ANNALES
POLONICI MATHEMATICI
117.1 (2016)

On the volume of a pseudo-effective class and semi-positive properties of the Harder–Narasimhan filtration on a compact Hermitian manifold

Zhiwei Wang (Beijing)

Abstract. This paper divides into two parts. Let (X,ω) be a compact Hermitian manifold. Firstly, if the Hermitian metric ω satisfies the assumption that $\partial\bar\partial\omega^k = 0$ for all k, we generalize the volume of the cohomology class in the Kähler setting to the Hermitian setting, and prove that the volume is always finite and the Grauert–Riemenschneider type criterion holds true, which is a partial answer to a conjecture posed by Boucksom. Secondly, we observe that if the anticanonical bundle K_X^{-1} is nef, then for any $\varepsilon > 0$, there is a smooth function ϕ_ε on X such that $\omega_\varepsilon := \omega + i\partial\bar\partial\phi_\varepsilon > 0$ and $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$. Furthermore, if ω satisfies the assumption as above, we prove that for a Harder–Narasimhan filtration of T_X with respect to ω, the slopes $\mu_\omega(\mathcal{F}_i/\mathcal{F}_{i-1})$ are nonnegative for all i; this generalizes a result of Cao which plays an important role in his study of the structures of Kähler manifolds.

1. Introduction. In this paper, we recall some results in Kähler geometry and study to what extent they can be generalized to the case of Hermitian manifolds.

Let L be a holomorphic line bundle on a compact complex manifold X. One defines the volume of L as

$$\mathrm{vol}(L) := \limsup_{k\to\infty} \frac{n!}{k^n} h^0(X, kL).$$

It is well-known that if $\mathrm{vol}(L) > 0$, then L is big. From [17], one knows that if $\mathrm{vol}(L) > 0$, the limsup is in fact a limit, so that $\mathrm{vol}(L)$ can be seen as a measure of the bigness of L. From the definition, $\mathrm{vol}(kL) = k^n \mathrm{vol}(L)$. Thus one can also define the volume of a $\mathbb{Q}$-line bundle by setting $\mathrm{vol}(L) = k^{-n}\mathrm{vol}(kL)$ for some k such that kL is an actual line bundle.

2010 *Mathematics Subject Classification*: Primary 53C55; Secondary 31C10, 32J25, 32Q26, 32S45.
Key words and phrases: volume, $\partial\bar\partial$-cohomology, nef class, pseudo-effective class, big class, closed positive current, Gauduchon metric, Monge–Ampère equation, Harder–Narasimhan filtration, stability.
Received 25 July 2015; revised 10 November 2015.
Published online 22 April 2016.

DOI: 10.4064/ap3780-11-2015

Boucksom [4] introduced a formula expressing the volume of L in terms of $c_1(L)$:

$$\mathrm{vol}(L) = \max_T \int_X T_{\mathrm{ac}}^n$$

for T ranging over the closed positive $(1,1)$-currents in the cohomology class $c_1(L)$, if L is not pseudo-effective, then we let $\mathrm{vol}(L) = 0$. Here T_{ac} is the absolutely continuous part of the Lebesgue decomposition of T on X. Furthermore, the volume of a line bundle is generalized to cohomology classes: to cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$, we define

$$\mathrm{vol}(\alpha) := \sup_T \int_X T_{\mathrm{ac}}^n$$

for T ranging over the closed positive $(1,1)$-currents in the class α, in case α is pseudo-effective; otherwise we let $\mathrm{vol}(\alpha) = 0$. The Kähler property plays an important role in the proof of the finiteness of the above volumes. Here we mention a couple of results in [4]:

(a) If $\alpha \in H^{1,1}(X, \mathbb{R})$ is nef, then $\mathrm{vol}(\alpha) = \alpha^n$.
(b) A class $\alpha \in H^{1,1}(X, \mathbb{R})$ is big if and only if $\mathrm{vol}(\alpha) > 0$.

In fact, (b) is a Grauert–Riemenschneider type criterion for bigness. For completeness let us recall the Grauert–Riemenschneider conjecture (now it is a theorem): a compact complex variety Y is Moishezon if and only if there is a proper nonsingular modification $X \to Y$ and a line bundle L over X such that the curvature $i\Theta_L$ is > 0 on a dense open subset. A compact complex manifold is said to be *Moishezon* if it is birational to a projective manifold. Siu [31] first proved this conjecture by getting a stronger result that X is Moishezon as soon as $i\Theta_L \geq 0$ everywhere and $i\Theta_L > 0$ at least at one point. Later Demailly [12, 13] gave another proof of a stronger result than the conjecture by using his holomorphic Morse inequalities. Also Berndtsson [1] gave another proof. It is proved in [24] that a compact complex manifold X is Moishezon if and only if X admits an integral Kähler current, i.e. there exists a big line bundle L on X. Now one can see that (b) above is obviously a generalization of the Grauert–Riemenschneider criterion. In fact, it gives a criterion for a transcendental class to be big rather than an integral class.

To conclude, the philosophy of the study of the volumes defined above is to ask for the existence of a Kähler current in a class α provided that $\mathrm{vol}(\alpha) > 0$. In [4], the following conjecture was posed.

Conjecture 1.1. *If a compact complex manifold X carries a closed positive $(1,1)$-current T with $\int_X T_{\mathrm{ac}}^n > 0$, then X is in the Fujiki class.*

A compact complex manifold X is said to be in the *Fujiki class* if it is bimeromorphic to a Kähler manifold. Demailly [15] proved that a compact complex manifold X is in the Fujiki class if and only if it carries a Kähler current.

Throughout this paper, we say that a Hermitian metric ω satisfies *assumption* $(*)$ if $\partial\overline{\partial}\omega^k = 0$ for all $k \in \{1, \dots, n-1\}$.

Now let (X, ω) be a compact Hermitian manifold, and α an arbitrary cohomology class α in $H^{1,1}_{\partial\overline{\partial}}(X, \mathbb{R})$. One defines the volume of α as

$$\mathrm{vol}(\alpha) := \sup_T \int_X T_{\mathrm{ac}}^n$$

for T ranging over the closed positive $(1,1)$-currents in the class α, in case α is pseudo-effective; if it is not, we set $\mathrm{vol}(\alpha) = 0$. We will see that the supremum involved is always finite under our assumption $(*)$. It is trivial that the volume $\mathrm{vol}(\alpha)$ of a big class α is nonzero. Firstly, we will prove that (a) also holds when (X, ω) is a compact Hermitian manifold endowed with a Gauduchon metric ω satisfying assumption $(*)$. Furthermore, by adapting arguments from [4] and [10], we are able to prove the following partial solution to Conjecture 1.1.

Theorem 1.2. *Let X be a compact complex manifold, and let ω be a Gauduchon metric on X satisfying assumption* $(*)$*. If X carries a pseudo-effective class $\alpha \in H^{1,1}_{\partial\overline{\partial}}(X, \mathbb{R})$ such that* $\mathrm{vol}(\alpha) > 0$*, then X is Kähler.*

Thus for the same reason as in [4], this definition is compatible with the previous one when X is assumed to satisfy assumption $(*)$.

Since every compact complex surface always carries a Gauduchon metric satisfying assumption $(*)$, Theorem 1.2 states that the Grauert–Riemenschneider type criterion always holds true on compact complex surfaces, which was proved in [4] by a different argument.

Chiose [10] proved that if X is a compact complex manifold, which admits a Gauduchon metric satisfying assumption $(*)$, and a nef class $\alpha \in H^{1,1}_{\partial\overline{\partial}}(X, \mathbb{R})$ has positive volume, then α is a big class and X is in the Fujiki class, and finally Kähler. The main difference between our Theorem 1.2 and Chiose's result is that we only assume that α is a pseudo-effective class. In general, the nef cone is only a subset of the pseudo-effective cone.

Recently, there has been important progress on the study of the structure of compact Kähler manifolds with nef anticanonical bundles. In [28, 29], it is proved that if X is a compact Kähler manifold with K_X^{-1} nef, then $\pi_1(X)$ has polynomial growth, and as a consequence it has a nilpotent subgroup of finite index. In [6, 7], it is proved that a compact Kähler manifold X with K_X^{-1} nef is projective and rationally connected if and only if $H^0(X, (T_X^*)^{\otimes m}) = 0$ for all $m \geq 1$. This result is a partial solution to a conjecture attributed to Mumford. The following two properties are crucial to prove the above results.

(1) Let (X,ω) be a compact Kähler manifold, where $\{\omega\}$ is the Kähler class on X. Then K_X^{-1} is nef if and only if for every $\varepsilon > 0$, there exists a Kähler metric $\omega_\varepsilon = \omega + i\partial\overline{\partial}\phi_\varepsilon$ in the cohomology class $\{\omega\}$ such that $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$.
(2) Let (X,ω) be a compact Kähler manifold with K_X^{-1} nef. Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s = T_X$$

be a Harder–Narasimhan filtration of T_X with respect to ω. Then

$$\mu_\omega(\mathcal{F}_i/\mathcal{F}_{i-1}) \geq 0 \quad \text{for all } i.$$

Since on a compact Hermitian manifold, K_X^{-1} can also be defined, and there is an analogue of the Harder–Narasimhan filtration, it is natural to ask whether we can get similar characterizations of nef K_X^{-1} and the Harder–Narasimhan filtration on a compact Hermitian manifold. In this paper, we prove

THEOREM 1.3. *Let (X,ω) be a compact Hermitian manifold. Then the following properties are equivalent:*

(i) *K_X^{-1} is nef.*
(ii) *For every $\varepsilon > 0$, there exists a smooth real function ϕ_ε such that $\omega_{\phi_\varepsilon} = \omega + i\partial\overline{\partial}\phi_\varepsilon > 0$ and $\mathrm{Ricci}(\omega_{\phi_\varepsilon}) \geq -\varepsilon\omega_{\phi_\varepsilon}$.*

THEOREM 1.4. *Let (X,ω) be a compact complex manifold with a Gauduchon metric ω satisfying assumption $(*)$. Assume that K_X^{-1} is nef. Let*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s = T_X$$

be a Harder–Narasimhan filtration of T_X with respect to ω. Then

$$\mu_\omega(\mathcal{F}_i/\mathcal{F}_{i-1}) \geq 0 \quad \textit{for all } i.$$

The structure of this paper is as follows. In Section 2, we prepare the technical preliminaries. In Section 3, we prove that for any nef class $\alpha \in H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$, the volume satisfies $\mathrm{vol}(\alpha) = \alpha^n$. This is a generalization of [4, Theorem 4.1]. In Sections 4, 5, and 6, we prove Theorems 1.2, 1.3, and 1.4, respectively.

2. Technical preliminaries. Let X be a compact complex n-fold. We will use dd^c to denote the operator $\frac{i}{\pi}\partial\overline{\partial}$.

DEFINITION 2.1. A closed real $(1,1)$-current T on X is said to be *almost positive* if $T \geq \gamma$ for some smooth real $(1,1)$-form γ. A function $\varphi \in L^1_{\mathrm{loc}}(X)$ is called *almost plurisubharmonic* if its complex Hessian $dd^c\varphi$ is an almost positive current.

We say that a function ϕ on X has *analytic singularities* along a subscheme $V(\mathscr{I})$ (corresponding to a coherent ideal sheaf $\mathscr{I}$) if there exists $c > 0$ such that ϕ is locally congruent to $\frac{c}{2}\log(\sum|f_i|^2)$ modulo smooth functions, where $f_1, \dots, f_r$ are local generators of $\mathscr{I}$. Note that a function with analytic singularities is automatically almost plurisubharmonic, and is smooth away from the support of $V(\mathscr{I})$.

We say an almost positive $(1,1)$-current has analytic singularities if we can find a smooth form θ and a function φ on X with analytic singularities such that $T = \theta + dd^c\varphi$. Note that one can always write $T = \theta + dd^c\varphi$ with θ smooth and φ almost plurisubharmonic on a compact complex manifold.

2.1. $\partial\overline{\partial}$-cohomology. Let X be an arbitrary compact complex manifold of complex dimension n. Since the $\partial\overline{\partial}$-lemma does not hold in general, it is better to work with $\partial\overline{\partial}$-cohomology which is defined as

$$H^{p,q}_{\partial\overline{\partial}}(X,\mathbb{C}) = \big(\mathcal{C}^\infty(X, \Lambda^{p,q}T^*_X) \cap \ker d\big)/\partial\overline{\partial}\mathcal{C}^\infty(X, \Lambda^{p-1,q-1}T^*_X).$$

By means of the Frölicher spectral sequence, one can see that $H^{p,q}_{\partial\overline{\partial}}(X,\mathbb{C})$ is finite-dimensional and can be computed either with spaces of smooth forms or with currents. In both cases, the quotient topology of $H^{p,q}_{\partial\overline{\partial}}(X,\mathbb{C})$ induced by the Fréchet topology of smooth forms or by the weak topology of currents is Hausdorff, and the quotient map under this Hausdorff topology is continuous and open.

In this paper, we will just need the $(1,1)$-cohomology space $H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{C})$. The real structure on the space of $(1,1)$-smooth forms (or $(1,1)$-currents) induces a real structure on $H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{C})$, and we denote by $H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$ the space of real points. A class $\alpha \in H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{C})$ can be seen as an affine space of closed $(1,1)$-currents. We denote by $\{T\} \in H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{C})$ the class of the current T. Since $i\partial\overline{\partial}$ is a real operator (on forms or currents), if T is a real closed $(1,1)$-current, its class $\{T\}$ lies in $H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$ and consists of all the closed currents $T + i\partial\overline{\partial}\varphi$ where φ is a real current of degree 0.

Definition 2.2. Let (X,ω) be a compact Hermitian manifold. A cohomology class $\alpha \in H^{1,1}_{\partial\overline{\partial}}(X,\mathbb{R})$ is said to be *pseudo-effective* if it contains a positive current; α is *nef* if, for each $\varepsilon > 0$, α contains a smooth form θ_ε with $\theta_\varepsilon \geq -\varepsilon\omega$; α is *big* if it contains a Kähler current, i.e. a closed $(1,1)$-current T such that $T \geq \varepsilon\omega$ for $\varepsilon > 0$ small enough. Finally, α is a *Kähler class* if it contains a Kähler form.

Since any two Hermitian forms ω_1 and ω_2 are commensurable (i.e. $C^{-1}\omega_2 \leq \omega_1 \leq C\omega_2$ for some $C > 0$), these definitions do not depend on the choice of ω.

2.2. Lebesgue decomposition of a current. In this subsection, we refer to [4, 26]. For a measure μ on a manifold M we denote by μ_{ac} and μ_{sing} the uniquely determined absolute continuous and singular measures (with respect to the Lebesgue measure on M) such that

$$\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}},$$

which is called the *Lebesgue decomposition* of μ. If T is a $(1,1)$-current of order 0 on X, written locally $T = i\sum T_{ij} dz_i \wedge d\bar{z}_j$, we define its absolutely continuous and singular components by

$$T_{\mathrm{ac}} = i\sum (T_{ij})_{\mathrm{ac}} dz_i \wedge d\bar{z}_j, \quad T_{\mathrm{sing}} = i\sum (T_{ij})_{\mathrm{sing}} dz_i \wedge d\bar{z}_j.$$

The Lebesgue decomposition of T is then

$$T = T_{\mathrm{ac}} + T_{\mathrm{sing}}.$$

If $T \geq 0$, it follows that $T_{\mathrm{ac}}, T_{\mathrm{sing}} \geq 0$. Moreover, if $T \geq \alpha$ for a continuous $(1,1)$-form α, then $T_{\mathrm{ac}} \geq \alpha$, $T_{\mathrm{sing}} \geq 0$. The Radon–Nikodym theorem ensures that T_{ac} is (the current associated to) a $(1,1)$-form with L^1_{loc} coefficients. The form $T_{\mathrm{ac}}(x)^n$ exists for almost all $x \in X$ and is denoted T^n_{ac}.

Note that in general T_{ac} is not closed, even when T is, so that the decomposition does not induce a significant decomposition at the cohomological level. However, when T is a closed positive $(1,1)$-current with analytic singularities along a subscheme V, the residual part R in the Siu decomposition (cf. [30]) of T is nothing but T_{ac}, and the divisorial part $\sum_k \nu(T, Y_k)[Y_k]$ is T_{sing}. The following facts are well-known.

Lemma 2.3 (cf. [4]). *Let $f : Y \to X$ be a proper surjective holomorphic map. If α is a locally integrable form of bidimension (k,k) on Y, then the push-forward current $f_*\alpha$ is absolutely continuous, hence a locally integrable form of bidimension (k,k). In particular, when T is a positive current on Y, the push-forward current $f_*(T_{\mathrm{ac}})$ is absolutely continuous, and $f_*(T_{\mathrm{ac}}) = (f_*T)_{\mathrm{ac}}$.*

The absolutely continuous part T_{ac} of a positive current T does not depend continuously on T, but we have the following semicontinuity property:

Lemma 2.4 (cf. [4]). *Let T_k be a sequence of positive $(1,1)$-currents converging weakly to T. Then*

$$T_{\mathrm{ac}}(x)^n \geq \limsup T_{k,\mathrm{ac}}(x)^n \quad \textit{for almost every } x \in X.$$

2.3. Regularization of currents. There are two basic types of regularizations (inside a fixed cohomology class) for closed $(1,1)$-currents, both due to J.-P. Demailly.

Theorem 2.5 (cf. [11, 14, 4]). *Let T be a closed almost positive $(1,1)$-current on a compact Hermitian manifold (X,ω). Suppose that $T \geq \gamma$ for some smooth $(1,1)$-form γ on X. Then:*

(i) *There exists a sequence of smooth forms θ_k in $\{T\}$ which converges weakly to T, and $\theta_k(x) \to T_{\mathrm{ac}}(x)$ a.e., such that $\theta_k \geq \gamma - C\lambda_k\omega$*

where $C > 0$ is a constant depending on the curvature of (T_X, ω) only, and λ_k is a decreasing sequence of continuous functions such that $\lambda_k(x) \to \nu(T, x)$ for every $x \in X$.

(ii) *There exists a sequence T_k of currents with analytic singularities in $\{T\}$ which converges weakly to T, and $T_{k,\mathrm{ac}}(x) \to T_{\mathrm{ac}}(x)$ a.e., such that $T_k \geq \gamma - \varepsilon_k \omega$ for some sequence $\varepsilon_k > 0$ decreasing to 0, and $\nu(T_k, x)$ increases to $\nu(T, x)$ uniformly with respect to $x \in X$.*

2.4. Resolution of singularities

Definition 2.6. Let $f : Y \to X$ be a surjective holomorphic map between compact complex manifolds and T be a closed almost positive $(1,1)$-current on X. Write $T = \theta + dd^c\varphi$ for some smooth form $\theta \in \{T\}$ and φ an almost plurisubharmonic function on X. We define its pull-back f^*T by f to be $f^*\theta + dd^c f^*\varphi$. Note that this definition is independent of the choices made, and we have $\{f^*T\} = f^*\{T\}$.

We now use the notation of Definition 2.1. From [23, 2, 3], one can blow up X along $V(\mathscr{I})$ and resolve the singularities to get a smooth modification $\mu : \widetilde{X} \to X$, where $\widetilde{X}$ is a compact complex manifold, such that $\mu^{-1}\mathscr{I}$ is just $\mathcal{O}(-D)$ for some simple normal crossing divisor D upstairs. The pull-back μ^*T clearly has analytic singularities along $V(\mu^{-1}(\mathscr{I})) = D$, thus its Siu decomposition reads

$$\mu^*T = \theta + cD,$$

where θ is a smooth $(1,1)$-form. If $T \geq \gamma$ for some smooth form γ, then $\mu^*T \geq \gamma$, and thus $\theta \geq \mu^*\gamma$. We call this operation *resolution of the singularities* of T.

2.5. Lamari's criterion

Theorem 2.7. *Let X be an n-dimensional compact complex manifold and let Φ be a real (k,k)-form. Then there exists a real $(k-1, k-1)$-current Ψ such that $\Phi + dd^c\Psi$ is positive iff for any strictly positive $\partial\overline{\partial}$-closed $(n-k, n-k)$-form Υ we have $\int_X \Phi \wedge \Upsilon \geq 0$.*

2.6. Gauduchon metrics. Gauduchon [19] proved that for any n-dimensional compact complex manifold X, there always exists a metric ω such that $\partial\overline{\partial}\omega^{n-1} = 0$. These metrics are called *Gauduchon metrics*. Actually, from [20] we know that in the conformal class of every Hermitian metric, there is a Gauduchon metric. As a consequence, if the Gauduchon metric ω satisfies assumption $(*)$, then for any closed $(1,1)$-current T and $k \in \{1, \ldots, n\}$, the integral $\int_X T^k \wedge \omega^{n-k}$ only depends on the class of T and the metric ω, provided that T^k is well-defined.

The following two theorems, which we will state without proof, will play key roles in this paper.

Theorem 2.8 ([8]). *Let (X,ω) be a compact Hermitian manifold. The complex Monge–Ampère equation*

$$(\omega + i\partial\overline{\partial}\phi)^n = e^{\varepsilon\phi - F_\varepsilon}\omega^n, \tag{2.1}$$

where $\varepsilon > 0$ and F_ε is a smooth function on X, has a smooth solution ϕ such that $\omega_\phi := \omega + i\partial\overline{\partial}\phi > 0$.

Theorem 2.9 ([33]). *Let (X,ω) be a compact Hermitian manifold. For any smooth real-valued function F on X, there exist a unique real number $C > 0$ and a unique smooth real-valued function ϕ on X solving*

$$(\omega + i\partial\overline{\partial}\phi)^n = Ce^F\omega^n$$

with $\omega + i\partial\overline{\partial}\phi > 0$ and $\sup_X \phi = 0$. Furthermore, if $\partial\overline{\partial}\omega^k = 0$ for $1 \le k \le n-1$, then

$$C = \frac{\int_X \omega^n}{\int_X e^F\omega^n}.$$

Remark 2.10. Assumption $(*)$ is also used in [21] to solve the complex Monge–Ampère equation.

Remark 2.11. Note that if ω is Gauduchon, and ϕ is a smooth function on X such that $\omega_\phi := \omega + i\partial\overline{\partial}\phi > 0$, then ω_ϕ is not Gauduchon in general.

Lemma 2.12. *Suppose ω is a Hermitian form on X satisfying assumption $(*)$. Then for any smooth function ϕ on M, ω_ϕ also satisfies $(*)$.*

Proof. It is a direct and easy computation. ∎

2.7. Finiteness of volume. The following two lemmas are small generalizations of those in [4]. The proofs are similar with minor modifications, but we give them for completeness.

Lemma 2.13. *Let T be any closed $(1,1)$-current on a compact Hermitian manifold (X,ω) with $T \ge \gamma$, where ω is the Gauduchon metric satisfying assumption $(*)$ and γ is a continuous $(1,1)$-form on X. Then one can define the Lelong number $\nu(T,x)$ for T at x to be $\nu(T+\beta,x)$, where β is a smooth closed $(1,1)$-form near x such that $T+\beta \ge 0$ and $\nu(T,x)$ can be bounded by a constant depending only on the $\partial\overline{\partial}$-cohomology class of T.*

Proof. By definition $\nu(T+\beta,x)$ is (up to a constant depending on ω near x) the limit as $r \to 0_+$ of

$$\nu(T+\beta,x,r) := \frac{(n-1)!}{(\pi r^2)^{n-1}} \int_{B(x,r)} (T+\beta)\wedge\omega^{n-1},$$

which is known to be an increasing function of r. Since β is smooth, one can see that the limit is independent of β, which means that $\nu(T,x)$ is well-defined. Choose a constant C such that $C\omega \ge -\gamma$; then $T + C\omega \ge 0$

on X. Thus if we choose r_0 small enough to ensure that each ball $B(x, r_0)$ is contained in a coordinate chart, we get

$$\nu(T,x) \le \nu(T + C\omega, x, r_0) \le \int_X (T + C\omega) \wedge \omega^{n-1} = \int_X T \wedge \omega^{n-1} + C \int_X \omega^n.$$

But the last term is a quantity depending on the cohomology class $\{T\}$ since ω satisfies $(*)$. ∎

LEMMA 2.14. *Under the same assumption as in Lemma* 2.13, *the integrals* $\int_X T_{\mathrm{ac}}^k \wedge \omega^{n-k}$ *are finite for all* $k = 0, 1, \dots, n$ *and can be bounded in terms of* ω *and the* $\partial\bar{\partial}$*-cohomology class of* T *only.*

Proof. Since γ is continuous and X is compact, there exists a constant $C > 0$ such that $T \ge -C\omega$, and thus $T_{\mathrm{ac}} \ge -C\omega$. Let $-C \le \lambda_1 \le \dots \le \lambda_n$ be the eigenvalues of T_{ac}. It is easy to observe that whenever λ_k is negative or positive, $|\lambda_k| \le \lambda_k + 2C$ always holds. Thus

$$\Big|\int_X T_{\mathrm{ac}}^k \wedge \omega^{n-k}\Big| \le \int_X (T_{\mathrm{ac}} + 2C\omega)^k \wedge \omega^{n-k}.$$

It suffices to prove that the right hand side is uniformly bounded. Choose a sequence T_k of smooth forms approximating T as in Theorem 2.5. Since $T_k \ge -C\omega - C\lambda_k\omega = -C(1 + \lambda_k)\omega$ for some constant $C > 0$ depending on (X, ω) only and for some continuous functions $\lambda_k(x)$ decreasing to $\nu(T, x)$, we find, using Lemma 2.13, a constant also denoted by C and depending on (X, ω) and the cohomology class $\{T\}$ only such that $T_k + C\omega \ge 0$. But now the quantity

$$\int_X (T_k + C\omega)^l \wedge \omega^{n-l} = \int_X (\{T\} + C\omega)^l \omega^{n-l}$$

does not depend on k since ω satisfies $(*)$, so the result follows by Fatou's lemma, as $T_k + C\omega$ is a smooth form converging to $T_{\mathrm{ac}} + C\omega$ a.e. ∎

2.8. Harder–Narasimhan filtrations on compact Hermitian manifolds. In this subsection, we refer to Bruasse [5]. Let (X, ω) be a compact Hermitian manifold endowed with a Gauduchon metric ω. Let L be a holomorphic line bundle on X and h be a Hermitian metric on L. Let $\Theta_{L,h}$ be the Chern curvature form of L associated to h. Since it is independent of h up to a $\partial\bar{\partial}$-exact term, and ω is Gauduchon, the ω-*degree* of L given by

$$\deg_\omega(L) = \int_X \Theta_{L,h} \wedge \omega^{n-1}$$

is a well-defined real number independent of h.

Now if $\mathcal{F}$ is a rank p coherent sheaf of $\mathcal{O}_X$-modules, consider the holomorphic line bundle $\det \mathcal{F} = (\bigwedge^p \mathcal{F})^{**}$.

Definition 2.15.

(i) The *ω-degree* of $\mathcal{F}$ is $\deg_\omega(\mathcal{F}) := \deg_\omega(\det \mathcal{F})$.
(ii) If $\mathcal{F}$ is nontrivial and torsion-free, then we define its *slope* (or *ω-slope*) by

$$\mu(\mathcal{F}) := \frac{\deg_\omega(\mathcal{F})}{\operatorname{rank}(\mathcal{F})}.$$

Definition 2.16. A torsion-free coherent sheaf $\mathcal{E}$ is called *ω-[semi] stable* if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}$, one has $\mu(\mathcal{F}) < [\leq]\ \mu(\mathcal{E})$.

Definition 2.17. Let (X, ω) be a compact complex manifold of dimension n endowed with a Gauduchon metric ω. Let $\mathcal{F}$ be a torsion-free coherent sheaf over X. A *Harder–Narasimhan filtration* for $\mathcal{F}$ is a flag

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s = \mathcal{F}$$

of subsheaves of $\mathcal{F}$ with the following properties:

(1) $\mathcal{F}_i/\mathcal{F}_{i-1}$ is ω-semistable for $1 \leq i \leq s-1$,
(2) $\mu(\mathcal{F}_{j+1}/\mathcal{F}_j) < \mu(\mathcal{F}_j/\mathcal{F}_{j-1})$ for $1 \leq j \leq s-1$.

In fact, $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the maximal ω-semistable subsheaf of $\mathcal{F}/\mathcal{F}_{i-1}$ for $1 \leq i \leq s-1$.

Theorem 2.18 ([5]). *Let (X, ω) be a compact complex manifold of dimension n endowed with a Gauduchon metric ω. Let E be a holomorphic vector bundle of rank r over X. Then E has a unique Harder–Narasimhan filtration.*

3. Volume of a nef class

Theorem 3.1. *Let (X, ω) be a compact Hermitian manifold endowed with a Gauduchon metric ω satisfying assumption $(*)$. If $\alpha \in H^{1,1}_{\partial\bar\partial}(X, \mathbb{R})$ is a nef class, then* $\operatorname{vol}(\alpha) = \alpha^n$.

Proof. The proof is a small modification of that in [4]; we give it for completeness. Firstly, we prove that for every positive $T \in \alpha$, we have $\int_X T^n_{\mathrm{ac}} \leq \alpha^n$, which will certainly imply $\operatorname{vol}(\alpha) \leq \alpha^n$. Write $T = \theta + dd^c\varphi$ with θ a smooth form. We consider a sequence $T^{(1)}_k = \theta + dd^c\varphi^{(1)}_k$ of smooth forms given by Theorem 2.5(i). Since α is nef, by definition, there exists a sequence of smooth functions $\varphi^{(2)}_k$ and a sequence of positive numbers $\varepsilon_k \to 0$ such that $T^{(2)}_k = \theta + dd^c\varphi^{(2)}_k$ satisfies $T^{(2)}_k \geq -\varepsilon_k\omega$. Set $\varphi^{(3)}_k := \max_\eta(\varphi^{(2)}_k - C_k, \varphi^{(1)}_{j_k})$, where $C_k \to \infty$ as $k \to \infty$ and $\varphi^{(1)}_{j_k}$ is a suitable subsequence of $\varphi^{(1)}_k$ (cf. [4, p. 1050]). Then $\varphi^{(3)}_k$ is a smooth function, and it is proved in [4, Lemma 4.2] that $T^{(3)}_k := \theta + dd^c\varphi^{(3)}_k$ is a smooth form such that $T_k^{(3)}(x) \to T_{\rm ac}(x)$ a.e., and $T_k^{(3)} \geq -\delta_k\omega$ for some sequence $\delta_k > 0$ converging to 0. Since $T_k^{(3)} + \delta_k\omega$ also converges to $T_{\rm ac}$ a.e., Fatou's lemma gives

$$\int_X T_{\rm ac}^n \leq \liminf_{k\to\infty} \int_X (T_k^{(3)} + \delta_k\omega)^n,$$

and the latter integral depends only on the class α and δ_k, thus it converges to α^n. That is, $\int_X T_{\rm ac}^n \leq \alpha^n$.

Secondly, we want to show $\mathrm{vol}(\alpha) \geq \int \alpha^n$. Normalize our Gauduchon metric ω in $(*)$ so that $\int_X \omega^n = 1$. For $\varepsilon > 0$, there exists a closed form $T \in \alpha$ such that $T + \varepsilon\omega > 0$. Using Theorem 2.9, one can solve the equation

$$\tau_\varepsilon^n = \Big(\int_X (T + \varepsilon\omega)^n\Big)\omega^n,$$

where $\tau_\varepsilon = T + \varepsilon\omega + dd^c\varphi_\varepsilon > 0$, and φ_ε is normalized so that $\sup_X \varphi_\varepsilon = 0$. Since the family $\tau_\varepsilon - \varepsilon\omega \in \alpha$ represents a bounded set of cohomology classes, it is bounded in mass, so we can extract some weak limit $T = \lim_{\varepsilon\to 0}(\tau_\varepsilon - \varepsilon\omega) = \lim_{\varepsilon\to 0}\tau_\varepsilon$, where the second equality holds because $\lim_{\varepsilon\to 0}\varepsilon\omega = 0$ in the strong sense. By Lemma 2.4, we get $T_{\rm ac}^n \geq (\int \alpha^n)\omega^n$, and integrating gives $\mathrm{vol}(\alpha) \geq \alpha^n$. ∎

4. The Grauert–Riemenschneider criterion: Proof of Theorem 1.2. From the definition of $\mathrm{vol}(\alpha)$, one can find a positive closed current $S \in \alpha$ such that $\int_X S_{\rm ac}^n > \mathrm{vol}(\alpha)/2 > 0$. By Theorem 2.5(ii), combined with Fatou's lemma, we can find a sequence T_k of closed currents with analytic singularities in α such that $T_k \geq -\varepsilon_k\omega$ and $\int_X T_{k,\rm ac}^n \geq c$ for some uniform lower bound $c > 0$, where $\varepsilon_k \to 0$ as $k \to \infty$. In fact, $0 < \int_X S_{\rm ac}^n \leq \liminf_k \int_X (T_{k,\rm ac} + \varepsilon_k\omega)^n$, thus one can extract a subsequence which is denoted by $T_{k,\rm ac} + \varepsilon_k\omega$ such that $\int_X (T_{k,\rm ac} + \varepsilon_k\omega)^n > C$ for some uniform constant C. But

$$\int_X (T_{k,\rm ac} + \varepsilon_k\omega)^n = \int_X T_{k,\rm ac}^n + \sum_{l=1}^n \varepsilon_k^l \binom{n}{l} \int_X T_{k,\rm ac}^{n-l} \wedge \omega^l$$

where the second term is uniformly bounded for k large enough (say $0 < \varepsilon_k \ll 1$) by Lemma 2.14. For each k, we choose a smooth proper modification $\mu_k : X_k \to X$ such that $\mu_k^* T_k = \theta_k + D_k$ with θ_k $(\geq -\varepsilon_k\mu_k^*\omega)$ a smooth closed form and E_k a real effective divisor. Set $\Omega_k = \varepsilon_k\mu_k^*\omega$. It is easy to see that $0 < c \leq \int_X T_{k,\rm ac}^n = \int_{X_k} (\mu_k^* T_k)_{\rm ac}^n = \int_{X_k} \theta_k^n$. Select on each X_k a Gauduchon metric $\widetilde{\omega}_k$ which also satisfies $(*)$ by the following

Lemma 4.1 (cf. [15]). *Suppose that (X, ω) is a compact complex manifold satisfying assumption $(*)$. Let $\mu : \widetilde{X} \to X$ be a smooth modification (a tower of blow-ups). Then there exists a Gauduchon metric Ω satisfying $(*)$ on $\widetilde{X}$.*

Proof. Suppose that $\widetilde{X}$ is obtained as a tower of blow-ups

$$\widetilde{X} = X_N \to X_{N-1} \to \cdots \to X_1 \to X_0 = X, \tag{4.1}$$

where X_{j+1} is the blow-up of X_j along a smooth center $Y_j \subset X_j$. Denote by $E_{j+1} \subset X_{j+1}$ the exceptional divisor, and let $\mu_j : X_{j+1} \to X_j$ be the blow-up map. The line bundle $\mathcal{O}(-E_{j+1})|_{E_{j+1}}$ is equal to $\mathcal{O}_{P(N_j)}(1)$ where N_j is the normal bundle to Y_j in X_j. Pick an arbitrary smooth Hermitian metric on N_j, use this metric to get an induced Fubini–Study metric on $\mathcal{O}_{P(N_j)}(1)$, and finally extend this metric as a smooth Hermitian metric on the line bundle $\mathcal{O}(-E_{j+1})$. Such a metric has positive curvature along tangent vectors of X_{j+1} which are tangent to the fibers of $E_{j+1} = P(N_j) \to Y_j$. Assume further that ω_j is a Gauduchon metric satisfying $(*)$ on X_j. Then

$$\Omega_{j+1} = \mu_j^*\omega_j - \varepsilon_{j+1}u_{j+1} \tag{4.2}$$

where $\mu_j^*\omega_j$ is semipositive on X_{j+1}, positive definite on $X_{j+1} \setminus E_{j+1}$, and also positive definite on tangent vectors of $T_{X_{j+1}}|_{E_{j+1}}$ which are not tangent to the fibers of $E_{j+1} \to Y_j$. It is then easy to see that $\Omega_{j+1} > 0$ by taking $\varepsilon_{j+1} \ll 1$. Thus our final candidate Ω on $\widetilde{X}$ has the form $\Omega = \mu^*\omega - \sum \varepsilon_j \widetilde{u}_j$, where $\widetilde{u}_j = (\mu_{N-1} \circ \cdots \circ \mu_j)^* u_j$. Since every u_j is a curvature term of a line bundle, the term $\sum \varepsilon_j \widetilde{u}_j$ is d-closed. Now Ω satisfies $(*)$ by Lemma 2.12. ∎

Now we want to show that the class $\{\theta_k\}$ is big for k large. It suffices to show that there exists $\varepsilon_0 > 0$ and a distribution χ such that $\theta_k + dd^c\chi \geq \varepsilon_0 \widetilde{\omega}_k$. According to Lamari's criterion [25] (cf. Section 2.5), this is equivalent to showing that

$$\int_X \theta_k \wedge g^{n-1} \geq \varepsilon_0 \int_X \widetilde{\omega}_k \wedge g^{n-1}$$

for any Gauduchon metric g on X_k. Here we use a theorem of Michelsohn [27] which states that every strictly positive $(n-1, n-1)$-form β has a $(1,1)$ root g such that $\beta = g^{n-1}$. Suppose to the contrary that for any $m \in \mathbb{N}$, there exists a Gauduchon metric ω_m on X_k such that

$$\int_{X_k} \theta_k \wedge \omega_m^{n-1} \leq \frac{1}{m} \int_{X_k} \widetilde{\omega}_k \wedge \omega_m^{n-1}.$$

We can assume that

$$\int_{X_k} \widetilde{\omega}_k \wedge \omega_m^{n-1} = 1 \quad \text{and therefore} \quad \int_{X_k} \theta_k \wedge \omega_m^{n-1} \leq \frac{1}{m}.$$

From Theorem 2.9, we can solve the equation

$$\left(\theta_k + \Omega_k + \frac{1}{m}\widetilde{\omega}_k + dd^c\varphi_m\right)^n = C_m \omega_m^{n-1} \wedge \widetilde{\omega}_k \tag{4.3}$$

for a function $\varphi_m \in \mathcal{C}^\infty(X_k, \mathbb{R})$ such that if we set

$$\alpha_m = \theta_k + \Omega_k + \frac{1}{m}\widetilde{\omega}_k + dd^c\varphi_m,$$

then $\alpha_m > 0$. The constant C_m is given by

$$C_m = \int_{X_k} \left(\theta_k + \Omega_k + \frac{1}{m}\widetilde{\omega}_k\right)^n \geq \int_{X_k} (\theta + \Omega_k)^n \\ = \int_{X_k} \theta_k^n + O(\varepsilon_k) \int_X \sum_{p\geq 1} T_{k,\mathrm{ac}}^p \wedge \omega^{n-p} \geq C > 0. \tag{4.4}$$

The second inequality in (4.4) follows from Lemma 2.14 for k sufficiently large. Here C is a uniform constant that depends only on the cohomology class α and ω. Now

$$\int_{X_k} \alpha_m^{n-1} \wedge \widetilde{\omega}_k = \int_{X_k} \widetilde{\omega}_k \wedge (\theta_k + \frac{1}{m}\widetilde{\omega}_k + \Omega_k)^{n-1} \leq \int_{X_k} \widetilde{\omega}_k \wedge (\theta_k + \widetilde{\omega}_k)^{n-1}.$$

But

$$\int_{X_k} \widetilde{\omega}_k \wedge (\theta_k + \widetilde{\omega}_k)^{n-1} \leq C' \int_X \omega \wedge (T_{k,\mathrm{ac}} + \omega)^{n-1},$$

since $\widetilde{\omega}_k = \mu_k^*\omega + \sum \varepsilon_j \widetilde{u}_j$ and ε_j can be chosen sufficiently small, which can be easily seen from Lemma 4.1 and a similar argument to (4.4). From Lemma 2.14, $\int_X \omega \wedge (T_{k,\mathrm{ac}} + \omega)^{n-1}$ is bounded by a uniform constant which depends only on the cohomology class α and the metric ω on X. Therefore we get

$$\int_{X_k} \widetilde{\omega}_k \wedge (\theta_k + \widetilde{\omega}_k)^{n-1} \leq M,$$

where M is also a uniform constant which depends only on the cohomology class α and the metric ω on X.

Set

$$E = \left\{ \frac{\alpha_m^{n-1} \wedge \widetilde{\omega}_k}{\omega_m^{n-1} \wedge \widetilde{\omega}_k} > 2M \right\}.$$

Then

$$\int_E \omega_m^{n-1} \wedge \widetilde{\omega}_k \leq \frac{1}{2}. \tag{4.5}$$

Therefore on $X_k \setminus E$, we have $\alpha_m^{n-1} \wedge \widetilde{\omega}_k \leq 2M\omega_m^{n-1} \wedge \widetilde{\omega}_k$. By looking at the eigenvalues of α_m with respect to ω, from (4.3) it follows that on $X_k \setminus E$ we have

$$\alpha_m \geq \frac{C_m}{2nM}\widetilde{\omega}_k.$$

Therefore

$$\int_{X_k} \alpha_m \wedge \omega_m^{n-1} \ge \int_{X_k\setminus E} \alpha_m \wedge \omega_m^{n-1} \ge \frac{C_m}{2nM} \int_{X_k\setminus E} \widetilde{\omega}_k \wedge \omega_m^{n-1} \tag{4.6}$$
$$= \frac{C_m}{2nM}\Big(\int_{X_k} \widetilde{\omega}_k \wedge \omega_m^{n-1} - \int_E \widetilde{\omega}_k \wedge \omega_m^{n-1}\Big) \ge \frac{C}{4nM}.$$

On the other hand,

$$\int_{X_k} \alpha_m \wedge \omega_m^{n-1} \le \int_{X_k} \theta_k \wedge \omega_m^{n-1} + \frac{2}{m}\int_{X_k} \widetilde{\omega}_k \wedge \omega_m^{n-1} \le \frac{3}{m}, \tag{4.7}$$

which is a contradiction for $m \gg 0$. Here the first inequality in (4.7) holds for k sufficiently large such that $\Omega_k = \varepsilon_k \mu_k^* \omega \le \frac{1}{m}\widetilde{\omega}_k$. Therefore θ_k is big, i.e. there exists a Kähler current $\Theta \in \{\theta_k\}$ on X_k, hence a Kähler current $(\mu_k)_*(\Theta + E_k)$ (see Lemma 2.3) on X. Thus it follows that X is in the Fujiki class. Theorem 2.2 in [9] implies that a manifold in the Fujiki class and which is strong Kähler with torsion (i.e. it supports a $\partial\overline{\partial}$-closed Hermitian metric) is in fact Kähler. ∎

Remark 4.2. In general, the pseudo-effective cone (even the big cone) and the nef cone on a compact complex manifold are not the same. In general, the nef cone is contained in the pseudo-effective cone. But the converse is not true. For example, the exceptional divisor of a blowing-up along one point in $\mathbb{C}P^2$ is pseudo-effective but not nef. To characterize the pseudo-effective cone is an important question in complex geometry.

Remark 4.3. Recently, Tosatti [32] gave a proof of (b) using ideas very close to our proof of Theorem 1.2.

Remark 4.4. In Demailly's book [15], Demailly introduced the following definition of $\mathrm{vol}_D(\alpha)$ for a pseudo-effective class α on Kähler manifolds.

Definition 4.5 (cf. [15]). Let X be a compact Kähler manifold. The volume, or mobile self-intersection, of a class $\alpha \in H^{1,1}(X, \mathbb{R})$ is defined to be

$$\mathrm{vol}_D(\alpha) = \sup_{T\in\alpha} \int_{X\setminus \mathrm{sing}(T)} T^n = \sup_{T\in\alpha} \int_{\widetilde{X}} \beta^n > 0$$

where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles, and $\mu^* T = [E] + \beta$ with respect to some modification $\mu : \widetilde{X} \to X$. Correspondingly, we set $\mathrm{vol}_D(\alpha) = 0$ if $\alpha \notin \mathcal{E}^o$.

It is almost trivial that $\mathrm{vol}_D(\alpha) \le \mathrm{vol}(\alpha)$. From Theorem 1.2, one can now see that if the compact Hermitian manifold (X, ω) satisfies assumption $(*)$ and $\mathrm{vol}(\alpha) > 0$, then α is big and X is Kähler. Thus it is natural to ask whether $\mathrm{vol}_D(\alpha) = \mathrm{vol}(\alpha)$ on a Kähler manifold.

Firstly, it is easy to see that $\mathrm{vol}_D(\alpha) = \sup_{T\in\alpha} \int_X T_{\mathrm{ac}}^n$, where the supremum is taken over all Kähler currents $T \in \alpha$ with logarithmic poles. In particular, $\mathrm{vol}_D(\alpha) \leq \mathrm{vol}(\alpha)$.

Secondly, $\mathrm{vol}(\alpha) \leq \mathrm{vol}_D(\alpha)$. In fact, it is trivial that $\mathrm{vol}(\alpha) = 0 \Leftrightarrow \mathrm{vol}_D(\alpha) = 0$. Otherwise it is a direct consequence of the following version of Fujita's theorem due to Boucksom.

Theorem 4.6 (cf. [4]). *Let X be a compact Kähler manifold, and let $\alpha \in H^{1,1}(X,\mathbb{R})$ be a big class on X. Then for every $\varepsilon > 0$, there exists a modification $\mu : \widetilde{X} \to X$, a Kähler class ω and an effective real divisor D on $\widetilde{X}$ such that*

- *$\mu^*\alpha = \omega + \{D\}$ as cohomology classes,*
- *$|\mathrm{vol}(\alpha) - \mathrm{vol}(\omega)| < \varepsilon$.*

To conclude, we have proved the following

Proposition 4.7. *Let X be a compact Kähler manifold. Then* $\mathrm{vol}(\alpha) = \mathrm{vol}_D(\alpha)$ *for any α in $H^{1,1}(X,\mathbb{R})$.*

5. A characterization of the nef anti-canonical bundle on X: Proof of Theorem 1.3. Let (X,ω) be a compact complex manifold with a Hermitian metric ω. Denote by $\mathrm{Ricci}(\omega)$ the Chern Ricci curvature of (X,ω), i.e. the Chern curvature of K_X^{-1} corresponding to the Hermitian metric induced by the Hermitian metric ω of X.

Firstly, we prove that (i) implies (ii). Suppose $L := K_X^{-1}$ is nef, that is, for any $\varepsilon > 0$, there is a smooth Hermitian metric h_ε of L, such that $\Theta_{L,h_\varepsilon} \geq -\varepsilon\omega$. Since Θ_{L,h_ε} is the Chern Ricci curvature of X, it is a representative of the first Chern class of X. One asks for a ϕ such that $\omega_\varepsilon := \omega + i\partial\overline{\partial}\phi > 0$ and $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$. Let us find the equation such a ϕ should satisfy. Let $u_\varepsilon := \Theta_{L,h_\varepsilon} \geq -\varepsilon\omega$. Then $u_\varepsilon = \mathrm{Ricci}(\omega) + i\partial\overline{\partial}F_\varepsilon$. It thus suffices to find a ϕ such that

$$\mathrm{Ricci}(\omega_\varepsilon) = -\varepsilon\omega_\varepsilon + \varepsilon\omega + u_\varepsilon, \tag{5.1}$$

which is equivalent to equation (2.1). In fact,

$$\begin{aligned} i\partial\overline{\partial}\log\omega_\varepsilon^n - i\partial\overline{\partial}\log\omega^n &= \mathrm{Ricci}(\omega) - \mathrm{Ricci}(\omega_\varepsilon) \\ &= \varepsilon(\omega_\varepsilon - \omega) + \mathrm{Ricci}(\omega) - u_\varepsilon \\ &= i\partial\overline{\partial}(\varepsilon\phi - F_\varepsilon). \end{aligned}$$

From Theorem 2.8, for any $\varepsilon > 0$, there is a smooth function ϕ such that $\omega_\varepsilon := \omega + i\partial\overline{\partial}\phi > 0$ and $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$.

Conversely, since $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$ and $\omega_\varepsilon := \omega + i\partial\overline{\partial}\phi > 0$, one can easily conclude that $\mathrm{Ricci}(\omega_\varepsilon) + i\partial\overline{\partial}(\varepsilon\phi) \geq -\varepsilon\omega$. But $\mathrm{Ricci}(\omega_\varepsilon) + i\partial\overline{\partial}(\varepsilon\phi)$

is precisely the curvature form associated to a Hermitian metric on K_X^{-1}. Thus one gets the nefness of K_X^{-1}.

6. A semipositive property of the Harder–Narasimhan filtration of T_X: Proof of Theorem 1.4. The proof follows the lines of the proof in Demailly [16]. First consider the case where the filtration is *regular*, i.e. all sheaves $\mathcal{F}_i$ and their quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are vector bundles. By the stability condition, it is sufficient to prove that

$$\int_X c_1(T_X/\mathcal{F}_i) \wedge \omega^{n-1} \geq 0$$

for all i. From Theorem 1.3, for each $\varepsilon > 0$, there is a smooth function ϕ_ε such that $\omega_\varepsilon := \omega + i\partial\overline{\partial}\phi_\varepsilon > 0$ and $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$. This is equivalent to the pointwise estimate

$$i\Theta_{T_X,\omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \geq -\varepsilon \cdot \mathrm{Id}_{T_X}\, \omega_\varepsilon^n.$$

Taking the induced metric on $T_X/\mathcal{F}_i$ (which we also denote by ω_ε), the second fundamental form contributes nonnegative terms on the quotient, hence the ω_ε-trace yields

$$\mathrm{Trace}(i\Theta_{T_X/\mathcal{F}_i,\omega_\varepsilon} \wedge \omega_\varepsilon^{n-1}) \geq -\varepsilon\, \mathrm{rank}(T_X/\mathcal{F}_i)\omega_\varepsilon^n.$$

Therefore, setting $r_i = \mathrm{rank}(T_X/\mathcal{F}_i)$, since for a line bundle, the curvatures differ by $\partial\overline{\partial}$-exact terms for different choices of the Hermitian metric, and both ω and ω_ε satisfy $(*)$, we get

$$\int_X c_1(T_X/\mathcal{F}_i) \wedge \omega^{n-1} = \int_X c_1(T_X/\mathcal{F}_i) \wedge \omega_\varepsilon^{n-1} \geq -\varepsilon r_i \int_X \omega_\varepsilon^n = -\varepsilon r_i \int_X \omega^n,$$

and we are done. In case there are singularities, from the construction in [5] they occur only on some analytic subset $S \subset X$ of codimension 2. The first Chern forms calculated on $X \setminus S$ extend as locally integrable currents on X and do not contribute any mass on S. The above calculations are still valid. Thus we have completed the proof of Theorem 1.4.

Acknowledgements. The author would like to thank Professor Jean-Pierre Demailly for his talk about structures of Kähler manifolds given at Peking University, pointing out to him the key observations in the proof of Lemma 2.14, and many inspiring discussions from which the author benefited a lot. The author is grateful to Professor Xiangyu Zhou and Kefeng Liu for their constant support, encouragement, and helpful discussions; and to Professor Valentino Tosatti for many helpful discussions. Part of this research was done at the Department of Mathematics at the University of California at Los Angeles under the support of Chinese Scholarship Council.

References

[1] B. Berndtsson, *An eigenvalue estimate for the $\overline{\partial}$-Laplacian*, J. Differential Geom. 60 (2003), 295–313.

[2] E. Bierstone and P. Milman, *A simple constructive proof of canonical resolution of singularities*, in: Effective Methods in Algebraic Geometry, Progr. Math. 94, Birkhäuser, Boston, 1991, 11–30.

[3] E. Bierstone and P. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128 (1997), 207–302.

[4] S. Boucksom, *On the volume of a line bundle*, Int. J. Math. 13 (2002), 1043–1063.

[5] L. Bruasse, *Harder–Narasimhan filtration on non Kähler manifolds*, Int. J. Math. 12 (2001), 579–594.

[6] J. Cao, *A remark on compact Kähler manifolds with nef anticanonical bundles and its applications*, arXiv:math.AG/1305.4397 (2013).

[7] J. Cao, *Vanishing theorems and structure theorems on compact Kähler manifolds*, PhD thesis, Univ. de Grenoble, 2013; https://tel.archives-ouvertes.fr/tel-00919536/.

[8] P. Cherrier, *Équations de Monge–Ampère sur les variétés hermitiennes compactes*, Bull. Sci. Math. (2) 111 (1987), 343–385.

[9] I. Chiose, *Obstructions to the existence of Kähler structures on compact complex manifolds*, Proc. Amer. Math. Soc. 142 (2014), 3561–3568.

[10] I. Chiose, *The Kähler rank of compact complex manifolds*, J. Geom. Anal. 26 (2016), 603–615.

[11] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. 15 (1982), 457–511.

[12] J.-P. Demailly, *Champs magnétiques et inégalités de Morse pour la d''-cohomologie*, Ann. Inst. Fourier (Grenoble) 35 (1985), no. 4, 189–229.

[13] J.-P. Demailly, *Holomorphic Morse inequalities*, in: Several Complex Variables and Complex Geometry (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math. 52, Part 2, Amer. Math. Soc., 1991, 93–114.

[14] J.-P. Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. 1 (1992), 361–409.

[15] J.-P. Demailly, *Analytic Methods in Algebraic Geometry. Surveys of Modern Mathematics, 1*, Int. Press, Somerville, MA, and Higher Education Press, Beijing, 2012.

[16] J.-P. Demailly, *Structure theorems for compact Kähler manifolds with nef anticanonical bundles*, in: Complex Analysis and Geometry, Springer Proc. Math. Statist. 144, Springer, Tokyo, 2015, 119–133.

[17] J.-P. Demailly, L. Ein and R. Lazarsfeld, *A subadditivity property of multiplier ideals*, Michigan Math. J. 48 (2000), 137–156.

[18] J.-P. Demailly, T. Peternell and M. Schneider, *Kähler manifolds with numerically effective Ricci class*, Compos. Math. 89 (1993), 217–240.

[19] P. Gauduchon, *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris Sér. A 285 (1977), 387–390.

[20] P. Gauduchon, *La 1-forme de torsion d'une variété hermitienne compacte*, Math. Ann. 267 (1984), 495–518.

[21] B. Guan and Q. Li, *Complex Monge–Ampère equations and totally real submanifolds*, Adv. Math. 225 (2010), 1185–1223.

[22] G. Harder and M. Narasimhan, *On the cohomology groups of moduli spaces*, Math. Ann. 212 (1975), 215–248.

[23] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, Ann. of Math. 79 (1964), 109–326.

[24] S. Ji and B. Shiffman, *Properties of compact complex manifolds carrying closed positive currents*, J. Geom. Anal. 3 (1993), 37–61.

[25] A. Lamari, *Courants kählériens et surfaces compactes*, Ann. Inst. Fourier (Grenoble) 49 (1999), 263–285.

[26] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, Progr. Math. 254, Birkhäuser, Basel, 2007.

[27] M. L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. 143 (1983), 261–295.

[28] M. Păun, *Sur le groupe fondamental des variétés kählériennes compactes à classe de Ricci numériquement effective*, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997), 1249–1254.

[29] M. Păun, *Sur les variétés Kählériennes compactes à classe de Ricci numériquement effective*, Bull. Sci. Math. 122 (1998), 83–92.

[30] Y.-T. Siu, *Analyticity of sets associated to Lelong numbers and the existence of closed positive currents*, Invent. Math. 27 (1974), 53–156.

[31] Y.-T. Siu, *A vanishing theorem for semi-positive line bundles over non-Kähler manifolds*, J. Differential Geom. 19 (1984), 431–452.

[32] V. Tosatti, *The Calabi–Yau theorem and Kähler currents*, arXiv:1505.02124 (2015).

[33] V. Tosatti and B. Weinkove, *The complex Monge–Ampère equation on compact Hermitian manifolds*, J. Amer. Math. Soc. 23 (2010), 1187–1195.

Zhiwei Wang
School of Mathematical Sciences
Peking University
Beijing 100871, China
E-mail: wangzw@amss.ac.cn