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# Sphere equivalence, Property H, and Banach expanders

by

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**Abstract.** We study the uniform classification of the unit spheres of general Banach sequence spaces. In particular, we obtain some interesting applications involving Property H introduced by Kasparov and Yu, and Banach expanders.

**1. Introduction.** Motivated by recent work [6, 14, 21], we study the sphere equivalence between Banach spaces and give its applications involving Property H introduced by Kasparov and Yu [14], and Banach expanders.

Let $(M,d)$ and $(M',d')$ be metric spaces and let $f : M \to M'$ be any map. The *modulus of continuity* of $f$ is the function $\omega_f : [0,\infty) \to [0,\infty]$ defined by

$$\omega_f(t) = \sup\{d'(f(x), f(y)) : x, y \in M \text{ and } d(x,y) \le t\}. \tag{1.1}$$

The map $f$ is said to be *uniformly continuous* if $\lim_{t\to+0} \omega_f(t) = 0$, and a *uniform homeomorphism* if $f$ is a bijection and $f$ and $f^{-1}$ are both uniformly continuous. The metric spaces $M$ and $M'$ are called *uniformly homeomorphic* provided there is a uniform homeomorphism between them.

In particular, Banach spaces $X$ and $Y$ are called *sphere uniformly homeomorphic* or *sphere equivalent*, written $X \sim_S Y$, if there exists a uniform homeomorphism between the unit spheres $S(X)$ and $S(Y)$. We write $[X]_S$ for the sphere equivalence class of $X$ (see [21]).

Concerning the uniform classification of the unit spheres of infinite-dimensional Banach spaces (see [2, Chapter 9]), the earliest result involves the so-called Mazur map, first used by Mazur in 1929 [19]. Recall that the *Mazur map* $M_{p,q}$ from $l_p$ to $l_q$ $(1 \le p, q < \infty)$, defined by

$$M_{p,q}(a_j) = (\mathrm{sign}(a_j)|a_j|^{p/q}),$$

is a uniform homeomorphism between the unit spheres of these spaces. Thus

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the unit spheres of $l_p$ spaces, $1 \le p < \infty$, are uniformly homeomorphic to the unit sphere of $l_2$. Therefore the Mazur map provides the uniform classification of the unit spheres of the classical sequence spaces $l_p$ for $1 \le p < \infty$.

THEOREM 1.1 (Mazur). *$l_p \in [l_2]_S$ for all $1 \le p < \infty$.*

Our first aim is to generalize the classification to a wider class of Banach spaces, namely to Banach sequence spaces.

THEOREM 1.2. *For any Banach spaces $X, Y$ if $X \sim_S Y$ then $l_p(Y) \in [l_2(X)]_S$ for all $1 \le p < \infty$.*

In fact, we obtain a more general result (Theorem 2.6). In [22, Theorem 3.1], Nahum established a connection between uniform classification of Banach spaces and that of their unit spheres.

THEOREM 1.3 (Nahum). *Assume that the Banach spaces $X$ and $Y$ are uniformly homeomorphic. Then $X \oplus \mathbb{R} \sim_S Y \oplus \mathbb{R}$.*

Theorem 1.2 together with Nahum's theorem gives the following uniform classification of spheres.

THEOREM 1.4. *Assume that the Banach spaces $X$ and $Y$ are uniformly homeomorphic. Then $l_p(Y) \in [l_2(X)]_S$ for all $1 \le p < \infty$.*

We review previous work in this direction. Let $X$ and $Y$ be Banach spaces with $X \sim_S Y$. Firstly, [2, Lemma 9.9] showed that $l_p(X) \sim_S l_p(Y)$ for $p = 2$, but the proof is still valid for all $1 \le p < \infty$ (see also Mimura [21, Proposition 3.9]). In particular, Mimura established quantitatively sharp estimates for this result (see Remark 2.7 below for details). On the other hand, for $1 < p, q < \infty$ he further proved by complex interpolation that $l_p(X) \sim_S l_q(X)$ when $X$ is uniformly convex (see [21, Theorem 3.8]). But the technique used there does not cover the case $p = 1$ (or $q = 1$) and the uniform convexity assumption of $X$ cannot be removed. Thus Theorem 1.2 improves and extends the previous work (in the qualitative sense of sphere equivalence). This will be treated in Section 2.

Sections 3 and 4 provide applications of the sphere equivalence results obtained in Section 2, which is our second aim in this paper.

In particular, our focus in Section 3 is on Property H, a new property of Banach spaces introduced by Kasparov and Yu [14] to study the Novikov conjecture. They proved that any discrete group which is coarsely embeddable into a Banach space with Property H satisfies the strong Novikov conjecture [14]. More recently, Chen, Wang and Yu [6] proved that any discrete metric space with bounded geometry which is coarsely embeddable into a Banach space with Property H satisfies the coarse Novikov conjecture. In Section 3 we mainly prove a basic permanence result for Property H

(Theorem 3.6), which seems to be particularly interesting in this field. For example, this result together with the work in [4, 5, 7] implies another permanence result: if two discrete groups are coarsely embeddable into a Property H Banach space, then so is their free product. We also give some simple remarks on this result.

In Section 4 we combine Theorem 1.1 with Mimura's results to establish the stability in $p$ of $(X, p)$-anders, which improves and extends Mimura's main results of [21].

Our notation and terminology for Banach spaces are standard, as may be found for example in [16] and [17]. All Banach spaces throughout the paper are supposed to be real.

**2. Sphere equivalence.** In this section, we first introduce the concept of "$r$-Hölder extension" for $0 < r < \infty$ and then extend the Odell and Schlumprecht theorem [23, Proposition 2.9]. Finally making use of the extension theorem and the proof method of [2, Lemma 9.9], we obtain several results on uniform classification of spheres in Banach sequence spaces.

**2.1. $r$-Hölder extension.** Recall the following uniform homeomorphism extension theorem due to Odell and Schlumprecht [23].

THEOREM 2.1 (Odell and Schlumprecht). *Let $X$ and $Y$ be Banach spaces and let $f : S(X) \to S(Y)$ be a uniform homeomorphism. The canonical extension $\widetilde{f}$ of $f$ is defined by $\widetilde{f}(x) = \|x\| f(x/\|x\|)$ if $x \neq 0$ and $\widetilde{f}(0) = 0$. Then $\widetilde{f} : B(X) \to B(Y)$ is also a uniform homeomorphism and*

$$\omega_{\widetilde{f}}(t) \leq \max\{t + \omega_f(2\sqrt{t}), t + 2t^{1/4}\} \quad \text{for } t > 0.$$

Motivated by the theorem of Odell and Schlumprecht, we introduce the following definition.

DEFINITION 2.2. Assume that $X$ and $Y$ are Banach spaces and $f$ is a map between their unit spheres. For any $r > 0$ we define the *$r$-Hölder extension* $\widetilde{f}_r$ of $f$ by $\widetilde{f}_r(x) = \|x\|^r f(x/\|x\|)$ if $x \neq 0$ and $\widetilde{f}_r(0) = 0$.

Our current goal is to extend the Odell and Schlumprecht theorem but first we need the following lemma. For completeness, we give proofs of some known results which are used in our proof.

LEMMA 2.3 ([19]). *Let $x \geq y \geq 0$ and $p \geq 0$. Then*

$$x^p - y^p \leq \max\{(x - y)^p, px^{p-1}(x - y)\}.$$

*Proof.* We assume that $y > 0$; the case of $y = 0$ is trivial. Let

$$f(t) = (1 + t)^p - (1 + t^p).$$

Then $f'(t) = p[(1+t)^{p-1} - t^{p-1}]$ and so $f'(t) \le 0$ for $t \ge 0$ and $0 \le p \le 1$. This gives $(1+t)^p - t^p \le 1$. Replacing $t$ by $(x-y)/y$ in the above inequality yields $x^p - y^p \le (x-y)^p$. If $p > 1$, then using the Mean Value Theorem, one can find $\xi \in (y, x)$ such that

$$x^p - y^p = p\xi^{p-1}(x-y) \le px^{p-1}(x-y). \ \blacksquare$$

THEOREM 2.4. *Let $X, Y$ be Banach spaces and let $f : S(X) \to S(Y)$ be a map. For $0 < r < \infty$ let $\widetilde{f}_r : X \to Y$ be the $r$-Hölder extension of $f$.*

(i) *If $f$ is injective (resp. surjective) and uniformly continuous then $\widetilde{f}_r : B(X) \to B(Y)$ is injective (resp. surjective) and uniformly continuous, and*

$$\omega_{\widetilde{f}_r}(t) \le \max\{t^r + 2t^{r/4}, t^r + \omega_f(2\sqrt{t}), rt + 2t^{r/4}, rt + \omega_f(2\sqrt{t})\}, \quad 0 \le t \le 1. \tag{2.1}$$

(ii) *If $f$ is a uniform homeomorphism then so is $\widetilde{f}_r : B(X) \to B(Y)$.*

*Proof.* (i) Clearly, $\widetilde{f}_r$ is injective (resp. surjective) if $f$ is injective (resp. surjective). Suppose that $f$ is uniformly continuous. Let $x_1, x_2 \in B(X)$ with $\|x_1 - x_2\| = \delta$ $(0 \le \delta < 1)$, $\|x_1\| = \alpha_1$, $\|x_2\| = \alpha_2$ and $\alpha_1 \ge \alpha_2$. Then

$$\begin{aligned}
\|\widetilde{f}_r(x_1) - \widetilde{f}_r(x_2)\| &= \left\|\alpha_1^r f\left(\frac{x_1}{\|x_1\|}\right) - \alpha_2^r f\left(\frac{x_2}{\|x_2\|}\right)\right\| \\
&= \left\|(\alpha_1^r - \alpha_2^r) f\left(\frac{x_1}{\|x_1\|}\right) - \alpha_2^r\left(f\left(\frac{x_2}{\|x_2\|}\right) - f\left(\frac{x_1}{\|x_1\|}\right)\right)\right\| \\
&\le (\alpha_1^r - \alpha_2^r) + \alpha_2^r\left\|f\left(\frac{x_1}{\|x_1\|}\right) - f\left(\frac{x_2}{\|x_2\|}\right)\right\|.
\end{aligned}$$

The first term on the right-hand side is bounded by $\max\{r\delta, \delta^r\}$ from Lemma 2.3, and we claim that the second one is bounded by $\max\{2\delta^{r/4}, \omega_f(2\sqrt{\delta})\}$. Indeed, if $\alpha_2 < \delta^{1/4}$, then the right-hand side above is bounded by $\max\{r\delta, \delta^r\} + 2\delta^{r/4}$. Otherwise

$$\begin{aligned}
\left\|\frac{x_1}{\alpha_1} - \frac{x_2}{\alpha_2}\right\| &= \frac{1}{\alpha_1\alpha_2}\|\alpha_2 x_1 - \alpha_1 x_2\| \\
&\le \frac{1}{\alpha_1\alpha_2}\big(\alpha_1\|x_1 - x_2\| + \alpha_1 - \alpha_2\big) \\
&\le \frac{\delta}{\alpha_2} + \frac{\delta}{\alpha_1\alpha_2} \le \frac{2\delta}{\alpha_1\alpha_2} \le \frac{2\delta}{\sqrt{\delta}} = 2\sqrt{\delta}.
\end{aligned}$$

Thus

$$\left\|f\left(\frac{x_1}{\|x_1\|}\right) - f\left(\frac{x_2}{\|x_2\|}\right)\right\| \le w_f(2\sqrt{\delta}).$$

Therefore, we finally get (2.1)

(ii) This follows directly from (i) and from $\widetilde{f}_r^{\,-1}(y) = \|y\|^{1/r} f^{-1}(y/\|y\|)$. $\blacksquare$

Let $\{M_j\}_{j\in\mathbb{N}}$ and $\{M'_j\}_{j\in\mathbb{N}}$ be sequences of metric spaces and let $f_j : M_j \to M'_j$ be a map. The sequence $\{f_j\}_{j\in\mathbb{N}}$ is said to be *equi-uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $j \in \mathbb{N}$, if $x, y \in M_j$ and $d_{M_j}(x,y) < \delta$ then $d_{M'_j}(f_j(x), f_j(y)) < \varepsilon$. This is equivalent to saying that the function $\omega : [0,\infty) \to [0,\infty]$ satisfies $\lim_{t\to+0}\omega(t) = 0$ where $\omega(t) = \sup\{\omega_{f_j}(t) : j \in \mathbb{N}\}$. Furthermore the sequence $\{f_j\}_{j\in\mathbb{N}}$ is called *equi-uniformly homeomorphic* if every $f_j$ is a bijection and the sequences $\{f_j\}_{j\in\mathbb{N}}$ and $\{f_j^{-1}\}_{j\in\mathbb{N}}$ are equi-uniformly continuous.

The following result follows immediately from Theorem 2.4.

LEMMA 2.5. *Let $\{X_j\}_{j\in\mathbb{N}}$ and $\{Y_j\}_{j\in\mathbb{N}}$ be sequences of Banach spaces and let $0 < r < \infty$. Assume that the sequence $\{f_j : S(X_j) \to S(Y_j)\}_{j\in\mathbb{N}}$ of maps is equi-uniformly continuous (resp. equi-uniformly homeomorphic). Then the sequence $\{\widetilde{f}_{j,r} : B(X_j) \to B(Y_j)\}_{j\in\mathbb{N}}$ is also equi-uniformly continuous (resp. equi-uniformly homeomorphic), where $\widetilde{f}_{j,r}$ is the r-Hölder extension of $f_j$.*

**2.2. Uniform classification of spheres for Banach sequence spaces.** Let $1 \le p < \infty$ and let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces. The $l_p$-*sum* of $X_n$ is defined as follows:

$$\Big(\sum_{n=1}^{\infty} X_n\Big)_{l_p} = \Big\{(x_n) : x_n \in X_n, \sum_{n=1}^{\infty} \|x_n\|^p < \infty\Big\}, \quad \|x\|_p = \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p}.$$

In the special case when $X_n = X$ is a constant sequence we will write $l_p(X) = (\sum_{n=1}^{\infty} X_n)_{l_p}$. It is well-known that $l_p$ is linearly isometric to $l_p(l_p)$ and these Banach sequence spaces contain the classical Banach sequence spaces as closed subspaces.

Now we turn to the uniform classification of spheres for general Banach sequence spaces.

THEOREM 2.6. *Let $\{X_j\}_{j=1}^{\infty}$ and $\{Y_j\}_{j=1}^{\infty}$ be sequences of Banach spaces. Assume that the sequence $\{f_j : S(X_j) \to S(Y_j)\}_{j=1}^{\infty}$ of maps is equi-uniformly continuous (resp. equi-uniformly homeomorphic). Then for any $1 \le p, q < \infty$, the map*

$$F_{p,q} : \Big(\sum_{j=1}^{\infty} X_j\Big)_{l_p} \to \Big(\sum_{j=1}^{\infty} Y_j\Big)_{l_q}, \qquad (x_j)_j \mapsto (\widetilde{f}_{j,p/q}(x_j))_j,$$

*is uniformly continuous (resp. uniformly homeomorphic) between the unit spheres. Here $\widetilde{f}_{j,p/q}$ is the $p/q$-Hölder extension of $f_j$.*

*Proof.* Since the sequence $\{f_j : S(X_j) \to S(Y_j)\}_{j=1}^{\infty}$ is equi-uniformly continuous, it follows from Lemma 2.5 that the $p/q$-Hölder extension sequence $\{\widetilde{f}_{j,p/q} : B(X_j) \to B(Y_j)\}_{j\in\mathbb{N}}$ is also equi-uniformly continuous. This implies that for each $\varepsilon > 0$ we can pick $\eta > 0$ such that $2^q\eta^{p/2} < \varepsilon$ and for all $j \in \mathbb{N}$, if $x_j, y_j \in B(X_j)$ and $\|x_j - y_j\| \le \sqrt{\eta}$, then $\|\widetilde{f}_{j,p/q}(x_j) - \widetilde{f}_{j,p/q}(y_j)\| \le \varepsilon$.

Suppose that $x=(x_n), y=(y_n) \in (\sum_{j=1}^{\infty} X_j)_{l_p}$ are such that $\sum_{n=1}^{\infty} \|x_n\|^p = 1$, $\sum_{n=1}^{\infty} \|y_n\|^p = 1$ and $\|x-y\|^p = \sum_{n=1}^{\infty} \|x_n - y_n\|^p \le \eta^p$. Let

$$\Omega = \{j \in \mathbb{N} : \|x_j - y_j\| \le \sqrt{\eta}\max\{\|x_j\|, \|y_j\|\}.$$

Set $\lambda_j = \max\{\|x_j\|, \|y_j\|\}$. Then

$$\begin{aligned}&\|F_{p,q}(x) - F_{p,q}(y)\|^q \\ &\qquad = \sum_{j\in\Omega} \|\widetilde{f}_{j,p/q}(x_j) - \widetilde{f}_{j,p/q}(y_j)\|^q + \sum_{j\notin\Omega} \|\widetilde{f}_{j,p/q}(x_j) - \widetilde{f}_{j,p/q}(y_j)\|^q.\end{aligned}$$

Note that for $\lambda > 0$ and $x \in X_j$, $\widetilde{f}_{j,p/q}(\lambda x) = \lambda^{p/q}\widetilde{f}_{j,p/q}(x)$ and

$$\begin{aligned}\sum_{j\in\Omega} \|\widetilde{f}_{j,p/q}(x_j) - \widetilde{f}_{j,p/q}(y_j)\|^q &= \sum_{j\in\Omega} \left\| \lambda_j^{p/q}\left[\widetilde{f}_{j,p/q}\left(\frac{x_j}{\lambda_j}\right) - \widetilde{f}_{j,p/q}\left(\frac{y_j}{\lambda_j}\right)\right]\right\|^q \\ &\le \sum_{j\in\Omega} \varepsilon^q\lambda_j^p \le \sum_{j\in\Omega} \varepsilon^q(\|x_j\|^p + \|y_j\|^p) \le 2\varepsilon^q.\end{aligned}$$

On the other hand, by our hypothesis,

$$\sum_{j\notin\Omega} 2^q\lambda_j^p \le 2^q\eta^{-p/2}\sum_{j\notin\Omega} \|x_j - y_j\|^p \le 2^q\eta^{-p/2}\|x-y\|^p \le 2^q\eta^{p/2} < \varepsilon,$$

and so

$$\begin{aligned}\sum_{j\notin\Omega} \|\widetilde{f}_{j,p/q}(x_j) - \widetilde{f}_{j,p/q}(y_j)\|^q &\le \sum_{j\notin\Omega} (\|\widetilde{f}_{j,p/q}(x_j)\| + \|\widetilde{f}_{j,p/q}(y_j)\|)^q \\ &= \sum_{j\notin\Omega} (\|x_j\|^{p/q} + \|y_j\|^{p/q})^q \\ &\le \sum_{j\notin\Omega} 2^q \max\{\|x_j\|, \|y_j\|\}^p \le \varepsilon.\end{aligned}$$

Thus $\|F_{p,q}(x) - F_{p,q}(y)\|^q \le 2\varepsilon^q + \varepsilon \le 3\varepsilon$.

Now, assume that the sequence $\{f_{j,p/q}^{-1} : S(Y_j) \to S(X_j)\}_{j\in\mathbb{N}}$ is also equi-uniformly continuous. It is easy to see that $F_{p,q}^{-1} = F_{q,p}$, and therefore the same calculation for $F_{p,q}^{-1}$ shows that $F_{p,q}^{-1}$ is again uniformly continuous on the unit sphere of $(\sum_{j=1}^{\infty} Y_j)_{l_q}$. ■

REMARK 2.7. For $p = q$ and $X_j = X$ $(j = 1, 2, \ldots)$ in Theorem 2.6(i), Mimura [21, Proposition 3.9] established quantitatively sharp estimates of the modulus of continuity of $F_{p,p}$ in the sense of order estimations. More precisely, he proved that if the original map $f$ from $S(X)$ to $S(Y)$ is $\alpha$-Hölder continuous for some $\alpha \in (0, 1]$, then so is the map $F_{p,p}$ in Theorem 2.6(i) from $S(l_p(X))$ to $S(l_p(Y))$. However, applying our proof of Theorem 2.6(i), we do not get such quantitative estimates when $1 \le p \ne q < \infty$.

As corollaries of Theorem 2.6, we obtain several results on uniform classification of spheres for general Banach sequence spaces.

COROLLARY 2.8. *Let* $\{X_n\}_{n\in\mathbb{N}}$ *be a sequence of Banach spaces and* $1 \le p < \infty$. *Then* $(\sum_{n\in\mathbb{N}} X_n)_{l_p} \in [(\sum_{n\in\mathbb{N}} X_n)_{l_2}]_S$.

COROLLARY 2.9. *For any Banach spaces* $X, Y$ *and* $1 \le p < \infty$, *if* $X \sim_S Y$ *then* $l_p(Y) \in [l_2(X)]_S$.

THEOREM 2.10. *Assume that the Banach spaces* $X$ *and* $Y$ *are uniformly homeomorphic. Then* $l_p(Y) \in [l_2(X)]_S$ *for all* $1 \le p < \infty$.

*Proof.* By [22, Theorem 3.1], $X \oplus \mathbb{R} \sim_S Y \oplus \mathbb{R}$. Again, according to Corollary 2.8, $l_p(X \oplus \mathbb{R}) \sim_S l_q(Y \oplus \mathbb{R})$. Note that $l_p(X \oplus \mathbb{R})$ and $l_q(Y \oplus \mathbb{R})$ are linearly isomorphic to $l_p(X)$ and $l_q(Y)$, respectively. ■

**3. Sphere equivalence and Property H.** A map $f$ from a metric space $M$ to another metric space $M'$ is said to be a *coarse embedding* [9] (also referred to as *uniform embedding* in the literature, e.g., in [3, 4, 5, 7, 9, 10]) if

(i) $\omega_f(t) < \infty$ for each $t > 0$, and
(ii) there exists a non-decreasing function $\rho$ on $[0, \infty)$ with $\lim_{r\to\infty} \rho(r) = \infty$ such that

$$\rho(d(x, y)) \le d(f(x), f(y)) \le \omega_f(d(x, y)) \quad \text{for all } x, y \in M.$$

Gromov [9] suggested using coarse embeddings of a discrete group into a Hilbert space or even into a certain Banach space as a tool for working on such well-known conjectures as the Novikov conjecture and the Baum–Connes conjecture. Yu [27] proved the coarse Baum–Connes conjecture (respectively the strong Novikov conjecture) for discrete metric spaces with bounded geometry (respectively discrete groups) which are coarsely embeddable into a Hilbert space. Recall that a discrete metric space $M$ is said to have *bounded geometry* if for any $r > 0$ there is $N(r) > 0$ such that any ball of radius $r$ in $M$ contains at most $N(r)$ elements. Later on, Kasparov and Yu [13] proved the coarse Novikov conjecture for discrete metric spaces with bounded geometry which are coarsely embeddable into a uniformly convex Banach space. On the other hand, people have already found some expanders which are not coarsely embeddable into Hilbert spaces and uniformly convex spaces (see [10, 15, 20]). So far, no counterexample to the coarse Novikov conjecture has been found.

Recently, Kasparov and Yu [14] introduced a new geometric property of Banach spaces, called Property H, and proved the strong Novikov conjecture for discrete groups which are coarsely embeddable into Property H Banach spaces. Still more recently, Chen, Wang and Yu [6] proved the coarse Novikov conjecture for discrete metric spaces with bounded geometry which are coarsely embeddable into Property H Banach spaces.

**3.1. The definition of Property H.** For convenience, we start with a notion defined in [12]. A *paving* of a (separable) Banach space $X$ is a sequence $E_1 \subset E_2 \subset \cdots$ of finite-dimensional subspaces of $X$ whose union is dense in $X$. Now we can define Property H by using the above notion.

Definition 3.1 ([6, 14]). A Banach space $X$ is said to have *Property H* if there exist pavings $\{E_n\}_{n=1}^{\infty}$ of $X$ and $\{F_n\}_{n=1}^{\infty}$ of $l_2$ and a uniformly continuous map $f : S(X) \to S(l_2)$ such that the restriction of $f$ to $S(E_n)$ is a homeomorphism onto $S(F_n)$ for all $n \in \mathbb{N}$.

Remark 3.2. (i) By the definition above, Property H of a Banach space $X$ automatically implies the separability of $X$. Therefore we will omit mentioning the separability of a Banach space when we consider its Property H.

(ii) Let the Banach space $X$, the sequences of finite-dimensional subspaces $\{E_n\}$ and $\{F_n\}$, and the map $f$ be as in Definition 3.1. By a standard topological argument, existence of $f$ ensures that $\dim E_n = \dim F_n$ for all $n \in \mathbb{N}$. Later we call $f$ a *paving map* between $X$ and $l_2$ (associated with $\{E_n\}$ and $\{F_n\}$). Thus a paving map is always (finite-)dimension preserving with respect to some chosen paving sequence, which is an important difference from a general uniformly continuous map or even a general uniform homeomorphism between unit spheres.

Let us list some Banach spaces that are known to have Property H [6, 14]. The $l_p$ space has Property H for any $1 \le p < \infty$. Indeed, let $E_n$ and $F_n$ be the subspaces of $l_p$ and $l_2$ respectively consisting of all sequences whose coordinates are zero after the $n$th place. Then $\{E_n\}_{n\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$ obviously pave $l_p$ and $l_2$, respectively. The Mazur map

$$M_{p,2}(\alpha_1, \ldots, \alpha_k, \ldots) = (\alpha_1|\alpha_1|^{p/2-1}, \ldots, \alpha_k|\alpha_k|^{p/2-1}, \ldots)$$

is the desired paving map. Similarly, the Banach space of all Schatten $p$-class operators on a separable Hilbert space has Property H. By using [2, Lemma 9.5 and Corollary 9.6], one can also prove that uniformly convex Banach spaces with an unconditional basis have Property H. In particular, this implies that the classical $L_p$-space has Property H for all $1 < p < \infty$.

We will next use the Mazur map to show that a paving map is not always (finite-)dimension preserving, which implies that we need to choose the paving sequence carefully when we hope to prove that a Banach space has Property H.

EXAMPLE 3.3. In the unit sphere $S(l_1)$ of $l_1$, take a sequence $\{x_j\}_{j\in\mathbb{N}}$ such that the only nonzero coordinates of $x_j$ are $1/4$ and $3/4$ at the $j$th and $(j+1)$th place, respectively. Set $V_n = [x_1, \dots, x_n]$. Then $V_1 \subset V_2 \subset \cdots$ and $\dim V_n = n$. Here $[A]$ denotes the closed linear span of a subset $A$ in a Banach space. We claim that the Mazur map between $S(l_1)$ and $S(l_2)$ does not preserve the dimensions of $V_n$. More precisely, we will prove that for any $n > 1$,

$$\dim[M_{1,2}(S(V_n))] \geq n + 1.$$

To show this claim, let

$$y_{n+1} = \frac{1}{n}(x_1 + \cdots + x_n) = \frac{1}{n}\left(\frac{1}{4}, 1, \dots, 1, \frac{3}{4}, 0, \dots\right) \in S(l_1).$$

Then $M_{1,2}(x_1), M_{1,2}(x_2), \dots, M_{1,2}(x_n), M_{1,2}(y_{n+1})$ are $n+1$ linearly independent vectors contained in $S(l_2)$. Indeed,

$$\begin{pmatrix} M_{1,2}(x_1) \\ M_{1,2}(x_2) \\ \vdots \\ M_{1,2}(x_n) \\ M_{1,2}(y_{n+1}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & \cdots \\ \frac{1}{2\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{\sqrt{3}}{2\sqrt{n}} & 0 & \cdots \end{pmatrix}.$$

Denote the first $n+1$ columns of the matrix above by $A_{n+1}$. By calculating its determinant, we have

$$|A_{n+1}| = L \begin{vmatrix} 1 & \sqrt{3} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \sqrt{3} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \sqrt{3} \\ 1 & 0 & 0 & 0 & \cdots & 0 & \sqrt{3} \end{vmatrix}$$

$$= L(\sqrt{3} + (-\sqrt{3})^n) \neq 0 \quad (n > 1).$$

Here $L = \frac{1}{2^{n+1}} \frac{1}{\sqrt{n}} \big(1 - \frac{2}{1+\sqrt{3}}\big)$.

**3.2. The stability of Property H.** In this section we mainly aim to provide a permanence result for Property H, which may be viewed as an analogue of the well-known Day theorem [8].

Theorem 3.4 (Day's theorem). *For every $1 < p < \infty$ the Banach sequence space $(\sum \oplus E_n)_{l_p}$ is uniformly convex provided that the sequence $\{E_n\}_{n\in\mathbb{N}}$ has a common modulus of convexity. In particular, $l_p(X)$ is uniformly convex whenever $X$ is uniformly convex.*

We first introduce the following definition.

Definition 3.5. We say that a sequence $\{X_j\}_{j\in\mathbb{N}}$ of real Banach spaces has *uniform Property H* or *equi-Property H* provided that for every $j$ the space $X_j$ has Property H and we can choose a corresponding paving map $f_j$ between $X_j$ and $l_2$ such that the sequence $\{f_j : S(X_j) \to S(l_2)\}_{j\in\mathbb{N}}$ is equi-uniformly continuous.

Theorem 3.6. *Let $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of real Banach spaces. If $\{X_j\}_{j\in\mathbb{N}}$ has equi-Property H then $(\sum_{j=1}^{\infty} X_j)_{l_p}$ has Property H for all $1 \le p < \infty$. In particular, $l_p(X)$ has Property H whenever $X$ does.*

*Proof.* For every $j$, we choose pavings $\{V_n^j\}_{n\in\mathbb{N}}$ of $X_j$, $\{W_n^j\}_{n\in\mathbb{N}}$ of $l_2$ and the paving map $f_j$ such that the sequence $\{f_j : S(X_j) \to S(l_2)\}_{j\in\mathbb{N}}$ is equi-uniformly continuous.

Define $F_{p,2} : (\sum X_j)_{l_p} \to (\sum l_2)_{l_2}$ by

$$F_{p,2}(x_1, x_2, \ldots) = (\widetilde{f}_{1,p/2}(x_1), \widetilde{f}_{2,p/2}(x_2), \ldots)$$

for $(x_1, x_2, \ldots) \in (\sum X_j)_{l_p}$. Here $\widetilde{f}_{j,p/2}$ is the aforementioned $p/2$-Hölder extension of $f_j$.

Take

$$V_n = (V_n^1 + V_n^2 + \cdots + V_n^n + 0 + \cdots)_{l_p},$$
$$W_n = (W_n^1 + W_n^2 + \cdots + W_n^n + 0 + \cdots)_{l_2}.$$

We claim that:

(i) $\{V_n\}_{n\in\mathbb{N}}$ and $\{W_n\}_{n\in\mathbb{N}}$ pave $(\sum X_j)_{l_p}$ and $(\sum l_2)_{l_2} = l_2$, respectively;

(ii) $F_{p,2}$ is uniformly continuous between the unit spheres of $(\sum X_j)_{l_p}$ and $(\sum l_2)_{l_2}$; and

(iii) the restricted map $F_{p,2} : S(V_n) \to S(W_n)$ is a homeomorphism for every $n \in \mathbb{N}$.

This means that $(\sum X_j)_{l_p}$ has Property H.

Now we prove the three assertions above.

(i) Obviously, $V_n$ is an increasing sequence of finite-dimensional subspaces of $X$. Furthermore $\bigcup_{n\in\mathbb{N}} V_n$ is dense in $(\sum X_j)_{l_p}$. Indeed, for every $\varepsilon > 0$ and $x = (x_j) \in (\sum X_j)_{l_p}$, choose $x^{n_0} = (x_1, \ldots, x_{n_0}, 0, \ldots)$ such that $\|x - x^{n_0}\| < \varepsilon/2$. For every $j \in \{1, \ldots, n_0\}$ the density of $\bigcup_{n=1}^{\infty} V_n^j$ in $X_j$ implies that there is $n_j \in \mathbb{N}$ such that $\operatorname{dist}(x_j, V_{n_j}^j) < \varepsilon/(2n_0)$. Set

$N_0 = \max_{1\le j\le n_0} n_j$. Then $\operatorname{dist}(x^{n_0}, V_{N_0}) < \varepsilon/2$ where $V_{N_0} = (V_{N_0}^1 + \cdots + V_{N_0}^{N_0} + 0 + \cdots)_{l_p}$. Thus

$$\operatorname{dist}(x, V_{N_0}) \le \|x - x^{n_0}\| + \operatorname{dist}(x^{n_0}, V_{N_0}) \le \varepsilon.$$

(ii) This follows directly from Theorem 2.6, since the sequence $\{f_j : S(X_j) \to S(l_2)\}_j$ is equi-uniformly continuous.

(iii) To prove this item, it suffices to note that given an $n \in \mathbb{N}$, the restriction of $F_{p,2}$ to $S(V_n)$ is a bijection onto $S(W_n)$. Indeed, by the hypothesis and by Theorem 2.4 the component $\widetilde{f}_{j,p/2}$ of $F_{p,2}$ is a homeomorphism from $B(V_n^j)$ onto $B(W_n^j)$ for all $j = 1, \ldots, n$; and obviously

$$F_{p,2}^{-1}(w_n^1, \ldots, w_n^n, 0, \ldots) = (\widetilde{f}_{1,p/2}^{-1}(w_n^1), \ldots, \widetilde{f}_{n,p/2}^{-1}(w_n^n), 0, \ldots) \in S(V_n)$$

for every $(w_n^1, \ldots, w_n^n, 0, \ldots) \in S(W_n)$. ∎

In contrast to Theorem 3.6, Brown and Guentner [3] proved that for each discrete metric space $A$ with bounded geometry there is a sequence $\{p_n\}$ with $p_n > 1$ and $p_n \to \infty$ such that $A$ is coarsely embeddable into $(\sum l_{p_n})_{l_2}$. This result was strengthened in [1] and [24]. (Observe that $(\sum l_{p_n})_{l_2}$ has trivial cotype.) The following question naturally arises:

QUESTION 3.7. *Does $(\sum l_{p_n})_{l_2}$ have Property H for some sequence $\{p_n\}_{n\in\mathbb{N}}$ with $p_n > 1$ and $p_n \to \infty$?*

According to [3] and by recent work in [6, 14], the positive answer to this question will imply the (strong, coarse) Novikov conjecture for all discrete groups with bounded geometry.

In the following, we give several simple remarks concerning the above question.

REMARK 3.8. (i) For any sequence $\{p_n\}_{n\in\mathbb{N}}$ as in Question 3.7, although the sequence $\{M_{p_n,2}\}_{n\in\mathbb{N}}$ of Mazur maps ensures that $l_{p_n}$ has Property H for all $n$, this sequence is not equi-uniformly continuous. Indeed, it is well-known that

$$c\|x-y\|^{p/q} \le \|M_{p,q}(x) - M_{p,q}(y)\| \le \frac{p}{q}\|x-y\| \tag{3.1}$$

for all $x, y \in S(l_p)$ and $p > q$, and the opposite inequalities hold if $p < q$ (note that $M_{p,q}^{-1} = M_{q,p}$), where the constant $c$ depends only on $p/q$. See more details in [2, Chapter 9].

(ii) It is also an open question whether $c_0$ has Property H. By recent work [6, 14], the positive answer to this question will imply the (coarse, strong) Novikov conjecture for all discrete groups since every discrete group admits a coarse embedding into $c_0$. In fact, we do not even know whether there exists a uniformly continuous injective map from some trivial cotype Banach space (respectively, $c_0$) sphere to the $l_2$ sphere. Recall that a Banach space $X$ has *trivial cotype* if it contains uniformly isomorphic copies of $\{l_\infty^n\}_n$. Here $l_\infty^n$ denotes $\mathbb{R}^n$ equipped with the $l_\infty$ norm.

(iii) Since the sequence $\{l_{p_n}\}_{n\in\mathbb{N}}$ has equi-Property H for every bounded sequence $\{p_n\}_{n\in\mathbb{N}}$ with $p_n \geq 1$, by Theorem 3.6 we immediately obtain a positive result.

THEOREM 3.9. *For any bounded sequence $\{p_n\}_{n\in\mathbb{N}}$ with $p_n \geq 1$ and for all $1 \leq p < \infty$, the sequence space $(\sum_{n=1}^{\infty} l_{p_n})_{l_p}$ has Property H.*

**3.3. Coarse embeddings of free products.** We conclude this section with a direct application of Theorem 3.6. It is usually of interest to find permanence properties of the class of discrete groups that are coarsely embeddable into a certain type of Banach spaces. In particular, it is shown in [4] that if two countable discrete groups $A$ and $B$ are coarsely embeddable into a Hilbert space, then so is their free product $A * B$. This has been generalized to the uniformly convex case by applying Day's theorem (see [5]). We shall prove an analogous result for coarse embeddability into Property H Banach spaces:

THEOREM 3.10. *Let $A$ and $B$ be countable discrete groups, and let $\Gamma = A * B$ be their free product. If $A$ and $B$ are coarsely embeddable into Property H Banach spaces, then so is $\Gamma$.*

*Proof.* Let $f_A : A \to X_A$ and $f_B : B \to X_B$ be coarse embeddings, where $X_A$ and $X_B$ have Property H. Then much as in the proofs of [4, Theorem] and [5, Theorem], we can construct a coarse embedding $F : \Gamma \to X_\Gamma$, where

$$X_\Gamma = \Big(\sum_{W_A} X_A\Big)_{l_2} \oplus \Big(\sum_{W_B} X_B\Big)_{l_2},$$

where $W_A$ (resp. $W_B$) denotes the set of those elements of $\Gamma$ whose expression as a reduced word begins with $A$ (resp. $B$) with $\{e\} = W_A \cap W_B$. By Theorem 3.6, $X_\Gamma$ also has Property H. ∎

**4. Sphere equivalence and Banach expanders.** Expanders are important examples of spaces which are not coarsely embeddable into "nice" Banach spaces. For example, Lafforgue [15] showed that there is a family of expanders which is not coarsely embedded into any uniformly convex Banach space. Recently Mendel and Naor [20] introduced the nonlinear spectral gap on general metric spaces and studied that notion in detail. In particular, they constructed a "super-expander" that does not coarsely embed into any uniformly convex Banach space, which is another construction of such graphs, entirely different from Lafforgue's. More recently, Mimura [21] specialized the concept of the nonlinear spectral gap to the Banach space setting and introduced and studied "$(X, p)$-anders for a sequence of finite graphs" to attack the following open problem: Are any expanders automatically $(X,p)$-anders for all $X$ of nontrivial cotype and for all $p \in [1,\infty)$?

In this section, we combine Theorem 2.6 with Mimura's [21, Theorem 4.1] to show directly that having $(X,p)$-anders is stable under varying the exponent $p$ over $[1,\infty)$.

Let us recall some notation and concepts used in [21]. Let $G=(V,E)$ be a finite connected undirected graph (here $V$ is the set of vertices and $E$ is the set of oriented edges), possibly with multiple edges and self-loops. Then $G$ can be regarded as a metric space equipped with the path metric $d_G$ (namely, the distance $d_G(v,w)$ between two vertices $v$ and $w$ in $V$ is the shortest length of a path connecting $v$ and $w$, and we set $d_G(v,v)=0$). The diameter of $G$ ($\operatorname{diam}(G)$) is the largest distance of vertices in $G$. Let $\Delta(G)$ denote the maximal degree of $G$. Assume that $(X,p)$ is a pair of a Banach space $X$ and an exponent $p$.

Definition 4.1. The *Banach spectral gap* is defined as follows: the *$(X,p)$-spectral gap* of $G$, denoted by $\lambda_1(G;X,p)$, is

$$\lambda_1(G;X,p)=\frac{1}{2}\inf_{f:V\to X}\frac{\sum_{v\in V}\sum_{e=(v,w)\in E}\|f(w)-f(v)\|^p}{\sum_{v\in V}\|f(v)-m(f)\|^p}. \tag{4.1}$$

Here, $m(f)=\sum_{v\in V}f(v)/|V|$ and $f$ runs over all nonconstant maps.

Definition 4.2 (Banach expanders). A sequence $\{G_n\}_{n\in\mathbb{N}}$ of finite connected graphs is called a *family of $(X,p)$-anders* if

(i) $\sup_n \Delta(G_n)<\infty$;
(ii) $\lim_{n\to\infty}\operatorname{diam}(G_n)=\infty$; and
(iii) $\inf_n \lambda_1(G_n;X,p)>0$.

Let us mention several known results:

(i) The existence of $(X,p)$-anders for some fixed $p$ implies poor embeddability into $X$: if $\{G_n\}_n$ is a sequence of $(X,p)$-anders for some $p$, then there is no family of coarse embeddings of $G_n, n\in\mathbb{N}$, into $X$ with common $\rho$ and $\omega$.
(ii) For all $1\le p<\infty$, $(X,p)$-anders are (classical) expanders (see [11, 18, 21, 25]).
(iii) Pisier [26] showed that expanders are $(X,2)$-anders for uniformly curved Banach spaces $X$.

Let us recall the notion of uniformly curved Banach spaces.

Definition 4.3 (Pisier [26]). A Banach space $X$ is said to be *uniformly curved* [26] if $\lim_{\varepsilon\to+0}D_X(\varepsilon)=0$, where $D_X(\varepsilon)$ denotes the infimum of $D\in(0,\infty)$ such that for every $n\in\mathbb{N}$, every matrix $T=(t_{ij})_{i,j}\in \mathrm{M}_n(\mathbb{R})$ with

$$\|T\|_{L_2^n(\mathbb{R})\to L_2^n(\mathbb{R})}\le\varepsilon \quad\text{and}\quad \|\operatorname{abs}(T)\|_{L_2^n(\mathbb{R})\to L_2^n(\mathbb{R})}\le 1,$$

where $\mathrm{abs}(T) = (|t_{ij}|)_{i,j}$ is the entrywise absolute value matrix of $T$, satisfies

$$\|T \otimes I_X\|_{L_2^n(X)\to L_2^n(X)} \le D.$$

We also need the following definitions.

Definition 4.4 ([21]). Let $X, Y$ be Banach spaces and $p, q \in [1, \infty)$. Let $\Omega$ be an at most countable set. For a map $\phi\colon S(l_p(\Omega, X)) \to S(l_q(\Omega, Y))$, we say that $\phi$ is $\mathrm{Sym}(\Omega)$-*equivariant* if $\phi \circ \sigma_{X,p} = \sigma_{Y,q} \circ \phi$ for any $\sigma \in \mathrm{Sym}(\Omega)$. Here for a Banach space $E$ and $r \in [1, \infty)$, the symbol $\sigma_{E,r}$ denotes the isometry $\sigma_{E,r}$ on $l_r(\Omega, E)$ induced by $\sigma$, namely, $(\sigma_{E,r}\xi)(a) := \xi(\sigma^{-1}(a))$ for $\xi \in l_r(\Omega, E)$ and $a \in \Omega$; and $\mathrm{Sym}(\Omega)$ is the group of all permutations on $\Omega$, *including those of infinite support.*

In this section, let $\Gamma$ be a finitely generated group, $S \not\ni e$ be a symmetric finite generating set, and $H$ be a subgroup of $\Gamma$ of finite index.

Definition 4.5 ([21]). Let $\Gamma, H, S$ be as above.

(1) Let $\lambda_{\Gamma,H,X,p}$ be the *quasi-regular representation* of $\Gamma$ on $l_p(\Gamma/H, l_p(X))$, namely, for $\gamma \in \Gamma$ and $\xi \in l_p(\Gamma/H, l_p(X))$, $\lambda_{\Gamma,H,X,p}(\gamma)\xi(xH) := \xi(\gamma^{-1}xH)$. Then $l_p(\Gamma/H, l_p(X))$ decomposes into $\Gamma$-representation spaces

$$l_p(\Gamma/H, l_p(X)) = l_p(\Gamma/H, l_p(X))^{\lambda_{\Gamma,H,X,p}(\Gamma)} \oplus l_{p,0}(\Gamma/H, l_p(X)).$$

Here the first summand is the space of $\lambda_{\Gamma,H,X,p}(\Gamma)$-invariant vectors, and the second is the space of "zero-sum" functions,

$$l_{p,0}(\Gamma/H, l_p(X)) := \Big\{\xi \in l_p(\Gamma/H, l_p(X)) : \sum_{v\in\Gamma/H} \xi(v) = 0\Big\}.$$

We use the same symbol $\lambda_{\Gamma,H,X,p}$ for the restriction to $l_{p,0}(\Gamma/H, l_p(X))$.

(2) The *p-displacement constant* on $X$ with respect to $\Gamma, H$ and $S$, written $\kappa_{X,p}(\Gamma, H, S)$, is defined as

$$\kappa_{X,p}(\Gamma, H, S) := \inf_{0\ne\xi\in l_{p,0}(\Gamma/H, l_p(X))} \sup_{s\in S} \frac{\|\lambda_{\Gamma,H,X,p}(s)\xi - \xi\|}{\|\xi\|}.$$

The following lemma plays a fundamental role in [21]; it relates the $p$-displacement constant on $X$ to the $(X, p)$-spectral gap for a Schreier graph.

Lemma 4.6 ([21]). *Let $G$ be a Schreier graph with respect to $\Gamma$, $H$ and $S$ and let $(X, p)$ be a pair as above. Then*

$$\kappa_{X,p}(\Gamma, H, S)^p \le \lambda_1(G; X, p) \le \frac{|S|}{2}\kappa_{X,p}(\Gamma, H, S)^p.$$

We now extend [21, Proposition 4.2].

Proposition 4.7. *Let $X \sim_S Y$ and let $f\colon S(X) \to S(Y)$ be a uniform homeomorphism. Let $\Gamma$ be a finitely generated group, $S \not\ni e$ be a symmetric* *finite subset, and $H$ be a subgroup of $\Gamma$ of finite index. Then for any $p, q \in [1, \infty)$,*

$$\kappa_{X,p}(\Gamma, H, S) \geq \delta_1^{-1}\left(\frac{1}{2}\delta_2^{-1}\left(\frac{1}{2}\right)\kappa_{Y,q}(\Gamma, H, S)\right),$$

*where $\delta_1 = \omega_{F_{p,q}}$, $\delta_2 = \omega_{F_{p,q}^{-1}}$ and $F_{p,q} : l_p(X) \to l_q(Y)$, $(x_j)_j \mapsto (\widetilde{f}_{p/q}(x_j))_j$, where $\widetilde{f}_{p/q}$ is the aforementioned $p/q$-Hölder extension of $f$.*

*Proof.* We consider the quasi-regular representations $\lambda_{\Gamma,H,X,p}$ and $\lambda_{\Gamma,H,Y,q}$ of $\Gamma$ on $l_p(\Gamma/H, l_p(X))$ and $l_q(\Gamma/H, l_q(Y))$, respectively. Note that $l_p(\Gamma/H, l_p(X))$ is linearly isometric to $l_p(X)$. We may regard $F_{p,q}$ as

$$F_{p,q} : S(l_p(\Gamma/H, l_p(X))) \to S(l_q(\Gamma/H, l_q(Y))),$$

which is $\mathrm{Sym}(\Gamma/H)$-equivariant (by construction) and is uniformly homeomorphic by Theorem 2.6. Choose any $\xi \in S(l_{p,0}(\Gamma/H, l_p(X)))$ and set $\eta =: F_{p,q}(\xi) \in S(l_q(\Gamma/H, l_q(Y)))$. Then continue the proof as in [21, Proposition 4.2]. ∎

THEOREM 4.8. *Let $X \sim_S Y$ and let $f : S(X) \to S(Y)$ be a uniform homeomorphism. Let $G = (V, E)$ be a finite graph. Then for any $p, q \in [1, \infty)$,*

$$\lambda_1(G; X, p) \geq \frac{1}{2}\left\{\delta_1^{-1}\left(\frac{1}{2}\left(\frac{2}{\Delta(G)}\right)^{1/q}\delta_2^{-1}\left(\frac{1}{2}\right)\lambda_1^{1/q}(G; Y, q)\right)\right\}^p.$$

*Here $\delta_i$ $(i = 1, 2)$ and $F_{p,q}$ are defined as in Proposition 4.7.*

*Proof.* If $G$ is a Schreier graph with respect to $\Gamma, H$ and $S$ then Proposition 4.7 and Lemma 4.6 end the proof. For general finite graphs, use the Gross trick of Mimura [21, Section 4.2] to reduce the argument to the previous case. ∎

From Theorem 4.8 we immediately obtain the following stability results.

THEOREM 4.9. *Let $X, Y$ be Banach spaces.*

(i) *If $X \sim_S Y$, then for any $p, q \in [1, \infty)$ and any sequence $\{G_n\}_{n\in\mathbb{N}}$ of graphs, $\{G_n\}_{n\in\mathbb{N}}$ is a family of $(X, p)$-anders if and only if it is a family of $(Y, q)$-anders.*
(ii) *In particular, for any sequence $\{G_n\}_{n\in\mathbb{N}}$ of graphs, $\{G_n\}_{n\in\mathbb{N}}$ is a family of $(X, p)$-anders for some $p \in [1, \infty)$ if and only if it is so for all $p \in [1, \infty)$.*

COROLLARY 4.10. *For a Banach space $X$ sphere equivalent to a uniformly curved Banach space and for all $p \in [1, \infty)$, any expanders are automatically $(X, p)$-anders.*

Obviously, these results improve and extend Mimura's main results in [21] (e.g., [21, Theorems A and B, Corollary C]). In particular, Theorem 4.9(ii) settles a natural question of whether the property of having $(X, p)$-anders for a sequence of finite connected graphs of uniformly bounded degrees depends on the exponent $p \in [1, \infty)$.

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