$ for some $\beta \in \operatorname{ext}(T)$ and $\widetilde{B}^{\mathrm{rad}} \neq 0$ almost everywhere on $\partial\mathbb{D}$. Then by passing to Berezin symbols, we have

$$\begin{aligned}\widetilde{T}(\lambda)\widetilde{B}(\lambda) &= \beta\widetilde{T}(\lambda)\widetilde{B}(\lambda) + \beta\langle B\widehat{k}_{\mathcal{H},\lambda}, T^*\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\rangle \\ &\quad - \langle T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}, B^*\widehat{k}_{\mathcal{H},\lambda}\rangle\end{aligned}$$

for all $\lambda \in \mathbb{D}$. Hence

$$\begin{aligned}|\widetilde{T}(\lambda)|\,|\widetilde{B}(\lambda)| &\le |\beta|\,|\widetilde{T}(\lambda)|\,|\widetilde{B}(\lambda)| + |\beta|\,\|B\|\,\|T^*\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T^*}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\| \\ &\quad + \|B\|\,\|T\widehat{k}_{\mathcal{H},\lambda} - \widetilde{T}(\lambda)\widehat{k}_{\mathcal{H},\lambda}\|,\end{aligned}$$

which implies that

$$|\widetilde{T}^{\mathrm{rad}}(\xi)|\,|\widetilde{B}^{\mathrm{rad}}(\xi)| \le |\beta|\,|\widetilde{T}^{\mathrm{rad}}(\xi)|\,|\widetilde{B}^{\mathrm{rad}}(\xi)|$$

for almost all $\xi \in \partial\mathbb{D}$, because $T \in \widetilde{\mathcal{A}}^{\mathrm{rad}}_{\mathcal{H}}$. Since $\widetilde{B}^{\mathrm{rad}}(\xi) \neq 0$ for almost all $\xi \in \partial\mathbb{D}$, it follows that $|\beta| \ge 1$, which contradicts $\beta \in \mathbb{D}$. ∎

**9. Reproducing kernels, Duhamel operator and existence of invariant subspaces.** Let $\mathrm{Hol}(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. The *Duhamel product* in $\mathrm{Hol}(\mathbb{D})$ is defined by

$$(f \circledast g)(z) := \frac{d}{dz}\int_0^z f(z-t)g(t)\,dt = \int_0^z f'(z-t)g(t)\,dt + f(0)g(z)$$

for $f, g \in \mathrm{Hol}(\mathbb{D})$ (see Wigley [W1]). This product is also known to be commutative and associative and has the identity $f(z) \equiv \mathbf{1}$.

LEMMA 9.1 (Wigley [W2]). *Let $1 \le p < \infty$ and let $f, g \in H^p(\mathbb{D})$ (the Hardy space). Then $f \circledast g \in H^p$ and there exists a constant $C_p$, depending only on $p$, such that*

$$\|f \circledast g\|_p \le C_p \|f\|_p \|g\|_p.$$

*Moreover, given $f \in H^p$, there exists $g \in H^p$ such that $(f \circledast g)(z) \equiv 1$ if and only if $f(0) \neq 0$.*

In particular, it follows from this lemma that $H^2$ becomes a Banach algebra. For $f \in H^2$, we define the *Duhamel operator* by $\mathcal{D}_f g := f \circledast g$, $g \in H^2$. So, $\mathcal{D}_f$ is invertible on $H^2$ if and only if $f(0) \neq 0$. This means that there is only one maximal ideal, namely, the set of functions which vanish at the origin, and the spectrum of each $\mathcal{D}_f$ is the singleton $\{f(0)\}$.

The following lemma can be easily proved by similar arguments to those in the first author's papers [K4, K5].

LEMMA 9.2. *Let $f \in H^2$ be a nonzero function and $\mathcal{D}_f$ be the associated Duhamel operator on $H^2$. Then $\mathcal{D}_f$ is compact if and only if $f(0) = 0$.*

Now we state the main result of this section, which gives some sufficient conditions in terms of reproducing kernels and Duhamel operators for the existence of a nontrivial invariant subspace (briefly, n.i.s.) in $H^2$. Let $S$ denote the shift operator on $H^2$ defined by $Sf = zf$, and let $S^*$ be a backward shift operator.

THEOREM 9.3. *Let $T : H^2 \to H^2$ be an operator. Suppose that there exists a nonzero operator $B \in \{T\}'$ such that:*

(i) $|(S^{*\beta_\lambda} B)^\sim(\lambda)| = o\left(\dfrac{|((I - SS^*)S^{*\beta_\lambda} B)^\sim(\lambda)|}{1 - |\lambda|^2}\right)$ *as $\lambda \to \partial\mathbb{D}$, where $\beta_\lambda := \mathsf{k}_{B\widehat{k}_\lambda}(0)$ is the order of zero of the function $B\widehat{k}_\lambda$ at $z = 0$;*

(ii) *there exists a sequence $(\lambda_n)_{n\ge1} \subset \mathbb{D}$ tending to a point $\xi_0$ in $\partial\mathbb{D}$ such that*

$$\mathcal{D}_{\widehat{k}_{\lambda_n} \circledast (S^{*\beta_{\lambda_n}} B\widehat{k}_{\lambda_n})^{-1\circledast} - (\widehat{k}_{\lambda_n} \circledast (S^{*\beta_{\lambda_n}} B\widehat{k}_{\lambda_n}))^{-1\circledast}(0)} S^{*\beta_{\lambda_n}}$$

*converges in the uniform operator topology to some operator $\mathcal{K}$ on $H^2$.*

*Then $T$ has a n.i.s.*

*Proof.* Suppose that $T$ has no nontrivial invariant subspace in $H^2$. Then $\ker(A) = \{0\}$ for any nonzero operator $A$ in $\{T\}'$, and therefore $Ah \neq 0$ for any nonzero function $h$ in $H^2$. In particular, $A\widehat{k}_\lambda \neq 0$ for all $A \in \{T\}' \setminus \{0\}$ and $\lambda \in \mathbb{D}$; here $\widehat{k}_\lambda := \widehat{k}_{H^2,\lambda}$ denotes the normalized reproducing kernel of $H^2$. Let $A\widehat{k}_\lambda = f_{A,\lambda}$ for any $\lambda \in \mathbb{D}$. Then $f_{A,\lambda} = z^{\alpha_\lambda} g_{A,\lambda}$, where $\alpha_\lambda := \mathsf{k}_{f_{A,\lambda}}(0)$ is the order of zero of $f_{A,\lambda}$ at $z = 0$, $g_{A,\lambda} \in H^2$, and $g_{A,\lambda}(0) \neq 0$. Hence $A\widehat{k}_\lambda = z^{\alpha_\lambda} g_{A,\lambda}$, and therefore $S^{*\alpha_\lambda} A\widehat{k}_\lambda = g_{A,\lambda}$, where $S^*$ is the backward shift operator on $H^2$ (which is just a co-analytic Toeplitz operator $T_{\overline{z}}$ on $H^2$). Since $g_{A,\lambda}(0) \neq 0$ by Lemma 9.1, there exists a function $\mathcal{G}_{A,\lambda} \in H^2$ (which is the $\circledast$-inverse of $S^{*\alpha_\lambda} A\widehat{k}_\lambda$, i.e., $\mathcal{G}_{A,\lambda} = (S^{*\alpha_\lambda} A\widehat{k}_\lambda)^{-1\circledast}$) such that $\mathcal{G}_{A,\lambda} \circledast g_{A,\lambda} = \mathbf{1}$ for all $\lambda \in \mathbb{D}$. Then

$$(\widehat{k}_\lambda \circledast \mathcal{G}_{A,\lambda}) \circledast S^{*\alpha_\lambda} A\widehat{k}_\lambda = \widehat{k}_\lambda \circledast \mathbf{1} = \widehat{k}_\lambda,$$

and hence

$$\mathcal{D}_{\widehat{k}_\lambda \circledast \mathcal{G}_{A,\lambda}} S^{*\alpha_\lambda} A\widehat{k}_\lambda = \widehat{k}_\lambda,$$

or by setting $F_{A,\lambda} := \widehat{k}_\lambda \circledast \mathcal{G}_{A,\lambda} = \widehat{k}_\lambda \circledast (S^{*\alpha_\lambda} A\widehat{k}_\lambda)^{-1\circledast}$, we obtain

$$\begin{aligned} F_{A,\lambda}(0) &= (\widehat{k}_\lambda \circledast (S^{*\alpha_\lambda} A\widehat{k}_\lambda)^{-1\circledast})(0) = \widehat{k}_\lambda(0)((S^{*\alpha_\lambda} A\widehat{k}_\lambda)^{-1\circledast})(0) \\ &= (1 - |\lambda|^2)^{1/2}((S^{*\alpha_\lambda} A\widehat{k}_\lambda)^{-1\circledast})(0) \neq 0 \end{aligned}$$

and

$$\mathcal{D}_{F_{A,\lambda}} S^{*\alpha_\lambda} A\widehat{k}_\lambda = \widehat{k}_\lambda \tag{11}$$

for all $A \in \{T\}' \setminus \{0\}$ and all $\lambda \in \mathbb{D}$. It is easy to see from (11) that

$$\widehat{k}_\lambda - F_{A,\lambda}(0) S^{*\alpha_\lambda} A\widehat{k}_\lambda = \mathcal{D}_{F_{A,\lambda} - F_{A,\lambda}(0)} S^{*\alpha_\lambda} A\widehat{k}_\lambda \tag{12}$$

for any $A \in \{T\}' \setminus \{0\}$ and any $\lambda \in \mathbb{D}$, where $\mathcal{D}_{F_{A,\lambda} - F_{A,\lambda}(0)}$ is a compact Duhamel operator on $H^2$ (see Lemma 9.2). In particular, from (12) we have

$$\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0) S^{*\beta_{\lambda_n}} B\widehat{k}_{\lambda_n} = \mathcal{D}_{F_{B,\lambda_n} - F_{B,\lambda_n}(0)} S^{*\beta_{\lambda_n}} B\widehat{k}_{\lambda_n} \tag{13}$$

for all $n \geq 1$, where $B \in \{T\}'$ satisfies the conditions of the theorem and $\beta_{\lambda_n} := \mathsf{k}_{B\widehat{k}_{\lambda_n}}(0)$, $n \geq 1$. By condition (ii), there exists an operator $\mathcal{K}$ on $H^2$ such that

$$\lim_{n\to\infty} \|\mathcal{D}_{F_{B,\lambda_n} - F_{B,\lambda_n}(0)} S^{*\beta_{\lambda_n}} - \mathcal{K}\|_{\mathcal{B}(H^2)} = 0.$$

Clearly, $\mathcal{K}$ is compact. Then by using (ii) and the last equality (13), we have

$$\begin{aligned} &\|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0) S^{*\beta_{\lambda_n}} B\widehat{k}_{\lambda_n}\| \\ &\quad = \|(\mathcal{D}_{F_{B,\lambda_n} - F_{B,\lambda_n}(0)} S^{*\beta_{\lambda_n}} - \mathcal{K}) B\widehat{k}_{\lambda_n} + \mathcal{K}B\widehat{k}_{\lambda_n}\| \\ &\quad \leq \|\mathcal{D}_{F_{B,\lambda_n} - F_{B,\lambda_n}(0)} S^{*\beta_{\lambda_n}} - \mathcal{K}\| \, \|B\| + \|\mathcal{K}B\widehat{k}_{\lambda_n}\| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

because $(\widehat{k}_{\lambda_n})_{n\geq 1}$ is a weak null sequence and $\mathcal{K}B$ is a compact operator on $H^2$. Thus

$$\lim_{n\to\infty} \|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0)S^{*\beta_{\lambda_n}}B\widehat{k}_{\lambda_n}\| = 0. \tag{14}$$

On the other hand,

$$\begin{aligned}
&\|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0)S^{*\beta_{\lambda_n}}B\widehat{k}_{\lambda_n}\|^2 \\
&\qquad = 1 - 2\,\mathrm{Re}[F_{B,\lambda_n}(0)(S^{*\beta_{\lambda_n}}B)^{\sim}(\lambda_n)] + \|F_{B,\lambda_n}(0)S^{*\beta_{\lambda_n}}B\widehat{k}_{\lambda_n}\|^2 \\
&\qquad \geq 1 - 2|F_{B,\lambda_n}(0)(S^{*\beta_{\lambda_n}}B)^{\sim}(\lambda_n)| + |F_{B,\lambda_n}(0)(S^{*\beta_{\lambda_n}}B)^{\sim}(\lambda_n)|^2 \\
&\qquad = 1 - 2\left|\frac{(S^{*\beta_{\lambda_n}}B)^{\sim}(\lambda_n)}{(S^{*\beta_{\lambda_n}}Bk_{\lambda_n})(0)}\right| + \left|\frac{(S^{*\beta_{\lambda_n}}B)^{\sim}(\lambda_n)}{(S^{*\beta_{\lambda_n}}Bk_{\lambda_n})(0)}\right|^2.
\end{aligned}$$

Also we have

$$\begin{aligned}
\frac{1}{(S^{*\beta_{\lambda_n}}Bk_{\lambda_n})(0)} &= \frac{1}{\langle S^{*\beta_{\lambda_n}}Bk_{\lambda_n}, \mathbf{1}\rangle} = \frac{1}{\langle S^{*\beta_{\lambda_n}}Bk_{\lambda_n}, (I-SS^*)k_{\lambda_n}\rangle} \\
&= \frac{1-|\lambda_n|^2}{\langle (I-SS^*)S^{*\beta_{\lambda_n}}B\widehat{k}_{\lambda_n}, \widehat{k}_{\lambda_n}\rangle} \\
&= \frac{1-|\lambda_n|^2}{[(I-SS^*)S^{*\beta_{\lambda_n}}B]^{\sim}(\lambda_n)},
\end{aligned}$$

and hence

$$\begin{aligned}
\|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0)S^{*\beta_{\lambda_n}}B\widehat{k}_{\lambda_n}\|^2 &\geq 1 - 2\left|\frac{(1-|\lambda_n|^2)[S^{*\beta_{\lambda_n}}B]^{\sim}(\lambda_n)}{[(I-SS^*)S^{*\beta_{\lambda_n}}B]^{\sim}(\lambda_n)}\right| \\
&\quad + \left|\frac{(1-|\lambda_n|^2)(S^{*\beta_{\lambda_n}}B)^{\sim}(\lambda_n)}{[(I-SS^*)(S^{*\beta_{\lambda_n}}B)]^{\sim}(\lambda_n)}\right|^2.
\end{aligned} \tag{15}$$

Since by condition (i) of the theorem,

$$\lim_{n\to\infty}\left|\frac{(1-|\lambda_n|^2)[S^{*\beta_{\lambda_n}}B]^{\sim}(\lambda_n)}{[(I-SS^*)S^{*\beta_{\lambda_n}}B]^{\sim}(\lambda_n)}\right| = 0,$$

we deduce from (15) that

$$\lim_{n\to\infty} \|\widehat{k}_{\lambda_n} - F_{B,\lambda_n}(0)S^{*\beta_{\lambda_n}}B\widehat{k}_{\lambda_n}\|^2 \geq 1,$$

which contradicts (14). ■

COROLLARY 9.4. *Let $T \in \mathcal{B}(H^2)$. Suppose that there exists a nonzero $B \in \{T\}'$ such that:*

(i) $(Bk_\lambda)(0) \neq 0$ *for all* $\lambda \in \mathbb{D}$ *and* $\lim_{\lambda\to\xi\in\partial\mathbb{D}} \widetilde{B}(\lambda)/(Bk_\lambda)(0) = 0$;
(ii) *there exists a sequence* $(\lambda_n)_{n\geq 1} \subset \mathbb{D}$ *converging to some* $\xi_0 \in \partial\mathbb{D}$ *and an operator* $\mathcal{K}$ *such that*

$$\lim_{n\to\infty} \|\mathcal{D}_{\widehat{k}_{\lambda_n}\circledast(B\widehat{k}_{\lambda_n})^{-1\circledast} - (\widehat{k}_{\lambda_n}\circledast(B\widehat{k}_{\lambda_n})^{-1\circledast})(0)} - \mathcal{K}\| = 0.$$

*Then $T$ has a n.i.s.*

*Proof.* As in the proof of (15), we obtain

$$
\begin{aligned}
(16)\qquad (Bk_\lambda)(0) &= \langle Bk_\lambda, k_0\rangle = \langle Bk_\lambda, \mathbf{1}\rangle = \langle Bk_\lambda, k_\lambda\rangle + \langle Bk_\lambda, 1-k_\lambda\rangle \\
&= \langle Bk_\lambda, k_\lambda\rangle + \left\langle Bk_\lambda, 1-\frac{1}{1-\overline{\lambda}z}\right\rangle = \langle Bk_\lambda, k_\lambda\rangle - \left\langle Bk_\lambda, \frac{\overline{\lambda}z}{1-\overline{\lambda}z}\right\rangle \\
&= \langle Bk_\lambda, k_\lambda\rangle - \langle Bk_\lambda, SS^*k_\lambda\rangle = \langle (I-SS^*)Bk_\lambda, k_\lambda\rangle \\
&= \|k_\lambda\|^2\langle (I-SS^*)B\widehat{k}_\lambda, \widehat{k}_\lambda\rangle = (1-|\lambda|^2)^{-1}((I-SS^*)B)^{\sim}(\lambda)
\end{aligned}
$$

for all $\lambda\in\mathbb{D}$. On the other hand, since $(Bk_\lambda)(0)\neq 0$ if and only if $(B\widehat{k}_\lambda)(0)\neq 0$, we have $\mathsf{k}_{B\widehat{k}_{\lambda_n}}(0)=0$. Now the remainder of the proof is immediate from the proof of Theorem 9.3. ∎

Corollary 9.5. *Let $T\in\mathcal{B}(H^2)$ be an operator such that*

(i) $(Tk_\lambda)(0)\neq 0$ *for all* $\lambda\in\mathbb{D}$ *and* $\lim_{\lambda\to\partial\mathbb{D}}\widetilde{T}(\lambda)/(Tk_\lambda)(0)=0$;
(ii) *there exists a sequence* $(\lambda_n)_{n\geq 1}\subset\mathbb{D}$ *tending to some point in* $\partial\mathbb{D}$ *and an operator* $\mathcal{K}$ *such that*

$$\lim_{n\to\infty}\|\mathcal{D}_{\widehat{k}_{\lambda_n}\circledast(T\widehat{k}_{\lambda_n})^{-1\circledast}-(\widehat{k}_{\lambda_n}\circledast(T\widehat{k}_{\lambda_n})^{-1\circledast})(0)}-\mathcal{K}\|=0.$$

*Then $T$ has a n.i.s.*

The proof of this corollary uses the same method as that of Theorem 9.3, and therefore is omitted.

Remark 9.6. (a) By considering formula (16), and compactness of the (one-dimensional) operator $(I-SS^*)B$, note that condition (i) in Corollary 9.4 means some growth condition for the Berezin symbol $\widetilde{B}$ of the operator $B$ in the commutant of $T$ at the boundary $\partial\mathbb{D}$.

(b) Also it is easy to see that if $T\in\mathcal{B}(H^2)$ and

$$(17)\qquad \|Tk_\lambda\|\to 0\quad \text{radially},$$

then obviously $T$ has a nontrivial invariant subspace, because in this case $T$ will just be the zero operator. Indeed, since

$$(T^*f)(\lambda)=\langle T^*f,k_\lambda\rangle=\langle f,Tk_\lambda\rangle$$

for all $f\in H^2$ and $\lambda\in\mathbb{D}$, by (17) we have

$$|(T^*f)(\lambda)|\leq\|f\|\,\|Tk_\lambda\|\to 0\quad \text{as } \lambda\to\partial\mathbb{D} \text{ radially},$$

which implies that $(T^*f)(e^{it})=0$ for almost all $t\in[0,2\pi)$. Then, by the Riesz brothers' theorem we have $T^*f=0$. Since $f\in H^2$ is arbitrary, this means that $T^*=0$, and hence $T=0$, as desired.

In general, the following two questions naturally arise:

QUESTION 9.7. Let $T : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ be a bounded operator on the standard RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ such that

$$\lim_{\lambda \to \partial\Omega} \|T\widehat{k}_{\mathcal{H},\lambda}\| = 0. \tag{18}$$

Is it true that $T$ has a nontrivial hyperinvariant subspace?

QUESTION 9.8. Let $T \in \mathcal{B}(\mathcal{H})$ be an operator on the standard RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ such that

$$\lim_{\lambda \to \partial\Omega} \widetilde{T}(\lambda) = 0. \tag{19}$$

Is it true that $T$ has a nontrivial hyperinvariant subspace?

Since $|\widetilde{T}(\lambda)| \le \|T\widehat{k}_{\mathcal{H},\lambda}\|$ for $\lambda \in \Omega$, it is clear that a positive answer to Question 9.8 will also give a positive answer to Question 9.7. Also note that, of course, for some special operators condition (18) or (19) implies compactness of $T$; see, for instance, Axler and Zheng [AZ2], where the authors characterize compact Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$. Moreover, since every compact operator on the standard RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ satisfies condition (18) (and hence also (19)), a positive answer to Question 9.8 will give an essential extension of the famous von Neumann and Lomonosov theorems in the case of RKHS.

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