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Borel equivalence relations and cardinal algebras

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Abstract. We show that Tarski's concept of cardinal algebra appears naturally in the context of the current theory of Borel equivalence relations. As a result one can apply Tarski's theory to discover a number of interesting laws governing the structure of Borel equivalence relations, which, in retrospect rather surprisingly, have not been realized before.

1. Introduction

(A) In the late 1940's Tarski published the book *Cardinal Algebras* [T], in which he developed an algebraic approach to the theory of cardinal addition, devoid of the use of the full Axiom of Choice, which of course trivializes it. A *cardinal algebra* is an algebraic system consisting of an abelian semigroup with identity (viewed additively) augmented with an infinitary addition operation for infinite sequences, satisfying certain axioms.

The theory of cardinal algebras seems to have been largely forgotten, but our goal in this paper is to show that they appear naturally in the context of the current theory of Borel equivalence relations, as can be verified by rather elementary considerations. Consequently, one can apply Tarski's theory to discover a number of interesting laws governing the structure of Borel equivalence relations, which, in retrospect rather surprisingly, have not been realized before.

Below, if E, F are Borel equivalence relations on standard Borel spaces X, Y, resp., a *Borel reduction* of E to F is a Borel function $f\colon X \to Y$ such that

$$xEy \iff f(x)Ff(y).$$

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Then f induces an injection $[f]\colon X/E \to Y/F$, defined by $[f]([x]_E) = [f(x)]_F$. We denote by $E \le_B F$ the pre-order of *Borel reducibility*, defined by

$$E \le_B F \iff \text{there is a Borel reduction of } E \text{ to } F.$$

We also let

$$E <_B F \iff E \le_B F \;\&\; F \not\le_B E,$$

and write

$$E \sim_B F \iff E \le_B F \;\&\; F \le_B E$$

for the associated notion of *Borel bireducibility.* Finally, we let

$$E \cong_B F$$

denote *Borel isomorphism.*

If $n > 0$ is a positive integer and E a Borel equivalence relation, then nE is the direct sum of n copies of E, i.e., the equivalence relation F on $X \times \{0, 1, \dots, n-1\}$ (where E lives on X) defined by $(x, i)F(y, j) \Leftrightarrow xEy \;\&\; i = j$.

Recall also that a Borel equivalence relation E is *countable* if every E-class is countable.

In order to give the flavor of the results one can obtain by applying Tarski's theory to cardinal algebras associated with Borel equivalence relations, we mention a few representative examples of results that will be discussed later (in much more general forms, see Theorem 2.2 and Section 3).

THEOREM 1.1.

(i) (Existence of least upper bounds) *Any increasing sequence $F_0 \le_B F_1 \le_B \cdots$ of countable Borel equivalence relations has a least upper bound (in the pre-order $\le_B$).*

(ii) (Interpolation) *If $\mathcal{S}, \mathcal{T}$ are countable sets of countable Borel equivalence relations and $\forall E \in \mathcal{S}\, \forall F \in \mathcal{T} (E \le_B F)$, then there is a countable Borel equivalence relation G such that $\forall E \in \mathcal{S}\, \forall F \in \mathcal{T} (E \le_B G \le_B F)$.*

(iii) (Cancelation) *If $n > 0$ and E, F are countable Borel equivalence relations, then*

$$nE \le_B nF \;\Rightarrow\; E \le_B F$$

and therefore

$$nE \sim_B nF \;\Rightarrow\; E \sim_B F.$$

(iv) (Dichotomy for integer multiples) *For any countable Borel equivalence relation E, exactly one of the following holds:*

(a) $E <_B 2E <_B 3E <_B \cdots$,

(b) $E \sim_B 2E \sim_B 3E \sim_B \cdots$.

(v) *The results in* (i)–(iv) *also hold for arbitrary Borel equivalence relations and* $\le_B$ *replaced by* $\sqsubseteq_B^i$ (*as defined after* 3.1).

(B) This paper is organized as follows. In Section 2, we review the theory of cardinal algebras. In Section 3, we discuss various cardinal algebras that arise in the theory of Borel equivalence relations and, in combination with the results mentioned in Section 2, we derive various consequences about the structure of certain classes of Borel equivalence relations. In Section 4, we show, using ergodic theory, that the multiplicative analog of the additive cancelation law in Theorem 1.1 fails for countable Borel equivalence relations: we prove that there are countable Borel equivalence relations $E <_B F$ such that $E^2 \sim_B F^2$.

2. Cardinal algebras

(A) A *cardinal algebra* (see [T]) is a system $\langle A, +, \sum\rangle$, where $\langle A, +\rangle$ is an abelian semigroup with identity, which will be denoted by 0, and $\sum\colon A^{\mathbb{N}} \to A$ is an infinitary operation, satisfying the following axioms, where we let $\sum_{n<\infty} a_n = \sum((a_n)_{n\in\mathbb{N}})$:

(A) $\sum_{n<\infty} a_n = a_0 + \sum_{n<\infty} a_{n+1}$.
(B) $\sum_{n<\infty}(a_n + b_n) = \sum_{n<\infty} a_n + \sum_{n<\infty} b_n$.
(C) If $a + b = \sum_{n<\infty} c_n$, then there are $(a_n), (b_n)$ such that

$$a = \sum_{n<\infty} a_n, \quad b = \sum_{n<\infty} b_n, \quad c_n = a_n + b_n.$$

(D) If $(a_n), (b_n)$ are such that $a_n = b_n + a_{n+1}$, then there is c such that, for each n, $a_n = c + \sum_{i<\infty} b_{n+i}$.

Remark 2.1. These axioms are slightly different than the ones in [T, Definition 1.1] but they are equivalent.

For any natural number n and finite sequence $(a_i)_{i<n}$ one can define $\sum_{i<n} a_i$ either by induction on n, using the addition operation $+$, or as $\sum_{i<\infty} b_i$, where $b_i = a_i$ for $i < n$ and $b_i = 0$ for $i \ge n$, and these turn out to be the same. By convention, when $n = 0$ this sum is equal to 0.

For a natural number n and any a, we define

$$na = \sum_{i<n} a,$$

so that in particular $0a = 0$. Furthermore, we let

$$\infty a = \sum_{n<\infty} a.$$

Let also

$$a \le b \iff \exists c(a + c = b).$$

It turns out that this is a partial ordering. Moreover all the expected commutativity and associativity laws for $+$, $\sum$ and monotonicity with respect to $\leq$ hold (see [T, Section 1]).

Finally, for any finite or infinite family $(a_i)_{i<n}$, where $n \leq \infty$, we let $\bigwedge_{i<n} a_i$ be the infimum of this family in the poset $\langle A, \leq \rangle$, if it exists, and we define similarly the supremum $\bigvee_{i<n} a_i$.

(B) In [T, Sections 2–4] Tarski derives various laws that hold in any cardinal algebra. We list below those laws that appear most interesting in the application to Borel equivalence relations in Section 3.

Theorem 2.2 (Tarski). *The following hold in any cardinal algebra* $\langle A, +, \sum \rangle$*:*

(1) [T, 2.24] *If* $a_0 \leq a_1 \leq a_2 \leq \cdots$, *then* $\bigvee_{n<\infty} a_n$ *exists.*
(2) [T, 2.21, 3.19] $\bigvee_{n<\infty} \sum_{i<n} a_i = \sum_{i<\infty} a_i$.
(3) [T, 3.4] *If* $a \wedge b$ *exists, then* $a \vee b$ *exists and* $(a \wedge b) + (a \vee b) = a + b$.
(4) [T, 3.23] *If* $n \leq \infty$ *and* $a_i \wedge a_j = 0$ *for all* $i \neq j < n$, *then* $\bigvee_{i<n} a_i = \sum_{i<n} a_i$.
(5) [T, 3.16, 3.17] *For any* $n \leq \infty$, $a = \bigvee_{i<n} a_i$ *iff for each* i, $a_i \leq a$, *and if* b *is such that for every* i, $a_i \leq b \leq a$, *then* $a = b$. *Similarly for* $\bigwedge_{i<n} a_i$.
(6) [T, 4.3] *For any* a, *we have either* $a = 2a = 3a = \cdots = \infty a$ *or* $a < 2a < 3a < \cdots < \infty a$.
(7) [T, 2.28] *If* $S, T \subseteq A$ *are non-empty countable and* $\forall a \in S \forall b \in T$ $(a \leq b)$, *then there is* c *such that* $\forall a \in S\, \forall b \in T (a \leq c \leq b)$.
[T, 2.30] *Moreover, if* $S, T \subseteq A$ *are non-empty countable and* $\forall a \in S$ $\forall b \in T (a \leq b)$ *and* $\forall a \in S \forall b \in T (a + d \leq e \leq b + d)$, *then there is* c *with* $e = c + d$ *such that* $\forall a \in S \forall b \in T (a \leq c \leq b)$.
(8) [T, 2.35] *If* $m \neq 0, n$ *are finite and* $ma + nc \leq mb + nc$, *then* $a + c \leq b + c$ *and similarly after replacing* $\leq$ *by* $=$. *In particular,* $ma \leq mb \Rightarrow a \leq b$ *and* $ma = mb \Rightarrow a = b$.
(9) [T, 2.37] *If* $m, n \geq 1$ *are finite and relatively prime, then* $ma = nb \Rightarrow \exists c (a = nc \;\&\; b = mc)$.
(10) [T, 1.37, 1.46, 1.47] *We say that* b absorbs a *iff* $a + b = b$. *Then* ∞a *is the smallest element that absorbs* a. *If* $0 < n \leq \infty$, *then* b *absorbs* a *iff* b *absorbs* na. *If* $n \leq \infty$, *then* b *absorbs* $\sum_{i<n} a_i$ *iff* $\forall i < n (b$ *absorbs* $a_i)$.
(11) [T, 2.16, 2.17] *If* $n \leq \infty$ *and* $a_i + c \leq b_i + c$, *then* $\sum_{i<n} a_i + c \leq \sum_{i<n} b_i + c$ *and similarly after replacing* $\leq$ *by* $=$.
(12) [T, 2.15] $a + c = b + c \;\&\; c \leq a, b \Rightarrow a = b$.

Some additional properties are established in [Tr].

Theorem 2.3 (Truss). *The following hold in any cardinal algebra $\langle A, +, \sum\rangle$:*

(1) [Tr, Theorem 3] *For $a_1, \dots, a_m$, $m < \infty$, there is $n < \infty$, a map φ from $\{a_1, \dots, a_m\}$ to the power set of $\{1, \dots, n\}$ and elements $b_1, \dots, b_n$ such that $a_i \le a_j \Leftrightarrow \varphi(a_i) \subseteq \varphi(a_j)$ and $a_i = \sum_{k\in\varphi(a_i)} b_k$.*

(2) [Tr, p. 582] *If $a\vee b$, $a\vee c$, $b\wedge c$ exist, then $a\vee(b\wedge c) = (a\vee b)\wedge(a\vee c)$ and similarly after switching $\vee$ and $\wedge$.*

(C) A *subalgebra* of a cardinal algebra $\langle A, +, \sum\rangle$ consists of a subset $B \subseteq A$ such that B is closed under $+, \sum$, and $\langle B, +, \sum\rangle$ is a cardinal algebra (where $+, \sum$ here are these operations restricted to B). For example, this is the case if B contains 0, is closed under $+, \sum$, and is downwards closed under $\le$.

We note that the following sets form subalgebras in any cardinal algebra $\langle A, +, \sum\rangle$. We say that $a \in A$ is *idempotent* if $2a = a$.

(i) For any idempotents a, b, the interval $I(a, b) = \{c\colon a \le c \le b\}$ and the "infinite interval" $I(a) = \{c\colon a \le c\}$. To see this, notice that if $a \le x$, then x absorbs a. So if in axiom (C) for cardinal algebras we have $a'+b' = \sum_{n<\infty} c'_n$, where $a \le a', b', c'_n$, and $(a'_n), (b'_n)$ are such that

$$a' = \sum_{n<\infty} a'_n, \qquad b' = \sum_{n<\infty} b'_n, \qquad c'_n = a'_n + b'_n,$$

then we can replace a'_n, b'_n by a'_n+a, b'_n+a without affecting these equalities. Similarly, in axiom (D) we can replace c by $c + a$.

(ii) The set $\mathrm{Id} = \{\infty a\colon a \in A\}$ of all idempotents of A. Again notice that if, in axiom (C), a, b, c_n are idempotents, then we can replace a_n, b_n by $\infty a_n, \infty b_n$. Also in axiom (D), if a_n, b_n are idempotents, we can replace c by ∞c.

It is easy to check that if a is idempotent, then $a \vee b = a + b$. Indeed, if $a, b \le c$, then $c = a + a'$ for some a'. Then $c = a + a' = a + a + a' = a + c$, so $a + b \le a + c = c$. Thus $\langle \mathrm{Id}, \le\rangle$ is an upper semilattice.

(D) If $\boldsymbol{S} = \langle S, +\rangle$ is an abelian semigroup with identity, a *finitely additive measure* (*fam*) on $\boldsymbol{S}$ is a homomorphism from $\boldsymbol{S}$ into the semigroup $\boldsymbol{R} = \langle[0, \infty], +\rangle$ (where $a + \infty = \infty + \infty = \infty$). We define again the partial pre-order $a \le b \Leftrightarrow \exists c(b = a + c)$ (this may not be a partial order). Also, for $n \in \mathbb{N}$, $a \in S$ let $na = a + \dots + a$ (n times). The following is a well-known result of Tarski (see, e.g., [W, Theorem 9.1]).

Theorem 2.4 (Tarski). *Let $\boldsymbol{S} = \langle S, +\rangle$ be an abelian semigroup with identity and $a \in S$. Then the following are equivalent:*

(i) $\forall n \in \mathbb{N}((n + 1)a \not\le na)$,

(ii) *there is a fam φ such that $\varphi(a) = 1$.*

In the particular case of a cardinal algebra $\langle A, +, \sum \rangle$, for the semigroup $\langle A, + \rangle$ condition (i) in Theorem 2.4 is equivalent to:

(i*) *a is not idempotent*,

and thus we have the following corollary:

COROLLARY 2.5. *Let $\langle A, +, \sum \rangle$ be a cardinal algebra and $a \in A$. Then the following are equivalent:*

(i) *a is not idempotent,*
(ii) *There is a fam φ on $\langle A, + \rangle$ such that $\varphi(a) = 1$.*

We also note the following result of Bhaskara Rao and Shortt [BRS]:

THEOREM 2.6 (Bhaskara Rao, Shortt). *If $\langle A, +, \sum \rangle$ is a cardinal algebra, then for any $a \neq b$ there is a fam φ such that $\varphi(a) \neq \varphi(b)$.*

3. Cardinal algebras in the theory of Borel equivalence relations

(A) Bireducibility types. In what follows, it will be convenient to admit the empty equivalence relation (on the empty space), denoted by $\emptyset$, as a Borel equivalence relation.

DEFINITION 3.1. Let $\mathcal{E}$ be a class of Borel equivalence relations. We denote by $[\mathcal{E}]$ the quotient space of $\mathcal{E}$ by $\sim_B$, i.e., $[\mathcal{E}] = \{[E] \colon E \in \mathcal{E}\}$, where $[E] = \{F \in \mathcal{E} \colon E \sim_B F\}$. We call $[E]$ the *bireducibility type* of E (in $\mathcal{E}$).

Given Borel equivalence relations E, F on standard Borel spaces X, Y, resp., we let $E \sqsubseteq_B^i F$ mean that there is a Borel F-invariant set $A \subseteq Y$ so that $E \cong_B F|A$. Then $E \sqsubseteq_B^i F$ & $F \sqsubseteq_B^i E \Rightarrow E \cong_B F$.

Finally, if E_i, $i < n$, where $n \leq \infty$, are Borel equivalence relations, with E_i living on X_i, then we let $\bigoplus_{i<n} E_i$ be the equivalence relation on $\bigsqcup_{i<n} X_i = \bigcup_{i<n} X_i \times \{i\}$ given by $(x, j) \bigoplus_{i<n} E_i (y, k) \Leftrightarrow j = k$ & $x E_j y$. In particular, $E \oplus \emptyset \cong_B E$ for any Borel equivalence relation E.

DEFINITION 3.2. Let $\mathcal{E}$ be a class of Borel equivalence relations such that:

(1) $\emptyset \in \mathcal{E}$.
(2) If $F \in \mathcal{E}$ and $E \sqsubseteq_B^i F$, then $E \in \mathcal{E}$. Equivalently, $\mathcal{E}$ is closed under $\cong_B$, and if $F \in \mathcal{E}$ lives on Y and $X \subseteq Y$ is Borel F-invariant, then $F|X \in \mathcal{E}$.
(3) If $F_0, F_1, F_2, \ldots \in \mathcal{E}$, then $\bigoplus_n F_n \in \mathcal{E}$.
(4) If $E, F_0, F_1, F_2, \ldots \in \mathcal{E}$ and $E \sim_B \bigoplus_n F_n$, then there are $E_n \in \mathcal{E}$ with $F_n \sim_B E_n$ such that $E \cong_B \bigoplus_n E_n$. Equivalently, if E lives on X, then there is a Borel partition $X = \bigsqcup_n X_n$, where X_n is E-invariant and Borel, such that $E|X_n \sim_B F_n$.

(5) If $E, F \in \mathcal{E}$ live on X, Y, resp., and $f\colon X \to Y$ is a Borel reduction of E to F, then the F-saturation $B = [f(X)]_F$ of $f(X)$ is a Borel subset of Y and $E \sim_B F|B$.

Then we say that $\mathcal{E}$ is a *Tarskian class* of Borel equivalence relations.

For any class $\mathcal{E}$ of Borel equivalence relations closed under $\bigoplus_{i<n}$, for $n \le \infty$, we can define on $[\mathcal{E}]$:

$$[E] + [F] = [E \oplus F], \qquad \sum_n [E_n] = \Big[\bigoplus_n E_n\Big].$$

It is easy to check that these are well-defined.

PROPOSITION 3.3. *If $\mathcal{E}$ is a Tarskian class of Borel equivalence relations, then $\langle [\mathcal{E}], +, \sum\rangle$ is a cardinal algebra. Moreover, for $E, F \in \mathcal{E}$, we have $E \le_B F \Leftrightarrow [E] \le [F]$.*

Proof. Axioms (A), (B) of Section 2(A) are trivial to verify. We now show that axiom (C) holds. Let $E, F, G_n \in \mathcal{E}$ be such that $[E]+[F] = \sum_n [G_n]$, i.e, $[E \oplus F] = \sum_n [G_n]$ or $E \oplus F \sim_B \bigoplus_n G_n$. By property (4) in Definition 3.2, if E lives on X, then F lives on Y, so that $E \oplus F$ lives on $X \sqcup Y$, we have $X \sqcup Y = \bigsqcup_n Z_n$, where Z_n is Borel $(E \oplus F)$-invariant, and $(E \oplus F)|Z_n \sim_B G_n$. Let $E_n = (E \oplus F)|(Z_n \cap X), F_n = (E \oplus F)|(Z_n \cap Y)$. Then $E_n, F_n \in \mathcal{E}$, $[E_n] + [F_n] = [G_n]$ and $[E] = \sum_n [E_n]$, $[F] = \sum_n [F_n]$.

Next we verify axiom (D). Let $F_n, G_n \in \mathcal{E}$ be such that $[F_n] = [G_n] + [F_{n+1}]$. Consider F_0, which lives on X_0. As $[F_0] = [G_0] + [F_1] = [G_0 \oplus F_1]$, by property (4) again, we have $X_0 = Y_0 \sqcup X_1$, where Y_0, X_1 are F_0-invariant, $F_0|Y_0 \sim_B G_0$ and $F_0|X_1 \sim_B F_1$. Since $[F_1] = [G_1 \oplus F_2]$, we have $X_1 = Y_1 \sqcup X_2$, where Y_1, X_2 are F_1-invariant, thus F_0-invariant, $F_1|Y_1 = F_0|Y_1 \sim_B G_1$ and $F_1|X_2 = F_0|X_2 \sim_B F_1$, etc. Proceeding this way, we can find pairwise disjoint F_0-invariant sets $Y_0, Y_1, Y_2, \ldots \subseteq X_0$ so that, if $X_1 = X_0 \setminus Y_0$, $X_2 = X \setminus (Y_0 \cup Y_1), \ldots,$ then $F_0|Y_n \sim_B G_n$ and $F_0|X_n \sim_B F_n$. Let $Y = X_0 \setminus \bigcup_n Y_n$. Then $G = F_0|Y \in \mathcal{E}$ and, for each n, $[F_n] = [F_0|X_n] = [F_0|Y] + [\bigoplus_{i<\infty} F_0|Y_{n+i}] = [G] + \sum_{i<\infty}[G_{n+i}]$.

That $E \le_B F \Leftrightarrow [E] \le [F]$ is obvious from Definition 3.2(5). ∎

REMARK 3.4. Note that when verifying in Proposition 3.3 that $\langle [\mathcal{E}], +, \sum\rangle$ is a cardinal algebra, we only used properties (1)–(4) of Definition 3.2. Property (5) is just used to verify the last statement in that proposition.

(B) Tarskian classes of Borel equivalence relations. We next verify that various classes of Borel equivalence relations are Tarskian.

The following concept was introduced in a stronger form (requiring a ccc condition) in [K1]:

DEFINITION 3.5. Let E be a Borel equivalence relation on X. Then E is *idealistic* if there is a map $C \in X/E \mapsto I_C$, assigning to each E-class C

a σ-ideal I_C of subsets of C, with $C \notin I_C$, such that $C \mapsto I_C$ is Borel in the following sense: For each Borel set $A \subseteq X^2$, the set $A_I \subseteq X$ defined by $x \in A_I \Leftrightarrow \{y \in [x]_E \colon (x,y) \in A\} \in I_{[x]_E}$ is Borel.

A typical example of an idealistic E is a Borel equivalence relation induced by a Borel action of a Polish group (see [K1, p. 285]).

By convention, we consider the empty equivalence relation to be countable (and so also idealistic). We now have the following result:

Theorem 3.6. *The class $\mathcal{I}$ of idealistic Borel equivalence relations is a Tarskian class, so $\langle [\mathcal{I}], +, \sum \rangle$ is a cardinal algebra.*

Proof. It is clear that $\mathcal{I}$ satisfies conditions (1)–(3) of Definition 3.2. We next verify condition (5).

Lemma 3.7. *Let $E \in \mathcal{I}$ live on a non-empty X, let F be a Borel equivalence relation living on Y and let $f \colon X \to Y$ be a Borel reduction of E to F. Then the F-saturation $B = [f(X)]_F$ of $f(X)$ is a Borel subset of Y and $E \sim_B F|B$.*

Proof. We will apply the "large section" uniformization theorem, see [K2, 18.6], in the form presented as Theorem 18.6* on p. 2 of: http://math.caltech.edu/˜kechris/papers/CDST-corrections.pdf. For the convenience of the reader we state it here:

Let X, Y be standard Borel spaces and $P \subseteq X \times Y$ be Borel with $A =$ $\mathrm{proj}_X(P) \subseteq X$*. Let $x \in A \mapsto I_x$ be a map assigning to each $x \in A$ a σ-ideal in Y such that:*

(i) *For each Borel $R \subseteq X \times Y$, there is a $\mathbf{\Sigma}^1_1$ set $S \subseteq X$ and a $\mathbf{\Pi}^1_1$ set $T \subseteq X$ such that*

$$x \in A \;\Rightarrow\; [R_x \in I_x \Leftrightarrow x \in S \Leftrightarrow x \in T],$$

(ii) $x \in A \Rightarrow P_x \notin I_x$.

Then there is a Borel uniformization of P and, in particular, A is Borel.

Define $P \subseteq Y \times X$ by $(y,x) \in P \Leftrightarrow f(x)Fy$. Then if $B = \mathrm{proj}_{\mathrm{Y}}(P)$, clearly $B = [f(X)]_F$. Let $C \mapsto I_C$ witness that E is idealistic, and for each $y \in B$, let $I_y = I_C$, where $C = f^{-1}([y]_F)$. Clearly, for any $y \in B$, $P_y = \{x \colon (y,x) \in P\} = C \notin I_y$, so condition (ii) in Theorem 18.6* is satisfied.

We next verify condition (i) in that theorem. Let $R \subseteq Y \times X$ be Borel and define $Q \subseteq X^2$ by $(x,x') \in Q \Leftrightarrow xEx'$ & $(f(x),x') \in R$. Then Q is Borel and for $y \in B$ we have $R_y \in I_y \Leftrightarrow \exists x[f(x)Fy \;\&\; x \in Q_I] \Leftrightarrow \forall x[f(x)Fy \Rightarrow x \in Q_I]$. Thus for $y \in B$, the condition $R_y \in I_y$ is both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, which verifies (i).

It follows that B is Borel and there is a Borel uniformization of P, which clearly gives a Borel reduction of $F|B$ to E, thus $E \sim_B F|B$. ∎

It is clear that the proof of Lemma 3.7 also shows the following:

LEMMA 3.8. *Let $E \in \mathcal{I}$ live on non-empty X, let F be a Borel equivalence relation living on Y and let $f: X \to Y$ be a Borel reduction of E to F. Then, for any E-invariant Borel set $A \subseteq X$, the set $B = [f(A)]_F$ is Borel and there is a function $g: B \to A$ which is a Borel reduction of $F|B$ to $E|A$ and if $[f]: X/A \to B/F$ and $[g]: B/F \to A/E$ are the induced functions, then $[g] = [f]^{-1}$.*

LEMMA 3.9. *Let $E, F \in \mathcal{I}$ live on non-empty X, Y and let $f: X \to Y$ be a Borel reduction of E to F and $g: Y \to X$ a Borel reduction of F to E. Then there is an E-invariant Borel set $A \subseteq X$ such that if $B = [f(A)]_F$, then B is Borel, $[f]: A/E \to B/F$ is* (*clearly*) *a bijection and $[g]: (Y \setminus B)/F \to (X \setminus A)/E$ is also a bijection.*

Proof. We follow the standard proof of the Schröder–Bernstein Theorem (see, e.g., [K2, Theorem 15.7]).

A subset $X' \subseteq X/E$ will be called "Borel" if $\{x \in X: [x]_E \in X'\}$ is Borel, and similarly for $Y' \subseteq Y/F$. By Lemma 3.8, if X' is "Borel", then $[f](X')$ is "Borel". Similarly, if Y' is "Borel", so is $[g](Y')$.

Define inductively $X'_n \subseteq X/E$, $Y'_n \subseteq Y/F$ as follows: $X'_0 = X/E$, $Y'_0 = Y/F$, $X'_{n+1} = [g][f](X'_n)$, $Y'_{n+1} = [f][g](Y'_n)$. Let also $X'_\infty = \bigcap X'_n$, $Y'_\infty = \bigcap Y'_n$ and set $A' = X'_\infty \cup \bigcup_n (X'_n \setminus [g](Y'_n))$ and $B' = Y'_\infty \cup \bigcup_n ([f](X'_n) \setminus Y'_{n+1})$. Then $[f](A') = B'$ and $[g]((Y/F) \setminus B') = (X/E) \setminus A'$. Finally, define $A = \{x \in X: [x]_E \in A'\}$ and $B = \{y \in Y: [y]_F \in B'\}$. ∎

To finish the proof of Theorem 3.6, we use the above lemmas to verify condition (4) of Definition 3.2. Let $E, F_0, F_1, F_2, \ldots \in \mathcal{I}$ and $E \sim_B \bigoplus_n F_n$. Say E lives on X. Then we can find Y, a Borel partition $Y = \bigsqcup_n Y_n$ and F a Borel equivalence relation on Y such that Y_n is F-invariant, $F|Y_n \sim_B F_n$ and $E \sim_B F$. Let $f: X \to Y$ and $g: Y \to X$ witness that $E \sim_B F$. By Lemma 3.9, there is an E-invariant Borel set $A \subseteq X$ such that if $B = [f(A)]_F$ then B is Borel, $[f]: E/A \to B/F$ is a bijection and $[g]: (Y \setminus B)/F \to (X \setminus A)/E$ is also a bijection. Let $X_n = f^{-1}(B \cap Y_n) \cup [g((Y \setminus B) \cap Y_n)]_E$. Then by Lemma 3.8, $E|X_n \sim_B F|Y_n \sim_B F_n$ and clearly $X = \bigsqcup_n X_n$. ∎

We next discuss various cardinal subalgebras of $\langle [\mathcal{I}], +, \sum \rangle$. We recall first some basic concepts and results from the theory of Borel equivalence relations.

A Borel equivalence relation is *smooth* if it has a Borel reduction to the equality relation on a Polish space. For countable Borel equivalence relations this is equivalent to the existence of a Borel transversal. The equivalence relation E_0 on $2^{\mathbb{N}}$ defined by $(x_n) E_0 (y_n) \Leftrightarrow \exists n \forall m \geq n (x_m = y_m)$ is

$\leq_B$-minimum among non-smooth Borel equivalence relations (see [HKL]). A Borel equivalence relation E is *treeable* if there is a Borel acyclic graph whose connected components are the E-classes. Among the treeable countable Borel equivalence relations there is a $\leq_B$-maximum, called the *universal treeable* countable Borel equivalence relation (see [JKL]). Also, among all countable Borel equivalence relations there is a $\leq_B$-maximum, called the *universal* countable Borel equivalence relation (see [DJK]). Finally, a Borel equivalence relation is called *essentially countable* if it has a Borel reduction to a countable Borel equivalence relation.

THEOREM 3.10.

(i) *Let $\mathcal{C}$ be the class of countable Borel equivalence relations. Then $\langle[\mathcal{C}], +, \sum\rangle$ is a cardinal algebra.*
(ii) *Let $\mathcal{NSC}$ be the class of non-smooth countable Borel equivalence relations. Then $\langle[\mathcal{NSC}], +, \sum\rangle$ is a cardinal algebra.*
(iii) *Let $\mathcal{T}$ be the class of treeable countable Borel equivalence relations. Then $\langle[\mathcal{T}], +, \sum\rangle$ is a cardinal algebra. Similarly for the class $\mathcal{NST}$ of non-smooth treeable countable Borel equivalence relations.*
(iv) *Let $\mathcal{NU}$ be the class of non-universal countable Borel equivalence relations. Then $\langle[\mathcal{NU}], +, \sum\rangle$ is a cardinal algebra.*
(v) *Let $\mathcal{I}d\mathcal{C}$ be the class of idempotent countable Borel equivalence relations, i.e., those satisfying $E \oplus E \sim_B E$. Then $\langle[\mathcal{I}d\mathcal{C}], +, \sum\rangle$ is a cardinal algebra.*

Proof. (i) It is easy to check that $\mathcal{C}$ is a Tarskian class. Alternatively note that we can view $[\mathcal{C}]$ as a subset of $[\mathcal{I}]$ by identifying the bireducibility type of E in $\mathcal{C}$ with its bireducibility type in $\mathcal{I}$. If e_∞ is the bireducibility type of a universal countable Borel equivalence relation, then, by Lemma 3.7, $[\mathcal{C}]$ is the interval $I(0, e_\infty)$ of $[\mathcal{I}]$ and thus a subalgebra of $[\mathcal{I}]$, since e_∞ is idempotent.

(ii) Clearly $[\mathcal{NSC}]$ is the "infinite interval" $I(e_0)$ of $[\mathcal{C}]$, where e_0 is the bireducibility type of E_0, which is idempotent.

(iii) Again $[\mathcal{T}]$ is the interval $I(0, e_{\infty T})$ of $[\mathcal{C}]$, where $e_{\infty T}$ is the bireducibility type of a universal treeable countable Borel equivalence relation, which is idempotent, and $[\mathcal{NST}]$ is the interval $I(e_0, e_{\infty T})$.

(iv) $[\mathcal{NU}]$ is a subalgebra of $[\mathcal{C}]$, as follows from the fact that the sum of a sequence of non-universal relations is non-universal, which is a result of Marks (see [MSS, Theorem 3.8]).

(v) See Section 2(C). ∎

In particular all the laws mentioned in Theorem 2.2 apply to all these cardinal algebras; this includes all the results in Theorem 1.1 (except for the last part of (iii) that will be dealt with in (C) below).

REMARK 3.11. Consider the class $\mathcal{T}^*$ of Borel equivalence relations which are treeable and essentially countable. Then by Hjorth [H1] every such relation E admits a Borel countable complete section A, and then by an argument similar to that in the proof of [JKL, Theorem 3.3(i)] it follows that $E|A$ is a treeable countable Borel equivalence relation and of course $E|A \sim_B E$. Therefore $[\mathcal{T}] = [\mathcal{T}^*]$ and Theorem 3.10(iii) holds as well for $\mathcal{T}^*$.

We will finish this subsection with some remarks and open questions concerning the structure of some of the cardinal algebras of bireducibility types discussed here.

Consider the cardinal algebra $\langle [\mathcal{NSC}], +, \sum \rangle$. Its identity element is $[E_0]$, which is of course its $\leq$-least element. It also has a $\leq$-largest element, namely $[E_\infty]$, where E_∞ is a universal countable Borel equivalence relation. It is known (see [AK]) that $\langle [\mathcal{NSC}], \leq \rangle$ is very complicated, e.g., one can embed in it every Borel poset. However the following is open:

PROBLEM 3.12. *Is $\langle [\mathcal{NSC}], \leq \rangle$ a lattice? Equivalently* (*by Theorem* 2.2(3)) *is it true that for any $a, b \in [\mathcal{NSC}]$, $a \wedge b$ exists?*

In an earlier version of this paper, we mentioned that in fact it seemed to be unknown whether there are any $\leq$-incomparable a, b for which $a \wedge b$ exists. Such examples have now been found in [CK].

Consider next the cardinal algebra $\langle [\mathcal{NU}], +, \sum \rangle$. We have here the following open problem:

PROBLEM 3.13. *Does $\langle [\mathcal{NU}], \leq \rangle$ have a $\leq$-largest element? If not, what is the shortest length of an unbounded wellordered subset of $\langle [\mathcal{NU}], \leq \rangle$* (*it is clearly at least $\aleph_1$*)*?*

We have seen that $\langle [\mathcal{I}d\mathcal{C}], \leq \rangle$ is an upper semilattice (see Section 2(C)).

PROBLEM 3.14. *Is $\langle [\mathcal{I}d\mathcal{C}], \leq \rangle$ a lattice?*

If this is the case, then by Theorem 2.3(2) it would be distributive. Note that, by [AK] again, $\langle [\mathcal{I}d\mathcal{C}], \leq \rangle$ also embeds any Borel poset.

It is known that there are non-idempotent elements in $[\mathcal{NSC}]$ (see S. Thomas [Th]). In fact [Th, Lemma 3.4] gives a countable Borel equivalence relation $E \in \mathcal{NSC}$ which is not divisible by any $n > 1$, i.e., there is no $F \in \mathcal{C}$ with $nF \sim_B F$. It follows, using Corollary 2.5, that there is a fam on $\langle [\mathcal{NSC}], + \rangle$ for which $\varphi([E]) = 1$, so φ takes a finite value. Of course no such φ can exist on $\langle [\mathcal{I}d\mathcal{C}], + \rangle$.

Finally, let $\mathcal{B}$ be the class of all Borel equivalence relations.

PROBLEM 3.15. *Is $\langle [\mathcal{B}], +, \sum \rangle$ a cardinal algebra?*

As opposed to the last statement in Proposition 3.3, however, it is not the case that for Borel equivalence relations E, F we have $E \leq_B F \Leftrightarrow [E] \leq [F]$. To see this we use the following result of Hjorth [H]: There is a

Borel equivalence relation E such that for some countable Borel equivalence relation F we have $E \leq_B F$, but for no countable Borel equivalence relation G do we have $E \sim_B G$. We claim that then $[E] \not\leq [F]$. Otherwise there is a Borel equivalence relation H such that $E \oplus H \sim_B F$. Let E, H, F live on X, Y, Z, resp., so that $E \oplus H$ lives on $X \sqcup Y$. Let $F\colon Z \to X \sqcup Y$ witness that $F \leq_B E \oplus H$. Set $W = f^{-1}(X)$. Then W is F-invariant and $F|W \sim_B E$, a contradiction.

(C) Borel isomorphism types. For each Borel equivalence relation $E \in \mathcal{B}$, denote by $[E]_{\cong} = \{F \in \mathcal{B} \colon E \cong_B F\}$ its Borel isomorphism type. Let $[\mathcal{B}]_{\cong} = \{[E]_{\cong} \colon E \in \mathcal{B}\}$ be the set of isomorphism types of Borel equivalence relations. We can define $+, \sum$ on $[\mathcal{B}]_{\cong}$ as before, and then it is not hard to check that $\langle [\mathcal{B}]_{\cong}, +, \sum \rangle$ is a cardinal algebra. It is also clear that in this cardinal algebra $[E]_{\cong} \leq [F]_{\cong} \Leftrightarrow E \sqsubseteq_B^i F$. In particular all the laws in Theorem 2.2 hold in $\langle [\mathcal{B}]_{\cong}, +, \sum \rangle$.

4. Cancelation fails for products. We show here that the Cancelation Law

$$n > 1,\, nE \sim_B nF \;\Rightarrow\; E \sim_B F$$

fails for products in the context of countable Borel equivalence relations. This answers a question of Andrew Marks, who raised it in connection with a discussion with Igor Pak on a related issue.

If E, F are Borel equivalence relations, on X, Y, resp., then their product $E \times F$ is the equivalence relation on $X \times Y$ defined by

$$(x, y) E \times F (x', y') \;\Leftrightarrow\; xEx' \;\&\; yFy'.$$

For $n \geq 1$ we let E^n be the product of n copies of E. We now have:

Theorem 4.1. *There are countable Borel equivalence relations $E <_B F$ such that $E^2 \sim_B F^2$.*

The proof was inspired by the result of Tarski in cardinal arithmetic that states that the Axiom of Choice is equivalent to the statement: For any two infinite cardinals κ, λ ($\kappa^2 = \lambda^2 \Rightarrow \kappa = \lambda$); see [J, Theorem 11.8]. As opposed to the proof of Tarski's Theorem, which makes use of the Hartogs number of an infinite cardinal to produce, assuming the Axiom of Choice fails, two infinite cardinals μ, ν such that if $\kappa = \mu + \nu$, $\lambda = \mu \cdot \nu$, then $\kappa^2 = \lambda^2$ but $\kappa < \lambda$, the proof of Theorem 4.1 uses ideas of ergodic theory and geometric group theory. The main idea for the construction of a pair E, F as in Theorem 4.1 is based on the following result:

Theorem 4.2. *Suppose R, S are Borel equivalence relations on standard Borel spaces X, Y, resp., such that:*

(i) *X/R and Y/S are infinite.*

(ii) *There are probability Borel measures μ, ν on X, Y, resp., such that R is μ-ergodic, S is ν-ergodic and, for any R-invariant Borel set A with $\mu(A) = 1$ and any S-invariant Borel set B with $\nu(B) = 1$, we have $R|A \not\leq_B S$ and $S|B \not\leq_B R$.*

(iii) $R^2 \sim_B R$ *and* $S^2 \sim_B S$.

Let $E = R \oplus S$ *and* $F = R \times S$. *Then* $E <_B F$ *but* $E^2 \sim_B F^2$.

Proof. First notice that $R \oplus S \leq_B R \times S$. Indeed $R \oplus S$ lives on the direct sum $X \sqcup Y$. Fix $(x_0, y_0) \in X \times Y$. Since clearly $R \leq_B R|(X \setminus [x_0]_R)$, and similarly for S, it is enough to show that $R|(X \setminus [x_0]_R) \oplus S|(Y \setminus [y_0]_S) \leq_B R \times S$. Let $Z = (X \setminus [x_0]_R) \sqcup (Y \setminus [y_0]_S)$.Then define $f\colon Z \to X \times Y$ by $f(x) = (x, y_0)$ and $f(y) = (x_0, y)$. Then f is Borel reduction of $R|(X \setminus [x_0]_R) \oplus S|(Y \setminus [y_0]_S)$ to $R \times S$.

Clearly, we have $(R \times S)^2 \sim_B R^2 \times S^2 \sim_B R \times S$. Also $(R \oplus S)^2 \sim_B R^2 \oplus 2R \times S \oplus S^2 \geq_B R \times S$. Observe that, denoting by 2 the equality relation on a set of cardinality 2, we have $2 \times R = 2R$ and $2 \leq_B R$, so $2R \leq_B R^2 \sim_B R$, therefore $2R \sim_B R$. Thus $(R \oplus S)^2 \sim_B R \oplus S \oplus (R \times S) \leq_B 2R \times S \sim_B R \times S$, so $(R \oplus S)^2 \sim_B (R \times S)^2$.

It remains to show that $R \oplus S <_B R \times S$. Otherwise, assume that $R \times S \leq_B R \oplus S$, towards a contradiction, and let $f\colon X \times Y \to X \sqcup Y$ witness that. Set $X_0 = f^{-1}(X)$ and $Y_0 = f^{-1}(Y)$, so that $X_0 \sqcup Y_0 = X \times Y$. Also, X_0, Y_0 are $(R \times S)$-invariant and $(R \times S)|X_0 \leq_B R$, $(R \times S)|Y_0 \leq_B S$.

Claim. *$R \times S$ is $(\mu \times \nu)$-ergodic.*

Proof of Claim. Let $A \subseteq X \times Y$ be $(R \times S)$-invariant. Then for each x the section A_x is S-invariant and $xRx' \Rightarrow A_x = A_{x'}$. Thus the function $x \mapsto \nu(A_x) \in \{0, 1\}$ is R-invariant, thus constant μ-a.e. If this constant value is 1, then by Fubini's Theorem $\mu \times \nu(A) = 1$, while if it is 0, $\mu \times \nu(A) = 0$.

So we have two possibilities: $\mu \times \nu(X_0) = 1$ or $\mu \times \nu(X_1) = 1$. In the first case, there is x such that $\nu((X_0)_x) = 1$ and of course $(X_0)_x$ is S-invariant. The map $y \in (X_0)_x \mapsto f(x, y)$ witnesses that $S|(X_0)_x \leq_B R$, which is a contradiction. The second case is similar.

This gives the Claim, and hence also Theorem 4.2. ∎

Thus to complete the proof of Theorem 4.1, it remains to construct examples of countable Borel equivalence relations R, S satisfying conditions (i)–(iii) of Theorem 4.2.

We will use the following result from group theory that was explained to us by Simon Thomas in response to a question by one of the authors.

Theorem 4.3 (Yu. A. Ol'shanskiĭ). *There is a countable, torsion free, simple group Γ with property* (T), *and an infinite countable, torsion, simple group Δ with property* (T).

Proof. For the convenience of the reader, we will give a sketch of the proof based on the results of Ol'shanskiĭ [O].

Fix a countable group G which is torsion free, hyperbolic and has property (T) (see, e.g., [DC, Proposition 2]). By [O, Corollary 1], there is a torsion free quotient Γ of G all of whose non-trivial proper subgroups are cyclic. Thus Γ has property (T) and we next check that it is simple. By the proof of Corollary 1 in [O, pp. 403–404], the center of Γ is trivial. If N is a non-trivial proper normal subgroup, then it is not contained in the center, so by looking at the conjugation action of Γ on N, we have a non-trivial homomorphism of Γ into the automorphism group of N, which is a 2-element group, so Γ has a subgroup of index 2, a contradiction.

To define Δ, use [O, Corollary 4] to find an infinite quotient G_1 of G which is quasi-finite (i.e., every proper subgroup is finite). If $N \lhd G_1$ is a proper normal subgroup, by looking again at the conjugation action of G_1 on the finite group N, we conclude that $N \leq Z(G_1)$. Define $\Delta = G_1/Z(G_1)$. Then Δ is infinite, torsion, simple and has property (T). ∎

Fix the groups Γ, Δ as in Theorem 4.3. Define $\Gamma^* = \Gamma \oplus \Gamma \oplus \cdots$ and $\Delta^* = \Delta \oplus \Delta \oplus \cdots$. Then $\Gamma^* \times \Gamma^* \cong \Gamma^*$ and $\Delta^* \times \Delta^* \cong \Delta^*$. Since every homomorphism from Γ to Δ is trivial and vice versa, it follows that every homomorphism of Γ to Δ^* is trivial and vice versa.

For any countable group G, consider the shift action of G on $[0,1]^G$ restricted to its free part and let F_G be the corresponding equivalence relation. Put now $R = F_{\Gamma^*}$, which lives in X, and $S = F_{\Delta^*}$, which lives in Y. We will verify that these satisfy the conditions of Theorem 4.2. Let μ be the product measure on $[0,1]^{\Gamma^*}$ restricted to X and similarly define ν on Y.

Condition (i) of Theorem 4.2 is obvious. Also, R is μ-ergodic and similarly S is ν-ergodic. We will next verify that if $A \subseteq X$ is R-invariant and has μ-measure 1, then $R|A \not\leq_B S$ (and vice versa).

For that we will use the superrigidity result of Popa [P] (see also [K3, Theorem 30.5] for an exposition), which asserts that if G is a countable infinite group with property (T), H is a countable group, and α is a Borel cocycle of the shift action of G on $[0,1]^G$ into H, then α is cohomologous to a homomorphism from G to H.

So assume that f is a Borel reduction of $R|A$ to S. Viewing Γ in the obvious way as a subgroup of Γ^*, we obtain the following Borel cocycle $\alpha(\gamma, x)$ from the restriction to Γ of the shift action of Γ^* on $[0,1]^{\Gamma^*}$ into Δ^*: $f(\gamma \cdot x) = \alpha(\gamma, x) \cdot f(x)$. Since this action of Γ is isomorphic to the shift action of Γ on $[0,1]^\Gamma$, by Popa's Theorem there is a Borel function $\pi\colon X \to \Gamma^*$ such that $\alpha(\gamma, x) = \pi(\gamma \cdot x)\pi(x)^{-1}$, μ-a.e. Let $g(x) = \pi(x)^{-1} \cdot f(x)$. Then g is also a reduction of $R|A$, μ-a.e., to S and $g(\gamma \cdot x) = g(x)$. By ergodicity, g must be constant μ-a.e., a contradiction.

Finally, we verify condition (iii) of Theorem 4.2. We will show $R^2 \sim_B R$ and similarly for S.

We have $R^2 = (F_{\Gamma^*})^2$, which is an equivalence relation on X^2 induced by the following free action of $\Gamma^* \times \Gamma^*$ on X^2: $(\delta, \epsilon) \cdot (x, y) = (\delta \cdot x, \epsilon \cdot y)$. Any free Borel action of a countable group G on an uncountable standard Borel space Z, which we can assume is the interval $[0, 1]$, can be embedded in a Borel way into the shift action of G on $[0, 1]^G$ via $z \mapsto (g \to g^{-1} \cdot z)$. Therefore $R^2 \leq_B F_{\Gamma^* \times \Gamma^*} \cong_B F_{\Gamma^*} = R$, so $R \sim_B R^2$.

Remark 4.4. There is a Baire category analog of Theorem 4.2, where X, Y are now Polish spaces and R, S are generically ergodic and, for any R-invariant Borel comeager set A and any every S-invariant Borel comeager set B, we have $R|A \not\leq_B S$ and $S|B \not\leq_B R$. Using this, one can show that for $R = E_0^{\mathbb{N}}$ and $S = E_1$, if $E = R \oplus S$ and $F = R \times S$, then $E <_B F$ but $E^2 \sim_B F^2$. (Here E_1 is the equivalence relation on $\mathbb{R}^{\mathbb{N}}$ defined by $(x_n) E_1 (y_n) \Leftrightarrow \exists n \forall m \geq n (x_m = y_m)$ and $E_0^{\mathbb{N}}$ is the equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined by $(x_n) E_0^{\mathbb{N}} (y_n) \Leftrightarrow \forall n (x_n E_0 y_n)$.)

Remark 4.5. One can also consider the set $[\mathcal{C}]$ as in Theorem 3.10(i), with the operation of multiplication $[E] \cdot [F] = [E \times F]$. It forms an abelian semigroup with identity (the equivalence relation on a singleton space). If $E_{\infty T}$ is the universal treeable countable Borel equivalence relation, then, by [HK, Theorem 8.1], we have $E_{\infty T} <_B E_{\infty T}^2 <_B E_{\infty T}^3 <_B \cdots$, so, by Theorem 2.4, there is a fam on $\langle [\mathcal{C}], \cdot \rangle$ such that $\varphi([E_{\infty T}]) = 1$.

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