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On arbitrary products of eigenforms

by

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1. Introduction. For any positive integer $k \geq 4$, let M_k be the space of modular forms of weight k for the group $\mathrm{SL}_2(\mathbb{Z})$. For even $k \geq 2$, the *Eisenstein series* of weight k is given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where B_k is the kth Bernoulli number, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, $q = e^{2\pi i z}$, and z is in the upper half-plane $\mathcal{H}$. For $k \geq 4$ and even, the Eisenstein series E_k defines a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$, and for $k = 2$, the Eisenstein series E_2 is not a modular form. However, E_2 is a quasimodular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$.

Identities among modular forms have attracted the attention of many mathematicians since they imply nice identities between the Fourier coefficients of modular forms. One such is the identity

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) \tag{1}$$

for $n \geq 1$. The above formula follows from the identity $E_4^2 = E_8$, in which the product of two eigenforms is an eigenform. Then to get identities like (1), we need to find all such relations among modular forms in which the product of two eigenforms is again an eigenform. This investigation was done by Duke [3] and Ghate [5] independently for modular forms on the full modular group $\mathrm{SL}_2(\mathbb{Z})$. In fact, they explicitly provided all of the cases in which the above phenomenon occurs.

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Another formula, which does not follow from such an identity in modular forms, is

$$n\tau(n) = \tau(n) - 24\sum_{m=1}^{n-1}\tau(m)\sigma_1(n-m) \tag{2}$$

for $n \geq 1$, where $\tau(n)$ is Ramanujan's tau function. The above identity follows from the relation $D\Delta = E_2\Delta$ between quasimodular forms for the group $\mathrm{SL}_2(\mathbb{Z})$, where $D = \frac{1}{2\pi i}\frac{d}{dz}$ is the differential operator and

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$$

is the Ramanujan delta function. Similarly to the case of modular forms, the relation $D\Delta = E_2\Delta$ is an identity between quasimodular forms in which the product of two quasimodular eigenforms results in an eigenform.

Therefore to get all identities like (2), we need to find all quasimodular eigenforms which can be written as products of two quasimodular eigenforms. This has been done for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ in [2] and [6]. In [1], the problem was considered for a class of nearly holomorphic modular forms for the group $\mathrm{SL}_2(\mathbb{Z})$. More precisely, [1] solved the problem for those nearly holomorphic modular forms which can be written as a Maass–Shimura operator applied to modular forms.

There is a possibility that products of more than two eigenforms result in an eigenform. Emmons and Lanphier [4] have provided all cases in which arbitrary products of eigenforms for the group $\mathrm{SL}_2(\mathbb{Z})$ result in eigenforms.

In this article, we first characterize all nearly holomorphic eigenforms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Let $\delta_k^{(p)}$ be the Maass–Shimura operator and E_2^* be the nearly holomorphic modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ defined in Section 2. Then we prove the following result.

THEOREM 1.1. *Let f be a nearly holomorphic eigenform of weight k and depth p for the full modular group* $\mathrm{SL}_2(\mathbb{Z})$. *If $p < k/2$ then $f = \delta_{k-2p}^{(p)} f_p$, where f_p is a modular form of weight $k-2p$ which is an eigenform, and if $p = k/2$ then $f \in \mathbb{C}\delta_2^{(k/2-1)}E_2^*$.*

Using the above theorem, we then extend the result given in [1] to all nearly holomorphic modular forms for the group $\mathrm{SL}_2(\mathbb{Z})$. From now on, Δ_k will denote the unique normalized cusp form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ for $k \in \{12, 16, 18, 20, 22, 26\}$.

THEOREM 1.2. *The product of two nearly holomorphic eigenforms for* $\mathrm{SL}_2(\mathbb{Z})$ *is never an eigenform except for:*

(1) *the* 16 *holomorphic cases presented in* [3] *and* [5], *namely*

$$E_4^2 = E_8, \qquad E_4E_6 = E_{10}, \qquad E_6E_8 = E_4E_{10} = E_{14},$$
$$E_4\Delta_{12} = \Delta_{16}, \quad E_6\Delta_{12} = \Delta_{18}, \quad E_4\Delta_{16} = E_8\Delta_{12} = \Delta_{20},$$
$$E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta_{12} = \Delta_{22},$$
$$E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta_{12} = \Delta_{26};$$

(2) *the cases*

$$(\delta_4E_4)E_4 = \tfrac{1}{2}\delta_8E_8, \qquad E_2^*\Delta_{12} = \delta_{12}\Delta_{12}.$$

As a consequence of the above theorem, we get the main result of [1]. We also get an extra identity in this case, apart from the identities given in [1].

We then consider the case of products of an arbitrary number of nearly holomorphic eigenforms and characterize all nearly holomorphic eigenforms, which can be written as products of finitely many nearly holomorphic eigenforms.

Theorem 1.3. *The product of finitely many nearly holomorphic eigenforms for* $\mathrm{SL}_2(\mathbb{Z})$ *is never an eigenform except for:*

(1) *the* 16 *holomorphic cases presented in Theorem* 1.2(1)*;*

(2) *other holomorphic cases which can be obtained from some of the identities presented in Theorem* 1.2(1), *namely*

$$E_4^2E_6 = E_{14}, \qquad E_4^2\Delta_{12} = \Delta_{20}, \qquad E_4E_6\Delta_{12} = \Delta_{22},$$
$$E_4^2\Delta_{18} = E_4E_6\Delta_{16} = E_4^2E_6\Delta_{12} = E_6E_8\Delta_{12} = E_4E_{10}\Delta_{12} = \Delta_{26};$$

(3) *the cases*

$$(\delta_4E_4)E_4 = \tfrac{1}{2}\delta_8E_8, \qquad E_2^*\Delta_{12} = \delta_{12}\Delta_{12}.$$

To prove the above, we use the following result on quasimodular forms that characterizes all quasimodular eigenforms which can be written as products of finitely many quasimodular eigenforms.

Theorem 1.4. *The product of finitely many quasimodular eigenforms for* $\mathrm{SL}_2(\mathbb{Z})$ *is never an eigenform except for:*

(i) *the holomorphic cases presented in Theorem* 1.3(1)–(2)*;*

(ii) *the cases*

$$(DE_4)E_4 = \tfrac{1}{2}DE_8, \qquad E_2\Delta_{12} = D\Delta_{12}.$$

2. Nearly holomorphic modular forms

Definition 2.1. A *nearly holomorphic modular form* f of weight k and depth $\le p$ for $\mathrm{SL}_2(\mathbb{Z})$ is a polynomial in $1/\Im(z)$ of degree $\le p$ whose coefficients are holomorphic functions on $\mathcal{H}$ with moderate growth such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathcal{H}$, where $\Im(z)$ is the imaginary part of z.

Let $\widehat{M}_k^{\leq p}$ denote the space of all nearly holomorphic modular forms of weight k and depth $\leq p$ for $\mathrm{SL}_2(\mathbb{Z})$. We denote by $\widehat{M}_k = \bigcup_p \widehat{M}_k^{\leq p}$ the space of all nearly holomorphic modular forms of weight k, and by $\widehat{M}_* = \bigoplus_k \widehat{M}_k$ the graded ring of all nearly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.2. The *Maass–Shimura operator* δ_k on $f \in \widehat{M}_k$ is defined by

$$\delta_k(f) = \left(\frac{1}{2\pi i}\left(\frac{k}{2i\Im(z)} + \frac{\partial}{\partial z}\right)f\right)(z).$$

The operator δ_k takes $\widehat{M}_k$ to $\widehat{M}_{k+2}$. We write $\delta_k^{(m)} := \delta_{k+2m-2} \circ \cdots \circ \delta_{k+2} \circ \delta_k$ with $\delta_k^{(0)} = \mathrm{id}$ and $\delta_k^{(1)} = \delta_k$. We state the following decomposition of the space of nearly holomorphic modular forms [9, Theorem 5.2].

Theorem 2.3. *Let $k \geq 2$ be even. If $f \in \widehat{M}_k^{\leq p}$ and $p < k/2$ then*

$$\widehat{M}_k^{\leq p} = \bigoplus_{r=0}^{p} \delta_{k-2r}^{(r)} M_{k-2r},$$

and if $p \geq k/2$ then

$$\widehat{M}_k^{\leq p} = \bigoplus_{r=0}^{k/2-1} \delta_{k-2r}^{(r)} M_{k-2r} \oplus \mathbb{C}\delta_2^{(k/2-1)} E_2^*,$$

where $E_2^*(z) = E_2(z) - \frac{3}{\pi\Im(z)}$ is a nearly holomorphic modular form of weight 2 for the group $\mathrm{SL}_2(\mathbb{Z})$.

For $f \in \widehat{M}_k$, the action of the nth Hecke operator on f is defined by

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right). \tag{3}$$

For each integer $n \geq 1$, T_n maps $\widehat{M}_k$ to itself. A nearly holomorphic modular form is called an *eigenform* if it is an eigenvector for each Hecke operator T_n. We recall the following commuting relation between Maass–Shimura operators and Hecke operators [1, Propositions 2.4, 2.5].

Proposition 2.4. *Let $f \in \widehat{M}_k$. Then*

$$(\delta_k^{(m)}(T_n f))(z) = \frac{1}{n^m}\big(T_n(\delta_k^{(m)}(f))\big)(z)$$

for $m \geq 0$. Moreover, $\delta_k^{(m)}(f)$ is an eigenform for T_n if and only if f is. In this case, if λ_n is the eigenvalue of T_n corresponding to f then the eigenvalue of T_n corresponding to $\delta_k^{(m)}(f)$ is $n^m\lambda_n$.

3. Quasimodular forms

Definition 3.1. A holomorphic function f on $\mathcal{H}$ is called a *quasimodular form* of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$ if there exist holomorphic functions $f_0, f_1, \ldots, f_p$ on $\mathcal{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{p} f_j(z)\left(\frac{c}{cz+d}\right)^j$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, f_p is not identically vanishing and f has no terms with negative exponents in its Fourier expansion.

Any quasimodular form f for $\mathrm{SL}_2(\mathbb{Z})$ of weight k and depth p can be written as

$$f(z) = g_0(z) + g_1(z)E_2(z) + \cdots + g_p(z)E_2^p(z), \tag{4}$$

where $z \in \mathcal{H}$, $g_i \in M_{k-2i}$ for $0 \le i \le p$ and $g_p \ne 0$. The action of the Hecke operator T_n on a quasimodular form is the same as the action on a nearly holomorphic modular form as given in (3). For each integer $n \ge 1$, T_n maps $\widetilde{M}_k$ to itself. A quasimodular form is called an *eigenform* if it is an eigenvector for each Hecke operator T_n.

4. Preparatory results. We first recall a well known result from linear algebra.

Lemma 4.1. *Let T be a linear operator defined on a finite-dimensional vector space over $\mathbb{C}$. Let $f = \sum_{i=1}^{r} c_i f_i$ be such that f and f_i are eigenvectors under T with eigenvalues a and a_i respectively. If all the f_i are linearly independent, then $a = a_i$ for all i.*

We have the following result for nearly holomorphic modular forms [1, Lemma 2.7].

Lemma 4.2. *Let $k > l$ and $f \in M_k$, $g \in M_l$ be eigenforms. Then for $m \ge 0$, $\delta_l^{((k-l)/2+m)}(g)$ and $\delta_k^{(m)}(f)$ do not have the same set of eigenvalues with respect to the Hecke operators.*

Similarly to the case of modular forms, we have the following two results for quasimodular eigenforms. The proofs go along the same lines as in the case of modular forms.

Lemma 4.3. *If $f = \sum_{n=0}^{\infty} a(n)q^n \in \widetilde{M}_k$ is a non-zero eigenform, then $a(1) \ne 0$.*

Lemma 4.4. *A quasimodular form $f \in \widetilde{M}_k$ with non-zero constant Fourier coefficient is an eigenform if and only if $f \in \mathbb{C}E_k$.*

Next, we recall the following result from [2, Lemma 3.5].

LEMMA 4.5. *For $r \geq 1$ and $h \in M_k$ let $D^r h = h_0 + h_1 E_2 + h_2 E_2^2 + \cdots + h_r E_2^r$ with $h_i \in M_{k+2r-2i}$. Then $h_r = \frac{r!}{12^r}\binom{k+r-1}{r} h$.*

We also prove the following result for derivatives of the Eisenstein series E_2.

LEMMA 4.6. *For $r \geq 1$ let $D^r E_2 = h_0 + h_1 E_2 + \cdots + h_{r+1} E_2^{r+1}$ with $h_i \in M_{2r+2-2i}$. Then $h_{r+1} = r!/12^r$.*

Proof. We apply induction on r. For $r = 1$, it is due to Ramanujan that

$$DE_2 = \frac{-E_4}{12} + \frac{E_2^2}{12}. \tag{5}$$

Now assume that the lemma is true for r. Let

$$D^{r+1}E_2 = D(D^r E_2) = f_0 + f_1 E_2 + \cdots + f_{r+2} E^{r+2}.$$

Then by using the induction hypothesis and (5), we see that

$$f_{r+2} = \frac{r!}{12^r}(r+1)\frac{1}{12} = \frac{(r+1)!}{12^{r+1}}. \blacksquare$$

The following result [2, Proposition 3.1] characterizes all quasimodular eigenforms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

PROPOSITION 4.7. *Let f be a quasimodular eigenform of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$. If $p < k/2$ then $f = D^p f_p$, where f_p is an eigenform of weight $k - 2p$, and if $p = k/2$ then $f \in \mathbb{C}D^{k/2-1}E_2$.*

Next we state the following result which has been proved in [6] and [2]. It gives all the cases in which the product of two quasimodular eigenforms for the group $\mathrm{SL}_2(\mathbb{Z})$ results in an eigenform.

THEOREM 4.8. *The product of two quasimodular eigenforms for $\mathrm{SL}_2(\mathbb{Z})$ is never an eigenform except for:*

(1) *the 16 holomorphic cases presented in [3] and [5], namely*

$$\begin{gathered} E_4^2 = E_8, \qquad E_4E_6 = E_{10}, \qquad E_6E_8 = E_4E_{10} = E_{14}, \\ E_4\Delta_{12} = \Delta_{16}, \quad E_6\Delta_{12} = \Delta_{18}, \quad E_4\Delta_{16} = E_8\Delta_{12} = \Delta_{20}, \\ E_4\Delta_{18} = E_6\Delta_{16} = E_{10}\Delta_{12} = \Delta_{22}, \\ E_4\Delta_{22} = E_6\Delta_{20} = E_8\Delta_{18} = E_{10}\Delta_{16} = E_{14}\Delta_{12} = \Delta_{26}; \end{gathered}$$

(2) *the cases*

$$(DE_4)E_4 = \tfrac{1}{2}DE_8, \qquad E_2\Delta_{12} = D\Delta_{12}.$$

5. Proof of Theorem 1.1. Let f be a nearly holomorphic modular form of weight k and depth p for the group $\mathrm{SL}_2(\mathbb{Z})$. If $p < k/2$, then by

Theorem 2.3 we have

$$(6)\qquad f = \sum_{r=0}^{p} \delta_{k-2r}^{(r)} f_r,$$

where $f_r \in M_{k-2r}$. Since the depth of f is p, $\delta_{k-2p}^{(p)} f_p$ is not identically equal to zero. Also, each f_r can be written as

$$(7)\qquad f_r = \sum_{j=1}^{d_r} b_{rj} h_{rj} + \beta_r E_{k-2r},$$

where $b_{rj}, \beta_r \in \mathbb{C}$, d_r is the dimension of S_{k-2r} and the set $\{h_{rj} : 1 \le j \le d_r\}$ is a Hecke basis of S_{k-2r} for each $0 \le r \le p$. By the hypothesis of the theorem, f is an eigenform. Therefore, by Lemmas 4.2 and 4.1,

$$f = \delta_{k-2p}^{(p)} f_p = \sum_{j=1}^{d_p} b_{pj} \delta_{k-2p}^{(p)} h_{pj} + \beta_p \delta_{k-2p}^{(p)} E_{k-2p}.$$

By Proposition 2.4 and Deligne's bounds for the eigenvalues of modular eigenforms, the nth Hecke eigenvalue of $\delta_{k-2p}^{(p)} h_{pj}$ is $O(n^{(k-1)/2+\epsilon})$ for any $\epsilon > 0$. Moreover, by Proposition 2.4, the nth Hecke eigenvalue of $\delta_{k-2p}^{(p)} E_{k-2p}$ is $n^p \sigma_{k-2p-1}$. Since

$$n^{k-p-1} \le n^p \sigma_{k-2p-1} \le C n^{k-p-1},$$

where C is some positive constant, there exist positive integers n such that the eigenvalues with respect to the Hecke operator T_n for $\delta_{k-2p}^{(p)} E_{k-2p}$ and $\delta_{k-2p}^{(p)} h_{pj}$ are different for each j. Then by Lemma 4.1, we have either $f = \sum_{j=1}^{d_p} b_{pj} \delta_{k-2p}^{(p)} h_{pj}$ or $f = \delta_{k-2p}^{(p)} E_{k-2p}$.

In the case $f = \sum_{j=1}^{d_p} b_{pj} \delta_{k-2p}^{(p)} h_{pj}$, we again apply Lemma 4.1 and use the fact that there are infinitely many n such that the eigenvalues of T_n with respect to any two h_{pj} are different.

Next, we consider $p = k/2$. In this case, one can write

$$f = \sum_{r=0}^{k/2-1} \delta_{k-2r}^{(r)} f_r + \alpha \delta_2^{(k/2-1)} E_2^*,$$

where $f_r \in M_{k-2r}$ and $\alpha \in \mathbb{C}$ is non-zero. The eigenvalue of T_n with respect to $\delta_2^{(k/2-1)} E_2^*$ is $n^{k/2-1}\sigma_1(n)$. Also, for $n > 1$, we have

$$n^{k/2} < n^{k/2-1}\sigma_1(n) \le n^{k/2}(\log n + 1).$$

For the latter inequality, we refer to [7, Exercise 1.3.4]. Again using Lemma 4.1 and comparing the eigenvalues as in the case when $p < k/2$, we conclude that $f = \alpha \delta_2^{(k/2-1)} E_2^*$. This proves the theorem.

6. Proof of Theorem 1.2. The idea of the proof is to use the ring isomorphism between $\widehat{M}_*$ and $\widetilde{M}_*$ and Theorem 4.8. By [8, Theorem 1], the space $\widehat{M}_*^{\leq p}$ of all nearly holomorphic modular forms of depth at most p is isomorphic to the space $\widetilde{M}_*^{\leq p}$ of all quasimodular forms of depth at most p, where the map is defined by

$$f(z) = \sum_{j=0}^{p} \frac{f_j(z)}{\Im(z)^j} \mapsto f_0(z).$$

This map induces a ring isomorphism between $\widehat{M}_*$ and $\widetilde{M}_*$. Also, if f is a nearly holomorphic modular form with constant coefficient f_0, then $\delta_k^{(r)} f$ has the constant coefficient $D^r f_0$. Therefore, by Theorem 1.1 and Proposition 4.7, we deduce that $\delta_k^{(r)} f$ is an eigenform if and only if $D^r f_0$ is an eigenform. By the above ring isomorphism, we see that the product relations among eigenforms in the space of quasimodular forms give rise to product relations among eigenforms in the space of nearly holomorphic modular forms and vice-versa. Theorem 4.8 gives all eigenforms in the space of quasimodular forms which are expressible as products of two eigenforms. Thus we get all the corresponding cases in the space of nearly holomorphic modular forms which are listed in Theorem 1.2.

7. Proof of Theorem 1.3. We assume Theorem 1.4 and prove this result. By Theorem 1.4, we have found all cases in the space of quasimodular forms which are expressible as products of finite numbers of eigenforms. Thus by the same argument as in Theorem 1.2, we get the corresponding result in the space of nearly holomorphic modular forms.

8. Proof of Theorem 1.4. By Lemmas 4.3, 4.4 and Proposition 4.7, we need to find out only in the following cases if the products result in eigenforms:

(1) $E_2^a E_{k_1} \ldots E_{k_m}$, where $k_i \geq 4$ for each i and $a + m \geq 2$,
(2) $E_2^a E_{k_1} \ldots E_{k_m} f$, where $k_i \geq 4$ for each i and $a + m \geq 1$, and f is a cusp form which is an eigenform,
(3) $E_2^a E_{k_1} \ldots E_{k_m} D^r E_2$, where $k_i \geq 4$ for each i, $r \geq 1$ and $a + m \geq 1$,
(4) $E_2^a E_{k_1} \ldots E_{k_m} D^r f$, where $k_i \geq 4$ for each i, $r \geq 1$ and $a + m \geq 1$, and f is a cusp form which is an eigenform,
(5) $E_2^a E_{k_1} \ldots E_{k_m} D^r E_k$, where k_i for each i, $k \geq 4$, $r \geq 1$ and $a+m \geq 1$.

In the above cases, we assume that the product $E_{k_1} \ldots E_{k_m}$ is 1 if $m = 0$.

In case (1), if $a = 0$, then the matter reduces to the case of a product of Eisenstein series which are modular forms. Then by [4], we have all the cases in which the product is again an eigenform, and these are listed in the statement of Theorem 1.4. If $a \neq 0$, then the constant term of the product is non-zero and the product is a non-modular quasimodular form which is not a constant multiple of E_2. Therefore by Lemma 4.4, this product is never an eigenform.

In case (2), if $a = 0$, then again the reasoning reduces to the modular case, and then by [4] we have all the cases in which the product is again an eigenform, which are listed in the statement of Theorem 1.4. If $a \neq 0$, let k be the weight of f and, without loss of generality, assume that f is normalized, i.e. the coefficient of q in the Fourier expansion of f is 1. Then the depth a of $E_2^a E_{k_1} \dots E_{k_m} f$ is strictly less than half of its weight $2a + k_1 + \cdots + k_m + k$. Thus by Proposition 4.7 we have

$$E_2^a E_{k_1} \dots E_{k_m} f = D^a h, \tag{8}$$

where h is a normalized modular eigenform. Using Lemma 4.5 and comparing the coefficients of E_2^a on both sides of (8), we obtain

$$E_{k_1} \dots E_{k_m} f = \frac{a!}{12^a} \binom{k_1 + \cdots + k_m + k + a - 1}{a} h.$$

Comparing the Fourier coefficients of q on both sides of the above identity, we get

$$\frac{a!}{12^a} \binom{k_1 + \cdots + k_m + k + a - 1}{a} = 1.$$

Simplifying the above identity yields

$$(k_1 + \cdots + k_m + k)(k_1 + \cdots + k_m + k + 1) \dots (k_1 + \cdots + k_m + k + a - 1) = 12^a.$$

Since $k \geq 12$, the above equality is valid only when $a = 1$ and $m = 0$. Then it implies that $k = 12$ and hence

$$E_2 \Delta_{12} = D \Delta_{12},$$

where Δ_{12} is the Ramanujan Delta function. The above identity is listed in the statement of Theorem 1.4.

Now consider case (3). If $m = 0$ then using Proposition 4.7 and comparing coefficients, we get

$$E_2^a D^r E_2 = D^{r+a} E_2.$$

Comparing the coefficients of q^2 on both sides, we get

$$2^r - 8a = 2^{r+a}.$$

This is not possible since $r, a \geq 1$. If $m \geq 1$ then by Proposition 4.7 we have

$$E_2^a E_{k_1} \dots E_{k_m} D^r E_2 = D^{r+a+1} g, \tag{9}$$

where g is a modular eigenform of weight $k = k_1 + \cdots + k_m$. Using (4) and Lemmas 4.6, 4.5, and then comparing the coefficients of E_2^{r+a+1} on both sides of the above equality, we obtain

$$\frac{r!}{12^r} E_{k_1} \dots E_{k_m} = dg, \tag{10}$$

where

$$d = \frac{(r+a+1)!}{12^{r+a+1}} \binom{k+r+a}{r+a+1}.$$

Since g is an eigenform of weight k with non-zero constant Fourier coefficient, by Lemma 4.4 we have $g = cE_k$ for some non-zero constant c. Substituting this value of g in (10) and comparing the constant Fourier coefficients on both sides of (10) we obtain

$$c = \frac{r!}{12^r d}.$$

Therefore, by (9) we have

$$E_2^a E_{k_1} \dots E_{k_m} D^r E_2 = \frac{r!}{12^r d} D^{r+a+1} E_k. \tag{11}$$

Comparing the Fourier coefficients of q on both sides of the above equation gives

$$-\frac{2k}{B_k} = -24 \frac{12^r}{r!} d. \tag{12}$$

If $m = 1$ we have

$$E_2^a E_k D^r E_2 = \frac{r!}{12^r d} D^{r+a+1} E_k.$$

Comparing the coefficients of q^2 in the above identity we get

$$\frac{-2k}{B_k} \frac{r!}{12^r d} 2^{r+a+1} \sigma_{k-1}(2) = -24 \left(3 \cdot 2^r - 24a - \frac{2k}{B_k} \right).$$

Using the fact that $\frac{-2k}{B_k} \frac{r!}{12^r d} = -24$ from (12) we arrive at

$$2^{r+a+1} \sigma_{k-1}(2) - 3 \cdot 2^r = -\left(24a + 24 \frac{12^r}{r!} d \right).$$

The above identity is not possible since the left hand side is a positive quantity, whereas the right hand side is negative.

If $m > 1$ then dg is, by (10) and up to a constant, a product of two or more Eisenstein series. Thus by [4], the possible values of k are $8, 10, 14$. From (12) we see that $-2k/B_k$ is negative. But for $k = 8$, $-2k/B_k = 480$ is positive. Thus $k = 8$ is not possible. If $k = 10$ then $-2k/B_k = -264$. From (12) we have

$$12^a \cdot 264 \cdot 9! = 2(r+1)(r+2) \dots (r+a+10).$$

From the above identity, we see that the right hand side is divisible by 25 but the left hand side is not. Thus the case $k = 10$ does not arise. Similarly, we get a contradiction if $k = 14$.

For case (4), without loss of generality, we assume that f is normalized. Suppose that the weight of f is k. By Proposition 4.7 we have

$$E_2^a E_{k_1} \dots E_{k_m} D^r f = D^{r+a} g,$$

where g is a normalized modular eigenform of weight $l = k + k_1 + \cdots + k_m$. Since both f and g are nomalized, applying Lemma 4.5 and (4) to both sides of the above equality and then comparing the coefficients of E_2^{r+a} we obtain

$$\frac{r!}{12^r}\binom{k+r-1}{r} = \frac{(r+a)!}{12^{r+a}}\binom{l+r+a-1}{r+a}.$$

This leads to

$$12^a k(k+1)\dots(k+r-1) = l(l+1)\dots(l+r+a-1).$$

Since $l \geq k \geq 12$, the above equality holds only when $a = 0$ and $l = k$. This implies that $a + m = 0$, which contradicts the assumption that $a + m \geq 1$.

Finally consider case (5). By Proposition 4.7, we have

$$E_2^a E_{k_1} \dots E_{k_m} D^r E_k = D^{r+a} h, \tag{13}$$

where h is a modular eigenform of weight $l = k + k_1 + \cdots + k_m$. Applying Lemma 4.5 and (4) to both $D^r E_k$ and $D^{r+a} h$, and then comparing the coefficients of E_2^{r+a} from both sides of the above equality, we have

$$d_1 E_{k_1} \dots E_{k_m} E_k = d_2 h, \tag{14}$$

where $d_1 = \frac{r!}{12^r}\binom{k+r-1}{r}$ and $d_2 = \frac{(r+a)!}{12^{r+a}}\binom{l+r+a-1}{r+a}$. Since $E_{k_1} \dots E_{k_m} E_k$ has non-zero constant Fourier coefficient, we deduce from (14) that the constant Fourier coefficient of h is non-zero. Hence by Lemma 4.4, $h = cE_l$ for some non-zero constant c. Substituting $h = cE_l$ in (13) we get the identity

$$E_2^a E_{k_1} \dots E_{k_m} D^r E_k = c D^{r+a} E_l. \tag{15}$$

From (14) we also see that

$$E_{k_1} \dots E_{k_m} E_k = E_l \tag{16}$$

and

$$c = \frac{d_1}{d_2}.$$

If $m = 0$, then $k = l$ and $c = d_1/d_2 = 1$. This implies that

$$12^a = (k+r)(k+r+1)\dots(k+r+a-1).$$

The above identity will hold only if $a = 1$ and $k + r = 12$. So we are left with the case

$$E_2 D^r E_k = D^{r+1} E_k, \quad \text{where} \quad k + r = 12.$$

By comparing the Fourier coefficients of q^2 on both sides, we see that the above identity cannot be true for $k+r=12$. Thus $m=0$ is not possible. Now let $m \geq 1$. Comparing the coefficients of q on both sides of (15) we see that

$$c = \frac{d_1}{d_2} = \frac{\left(\frac{-2k}{B_k}\right)}{\left(\frac{-2l}{B_l}\right)} = \frac{2kB_l}{2lB_k}. \tag{17}$$

By [4], the values of the tuple (k,l) for which (16) holds are

$$(4,8),(4,10),(6,10),(4,14),(6,14),(8,14),(10,14).$$

We see that $(4,10)$, $(4,14)$ and $(8,14)$ are ruled out since for these values, $\frac{2kB_l}{2lB_k}$ is negative but $\frac{d_1}{d_2}$ is always positive, which contradicts (17). So the remaining values of (k,l) to be checked are

$$(4,8),(6,10),(6,14),(10,14).$$

Also from (17), we see that

$$\frac{2l}{B_l}\frac{r!}{12^r}\binom{k+r-1}{r} = \frac{2k}{B_k}\frac{(r+a)!}{12^{r+a}}\binom{l+r+a-1}{r+a}.$$

Simplifying the above identity, we arrive at

$$12^a\frac{2l}{B_l}k(k+1)\ldots(l-1) = \frac{2k}{B_k}(k+r)(k+r+1)\ldots(l+r+a-1). \tag{18}$$

For $(k,l)=(4,8)$ we have $(-2k/B_k,-2l/B_l)=(240,480)$. Substituting the above values in (18) we obtain

$$12^a \times 5 \times 6 \times 7 \times 8 = (r+4)(r+5)\ldots(r+a+7). \tag{19}$$

If $r+a+7 \geq 11$, then we get a contradiction from (19) since the left hand side of (19) is not divisible by 11 whereas the right hand side is. Also, since $r \geq 1$, we deduce that $r+a+7 \geq 8$. Thus

$$1 \leq r+a \leq 3.$$

By taking values of r and a for which $r+a=2,3$, one sees that we get contradictions from (19). Thus the only case remaining is $r+a=1$. This implies that $r=1$ and $a=0$. Therefore from (15), we obtain the identity

$$E_4(DE_4) = \tfrac{1}{2}DE_8.$$

If $(k,l)=(6,10)$, then $(-2k/B_k,-2l/B_l)=(504,264)$. Substituting these values in (18) we obtain

$$12^a \times 11 \times 6 \times 7 \times 8 \times 9 = 21(r+6)(r+7)\ldots(r+9+a). \tag{20}$$

As in the previous case we get

$$1 \leq r+a \leq 3.$$

Inserting the above possible values of r and a in (20), we see that they contradict (20). Thus the case $(k,l) = (6,10)$ is not possible. Similarly, we get contradictions for the other possible values $(k,l) = (6,14)$ and $(10,14)$. This proves the theorem.

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