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Asymptotically conformal classes and non-Strebel points

by

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Abstract. Let $T(\Delta)$ be the universal Teichmüller space on the unit disk Δ and $T_0(\Delta)$ be the set of asymptotically conformal classes in $T(\Delta)$. Suppose that μ is a Beltrami differential on Δ with $[\mu] \in T_0(\Delta)$. It is an interesting question whether $[t\mu]$ belongs to $T_0(\Delta)$ for general $t \neq 0, 1$. In this paper, it is shown that there exists a Beltrami differential $\mu \in [0]$ such that $[t\mu]$ is a non-trivial non-Strebel point for any $t \in (-1/\|\mu\|_\infty, 1/\|\mu\|_\infty) \setminus \{0, 1\}$.

1. Introduction. Let S be a plane domain with at least two boundary points. The *Teichmüller space* $T(S)$ is the space of equivalence classes of quasiconformal maps f from S to a variable domain $f(S)$. Two quasiconformal maps f from S to $f(S)$ and g from S to $g(S)$ are said to be *equivalent*, denoted by $f \sim g$, if there is a conformal map c from $f(S)$ onto $g(S)$ and a homotopy through quasiconformal maps h_t mapping S onto $g(S)$ such that $h_0 = c \circ f$, $h_1 = g$ and $h_t(p) = c \circ f(p) = g(p)$ for every $t \in [0, 1]$ and every p in the boundary of S. Denote by $[f]$ the Teichmüller equivalence class of f; sometimes it is more convenient to use $[\mu]$ to express the equivalence class where μ is the Beltrami differential (or the complex dilatation) of f.

Denote by $\mathrm{Bel}(S)$ the Banach space of Beltrami differentials $\mu = \mu(z)d\bar{z}/dz$ on S with finite L^∞-norm and by $M(S)$ the open unit ball in $\mathrm{Bel}(S)$.

For $\mu \in M(S)$, define

$$k_0([\mu]) = \inf\{\|\nu\|_\infty : \nu \in [\mu]\}.$$

We say that μ is *extremal* in $[\mu]$ if $\|\mu\|_\infty = k_0([\mu])$ (the corresponding quasiconformal map f is said to be extremal for its boundary values as well), and *uniquely extremal* if $\|\nu\|_\infty > k_0(\mu)$ for any other $\nu \in [\mu]$.

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The cotangent space to $T(S)$ at the basepoint is the Banach space $Q(S)$ of integrable holomorphic quadratic differentials φ on S with L^1-norm

$$\|\varphi\| = \iint_S |\varphi(z)|\, dx\, dy < \infty.$$

In what follows, let $Q^1(S)$ denote the unit sphere of $Q(S)$.

For any μ, define $h^*(\mu)$ to be the infimum over all compact subsets E contained in S of the essential supremum norm of the Beltrami differential $\mu(z)$ as z varies over $S \setminus E$. Define $h([\mu])$ to be the infimum of $h^*(\nu)$ taken over all representatives ν of the class $[\mu]$. It is obvious that $h([\mu]) \le k_0([\mu])$. Following [5], $[\mu]$ is called a *Strebel point* if $h([\mu]) < k_0([\mu])$; otherwise, $[\mu]$ is a *non-Strebel point*. The basepoint $[0]$ is called the *trivial* non-Strebel point.

It is well known that the set of Strebel points is open and dense in $T(S)$ [16]. By Strebel's frame mapping criterion (see [8, Chapter 4]), every Strebel point $[\mu]$ is represented by the unique Beltrami differential of the form $k\overline{\varphi}/|\varphi|$, where $k = k_0([\mu]) \in (0, 1)$ and φ is a unit vector in $Q(S)$.

A quasiconformal mapping f on S is said to be *asymptotically conformal* if its Beltrami differential μ satisfies $h^*(\mu) = 0$. Hence we call $[\mu]$ an *asymptotically conformal class* if $h([\mu]) = 0$. Denote by $T_0(S)$ the set of asymptotically conformal classes in $T(S)$. It is a closed subspace of $T(S)$ (see [9]) and every point in $T_0(S)$ except the basepoint $[0]$ is a Strebel point.

$T_0(S)$ was introduced by Gardiner and Sullivan [9] for the unit disk and by Earle, Gardiner and Lakic for an arbitrary hyperbolic Riemann surface [3, 4, 8]. $T_0(S)$ is extensively studied and plays an important role in the theory of asymptotic Teichmüller spaces (see e.g. [1, 7, 14, 19, 26]).

The motivation for writing this note stems from two problems. The first one was posed by Kra in the 1960s.

PROBLEM 1.1. *If $\mu \in [0]$, is it true that $t\mu \in [0]$ for all $0 < t < 1$?*

By use of the Teichmüller shift mapping (see [15, 23]), Gehring [10] proved that there exists some $\mu \in [0]$ such that $t\mu \notin [0]$ for $0 < t < 1$ on the upper half-plane, which gives a negative answer to Problem 1.1. A more concrete example was given by Reich and Strebel [21] (or see [15]) with the help of the Reich–Strebel inequality, so-called Main Inequality (see [8, 21, 22]). Although we have a negative answer to Problem 1.1 generally, one might expect that if $\mu \in [0]$ then $[t\mu] \in T_0(S)$.

The second problem is related to geodesics in $T(S)$:

PROBLEM 1.2. *Let $\mu = k\overline{\varphi}/|\varphi|$ represent an asymptotically conformal class $[\mu]$ in $T_0(S)$ where $k \in (0, 1)$ is a constant and $\varphi \in Q^1(S)$. Is the geodesic $\{[t\mu] : t \in (0, 1)\}$ joining $[\mu]$ and $[0]$ in $T(S)$ contained in $T_0(S)$?*

Problem 1.2 is actually whether the subspace $T_0(S)$ is geodesic-connected. It was proved in [4] and [13] in a different way that the Teichmüller metric coincides with the Kobayashi metric on $T_0(S)$. If the answer to Problem 1.2 is affirmative, we can derive this fact immediately. The author got to know this problem from Professor Earle who visited Peking University in 2002. Up to the present, there is no substantial progress in solving the problem. So, it is regarded as a very difficult problem in the theory of Teichmüller spaces. Nevertheless, it is interesting to study the "starlikeness" for general Beltrami differentials representing asymptotically conformal classes. The "starlikeness" problem is posed in the following precise form.

PROBLEM 1.3. *Let* $\mu \in M(S)$ *be such that* $[\mu] \in T_0(S)$ *and* $\|\mu\|_\infty > 0$. *Does* $[t\mu]$ *belong to* $T_0(S)$ *for* $t \in (-1/\|\mu\|_\infty, 1/\|\mu\|_\infty) \setminus \{0, 1\}$*?*

Let Δ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $T(\Delta)$ be the universal Teichmüller space. Our first theorem answers Problem 1.3 on Δ negatively; of course, this shows that the aforementioned expectation is wrong.

THEOREM 1.4. *There exists a Beltrami differential* $\mu \in [0]$ *in* $T_0(\Delta)$ *such that* $[t\mu]$ *is a non-trivial non-Strebel point for any* $t \in (-1/\|\mu\|_\infty, 1/\|\mu\|_\infty) \setminus \{0, 1\}$.

Z. Zhou [27] claimed to have a negative answer to Problem 1.3 via a counterexample, but unfortunately, there is a substantial gap in his computation and his proof is problematic. Since his paper is in Chinese except for an English abstract, we would not like to go into the details of the gap. As far as the author knows, no examples can be found in the literature to hint at an answer to Problem 1.3.

Our second theorem generalizes Theorem 1.4 to general Teichmüller spaces $T(S)$ but under some assumption.

THEOREM 1.5. *Let* S *be a plane domain such that a piece of its boundary* (*not necessarily the whole*) *is a smooth curve. Then there exists a Beltrami differential* $\mu \in [0]$ *in* $T_0(S)$ *such that* $[t\mu]$ *is a non-trivial non-Strebel point for any* $t \in (-1/\|\mu\|_\infty, 1/\|\mu\|_\infty) \setminus \{0, 1\}$.

With the help of Theorem 1.4, we show that the Strebel points generally have no "starlikeness" in our third theorem.

THEOREM 1.6. *Let* S *be a plane domain such that a piece of its boundary is a smooth curve. Then there exists a Beltrami differential* $\mu \in M(S)$ *such that* $[\mu] \in T_0(S) \setminus \{[0]\}$ *while* $[t\mu]$ *is a non-trivial non-Strebel point for any* $t \in (-1/\|\mu\|_\infty, 1 - \epsilon) \cup (1 + \epsilon, 1/\|\mu\|_\infty) \setminus \{0\}$ *where* $\epsilon \in (0, 1)$ *is a small number.*

The following direct corollary shows that the non-Strebel points generally have no "starlikeness".

COROLLARY 1.7. *Let* S *be a plane domain such that a piece of its boundary is a smooth curve. Then there exists a Beltrami differential* $\mu \in M(S)$

such that $[\mu]$ *is a non-trivial non-Strebel point while* $[t\mu] \in T_0(S) \setminus \{[0]\}$ *for any* $t \in (a,b)$ *where* (a,b) *is a suitable interval.*

After some preparations in Section 2, we prove Theorem 1.4 and derive Theorems 1.5 and 1.6 from Theorem 1.4 in Section 3. The method used here can also be applied to deal with some more general cases, for example, S can be a Riemann surface with a certain boundary condition.

2. Some preliminaries. Let $p \in \partial S$ and $\mu \in M(S)$. Define

$$h_p^*(\mu) = \inf\Big\{\operatorname*{ess\,sup}_{z\in U\cap S} |\mu(z)| : U \text{ is an open neighborhood in } \mathbb{C} \text{ containing } p\Big\}$$

and

$$h_p([\mu]) = \inf\{h_p^*(\nu) : \nu \in [\mu]\}$$

to be the local boundary dilatation at p of μ and $[\mu]$ respectively.

It was proved by Fehlmann [6] for the unit disk and by Lakic [17] for general plane domains that

$$h([\mu]) = \max_{p\in\partial S} h_p([\mu]).$$

As is well known, a Beltrami differential μ is an extremal if and only if it has a so-called *Hamilton sequence*, that is, a sequence $\{\varphi_n \in Q^1(S)\}$ such that

$$\lim_{n\to\infty} \iint_S \mu\varphi_n(z)\, dx\, dy = \|\mu\|_\infty. \tag{2.1}$$

By the result in [5], $[\mu]$ is a non-Strebel point if and only if the extremal in $[\mu]$ has a degenerating Hamilton sequence. A sequence is called *degenerating* if it converges to 0 uniformly on compact subsets of S. In particular, a sequence $\{\varphi_n\} \subset Q^1(S)$ is called *degenerating towards* $p \in \partial S$ if it converges to 0 uniformly on compact subsets of $S \setminus \{p\}$.

The following lemma derives from [17, Theorem 6].

LEMMA 2.1. *Suppose that* μ *is an extremal and* $[\mu]$ *is a non-Strebel point in* $T(S)$. *The following two conditions are equivalent:*

(1) $h([\mu]) = h_p([\mu])$,
(2) *there exists a Hamilton sequence for* μ *degenerating towards* p.

If either of the two conditions in the lemma holds at some $p \in \partial S$, we call p a *substantial boundary point* of $[\mu]$.

LEMMA 2.2. *Suppose that* φ *is holomorphic in* Δ *and has a second order pole at* $p \in \partial\Delta$. *Let* $\mu = k\overline{\varphi}/|\varphi|$ *where* $k \in (0,1)$ *is a constant. Then* μ *is extremal and* $[\mu]$ *is a non-Strebel point in* $T(\Delta)$. *In particular* p *is a substantial boundary point of* $[\mu]$.

Proof. We can write

$$\varphi(z)=\widetilde{\varphi}(z)+\frac{A}{(z-p)^2},$$

where $A\neq 0$ is a constant, $\widetilde{\varphi}$ is holomorphic in Δ and also holomorphic in a neighborhood of p. Let $\phi(z)=A/(z-p)^2$ and $\varphi_n(z)=\phi((1-1/n)z)$. It is easy to check (see [24, Theorem 1] or [25, Theorem 1]) that $\{\varphi_n/\|\varphi_n\|\}$ is a Hamilton sequence for μ, that is,

$$\lim_{n\to\infty}\iint_{\Delta}\mu\frac{\varphi_n}{\|\varphi_n\|}\,dx\,dy=k.$$

On the other hand, since $\|\varphi_n\|\to\infty$, we derive that $\{\varphi_n/\|\varphi_n\|\}$ is degenerating towards p.

Therefore μ is extremal and $[\mu]$ is a non-Strebel point in $T(\Delta)$. In particular, p is a substantial boundary point of $[\mu]$ and $h_p^*(\mu)=h_p([\mu])=h([\mu])=k$ by Lemma 2.1. ∎

3. Proofs of main results

Proof of Theorem 1.4. Denote by Γ the collection of non-negative measurable functions γ in $L^\infty[0,1]$ satisfying the following conditions:

(a) $\operatorname{ess\,inf}_{x\in[0,1]}\gamma(x)=\rho>0$,
(b) $\int_0^1\gamma(x)\,dx=1$.

For any given $\gamma\in\Gamma$, define

$$\Phi(x)=\int_0^x\gamma(s)\,ds,\qquad x\in[0,1]. \tag{3.1}$$

Then $\Phi(x)$ is differentiable at almost every $x\in[0,1]$. In fact, we have $\Phi'(x)=\gamma(x)$ for almost all $x\in[0,1]$.

Let

$$\mathscr{S}=\{z=x+iy\in\mathbb{C}:x\in[0,1],\ y\in\mathbb{R}\}$$

and define a mapping σ from the strip $\mathscr{S}$ onto itself by

$$\sigma(z)=\Phi(x)+iy,\qquad z=x+iy\in\overline{\mathscr{S}}.$$

Observe that $\Phi(x)$ is a strictly increasing function of $x\in[0,1]$. In particular, $\Phi(0)=0$ and $\Phi(1)=1$. It is clear that σ is a self-homeomorphism of $\overline{\mathscr{S}}$ and keeps the boundary points fixed. Moreover, $\sigma(z)$ is differentiable at almost every $z=x+iy\in\mathscr{S}$. More precisely, for almost every $x\in(0,1)$, we have

$$\begin{aligned}\partial_z\sigma(x+iy)&=\frac{\gamma(x)+1}{2},\\ \partial_{\bar z}\sigma(x+iy)&=\frac{\gamma(x)-1}{2}.\end{aligned} \tag{3.2}$$

Let μ_σ denote the Beltrami differential of σ. Then

$$\mu_\sigma(z) = \frac{\partial_{\bar z}\sigma}{\partial_z\sigma} = \frac{\gamma(x)-1}{\gamma(x)+1}, \qquad z = x+iy \in \mathscr{S}. \tag{3.3}$$

It is evident that both $\partial_z\sigma$ and $\partial_{\bar z}\sigma$ are locally L^2-integrable on $\mathscr{S}$. On the other hand, conditions (a) and (b) imply that $\|\gamma\|_\infty \geq 1$ and $\rho \leq 1$. It is easy to verify that

$$\|\mu_\sigma\|_\infty = \max\left\{\frac{\|\gamma\|_\infty - 1}{\|\gamma\|_\infty + 1}, \frac{1-\rho}{1+\rho}\right\}. \tag{3.4}$$

Therefore, the homeomorphism σ is a generalized L^2-solution of the Beltrami equation

$$\partial_{\bar z} w = \mu_\sigma(z)\partial_z w.$$

By the classical characterization of quasiconformal mappings in the plane [18], we see that σ is a quasiconformal mapping.

Let $\phi : \Delta \to \mathscr{S}$ be a conformal mapping from Δ onto $\mathscr{S}$. Then $f_\sigma = \phi^{-1} \circ \sigma \circ \phi$ is a quasiconformal mapping from Δ onto itself. Obviously, f_σ keeps the boundary points fixed. Let μ_{f_σ} and μ_σ denote the Beltrami differentials of f_σ and σ. We have $\mu_{f_\sigma} \in [0]$. A simple computation shows that

$$\mu_{f_\sigma}(z) = \mu_\sigma(\phi(z))\frac{\overline{\phi'^2(z)}}{|\phi'^2(z)|}. \tag{3.5}$$

From now on, we assume $\|\gamma\|_\infty > 1$. Then $\|\mu_\sigma\|_\infty > 0$. We will choose $\gamma \in \Gamma$ such that if $t \in (-1/\|\mu_{f_\sigma}\|_\infty, 1/\|\mu_{f_\sigma}\|_\infty) \setminus \{0,1\}$, then $[t\mu_{f_\sigma}]$ is a non-trivial non-Strebel point. In terms of (3.3), we analyze $t\mu_\sigma$ first. Let $\beta_t(x)$ be the measurable function on $[0,1]$ satisfying

$$t\mu_\sigma(z) = t\frac{\gamma(x)-1}{\gamma(x)+1} = \frac{\beta_t(x)-1}{\beta_t(x)+1}, \qquad t \in (-1/\|\mu_\sigma\|_\infty, 1/\|\mu_\sigma\|_\infty). \tag{3.6}$$

Then

$$\beta_t(x) = \frac{1+t\mu_\sigma(z)}{1-t\mu_\sigma(z)} = \frac{\gamma(x)+1+t(\gamma(x)-1)}{\gamma(x)+1-t(\gamma(x)-1)}. \tag{3.7}$$

Since $\|t\mu_\sigma\|_\infty < 1$, we find that $\beta_t \in L^\infty[0,1]$ is a non-negative measurable function and

$$\frac{1+\|t\mu_\sigma\|_\infty}{1-\|t\mu_\sigma\|_\infty} \geq \operatorname*{ess\,sup}_{x\in[0,1]} |\beta_t(x)| \geq \operatorname*{ess\,inf}_{x\in[0,1]} |\beta_t(x)| \geq \frac{1-\|t\mu_\sigma\|_\infty}{1+\|t\mu_\sigma\|_\infty} > 0. \tag{3.8}$$

Define

$$\Psi_t(x) = \int_0^x \beta_t(s)ds = \int_0^x \frac{\gamma(s)+1+t(\gamma(s)-1)}{\gamma(s)+1-t(\gamma(s)-1)}\, ds, \qquad x \in [0,1]. \tag{3.9}$$

Then $\Psi_t(x)$ is differentiable at almost every $x \in [0,1]$ and $\Psi_t'(x) = \beta_t(x)$ for almost all $x \in [0,1]$. Set

$$\delta_t = \Psi_t(1) = \int_0^1 \beta_t(x)\,dx$$

and let

$$\mathscr{T}_t = \{z = x + iy \in \mathbb{C} : x \in [0, \delta_t],\ y \in \mathbb{R}\}.$$

Define a mapping σ_t from the strip $\mathscr{S}$ onto the strip $\mathscr{T}_t$ by

$$\sigma_t(z) = \Psi_t(x) + iy, \qquad z = x + iy \in \overline{\mathscr{S}}.$$

By the analysis similar to the one for σ, we see that σ_t is a quasiconformal mapping from $\mathscr{S}$ onto $\mathscr{T}_t$. Moreover, σ_t maps $\partial\mathscr{S}$ onto $\partial\mathscr{T}_t$ by the following equation:

$$\sigma_t(z) = \delta_t x + iy, \qquad z = x + iy,\ x = 0, 1,\ y \in \mathbb{R}. \tag{3.10}$$

The Beltrami differential μ_{σ_t} of σ_t on $\mathscr{S}$ is just $t\mu_\sigma$. Let $\phi_t : \mathscr{T}_t \to \Delta$ be a conformal mapping from $\mathscr{T}_t$ onto Δ and let $f_t = \phi_t \circ \sigma_t \circ \phi$. Then f_t is a quasiconformal mapping from Δ onto Δ. Let μ_{f_t} denote the Beltrami differential of f_t. Then

$$\mu_{f_t}(z) = \mu_{\sigma_t}(\phi(z)) \frac{\overline{\phi'^2(z)}}{|\phi'^2(z)|} = t\mu_{f_\sigma}(z).$$

Define another quasiconformal mapping F_t from $\mathscr{S}$ onto $\mathscr{T}_t$ by

$$F_t(z) = \delta_t x + iy, \qquad z = x + iy \in \mathscr{S}. \tag{3.11}$$

We see that F_t coincides with σ_t on $\partial\mathscr{S}$ and its Beltrami differential is $\mu_{F_t}(z) = (\delta_t - 1)/(\delta_t + 1)$. Hence $\mu_{F_t} \in [\mu_{\sigma_t}]$ in the Teichmüller space $T(\mathscr{S})$.

We now show that F_t is an extremal quasiconformal mapping in its class. For this, we transfer F_t together with its Beltrami differential μ_{F_t} to Δ. Let $\mathfrak{F}_t = \phi_t \circ F_t \circ \phi$. Then $\mathfrak{F}_t$ is a quasiconformal mapping from Δ onto Δ. Let $\mu_{\mathfrak{F}_t}$ denote the Beltrami differential of $\mathfrak{F}_t$. Then $\mu_{\mathfrak{F}_t} \in [\mu_{f_t}]$ in the Teichmüller space $T(\Delta)$ and

$$\mu_{\mathfrak{F}_t}(z) = \mu_F(\phi(z)) \frac{\overline{\phi'^2(z)}}{|\phi'^2(z)|}.$$

We will show that $\mu_{\mathfrak{F}_t}$ is an extremal (actually uniquely extremal, see Remark 3.1) Beltrami differential in $[\mu_{f_t}] = [t\mu_{f_\sigma}]$ and $[t\mu_{f_\sigma}]$ is a non-trivial non-Strebel point if $t \neq 0, 1$.

Now let $\phi : \Delta \to \mathscr{S}$ be the conformal mapping from Δ onto $\mathscr{S}$ defined by

$$\phi(z) = \frac{1}{i\pi} \log \frac{z+1}{i(z-1)},$$

where $\log z$ means the univalent branch subject to $\log 1 = 0$. By a simple computation, we have

$$\mu_{\mathfrak{F}_t}(z) = \mu_{F_t}(\phi(z)) \frac{\overline{\phi'^2(z)}}{|\phi'^2(z)|} = \frac{1-\delta_t}{1+\delta_t} \frac{\overline{\varphi(z)}}{|\varphi(z)|},$$

where

$$\varphi(z) = \frac{1}{(z-1)^2(z+1)^2}.$$

Choose $\gamma \in \Gamma$ such that $\delta_t \neq 1$, for example, let

$$\gamma(x) = \begin{cases} \xi, & x \in [0, 1/2], \\ \eta, & x \in (1/2, 1], \end{cases}$$

where ξ and η are constants satisfying $\xi > \eta > 0$ and $\xi + \eta = 2$. Note that $1 - \xi\eta = (\xi - 1)^2$. It is easy to check that $\gamma \in \Gamma$ and

$$\|\mu_{f_\sigma}\|_\infty = \|\mu_\sigma\|_\infty = \max\left\{\frac{\xi-1}{\xi+1}, \frac{1-\eta}{1+\eta}\right\} = \frac{1-\eta}{1+\eta}. \tag{3.12}$$

We now assume $t \in (-1/\|\mu_\sigma\|_\infty, 1/\|\mu_\sigma\|_\infty) \setminus \{0, 1\}$. Then

$$\begin{aligned}
\delta_t &= \int_0^1 \beta_t(x)\, dx = \int_0^{1/2} \frac{\xi + 1 + t(\xi - 1)}{\xi + 1 - t(\xi - 1)}\, dx + \int_{1/2}^1 \frac{\eta + 1 + t(\eta - 1)}{\eta + 1 - t(\eta - 1)}\, dx \\
&= \frac{1}{2}\left[\frac{\xi + 1 + t(\xi - 1)}{\xi + 1 - t(\xi - 1)} + \frac{\eta + 1 + t(\eta - 1)}{\eta + 1 - t(\eta - 1)}\right] \\
&= 1 + \frac{2t(t-1)(1-\xi\eta)}{(\xi + 1 - t(\xi - 1))(\eta + 1 - t(\eta - 1))} \\
&= 1 + \frac{2t(t-1)(\xi - 1)^2}{4 - (1-t)^2(\xi - 1)^2} \neq 1.
\end{aligned}$$

Since φ is holomorphic on $\overline{\Delta}$ except for two second order poles at $z = \pm 1$, by Lemma 2.2, $\mu_{\mathfrak{F}_t}$ is an extremal Beltrami differential in $[\mu_{f_t}]$ and $[\mu_{f_t}]$ is a non-Strebel point in $T(\Delta)$. In particular, $z = \pm 1$ are two substantial boundary points of $[\mu_{f_t}]$ and

$$h([\mu_{f_t}]) = k_0([\mu_{f_t}]) = \left|\frac{1-\delta_t}{1+\delta_t}\right| > 0.$$

Thus, whenever $t \in (-1/\|\mu_{f_\sigma}\|_\infty, 1/\|\mu_{f_\sigma}\|_\infty) \setminus \{0, 1\}$, $[t\mu_{f_\sigma}] = [\mu_{f_t}]$ is a non-trivial non-Strebel point. The completes the proof of Theorem 1.4.

Remark 3.1. Note that φ is holomorphic in Δ and satisfies

$$\int_0^{2\pi} |\varphi(re^{i\theta})|\, d\theta \leq \frac{C}{1-r}, \qquad 0 \leq r < 1,$$

where C is a universal constant. By the main results in [12, 20], $\mu_{\mathfrak{F}_t}$ is uniquely extremal.

Proof of Theorem 1.5. Assume that S is a plane domain with a piece of smooth boundary curve C. Let $p \in C$ be an interior point of C. Then there is a small neighborhood $B = \{z \in S : |z - p| < r\}$ of p in S such that B is a Jordan domain and the boundary curve ∂B is piecewise smooth.

Let $\chi \in M(\Delta)$ be provided by Theorem 1.4 such that $\chi \in [0]$ while $[t\chi]$ is a non-trivial non-Strebel point for any $t \in (-1/\|\chi\|_\infty, 1/\|\chi\|_\infty) \setminus \{0, 1\}$, for example, let $\chi = \mu_{f_\sigma}$ as above.

Let $\psi : B \to \Delta$ be a conformal mapping from B onto Δ. By Carathéodory's theorem ([2] or [11, Theorem 2, p. 41]), ψ can be extended to $\overline{B}$ such that ψ is a homeomorphism between $\overline{B}$ and $\overline{\Delta}$. Assume $\psi(p) = 1$. Let $g = \psi^{-1} \circ f_\sigma \circ \psi$. Then g is a homeomorphism from $\overline{B}$ onto itself. Let μ_g be the Beltrami differential of g. We have

$$\mu_g(z) = \mu_{f_\sigma}(\psi(z)) \frac{\overline{\psi'^2(z)}}{|\psi'^2(z)|}.$$

Then $\mu_g \in [0]$ in the Teichmüller space $T(B)$. Let $\mathscr{F}_t = \psi^{-1} \circ \mathfrak{F}_t \circ \psi$. Then $\mathscr{F}_t$ is also a homeomorphism from $\overline{B}$ onto itself. Let $\mu_{\mathscr{F}_t}$ be the Beltrami differential of $\mathscr{F}_t$. We have

$$\mu_{\mathscr{F}_t}(z) = \mu_{\mathfrak{F}_t}(\psi(z)) \frac{\overline{\psi'^2(z)}}{|\psi'^2(z)|} = \frac{1-\delta_t}{1+\delta_t} \frac{\overline{\varphi \circ \psi(z)}}{|\varphi \circ \psi(z)|} \frac{\overline{\psi'^2(z)}}{|\psi'^2(z)|} = \frac{1-\delta_t}{1+\delta_t} \frac{\overline{\omega(z)}}{|\omega(z)|},$$

where $\omega(z) = \psi'^2(z)[\varphi \circ \psi(z)]$. We conclude that $[t\mu_g] = [\mu_{\mathscr{F}_t}]$ is a non-trivial non-Strebel point for any $t \in (-1/\|\mu_g\|_\infty, 1/\|\mu_g\|_\infty) \setminus \{0, 1\}$ by Theorem 1.4. In particular, p is a substantial boundary point of the non-Strebel point $[\mu_{\mathscr{F}_t}]$.

Let $\mu \in M(S)$ be given by

$$\mu(z) = \begin{cases} \mu_g(z), & z \in B, \\ 0, & z \in S \setminus B. \end{cases}$$

It is clear that $\mu \in [0]$ in the Teichmüller space $T(S)$ and $t\mu$ is equivalent to the following Beltrami differential ν_t:

$$\nu_t(z) = \begin{cases} \mu_{\mathscr{F}_t}(z), & z \in B, \\ 0, & z \in S \setminus B. \end{cases}$$

We now assume $t \in (-1/\|\mu\|_\infty, 1/\|\mu\|_\infty) \setminus \{0, 1\}$. By a simple argument, we derive that ν_t is an extremal in $[t\mu]$ and p is a substantial boundary point of $[t\mu]$ in $T(S)$ since p is a substantial boundary of $[\mu_{\mathscr{F}_t}]$ in $T(B)$. Moreover,

$$h_p^*(t\mu) = h([t\mu]) = k_0([t\mu]) = \|\mu_{\mathscr{F}_t}\|_\infty = \left| \frac{1-\delta_t}{1+\delta_t} \right| > 0.$$

Thus, $[t\mu]$ is a non-trivial non-Strebel point whenever $t \in (-1/\|\mu\|_\infty, 1/\|\mu\|_\infty) \setminus \{0,1\}$. The proof of Theorem 1.5 is complete.

Proof of Theorem 1.6. We use a similar construction to what is done in the proof of Theorem 1.5. In the following argument, we use the same notation and assumptions as previously. For convenience, we assume $\overline{\Delta} \subset S$ and $\overline{\Delta} \cap \overline{B} = \emptyset$.

By a simple computation, we get

$$\frac{1-\delta_t}{1+\delta_t} = \frac{t(t-1)(\xi\eta-1)}{(1-t)\xi\eta+3+t} = \frac{t(t-1)(\xi\eta-1)}{4+(t-1)(1-\xi\eta)}. \tag{3.13}$$

For small $\epsilon \in (0, \min\{1/\|\mu_\sigma\|_\infty - 1, 1\})$, let

$$\mathcal{T}_\epsilon = (-1/\|\mu_\sigma\|_\infty, 1-\epsilon) \cup (1+\epsilon, 1/\|\mu_\sigma\|_\infty) \setminus \{0\},$$

where $\|\mu_\sigma\|_\infty$ is given by (3.12). Let

$$\lambda(t) = \frac{(t-1)(\xi\eta-1)}{4+(t-1)(1-\xi\eta)}, \qquad t \in \mathcal{T}_\epsilon,$$

$$\lambda_\epsilon = \inf_{t\in\mathcal{T}_\epsilon} |\lambda(t)|.$$

By (3.12), it follows readily that

$$\lambda_\epsilon \geq \frac{\epsilon(1-\xi\eta)}{4+(1/\|\mu\|_\infty - 1)(1-\xi\eta)} = \frac{\epsilon(1-\eta)^2}{2\xi(1+\eta)} > 0.$$

Choose $\alpha \in M(\Delta)$ such that $[\alpha] \neq [0]$ in $T(\Delta)$ and $k_0([\alpha]) \leq \lambda_\epsilon$. Set

$$\mu(z) = \begin{cases} \mu_g(z), & z \in B, \\ \alpha(z), & z \in \Delta, \\ 0, & z \in S \setminus (B \cup \Delta). \end{cases}$$

Since $[\mu_g] = [0]$ in $T(B)$ and $[\alpha] \neq [0]$ in $T(\Delta)$, it follows by a simple analysis that $[\mu] \neq [0]$ in $T(S)$. Observe that $h([\mu]) = 0$. We conclude that $[\mu] \in T_0(S) \setminus \{[0]\}$. It is clear that $t\mu$ is equivalent to the following Beltrami differential ν_t:

$$\nu_t(z) = \begin{cases} \mu_{\mathscr{F}_t}(z), & z \in B, \\ t\alpha(z), & z \in \Delta, \\ 0, & z \in S \setminus (B \cup \Delta). \end{cases}$$

Since

$$\|t\alpha\|_\infty \leq |t\lambda_\epsilon| \leq |t\lambda(t)| = \left|\frac{1-\delta_t}{1+\delta_t}\right| = \|\mu_{\mathscr{F}_t}\|_\infty, \qquad t \in \mathcal{T}_\epsilon,$$

by the previous analysis we conclude that when $t \in \mathcal{T}_\epsilon$, $p \in C$ is a substantial boundary point of $[t\mu]$ and

$$h_p^*(t\mu) = h([t\mu]) = k_0([t\mu]) = \|\mu_{\mathscr{F}_t}\|_\infty = \left|\frac{1-\delta_t}{1+\delta_t}\right| > 0.$$

Thus, $[t\mu]$ is a non-trivial non-Strebel point whenever $t \in \mathcal{T}_\epsilon$. This completes the proof of Theorem 1.6.

REMARK 3.2. Since the set of Strebel points is dense in $T(S)$, the conclusion in Theorem 1.6 is sharp in a certain sense. So, we should not expect that there is a Beltrami differential $\mu \in M(S)$ such that $[\mu] \in T_0(S) \setminus \{[0]\}$ while $[t\mu]$ is a non-trivial non-Strebel point for any $t \in (-1/\|\mu\|_\infty, 1/\|\mu\|_\infty) \setminus \{0, 1\}$.

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