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Muckenhoupt–Wheeden conjectures in higher dimensions

by

Alberto Criado (Bilbao) and Fernando Soria (Madrid)

Abstract. In recent work by Reguera and Thiele (2012) and by Reguera and Scurry (2013), two conjectures about joint weighted estimates for Calderón–Zygmund operators and the Hardy–Littlewood maximal function were refuted in the one-dimensional case. One of the key ingredients for these results is the construction of weights for which the value of the Hilbert transform is substantially bigger than that of the maximal function. In this work, we show that a similar construction is possible for classical Calderón–Zygmund operators in higher dimensions. This allows us to fully disprove the conjectures.

1. Introduction and statements of results. In this paper we will study joint weighted estimates for the Hardy–Littlewood maximal operator and classical Calderón–Zygmund operators. We consider the non-centered Hardy–Littlewood maximal operator over cubes, defined for a locally integrable function f as

$$Mf(x) = \sup_{x\in Q\subset \mathcal{Q}} \fint_Q |f(y)|\,dy,$$

where $\mathcal{Q}$ denotes the family of all cubes with sides parallel to the coordinate axes in $\mathbb{R}^d$. We will also consider classical Calderón–Zygmund singular integral operators, whose action on a smooth function f is defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^d} K(x,y)f(y)\,dy.$$

Here the kernel K has the form

$$K(x,y) = \frac{\Omega(x-y)}{|x-y|^d}, \tag{1.1}$$

with Ω a homogeneous function of degree 0 such that $\Omega \in C^1(\mathbb{S}^{d-1})$ and $\int_{\mathbb{S}^{d-1}} \Omega(x)\,d\sigma_{d-1}(x) = 0$. The Hilbert transform in one dimension and the

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Riesz transforms in higher dimensions are examples of such operators. We may also consider more general Calderón–Zygmund operators. In fact, our arguments work well for operators with variable kernels K satisfying standard size and regularity conditions. We will not pursue here these generalizations. Instead, we will make some comments on how to extend our results to this more general setting.

In this context, a *weight* simply means a non-negative, locally integrable function $w : \mathbb{R}^n \to [0, \infty]$. Such a w can be interpreted as the density of an absolutely continuous measure. This measure is usually denoted by the same letter as its density. That is, if w is a weight in $\mathbb{R}^n$, then for a measurable $E \subset \mathbb{R}^n$ one writes $w(E) = \int_E w(x)\,dx$ and for f a measurable function we say that $f \in L^p(w)$ if $\|f\|_{L^p(w)} = (\int |f|^p w)^{1/p} < \infty$.

In the 1970's B. Muckenhoupt and R. Wheeden among other authors began the study of weighted inequalities for maximal, Calderón–Zygmund and other operators. They defined the A_p class as the collection of weights w satisfying

$$\sup_{Q\in\mathcal{Q}} \fint_Q w(y)\,dy \left(\fint_Q w(y)^{-p'/p}\,dy\right)^{p/p'} < \infty \tag{1.2}$$

if $1 < p < \infty$, or

$$Mw(x) \le Cw(x) \quad \text{a.e. } x, \tag{1.3}$$

with $C > 0$ independent of x if $p = 1$. It is well known that $w \in A_p$ is equivalent to M being bounded on $L^p(w)$ if $p > 1$, and to M being weakly bounded on $L^1(w)$ if $p = 1$. It is also known that (1.2) and (1.3) are sufficient too for the same kind of estimates of a Calderón–Zygmund operator, but only necessary in the sense that if all the d Riesz transforms are weakly bounded on $L^p(w)$, then $w \in A_p$, for $1 \le p < \infty$. In the one-dimensional case this means in particular that the Hilbert transform is weakly bounded on $L^p(w)$ if and only if $w \in A_p$, for $1 \le p < \infty$. For a more complete account of these facts see [13] and [14].

The situation is more complicated when one considers norm estimates with two weights. A pair of weights (u, v) is in the A_p class if

$$\sup_{Q\in\mathcal{Q}} \fint_Q v(y)\,dy \left(\fint_Q u(y)^{1-p'}\,dy\right)^{1/p'} < \infty \tag{1.4}$$

for $p > 1$, and in A_1 if

$$Mv(x) \le Cu(x) \quad \text{a.e. } x, \tag{1.5}$$

with $C > 0$ independent of x. These conditions are equivalent to the mapping $M : L^p(u) \to L^{p,\infty}(v)$ being bounded for $1 \le p < \infty$, and necessary for the strong boundedness $M : L^p(u) \to L^p(v)$ if $p > 1$, but not sufficient for it.

The continuity of M from $L^p(u)$ to $L^p(v)$ was, nevertheless, characterized by E. Sawyer [25] to be equivalent to

$$\int_Q M(\chi_Q v^{1-p'})^p u \le C \int_Q v^{1-p'} < \infty$$

for all $Q \in \mathcal{Q}$. In the one-weight setting some of the norm estimates for Calderón–Zygmund operators were shown to be equivalent to the ones for M. This suggested that similar connections might be found in the two-weight setting. B. Muckenhoupt, R. Wheeden and others proposed several of them. For many years they could not be confirmed or refuted and became known as Muckenhoupt–Wheeden conjectures.

Perhaps the most famous one originates in a result by C. Fefferman and E. M. Stein [12] showing that there is an absolute constant such that for any weight w one has

$$w(\{x \in \mathbb{R}^d : Mf(x) > \lambda\}) \le \frac{C}{\lambda} \int f(x) Mw(x)\, dx. \tag{1.6}$$

To Muckenhoupt and Wheeden is attributed the conjecture that the same two-weight inequality should be true for a Calderón–Zygmund operator.

Conjecture 1. For each classical Calderón–Zygmund operator T, there exists a constant $C > 0$ such that for every weight w one has

$$w(\{x \in \mathbb{R}^d : |Tf(x)| > \lambda\}) \le \frac{C}{\lambda} \int |f(x)| Mw(x)\, dx \tag{1.7}$$

for all $\lambda > 0$ and $f \in L^1(Mw)$.

The question was extended to more general operators and the conjecture was shown to be true for some square functions in [2], but false for fractional integral operators in [1]. The closest approach, on the positive side, for Calderón–Zygmund operators is due to C. Pérez, who showed in [19] that (1.7) is true if M is replaced by the iterated operator M^2 or even by the operator $M_{L(\log L)^\varepsilon}$ with any $\varepsilon > 0$. Later, C. Pérez and D. Cruz-Uribe [7] used the extrapolation technique to show that if (1.7) holds for a sublinear operator T, then

$$\int |Tf(x)|^p w(x)\, dx \le C \int |f(x)|^p \left(\frac{Mw(x)}{w(x)}\right)^p w(x)\, dx, \tag{1.8}$$

for all $p > 1$. This necessary condition was disproved by M. C. Reguera and C. Thiele [24] for $p = 2$, thus showing the conjecture to be false. They gave a counterexample in the one-dimensional case, that is, when T is the Hilbert transform. The construction was based on a simplification of the technique used by M. C. Reguera [22] in order to refute the corresponding assertion in the dyadic setting.

Our first result shows that Conjecture 1 is false for all classical Calderón–Zygmund operators.

THEOREM 1. *Let T be a Calderón–Zygmund operator with an associated kernel satisfying* (1.1). *Then for each $N > 0$ there is a weight w, a function $f \in L^1(Mw)$ and a constant $\lambda > 0$ such that*

$$w(\{|Tf| > \lambda\}) \geq \frac{N}{\lambda}\int |f|Mw. \tag{1.9}$$

D. Cruz-Uribe, C. Pérez and J. M. Martell [5] considered another conjecture relating two-weight estimates for the maximal operator and Calderón–Zygmund operators. This conjecture is also attributed to Muckenhoupt and Wheeden and its precise statement is the following.

CONJECTURE 2. Let T be a Calderón–Zygmund operator as above. Then

$$\left.\begin{array}{r} M : L^p(u) \to L^p(v) \\ M : L^{p'}(v^{1-p'}) \to L^{p'}(u^{1-p'}) \end{array}\right\} \Rightarrow\ T : L^p(u) \to L^p(v).$$

REMARK. To simplify the notation, the symbol '$S : X \to Y$' will always mean that the operator S maps the elements of the space X into elements of Y in a continuous way. This notation has already been used in the statement of the above conjecture.

The motivation for the second condition on M is the following. A simple duality argument shows that since T is an essentially self-adjoint operator, $T : L^p(u) \to L^p(v)$ is equivalent to $T : L^{p'}(v^{1-p'}) \to L^{p'}(u^{1-p'})$.

This conjecture was refuted by M. C. Reguera and J. Scurry [23] for the Hilbert transform. Their counterexample is based on the one that disproved Conjecture 1 in [24]. We show that the conjecture is false for every classical Calderón–Zygmund operator.

THEOREM 2. *Fix $1 < p < \infty$, and let T be a Calderón–Zygmund operator as in Theorem* 1. *Then one can construct weights u and v such that $M : L^p(u) \to L^p(v)$ and $M : L^{p'}(v^{1-p'}) \to L^{p'}(u^{1-p'})$ but there exists an $f \in L^p(u)$ such that $\|Tf\|_{L^p(v)} = \infty$.*

One important observation is that while an A_p weight is a.e. positive, the previous results have no assumptions on the support of the weight. In fact, one of the key ingredients in the proofs in [24] and [23] is to consider weights with sparse support. At this point, it is not known whether Conjecture 2 holds for a.e. positive weights. In [23] it is shown that in the one-dimensional setting the weights considered there do not preserve the equivalence of the boundedness of M and H on weighted L^p. We will extend this by showing that, unlike for a.e. positive weights, in this setting the boundedness of M on $L^p(w)$ does not imply the same result for Calderón–Zygmund operators.

THEOREM 3. *Let T be a Calderón–Zygmund operator. Then there exist a weight u and a function $f \in L^p(u)$ such that M is bounded on $L^p(u)$ but $\|Tf\|_{L^p(u)} = \infty$.*

In order that the questions we are treating make sense, for w a weight vanishing in some set of positive Lebesgue measure, we will define $L^p(w)$ as the space of measurable functions f such that $\operatorname{supp} f \subset \operatorname{supp} w$ and $\|f\|_{L^p(w)} < \infty$ ([1]).

Although our work does not make any contribution to them, for completeness we briefly comment on some other important Muckenhoupt–Wheeden conjectures. Conjecture 2 had a weak version asserting that $M : L^{p'}(v^{1-p'}) \to L^{p'}(u^{1-p'})$ implies $T : L^p(u) \to L^{p,\infty}(v)$ for T a Calderón–Zygmund operator. This has been shown to be false for the Hilbert transform by D. Cruz-Uribe, A. Reznikov and A. Volberg [10]. By duality, Conjecture 1 implied this last conjecture. Thus, the argument in [10] also refutes the one-dimensional case of Conjecture 1 in an indirect way.

Finally, we mention a still open conjecture. It asserts that replacing the L^p or $L^{1-p'}$ integrability requirement in (1.4) by a slightly stronger one in the sense of Orlicz integrals will be enough to guarantee the L^p boundedness of Calderón–Zygmund operators. This is known as the bump conjecture and only partial results on it have been obtained so far. For more details see [3–10, 16–21, 27].

The rest of the paper is organized as follows. In Section 2 we prove Theorems 1–3 assuming the existence of some weights with certain specific properties. Section 3 is devoted to the construction of such weights. As usual, C and c will denote positive constants that may have different values at different occurrences. Also, given two quantities $A, B > 0$, by writing $A \lesssim B$ we mean that there exists a constant $C > 0$, which may depend on the dimension but is otherwise independent of the main parameters involved, such that $A \le CB$. We define $A \gtrsim B$ in the obvious way and we write $A \sim B$ if both $A \lesssim B$ and $A \gtrsim B$.

2. Proofs of the theorems. The proofs of the three theorems stated in the previous section are based on the construction of weights satisfying a local A_1 property but allowing large values under the action of a given Calderón–Zygmund operator.

([1]) In a similar fashion, the expression $w^\alpha(x)$ for negative α is set to be zero at the points x where $w(x) = 0$.

Proposition 1. *Let T be a Calderón–Zygmund operator with an associated kernel satisfying* (1.1). *Then, for each sufficiently large $N \in \mathbb{N}$, there exists a weight w_N such that if we denote $D_N := \operatorname{supp} w_N \subset [0,1]^d$ then both $w_N \geq 1$ and $Mw_N \lesssim w_N$ on D_N and $|Tw_N| \gtrsim Nw_N$ on some $\widehat{D_N} \subset D_N$ with $|\widehat{D_N}| \sim |D_N|$ and $w_N(\widehat{D_N}) \sim w_N(D_N) = 1$.*

The conclusion $Mw_N \lesssim w_N$ on the support of w_N is what makes w_N an A_1 weight in a local sense. We will first prove Theorems 1–3 assuming that Proposition 1 is true, leaving its proof for the next section.

Proof of Theorem 1. Let T^* be the adjoint operator of T. Note that T is an essentially self-adjoint operator, as $T^*f(x) = Tf(-x)$. Given $N > 0$ consider the weight w_N associated to T^* from Proposition 1. Taking $f = w_N T^* w_N/(Mw_N)^2$, we have

$$\begin{aligned}(2.1)\qquad \int Tf\, w_N = \int f\, T^* w_N &= \int \left|\frac{T^* w_N}{Mw_N}\right|^2 w_N \\ &\gtrsim N^2\, w_N(\widehat{D_N}) \gtrsim N^2 > 0.\end{aligned}$$

Letting F be the non-increasing rearrangement of $|Tf|$ with respect to w_N in $\mathbb{R}^d$, we also have

$$\begin{aligned}(2.2)\qquad \int \left|\frac{T^* w_N}{Mw_N}\right|^2 w_N &= \int Tf\, w_N \leq \int |Tf|\, w_N \\ &= \int_0^{w_N(\mathbb{R}^d)} F(t)\, dt \\ &\leq \int_0^1 \frac{dt}{t^{1/2}} \sup_{s>0} s^{1/2} F(s) \\ &= 2 \sup_{\lambda>0} \lambda w_N(\{|Tf| > \lambda\})^{1/2} \\ &\leq 3\lambda_0 w_N(\{|Tf| > \lambda_0\})^{1/2}\end{aligned}$$

for some λ_0. Combined with (2.1), this yields

$$\begin{aligned}(2.3)\qquad \left(\int \left|\frac{T^* w_N}{Mw_N}\right|^2 w_N\right)^{1/2} &\lesssim \frac{1}{N} \int \left|\frac{T^* w_N}{Mw_N}\right|^2 w_N \\ &\lesssim \frac{1}{N} \lambda_0\, w_N(\{|Tf| > \lambda_0\})^{1/2}.\end{aligned}$$

Now we define $E = \{|Tf| > \lambda_0\}$ and $w = \chi_E w_N$. Using Hölder's inequality and (2.3) we have

$$\begin{aligned}\int|f|\,Mw &= \int\frac{w_N T^* w_N}{(Mw_N)^2}Mw\\ &\le \left(\int\left|\frac{T^*w_N}{Mw_N}\right|^2 w_N\right)^{1/2}\left(\int\left|\frac{Mw}{Mw_N}\right|^2 w_N\right)^{1/2}\\ &\lesssim \frac{\lambda_0}{N}w_N(E),\end{aligned}$$

where the last inequality is a consequence of the following lemma.

Lemma 1. *There exists a constant $C>0$ such that for all weights v and all measurable sets $E\subset\mathbb{R}^n$ one has*

$$\left(\int\left|\frac{M(\chi_E v)}{Mv}\right|^2 v\right)^{1/2}\le Cv(E)^{1/2}. \tag{2.4}$$

Proof. Given a weight v we define the operator S_v for $f\in L^1_{\mathrm{loc}}(v)$ as

$$S_v f(x)=\frac{M(fv)(x)}{Mv(x)}.$$

We will prove indeed a stronger result: for all $p>1$,

$$\int|S_v f|^p v\le C\int|f|^p v.$$

Since $M(fv)\le\|f\|_{L^\infty(v)}Mv$, the operator S_v is bounded on $L^\infty(v)$ with operator norm 1. By interpolation, the result is proved if we show that S_v is of weak type $L^1(v)$ with a constant independent of v. Since it makes no essential difference, we will check it for $\widetilde{S}_v f=\widetilde{M}(fv)/\widetilde{M}v$, where $\widetilde{M}$ denotes the centered maximal operator. Let $f\in L^1(v)$ and $0<\lambda<1$. If $\widetilde{S}_v f(x)>\lambda$, there exists $R_x>0$ such that

$$\fint_{Q(x,R_x)}|f|v>\lambda\widetilde{M}v(x)\ge\lambda\fint_{Q(x,R_x)}v>0,$$

where by $Q(x,R)$ we mean the cube in $\mathcal{Q}$ of edge length R and centered at x. This implies that

$$v(Q(x,R_x))\le\frac{1}{\lambda}\int_{Q(x,R_x)}|f|v.$$

Observe that the cubes $Q(x,R_x)$ with $x\in A_\lambda:=\{x\in\mathbb{R}^d:\widetilde{S}f(x)>\lambda\}$ are a Besicovitch cover of A_λ. By the Besicovitch Covering Theorem (see [15]) there is a subcover by cubes $Q(x,R_x)$, with $x\in A_\star\subset A_\lambda$, such that each $x\in\mathbb{R}^d$ belongs to at most b_d cubes of the subcover, where b_d is a number that only depends on the dimension. Then we have

$$v(A_\lambda) \le v\Big(\bigcup_{x\in A_\star} Q(x,R_x)\Big) \le \sum_{x\in A_\star} v(Q(x,R_x))$$
$$\le \frac{1}{\lambda}\sum_{x\in A_\star}\int_{Q(x,R_x)} |f|v \le \frac{b_d}{\lambda}\int |f|v.$$

This proves the lemma, and hence Theorem 1 too. ∎

Proof of Theorem 3. We use the same 'gliding hump' argument as in [23]. Let $z\in\mathbb{R}^d$ be a unitary vector. We define $w := \sum_{N=N_0}^{\infty} \widetilde{w}_N$, where $\widetilde{w}_N(x) = w_N(x-3^N z)$ and w_N are the weights described in Proposition 1, starting at some N_0 large. We also define $g := \sum_{N=N_0}^{\infty} N^{-\varepsilon}\chi_{Q_N}$ with $Q_N = [0,1]^d + 3^N z$ and $1/p<\varepsilon<1$. Finally, we take $u = w^{1-p}$ and $f = gw$.

First, we check that $f\in L^p(u)$:

$$\int |f|^p u = \int g^p w = \sum_{N=N_0}^{\infty}\int_{Q_N} \frac{1}{N^{\varepsilon p}} w_N(x-3^N z)\,dx = \sum_{N=N_0}^{\infty}\frac{1}{N^{\varepsilon p}} < \infty.$$

Next, we see that $Tf\notin L^p(u)$. In order to do so, we write $\|Tf\|_{L^p(u)}$ as

$$\bigg(\sum_{N=N_0}^{\infty}\int_{Q_N}\bigg|\frac{1}{N^\varepsilon}T\widetilde{w}_N(x) + \sum_{J\ne N}\frac{1}{J^\varepsilon}T\widetilde{w}_J(x)\bigg|^p \widetilde{w}_N(x)^{1-p}\,dx\bigg)^{1/p}.$$

By the triangle inequality this is greater than or equal to $A-B$, where

$$A = \bigg(\sum_{N=N_0}^{\infty}\int_{Q_N}\bigg|\frac{1}{N^\varepsilon}T\widetilde{w}_N(x)\bigg|^p \widetilde{w}_N(x)^{1-p}\,dx\bigg)^{1/p},$$
$$B = \bigg(\sum_{N=N_0}^{\infty}\int_{Q_N}\bigg|\sum_{J\ne N}\frac{1}{J^\varepsilon}T\widetilde{w}_J(x)\bigg|^p \widetilde{w}_N(x)^{1-p}\,dx\bigg)^{1/p}.$$

We will see that $A=\infty$ and $B<\infty$. We begin with B. If $x\in Q_N$ and $J\ne N$ we have

$$|T\widetilde{w}_J(x)| \le \int_{Q_J} |K(x-y)| w_J(y-3^J z)\,dy$$
$$\lesssim \int_{R_J} \frac{1}{|3^N-3^J|^d} w_J(y-3^J z)\,dy$$
$$\lesssim \frac{1}{\max\{3^N,3^J\}^d} w_J([0,1]^d) \le \frac{1}{3^{dN/2}3^{dJ/2}}.$$

Here we have used the fact that for $y\in Q_J$ and $J\ne N$ one has $|x-y|\sim$

$|3^N - 3^J| \sim 3^N + 3^J$. Hence,

$$B^p \lesssim \sum_{N=N_0}^{\infty} \int_{Q_N} \Big| \sum_{J \neq N} \frac{1}{J^\varepsilon} \frac{1}{3^{dN/2} 3^{dJ/2}} \Big|^p w_N(x - 3^N z)^{1-p}\, dx$$

$$\leq \sum_{N=N_0}^{\infty} \Big| \sum_{J \neq N} \frac{1}{J^\varepsilon} \frac{1}{3^{dN/2} 3^{dJ/2}} \Big|^p < \infty.$$

Now we proceed with A. Using an obvious change of variables in the integration, and the property that $|Tw_N| \geq CNw_N$ in $\widehat{D_N}$, we obtain

$$A^p = \sum_{N=N_0}^{\infty} \frac{1}{N^{\varepsilon p}} \int_{[0,1]^d} |Tw_N(x)|^p w_N(x)^{1-p}\, dx$$

$$\geq \sum_{N=N_0}^{\infty} \frac{1}{N^{\varepsilon p}} \int_{\widehat{D_N}} |Tw_N(x)|^p w_N(x)^{1-p}\, dx$$

$$\gtrsim \sum_{N=N_0}^{\infty} \frac{N^p}{N^{\varepsilon p}} \int_{\widehat{D_N}} w_N(x)\, dx \gtrsim \sum_{N=N_0} N^{p(1-\varepsilon)} = \infty.$$

It remains to prove that M is bounded on $L^p(u)$. Since it makes no essential difference we will prove it for the centered maximal operator $\widetilde{M}$ again. We define $\mathcal{Q}_w = \{Q \in \mathcal{Q} : w(Q) > 0\}$. For $f \in L^p(u)$ and $Q \in \mathcal{Q}_w$ we have

$$\frac{1}{|Q|} \int |f| = \frac{w(Q)}{|Q|} \frac{1}{w(Q)} \int_Q |fw^{-1}| w.$$

This implies that

$$\widetilde{M} f \leq \widetilde{M} w \widetilde{M}_w (fw^{-1}),$$

where $\widetilde{M}_w$ is the centered maximal operator associated to w defined by

$$\widetilde{M}_w g(x) = \sup_{R>0,\, w(Q(x,R))>0} \frac{1}{w(Q(x,R))} \int_{Q(x,R)} |g| w.$$

It is easy to check that for $x \in Q_N$ one has $Mw(x) \sim Mw_N(x - 3^N z) \lesssim w_N(x - 3^N z) = w(x)$, that is,

$$Mw \sim w \quad \text{in } \operatorname{supp} w. \tag{2.5}$$

Hence, since the same is true for $\widetilde{M}$, we have

$$\int [\widetilde{M} f]^p w^{1-p} \leq \int [\widetilde{M} w]^p [\widetilde{M}_w (fw^{-1})]^p w^{1-p} \lesssim \int [\widetilde{M}_w (fw^{-1})]^p w.$$

A well-known consequence of the Besicovitch Covering Theorem is that $\widetilde{M}_w$ is bounded on $L^p(w)$. This, together with the observation that $f \in L^p(w^{1-p})$ if and only if $fw^{-1} \in L^p(w)$, finishes the proof. ∎

We now present the proof of Theorem 2. As we will see, everything reduces to the same arguments used in the proof of Theorem 3.

Proof of Theorem 2. At this point we assume that the reader is familiar with the notation and the circle of ideas surrounding the proof of Theorem 3. Taking again $w(x) = \sum_{N=N_0}^{\infty} w_N(x - 3^N z)$ we consider the weights $u = (Mw/w)^p w$ and w. In view of (2.5), we have $u \sim w$ in $W = \operatorname{supp} w$, which reduces the problem to the one-weight setting. The reason to choose these weights is that in this way we also disprove the necessary condition (1.8).

It is easy to see that for an essentially self-adjoint operator T, the following inequalities are equivalent:

$$\|Tf\|_{L^p(w)} \lesssim \|f\|_{L^p(u)}, \tag{2.6}$$
$$\|T(fu^{1-p'})\|_{L^p(w)} \lesssim \|f\|_{L^p(u^{1-p'})},$$
$$\|T(fw)\|_{L^{p'}(u^{1-p'})} \lesssim \|f\|_{L^{p'}(w)}. \tag{2.7}$$

Instead of (2.6) we will disprove (2.7). Taking again $g = \sum_{N=N_0}^{\infty} N^{-\varepsilon}\chi_{Q_N}$ with $1/p < \varepsilon < 1$, we have $g \in L^{p'}(w)$. On the other hand,

$$\|T(gw)\|_{L^{p'}(u^{1-p'})}^{p'} = \int |T(gw)|^{p'} \frac{w}{(Mw)^{p'}} \geq \int |T(gw)|^{p'} w^{1-p'},$$

and this last quantity was shown to be infinite in the proof of Theorem 3, except that the roles of p and p' were interchanged.

To prove $M : L^p(u) \to L^p(v)$ is easy. For $f \in L^p(u)$, using the Fefferman–Stein inequality (1.6) and (2.5), we have

$$\begin{aligned}\|Mf\|_{L^p(v)}^p &= \int |Mf|^p w \lesssim \int |f|^p Mw \lesssim \int |f|^p w \\ &\leq \int |f|^p \left(\frac{Mw}{w}\right)^p w = \|f\|_{L^p(v)}^p.\end{aligned}$$

We finish by showing that $M : L^{p'}(v^{1-p'}) \to L^{p'}(u^{1-p'})$. In much the same way as before, for $f \in L^{p'}(w^{1-p'})$ we have

$$\|Mf\|_{L^{p'}(u^{1-p'})}^{p'} = \int |Mf|^{p'} \frac{w}{(Mw)^{p'}} \leq \int |Mf|^{p'} w^{1-p'} \lesssim \int |f|^{p'} w^{1-p'},$$

where the last inequality was obtained in the proof of Theorem 3 for p instead of p'. ∎

3. The construction of the weights. The construction of the weights w_N in Proposition 1 is an extension to higher dimensions of the one by M. C. Reguera and C. Thiele [24], which in turn was a simplification of the construction by M. C. Reguera [22]. The argument is long and involves some technicalities.

Proof of Proposition 1. First we will give the basics of the construction of the weight w_N and of the sets D_N and $\widehat{D_N}$. Then we will proceed to estimate Mw_N on D_N and Tw_N on $\widehat{D_N}$, and we will complete the details of the construction of w_N so that the conclusion is reached.

The triadic decomposition. For $k \in \mathbb{Z}$, we say that Q is a *triadic cube of the kth generation* in $\mathbb{R}^n$ if Q has edge length 3^{-k} and its vertices are points of the grid $3^{-k}\mathbb{Z}^n$. For any cube $Q = Q(x,R)$ we define its *triadic middle child* as $\widehat{Q} = Q(x, R/3)$. For $k = 0, 1, \ldots$ we will define a family $\mathcal{T}_k$ of triadic cubes of the (Nk)th generation, with $N \in \mathbb{N}$ fixed. We define these families inductively. We begin with $\mathcal{T}_0 = \{[0,1]^d\}$. Once $\mathcal{T}_k$ is determined, for each $Q \in \mathcal{T}_k$ we select a family $\mathcal{T}_{k+1}(Q)$ of triadic subcubes so that $\mathcal{T}_{k+1}(Q) \subset \{\text{triadic } Q' \subset \widehat{Q} : |Q'| = 3^{-Nd}|Q|\}$ and $\sharp T_{k+1}(Q) = A \sim 3^{(N-1)d}$, with $A \in \mathbb{N}$ a fixed number depending neither on Q nor on k. The exact way of selecting these cubes will be explained later. Then we take $\mathcal{T}_{k+1} = \bigcup_{Q\in\mathcal{T}_k} \mathcal{T}_{k+1}(Q)$.

For each $Q \in \mathcal{T}_k$ we consider a triadic cube $J(Q)$ contained in Q such that $|J(Q)| = |Q'| = 3^{-Nd(k+1)}$ for any $Q' \in \mathcal{T}_{k+1}$. We will choose $J(Q)$ having disjoint interior from $\widehat{Q}$ but contiguous to it, in the sense that their boundaries intersect. In particular, the elements of the family $\{J(Q)\}_{Q\in\bigcup_{k=0}^{\infty}\mathcal{T}_k}$ are all disjoint. Moreover, if $N \geq 3$ and $Q_0 \in \mathcal{T}_{k_0}$ for some k_0, then

$$\operatorname{dist}\Big(J(Q_0), \Big[\bigcup_{k=0}^{\infty} \bigcup_{Q\in\mathcal{T}_k} J(Q)\Big] \setminus J(Q_0)\Big) \geq \frac{\ell}{3} - \frac{\ell}{3^N} \geq \frac{\ell}{4} \tag{3.1}$$

with $\ell = |J(Q_0)|^{1/d}$.

The construction of the weight. We define a weight w_N supported in

$$D_N = \bigcup_{k=0}^{\infty} \bigcup_{Q\in\mathcal{T}_k} J(Q)$$

so that w_N is constant over each $J(Q)$ and if $x \in J(Q)$ with $Q \in \mathcal{T}_k$ one has

$$\alpha_k = w_N(x) = \frac{w_N(J(Q))}{|J(Q)|} = \frac{w_N(Q')}{|Q'|} \tag{3.2}$$

for any $Q' \in \mathcal{T}_{k+1}$. In this way,

$$w_N(x) = \sum_{k=0}^{\infty} \alpha_k \sum_{Q\in\mathcal{T}_k} \chi_{J(Q)}.$$

Observe that for $Q \in \mathcal{T}_k$,

$$w_N(Q) = w_N(J(Q)) + w_N(\widehat{Q}) = w_N(J(Q)) + \sum_{Q'\in\mathcal{T}_{k+1}(Q)} w_N(Q').$$

Using (3.2), we can rewrite the previous formula as

$$\alpha_{k-1}|Q| = \alpha_k|J(Q)| + \alpha_k \,\sharp\mathcal{T}_{k+1}(Q)\,|J(Q)| = \alpha_k|J(Q)| + \alpha_k A|J(Q)|,$$

obtaining

$$\frac{\alpha_k}{\alpha_{k-1}} = \frac{3^{Nd}}{1+A} =: a.$$

Hence, $\alpha_k = a^k\alpha_0$ for a certain α_0 and

$$\begin{aligned} w_N([0,1]^d) &= \sum_{k=0}^{\infty}\sum_{Q\in\mathcal{T}_k} w_N(J(Q)) = \alpha_0\sum_{k=0}^{\infty}\sharp\mathcal{T}_k\,|J(Q)|a^k \\ &= \alpha_0\sum_{k=0}^{\infty}A^k 3^{-Nd(k+1)}a^k = \alpha_0 3^{-Nd}\sum_{k=0}^{\infty}\Big(\frac{A}{1+A}\Big)^k \\ &= \alpha_0 3^{-Nd}(1+A) = \alpha_0/a. \end{aligned}$$

We take $\alpha_0 = a$ so that w_N is a probability measure and $w_N \geq a > 1$ in D_N, as stated.

Controlling the maximal function. We prove here that $Mw_N \leq Cw_N$ in D_N, with a constant C independent of N. Fix $x \in J(Q)$ with $Q \in \mathcal{T}_k$ and take an arbitrary cube R containing x. We want to show that

$$\frac{w_N(R)}{|R|} \leq Cw(x).$$

If $|R|^{1/d} < \frac{1}{4}|J(Q)|^{1/d}$, then $R\cap D_N = R\cap J(Q)$ from (3.1). This says that w is constant in $R\cap J(Q)$ and the result is obvious. If, on the contrary, $|R|^{1/d} \geq \frac{1}{4}|J(Q)|^{1/d}$ and we consider

$$\mathcal{A} = \{\text{triadic } Q' : Q'\cap R \neq \emptyset,\ |Q'| = |J(Q)|\},$$

then $\sum_{Q'\in\mathcal{A}}|Q'| \leq 9^d|R|$. We claim that if $L \subset [0,1]^d$ is a triadic cube with size $|L| = |J(Q)|$ then

$$w_N(L) \leq \alpha_k|L|. \tag{3.3}$$

Using this, one has

$$\frac{w_N(R)}{|R|} \leq \frac{1}{|R|}\sum_{Q'\in\mathcal{A}} w_N(Q') \leq \frac{\alpha_k}{|R|}\sum_{Q'\in\mathcal{A}}|Q'| \leq 9^d w_N(x).$$

The proof of (3.3) is easy. We have three possible situations:

(i) $L\cap D_N = \emptyset$, and there is nothing to show.

(ii) $L \subset J(Q_0)$ for some $Q_0 \in \mathcal{T}_j$ with $j \leq k$. In this case w_N is constant in L with value α_j. Since $\alpha_j \leq \alpha_k$, the result follows immediately.

(iii) $L = Q'$ for some $Q' \in \mathcal{T}_{k+1}$. Here we have directly $w_N(L) = \alpha_k|L|$ by definition.

Splitting Tw_N into 'continuous' and 'discrete' pieces. Taking, with a slight abuse of notation, $\widehat{D_N} := \bigcup_{k=0}^{\infty} \bigcup_{Q \in \mathcal{T}_k} \widehat{J(Q)}$, we want to prove that $|Tw_N| \geq CNw_N$ in $\widehat{D_N}$.

Let $x \in \widehat{J(Q)}$ with $Q \in \mathcal{T}_k$. Then

$$Tw_N(x) = \int_{Q^c} K(x,y) w_N(y)\, dy + \int_{Q \setminus J(Q)} K(x,y) w_N(y)\, dy + \text{p.v.} \int_{J(Q)} K(x,y) w_N(y)\, dy = I + II + III.$$

We further split I and II into a 'continuous' and a 'discrete' part. Denoting by c_R the center of a cube R, we have

$$\begin{aligned} I &= \sum_{\substack{L \,\text{triadic} \\ |L|=|Q|,\, L \neq Q}} \int_L K(x,y) w_N(y)\, dy \\ &= \sum_{\substack{L \,\text{triadic} \\ |L|=|Q|,\, L \neq Q}} K(c_Q, c_L) w_N(L)\, dy \\ &\quad + \sum_{\substack{L \,\text{triadic} \\ |L|=|Q|,\, L \neq Q}} \int_L \big(K(x,y) - K(c_Q, c_L)\big) w_N(y)\, dy \\ &= I_1 + I_2, \end{aligned}$$

and

$$\begin{aligned} II &= \sum_{L \in \mathcal{T}_{k+1}(Q)} \int_L K(x,y) w_N(y)\, dy \\ &= \sum_{L \in \mathcal{T}_{k+1}(Q)} K(c_{J(Q)}, c_L) w_N(L)\, dy \\ &\quad + \sum_{L \in \mathcal{T}_{k+1}(Q)} \int_L \big(K(x,y) - K(c_{J(Q)}, c_L)\big) w_N(y)\, dy \\ &= II_1 + II_2. \end{aligned}$$

First, we will show that the 'continuous' parts I_2, II_2 and III are 'small' in the sense that $|I_2| + |II_2| + |III| \lesssim w_N(x)$. Then we will show that II_1 is much bigger than w_N by showing that $|II_1| \gtrsim Nw_N(x)$. Although we will not have any control on I_1, we will construct $J(Q)$ and $\mathcal{T}_{k+1}(Q)$ so that II_1 has the same sign as I_1. In this way, we will have $|I_1 + II_1| \geq |II_1| \gtrsim Nw_N(x)$. At that point we will get

$$|Tw_N(x)| \geq |I_1 + II_1| - |I_2 + II_2 + III| \geq (cN - C) w_N(x) \gtrsim Nw_N(x)$$

for sufficiently large N. This will prove the result.

The 'continuous' pieces. We recall the well-known fact (see [26] for instance) that our hypotheses on K imply the following estimates: there exist $\delta, \eta > 0$ such that

$$|K(x,y) - K(x,\bar{y})| \le C \frac{|y-\bar{y}|^\delta}{|x-y|^{d+\delta}} \tag{3.4}$$

if $|x-y| > (1+\eta)|y-\bar{y}|$, and

$$|K(x,y) - K(\bar{x},y)| \le C \frac{|x-\bar{x}|^\delta}{|x-y|^{d+\delta}} \tag{3.5}$$

if $|x-y| > (1+\eta)|x-\bar{x}|$. These estimates give rise to the so called δ-Calderón–Zygmund kernels. Although in our case we have $\delta = 1$, it is worth observing that this part of the construction works for these more general kernels too.

When estimating I_2, first we use the fact that $x \in \widehat{Q}$ and $y \in \widehat{L}$ to deduce

$$|x-y| \sim |x-c_L| \sim |c_Q - c_L|,$$

and as a consequence

$$\begin{aligned} |K(x,y) - K(c_Q,c_L)| & \\ &\le |K(x,y) - K(x,c_L)| + |K(x,c_L) - K(c_Q,c_L)| \\ &\lesssim \frac{|x-c_Q|^\delta}{|x-y|^{d+\delta}} + \frac{|y-c_L|^\delta}{|x-y|^{d+\delta}} \lesssim \frac{|Q|^{\delta/d}}{|y-x|^{d+\delta}}. \end{aligned} \tag{3.6}$$

Hence,

$$\begin{aligned} |I_2| &\lesssim |Q|^{\delta/d} \sum_{\substack{|L|=|Q| \\ [0,1]^n \supset L \neq Q}} \int_L \frac{w_N(y)}{|x-y|^{d+\delta}}\, dy \\ &\le |Q|^{\delta/d} \int_{|x-y|>|Q|^{1/\delta}/4} \frac{w_N(y)}{|x-y|^{d+\delta}}\, dy \lesssim M w_N(x). \end{aligned}$$

The last inequality follows from the fact that $x \mapsto |x|^{-d-\delta}$ is a radially decreasing function and

$$\int_{|x-y|>|Q|^{1/\delta}/4} \frac{1}{|x-y|^{d+\delta}}\, dy = |\mathbb{S}^{d-1}|_{d-1} \int_{|Q|^{1/d}/4}^{\infty} \frac{1}{t^{1+\delta}}\, dt \sim \frac{1}{|Q|^{\delta/d}}$$

(see [26]).

We estimate II_2 in a similar way. Since $J(Q)$ is not contained in $\widehat{Q}$, for $x \in J(Q)$, $y \in L \in \mathcal{T}_{k+1}(Q)$ and $v_{J(Q)} \in J(Q)$ to be determined later one has

$$|x-y| \sim |x-c_L| \sim |v_{J(Q)} - c_L|,$$

and

$$|K(x,y)-K(v_{J(Q)},c_L)| \le |K(x,y)-K(x,c_L)| + |K(x,c_L)-K(v_{J(Q)},c_L)| \lesssim \frac{|J(Q)|^{\delta/d}}{|y-x|^{d+\delta}}.$$

Then reasoning as before we obtain again

$$|II_2| \lesssim |J(Q)|^{\delta/d} \sum_{L\in\mathcal{T}_{k+1}(Q)} \int_L \frac{w_N(y)}{|x-y|^{d+\delta}}\,dy \le |J(Q)|^{\delta/d} \int_{|x-y|>|J(Q)|^{1/\delta}/3} \frac{w_N(y)}{|x-y|^{d+\delta}}\,dy \lesssim Mw_N(x).$$

In order to bound III, we use the fact that w_N is constant over $J(Q)$ and the cancellation property of K on $E=\{y:|x-y|<|\widehat{J(Q)}|^{1/d}\}$ to obtain

$$III = w_N(x)\ \text{p.v.} \int_{J(Q)} \frac{\Omega(x-y)}{|x-y|^d}\,dy = w_N(x) \int_{J(Q)\setminus E} \frac{\Omega(x-y)}{|x-y|^d}\,dy. \tag{3.7}$$

Hence

$$|III| \le w_N(x) \int_{J(Q)\setminus E} \frac{\|\Omega\|_{L^\infty}}{|\widehat{J(Q)}|}\,dy \lesssim w_N(x).$$

Remark. Observe that in none of the above estimates have we needed a precise description of the families $\mathcal{T}_{k+1}(Q)$ and the cubes $J(Q)$. The only information we have used so far is that each $Q'\in\mathcal{T}_{k+1}(Q)$ is a triadic subcube of $\widehat{Q}$ of size $3^{-Nd(k+1)}$ and that $J(Q)$ is of the same size and 'touches' $\widehat{Q}$ from the outside.

Another important observation is that for $Q\in\mathcal{T}_k$, the term $I_1=I_1(Q)$ does not depend on the triadic cubes of the next generation. In particular, $I_1(Q)$ is independent of $\mathcal{T}_i$ for all $i>k$. This is consistent with the inductive process that we use in order to define the weights w_N.

The 'discrete' pieces in a simpler case: Riesz transforms. To get some intuition of the construction, we will first consider a concrete example. Assume that T is a Riesz transform, that is, $T=R_j$ for some $j\in\{1,\dots,d\}$, where

$$R_jf(x) = c_d\,\text{p.v.} \int_{\mathbb{R}^n} \frac{x_j-y_j}{|x-y|^{d+1}} f(y)\,dy,$$

and c_d is a normalizing constant depending on the dimension. In this case, given $Q\in\mathcal{T}_k$ we choose $\mathcal{T}_{k+1}(Q)$ to consist of all the triadic subcubes of $\widehat{Q}$

of size $3^{-Nd}|Q|$. We take $J(Q)$ to be a triadic cube of size $3^{-N(k+1)d}$ contiguous to $\widehat{Q}$ so that their boundaries only share a point, hence a vertex. For $x \in \mathbb{R}^d$, we denote by x_j its jth coordinate. Now, if $I_1 \geq 0$ we place $J(Q)$ so that $\min_{x\in J(Q)} x_j \geq \max_{x\in\widehat{Q}} x_j$, and if $I_1 \leq 0$ we require instead $\max_{x\in J(Q)} x_j \geq \min_{x\in\widehat{Q}} x_j$. This makes the signs of I_1 and II_1 coincide. Calling

$$\mathcal{T}^i_{k+1}(Q) = \{L \in \mathcal{T}_{k+1}(Q) : |(c_L)_j - (c_{J(Q)})_j| = |c_L - c_{J(Q)}|_\infty = 3^{-N(k+1)}i\},$$

and taking $v_{J(Q)} = c_{J(Q)}$, we have

$$\begin{aligned}|II_1| &\gtrsim \sum_{L\in\mathcal{T}_{k+1}(Q)} \frac{|(c_L)_j - (c_{J(Q)})_j|}{|c_L - c_{J(Q)}|_\infty^{d+1}} w_N(L) \\ &= a^{k+1}|J(Q)| \sum_{i=1}^{3^{N-1}} \sum_{L\in\mathcal{T}^i_{k+1}(Q)} \frac{1}{|c_L - c_{J(Q)}|^d_\infty} \\ &= w_N(x)|J(Q)| \sum_{i=1}^{3^{N-1}} \frac{i^{d-1}}{(3^{-N(k+1)}i)^d} = w_N(x) \sum_{i=1}^{3^{N-1}} \frac{1}{i} \gtrsim N w_N(x).\end{aligned}$$

Observe also that in this case $A = 3^{(N-1)d}$, and therefore

$$a = \frac{3^{Nd}}{1+A} \sim 3^d.$$

Finishing the construction of the measure for a general operator. We will now explain how we choose $J(Q)$ and $\mathcal{T}_{k+1}(Q)$ so that II_1 behaves the way we need when T is a general Calderón–Zygmund operator. This choice will depend on T.

Since Ω is a continuous function over the sphere with null integral, there exist $\lambda, r > 0$ and two points z_+ and z_- in the sphere such that $\Omega(y) > \lambda$ for any $y \in B^+ = B(z_+, r) \cap \mathbb{S}^{d-1}$, and $\Omega(y) < -\lambda$ for any $y \in B^- = B(z_-, r) \cap \mathbb{S}^{d-1}$. We define the cones

$$U^+ = \{tx : t > 0,\, x \in B^+\}, \qquad U^- = \{tx : t > 0,\, x \in B^-\}.$$

Using a rotation if necessary, we can assume that from each coordinate axis we have the same distance to z_+ and to z_-. This is equivalent to saying that $|(z_+)_i| = |(z_-)_i| \neq 0$ for all $i = 1, \dots, d$, or that $z_+ = \tau z_-$, where τ is the $d \times d$ matrix

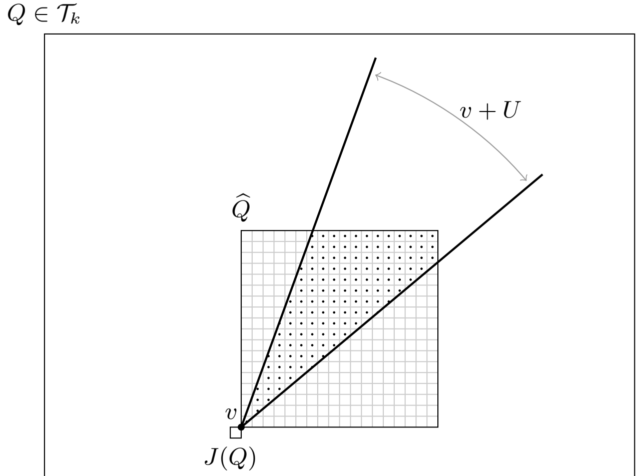
$$\tau = \left(\delta_{i,j}\frac{\operatorname{sign}(z_-)_i}{\operatorname{sign}(z_+)_j}\right)_{i,j=1,\dots,d} = \begin{bmatrix} \pm1 & 0 & \cdots & 0 \\ 0 & \pm1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm1 \end{bmatrix}.$$

Note that also $U^+ = \tau U^-$.

For a $Q \in \mathcal{T}_k$ we denote by v_+ (respectively, v_-) the only vertex of $\widehat{Q}$ such that the half-line $s_+ \equiv v_+ + tz_+$ (respectively, $s_- \equiv v_- + tz_-$) for $t > 0$ intersects the interior of $\widehat{Q}$. If $I_1 \geq 0$ we will choose $v = v_{J(Q)} := v_+$, $z = z_+$ and $U = U^+$. On the other hand, if $I_1 \leq 0$ we choose $v = v_{J(Q)} := v_-$, $z = z_-$ and $U = U^-$. Now we take $J(Q)$ to be the only triadic cube of size $3^{-Nd}|Q|$ such that the boundaries of $J(Q)$ and of $\widehat{Q}$ intersect only at v. Once this is done we take

$$\mathcal{T}_{k+1}(Q) = \{\text{triadic } R \subset \widehat{Q} : |R| = 3^{-Nd}|Q|,\, c_R \in v + U\}.$$

The construction guarantees that $A = \sharp\mathcal{T}_{k+1}(Q) \sim 3^{(N-1)d}$ is independent of k and Q, as required before.



The cubes in $\mathcal{T}_{k+1}(Q)$ are the triadic subcubes of $\widehat{Q}$ whose size equals that of $J(Q)$ and whose centers are in the cone $v + U$.

Estimating the 'discrete' pieces. Since $c_L \in W$ for all $L \in \mathcal{T}_{k+1}(Q)$, we have

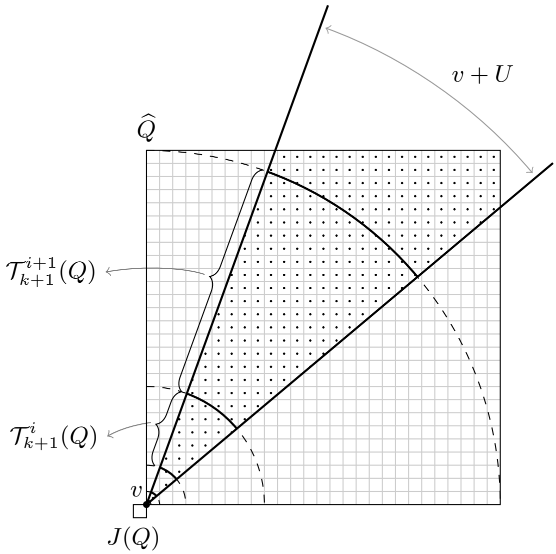
$$|II_1| = \Big|\sum_{L\in\mathcal{T}_{k+1}(Q)} K(v, c_L)w_N(L)\Big| \geq \lambda a^{k+1}|J(Q)| \sum_{L\in\mathcal{T}_{k+1}(Q)} \frac{1}{|c_L - v|^d}.$$

We want to find a lower estimate for the last sum. We could use an argument similar to the one for the Riesz transforms but we will use a more direct one. For a positive integer i we define

$$\varGamma_i = \{x \in v + U \cap \widehat{Q} : 3^{i-1} \cdot 3^{-N(k+1)} < |x - v| \leq 3^i \cdot 3^{-N(k+1)}\}.$$

We also define

$$\mathcal{T}^i_{k+1}(Q) = \{R \in \mathcal{T}_{k+1}(Q) : c_R \in \varGamma_i\}.$$



Now we choose N large enough to make $J(Q)$ very small compared to $\Gamma_{\lfloor N/2\rfloor}$, so that the measure of Γ_i is comparable to the sum of the measures of the cubes in $\mathcal{T}_{k+1}^i$ for $\lfloor N/2\rfloor \leq i \leq N-1$, that is,

$$|\Gamma_i| \sim \sum_{R\in\mathcal{T}_{k+1}^i(Q)} |R| = \sharp\mathcal{T}_{k+1}^i(Q)\,|J(Q)|.$$

Note that $|J(Q)| = 3^{-Nd(k+1)}$ and $|\Gamma_i| = \beta(3^{d(i-N(k+1))} - 3^{d(i-1-N(k+1))}) = 2\beta\cdot 3^{d(i-1-N(k+1))}$ for a certain $\beta > 0$ that depends on the opening of the cone U. The previous choice of N is possible since the quotient of the measures of $\Gamma_{\lfloor N/2\rfloor}$ and $J(Q)$ is of the order of $3^{dN/2}$. The conclusion is that for $\lfloor N/2\rfloor \leq i \leq N-1$ one has $\sharp\mathcal{T}_{k+1}^i(Q) \sim 3^{di}$ and as a consequence

$$\begin{aligned}|II_1| &\geq \lambda w_N(x)|J(Q)| \sum_{i=\lfloor N/2\rfloor}^{N-1}\ \sum_{L\in\mathcal{T}_{k+1}^i(Q)} \frac{1}{|c_L-v|^d}\\ &\gtrsim \lambda w_N(x)|J(Q)| \sum_{i=\lfloor N/2\rfloor}^{N-1} \frac{\sharp\mathcal{T}_{k+1}^i(Q)}{|3^i\cdot 3^{-N(k+1)}|^d} \gtrsim \lambda N w_N(x).\end{aligned}$$

This finishes the proof of Proposition 1. ■

4. Final remarks

Variable kernels. We point out that most of the arguments of the previous proof also work if K is a variable Calderón–Zygmund kernel with the standard size conditions. Thus, a similar construction is possible for such kernels if in addition they have a suitable distribution of signs so that one can find cones defining $\mathcal{T}_{k+1}(Q)$ as before.

Counterexamples for condition (1.8). It is implicit in the proof of Theorem 1 that the weights w_N together with the functions $f_N = w_N T^* w_N/(Mw_N)^2$ give counterexamples for the condition (1.8) established by C. Pérez and D. Cruz-Uribe. As already pointed out in [23] the selection of u and v in the proof of Theorem 2 gives again counterexamples for (1.8). The point in the given proof of Theorem 1 is to produce an explicit counterexample for Conjecture 1.

Local A_p weights. These weights share some of the properties of the usual Muckenhoupt A_p weights. For example, it is easy to see that conditions (1.3) and (1.2), satisfied on the support of the weight, are equivalent to the weak boundedness of M on weighted L^p. However, there are some important differences too. One of them is the non-existence of a reverse Hölder inequality for local weights. In fact, we have the following

LEMMA 2. *Let w be the local A_1 weight defined in the proofs of Theorems 2 and 3. Then $w^{1+\varepsilon}$ is not locally integrable whenever $\varepsilon > 0$.*

Proof. Observe that for each N,

$$\int_{Q_N} w^{1+\varepsilon} = \int_{D_N} w_N^{1+\varepsilon} = \sum_{k=0}^{\infty}\left(\frac{3^{Nd}}{1+A}\right)^{(k+1)(1+\varepsilon)} A^k 3^{-Nd(k+1)}$$
$$= \frac{1}{A}\sum_{k=0}^{\infty}\left(\frac{3^{\varepsilon Nd}A}{(1+A)^{1+\varepsilon}}\right)^{k+1}.$$

Since $A \le 3^{(N-1)d}$, by taking N large so that $3^{\varepsilon Nd}A/(1+A)^{1+\varepsilon} > 1$, we see that the series diverges to ∞. This shows that $w^{1+\varepsilon} \notin L^1_{\mathrm{loc}}(\mathbb{R}^d)$. ∎

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Alberto Criado
Departamento de Matemáticas
Facultad de Ciencia y Tecnología
de la Universidad del País Vasco
48080 Bilbao, Spain
E-mail: alberto.criado@ehu.es

Fernando Soria
Departamento de Matemáticas and
Instituto de Ciencias Matemáticas
CSIC–UAM–UC3M–UCM
Universidad Autónoma de Madrid
28049 Madrid, Spain
E-mail: fernando.soria@uam.es