STUDIA MATHEMATICA 233 (1) (2016)

On the diametral dimension of weighted spaces of analytic germs

by

Michael Langenbruch (Oldenburg)

Abstract. We prove precise estimates for the diametral dimension of certain weighted spaces of germs of holomorphic functions defined on strips near $\mathbb{R}$. This implies a full isomorphic classification for these spaces including the Gelfand–Shilov spaces S^1_α and S^α_1 for $\alpha > 0$. Moreover we show that the classical spaces of Fourier hyperfunctions and of modified Fourier hyperfunctions are not isomorphic.

1. Introduction. The isomorphic classification of linear topological spaces of analysis is a classical problem which has been studied intensively for spaces of analytic functions, (ultra)differentiable functions and (ultra)distributions and for solution spaces of partial differential equations (see e.g. [8–18, 20, 21, 23, 24] and the references cited there).

In the present paper we will study this problem for certain weighted spaces $\mathcal{H}_v(\mathbb{R})$ of analytic germs defined by

$$\mathcal{H}_v(\mathbb{R}) := \Big\{ f \ \Big|\ \exists n \in \mathbb{N} : f \in \mathcal{H}(V_{1/n}) \text{ and } \sup_{z \in V_{1/n}} |f(z)| e^{v(z)/n} < \infty \Big\}.$$

Here $V_{1/n}$ denotes the strip $\{z \in \mathbb{C} \mid |\operatorname{Im} z| < 1/n\}$ near $\mathbb{R}$ and v is a weight function satisfying some mild natural conditions (see Definition 2.1). We have shown in [17] that the space $\mathcal{H}_v(\mathbb{R})$ has a basis and in this way is isomorphic to some $\Lambda_0(\alpha_n)'_b$, i.e. to the strong dual of some power series space of finite type. The results from [17] needed here are collected in Section 2.

In the present paper we will calculate the coefficient space $\Lambda_0(\alpha_n)'$ for this basis. We thus obtain an isomorphism to a concrete sequence space and a precise isomorphic classification for the spaces $\mathcal{H}_v(\mathbb{R})$. Notice that the

2010 *Mathematics Subject Classification*: Primary 46A45, 46A63; Secondary 46E10, 46F15, 30H50.
Key words and phrases: holomorphic germs, diametral dimension, isomorphic classification, Gelfand–Shilov spaces, Fourier hyperfunctions, modified Fourier hyperfunctions.
Received 6 November 2015.
Published online 5 May 2016.

DOI: 10.4064/sm8447-4-2016

test function spaces for the Fourier hyperfunctions and also certain Gelfand–Shilov spaces are of this type (see the examples below).

In fact we can prove that

$$(\alpha_n)_{n\in\mathbb{N}} \text{ is equivalent to } (n/g(n))_{n\in\mathbb{N}}$$

where g is the inverse function of $f(t) := tv(t)$ (see Theorem 4.6). The function g can be easily calculated in many situations (see Section 5 for some instructive examples).

The main tools used in the present paper are the diametral dimension introduced by Bessaga, Pełczyński and Rolewicz [1] (see Section 3) and a modified diametral dimension introduced in [14] (see Section 4). The crucial analytical results are obtained in Theorems 4.2 and 4.4. Specifically, we prove the surprising result that the auxiliary weighted space

$$\mathcal{H}_v^\infty(\mathbb{R}) := \Big\{ f \;\Big|\; \exists n \in \mathbb{N} : f \in \mathcal{H}(V_{1/n}) \text{ and } \sup_{z\in V_{1/n}} |f(z)|e^{-nv(nx)} < \infty \Big\}$$

of holomorphic germs near $\mathbb{R}$ contains the weighted space of holomorphic functions on the unit disc $\mathbb{D}$,

$$\mathcal{H}_v^\infty(\mathbb{D}) := \Big\{ f \in \mathcal{H}(\mathbb{D}) \;\Big|\; \exists n \in \mathbb{N} : \sup_{z\in\mathbb{D}} |f(z)|e^{-nv(\frac{n}{1-|z|})} < \infty \Big\},$$

as a closed subspace.

Our results also imply that the spaces $\mathcal{H}_v(\mathbb{R})$ are always stable in the sense that $\mathcal{H}_v(\mathbb{R}) \times \mathcal{H}_v(\mathbb{R})$ is isomorphic to $\mathcal{H}_v(\mathbb{R})$.

Our calculations (together with the results from [17]) show that the Gelfand–Shilov spaces S_α^1 and S_1^α (see [4]) are isomorphic to $\Lambda_0(n^{1/(\alpha+1)})_b'$ for $\alpha > 0$. Hence these spaces are pairwise non-isomorphic for different α.

The special case $\alpha = 1$ shows that the space of test functions for the Fourier hyperfunctions (see [6] and Section 5) is isomorphic to $\Lambda_0(n^{1/2})_b'$ (see [16] for a different proof via Hermite functions). Fourier hyperfunctions and modified Fourier hyperfunctions (see [7, 22]) are a modern very general frame for Fourier transformation. Since these spaces are a flabby sheaf, this allows one to use Fourier methods for general distributions and hyperfunctions without imposing any growth conditions. We prove here that the space of test functions for the modified Fourier hyperfunctions (see [7, 22] and Section 5) is isomorphic to $\Lambda_0(n/\ln(n))_b'$. Though the spaces of Fourier hyperfunctions and of modified Fourier hyperfunctions are very similar by definition, we thus show that surprisingly, they are not isomorphic, on the contrary, the space of modified Fourier hyperfunctions is minimal in the class of spaces considered here in the sense that it is isomorphic to a closed subspace of any of the spaces $\mathcal{H}_v(\mathbb{R})_b'$ considered in this paper.

2. Weighted holomorphic germs. In this section we introduce the basic notions and recall the results from [17] that we need.

In this paper v always denotes a weight function in the following sense:

Definition 2.1. A continuous function $v : \mathbb{C} \to [0, \infty[$ is called a *weight function* if $v(x + iy) := \widetilde{v}(|x|)$ on $\mathbb{C}$ where $\widetilde{v} : [0, \infty[\to [0, \infty[$ is strictly increasing and satisfies

$$\ln(1 + |x|) = o(\widetilde{v}(x)) \tag{2.1}$$

and there are $\Gamma > 1$ and $C > 0$ such that

$$\widetilde{v}(x + 1) \le \Gamma \widetilde{v}(x) + C \quad \text{for } x \ge 0. \tag{2.2}$$

Without loss of generality we will always assume that $v(0) = 0$, i.e. v is bijective on $[0, \infty[$.

The spaces of holomorphic germs considered in this paper are inductive limits of the weighted Banach spaces

$$H_\tau(V_t) := \Big\{ f \in \mathcal{H}(V_t) \;\Big|\; \|f\|_{\tau,t} := \sup_{z \in V_t} |f(z)| e^{\tau v(z)} < \infty \Big\}$$

of holomorphic functions on the strips

$$V_t := \{ z \in \mathbb{C} \mid |\operatorname{Im} z| < t \}$$

where $t > 0$ and $\tau \in \mathbb{R}$.

The following quantitative decomposition theorem has been proved in [17, Theorem 2.2]. It will be a major tool in Section 4 (see Theorem 4.2).

Decomposition Theorem 2.2. *There are $\widetilde{t}, K_1, K_2 > 0$ such that for any $\tau_0 < \tau < \tau_2$ there are $C_0 = C_0(\operatorname{sign}(\tau_0)) > 0$ and $K_0 = K_0(\operatorname{sign}(\tau)) > 0$ such that for any $0 < 2t_0 < t < t_2 < \widetilde{t}$ with*

$$t_0 \le \min \left[K_1, K_2 \sqrt{\frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0}} \right]$$

there is $C_1 \ge 1$ such that for any $r \ge 0$ and any $f \in H_\tau(V_t)$ with $\|f\|_{\tau,t} \le 1$ the following holds: There are $f_2 \in \mathcal{H}(V_{t_2})$ and $f_0 \in \mathcal{H}(V_{t_0})$ such that $f = f_0 + f_2$ on V_{t_0} and

$$\|f_0\|_{K_0\tau_0, t_0} \le C_1 e^{-Gr} \quad \text{and} \quad \|f_2\|_{\tau_2, t_2} \le e^r \tag{2.3}$$

where

$$G := K_1 \min \left[1, \frac{t - t_0}{2\widetilde{t}}, \frac{\tau - C_0 \tau_0}{\tau_2 - C_0 \tau_0} \right].$$

We are mainly interested in the weighted spaces $\mathcal{H}_v(\mathbb{R})$ of holomorphic germs defined by

$$\mathcal{H}_v(\mathbb{R}) := \operatorname*{ind}_{n \to \infty} H_{1/n}(V_{1/n}).$$

Recall that the dual of a power series space of finite type is defined as follows: For $M \subset \mathbb{Z}^j$ let $(a_k)_{k \in M}$ be a set of positive numbers such that

$a_k \to \infty$ as $|k| \to \infty$ in M. Then

$$\Lambda_0((a_k)_{k\in M})'_b := \Big\{(c_k)_{k\in M} \;\Big|\; \exists j \in \mathbb{N} : |(c_k)|_j := \sum_{k\in M} |c_k| e^{a_k/j} < \infty\Big\}.$$

If $M = \mathbb{N}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ is a sequence we also write $\Lambda_0(\alpha)'_b$ or $\Lambda_0(\alpha_n)'_b$ instead. The following is the main result of [17] (see [17, Theorem 4.4]):

THEOREM 2.3. *$\mathcal{H}_v(\mathbb{R})$ is (tamely) isomorphic to some $\Lambda_0(\alpha_n)'_b$.*

Notice that by [5, 21.5.1 and 21.5.3] the spaces $\mathcal{H}_v(\mathbb{R})$ (and therefore also $\Lambda_0(\alpha_n)$) are nuclear by our definition of weight functions (use (2.1)) and therefore

$$\ln(n) = o(\alpha_n) \tag{2.4}$$

by [19, Theorem 29.6].

Since the notion of tameness is not needed in the present paper, we omit the corresponding definitions. The interested reader is referred to [17].

3. The diametral dimension. We know by Theorem 2.3 that the Fréchet space $\mathcal{H}_v(\mathbb{R})'_b$ is isomorphic to a power series space $\Lambda_0(\alpha)$. In this section we will estimate the sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ from below using the diametral dimension (see Theorem 3.2). For the convenience of the reader we recall this classical notion (see [5, p. 209]): Let E be a vector space and let $V \subset U \subset E$ be circled subsets. Let $D_n(E)$ be the set of at most n-dimensional subspaces of E and let

$$\delta_n(V,U) := \inf\{\delta > 0 \mid \exists L \in D_n(E) : V \subset \delta U + L\}$$

be the nth diameter of V with respect to U. For a topological vector space E with a basis $\mathcal{U}$ of absolutely convex zero neighborhoods the *diametral dimension* is the set

$$\Delta(E) := \{(c_n)_{n\in\mathbb{N}} \mid \forall U \in \mathcal{U}\ \exists V \in \mathcal{U}, V \subset U : c_n\delta_n(V,U) \to 0\}.$$

We need the following basic facts (see [5, 10.6.8 and 10.6.10]):

PROPOSITION 3.1.

(a) *Let E be a nuclear Fréchet space and let E_1 be isomorphic to a quotient of E. Then*

$$\Delta(E) \subset \Delta(E_1).$$

(b) *$\Delta(\Lambda_0(\alpha)) = \Lambda_0(\alpha)$ if $\ln(n) = o(\alpha_n)$.*

Let g be the inverse function of $f(t) := tv(t)$, which exists since v is bijective on $[0,\infty[$ by our general assumption. Then

$$v(g(t)) = t/g(t) = (\tau v^{-1}(\tau))^{-1}(t) \quad \text{for } t \geq 0. \tag{3.1}$$

Indeed, the first equality holds since $g(t)v(g(t)) = t$ by the definition of g, and the second equality follows from

$$\frac{t}{g(t)}v^{-1}\left(\frac{t}{g(t)}\right) = \frac{t}{g(t)}v^{-1}(v(g(t))) = t.$$

Notice that the function $t/g(t)$ is increasing and unbounded on $[0,\infty[$ by (3.1).

THEOREM 3.2.

(a) $(\Lambda_0((k+v(j))_{(k,j)\in\mathbb{N}_0\times\mathbb{Z}}))'_b$ *contains* $\mathcal{H}_v(\mathbb{R})$ *as a closed subspace.*

(b) *There is* $C > 0$ *such that* $n/g(n) \le C\alpha_n$ *for large* n.

Proof. (a) Let $K_n := \{x + z \mid x \in [-1,1],\ |z| < 1/n\}$ and

$$\mathcal{BH}(K_n) := \{f \in \mathcal{H}(K_n) \mid f \text{ is bounded on } K_n\}$$

endowed with the sup-norm $\|\!|\ \|\!|_n$ on K_n. Then there is $C > 0$ such that $H_{1/n}(V_{1/n})$ is continuously embedded in

$$E_{Cn} := \Big\{(f_j) \in \mathcal{BH}(K_n)^{\mathbb{Z}} \ \Big|\ \sup_{j\in\mathbb{Z}} \|\!|f_j\|\!|_{Cn} e^{v(j)/(Cn)} < \infty\Big\}$$

for any $n \in \mathbb{N}$ via the mapping

$$J(f) := (f_j)_{j\in\mathbb{Z}} := (f|_{j+K_{Cn}})_{j\in\mathbb{Z}}$$

(use (2.2)). Hence

$$J : \mathcal{H}_v(\mathbb{R}) = \operatorname*{ind}_{n\to\infty} H_{1/n}(V_{1/n}) \to E := \operatorname*{ind}_{n\to\infty} E_n$$

is continuous. If $J(M)$ is bounded in E then $J(M)$ is contained and bounded in E_n for some n since the spectrum $\{E_n \mid n \in \mathbb{N}\}$ is compact. As above, we conclude that M is contained and bounded in $H_{1/(Cn)}(V_{1/(Cn)})$ for some C independent of n, hence M is bounded in $\mathcal{H}_v(\mathbb{R})$. Since also the spectrum $\{H_{1/n}(V_{1/n}) \mid n \in \mathbb{N}\}$ is compact, the Baernstein Lemma [19, 26.26] shows that $\mathcal{H}_v(\mathbb{R})$ is (topologically isomorphic via J to) a closed subspace of E. Since it is well known that $\mathcal{H}([-1,1]) = \operatorname{ind}_{n\to\infty}\mathcal{BH}(K_n)$ is isomorphic to $\Lambda_0(k)'_b = \operatorname{ind}_{n\to\infty}\{(c_k) \mid \sum_{k\in\mathbb{N}_0}|c_k|e^{k/n} < \infty\}$ (see e.g. [2, Example 4.1(3)]), this implies (a).

(b) (i) We will need an estimate from below for the increasing rearrangement $(\beta_n)_{n\in\mathbb{N}}$ of $(k+v(j))_{(k,j)\in\mathbb{N}_0\times\mathbb{Z}}$. Notice that

$$h(t) := \sharp\{(k,j) \in \mathbb{N}_0\times\mathbb{Z} \mid k + v(|j|) \le t\} \ge n \quad \text{if } \beta_n = t. \tag{3.2}$$

We first estimate $h(t)$ from above. Let $[\ \ ]$ denote the Gauss bracket. Then

$$\begin{aligned} h(t) &= \sum_{|j|\le[v^{-1}(t)]} \sharp\{k\in\mathbb{N}_0 \mid k \le [t - v(|j|)]\} = \sum_{|j|\le[v^{-1}(t)]} ([t - v(|j|)] + 1) \\ &\le \sum_{|j|\le[v^{-1}(t)]} (t+1) \le (2v^{-1}(t)+1)(t+1) \le 3tv^{-1}(t) \quad \text{for large } t, \end{aligned}$$

that is,

$$n \le 3\beta_n v^{-1}(\beta_n) \quad \text{for large } n. \tag{3.3}$$

Since g is increasing, applying the inverse function in (3.3) we get

$$\frac{n}{3g(n)} \le \frac{n/3}{g(n/3)} = (3\tau v^{-1}(\tau))^{-1}(n) \le \beta_n \quad \text{for large } n \tag{3.4}$$

by (3.1) and (3.3).

(ii) By (a), $\mathcal{H}_v(\mathbb{R})'_b$ is isomorphic to a quotient of $\Lambda_0(\beta_n)$. Hence, Theorem 2.3 and Proposition 3.1 imply that

$$\Lambda_0(\beta_n) = \Delta(\Lambda_0(\beta_n)) \subset \Delta(\Lambda_0(\alpha_n)) = \Lambda_0(\alpha_n).$$

Notice that $\ln(n) = o(\alpha_n)$ by (2.4) and $\ln(n) = o(\beta_n)$. The latter is seen as follows. By (2.1) we have $\ln(t) \le \varepsilon v(t)$ for large t; taking inverses we get $v^{-1}(t) \le e^{\varepsilon t}$, hence $3tv^{-1}(t) \le e^{2\varepsilon t}$ for large t. Taking inverses again yields the claim by (3.4).

The inclusion $\Lambda_0(\beta_n) \hookrightarrow \Lambda_0(\alpha_n)$ is continuous by the closed graph theorem, hence there is $C > 0$ such that

$$n/(3g(n)) \le \beta_n \le C\alpha_n \quad \text{for large } n$$

by (3.4). ■

4. The generalized diametral dimension. To obtain an upper estimate for the sequence (α_n) from Theorem 2.3 we need a more sophisticated variant of Proposition 3.1 based on the results of [14] relating the calculation of a generalized diametral dimension to a $(\underline{\mathrm{DN}})$-type property for a pair of locally convex spaces which was introduced in [14, (1.1)] as follows:

Let X and $\widetilde{X}$ be locally convex spaces with bases $\mathcal{U}, \widetilde{\mathcal{U}}$ of absolutely convex closed 0-neighborhoods and corresponding seminorms p_U and $p_{\widetilde{U}}$ for $U \in \mathcal{U}$ and $\widetilde{U} \in \widetilde{\mathcal{U}}$, respectively. Let $d : X \to \widetilde{X}$ be linear and continuous. Then the triple $(d, X, \widetilde{X})$ has *property* $(\underline{\mathrm{DN}})$ if there is $\widetilde{U} \in \widetilde{\mathcal{U}}$ such that for any $U \in \mathcal{U}$ there are $V \in \mathcal{U}, 0 < \lambda < 1$ and $C > 0$ such that

$$p_U(x) \le C p_{\widetilde{U}}(d(x))^{\lambda} p_V(x)^{1-\lambda} \quad \text{for any } x \in X. \tag{4.1}$$

Switching to polars we see that property $(\underline{\mathrm{DN}})$ for $(d, X, \widetilde{X})$ is equivalent to a decomposition with bounds in X' as follows (see [14, (1.2)]): $(d, X, \widetilde{X})$ has property $(\underline{\mathrm{DN}})$ if and only if there is $\widetilde{U} \in \widetilde{\mathcal{U}}$ such that for any $U \in \mathcal{U}$ there are $V \in \mathcal{U}$ and $\delta, C > 0$ such that for any $\xi > 0$,

$$U^0 \subset \xi\, {}^t d(\widetilde{U}^0) + C(1/\xi)^{\delta} V^0 \tag{4.2}$$

where $(\ )^0$ denotes the polars in the respective dual spaces.

Using property ($\underline{\mathrm{DN}}$) we can estimate the following *generalized diametral dimension* introduced in [14, 1.2]:

$$\widetilde{\Delta}(E) := \{(c_n) \mid \forall U \in \mathcal{U}\ \exists V \in \mathcal{U}, V \subset U\ \exists C > 0 : c_n \delta_n(V,U)^C \to 0\}.$$

Notice that

$$\widetilde{\Delta}(\Lambda_\infty(\beta)) = \Lambda_\infty(\beta)' \tag{4.3}$$

if $\ln(n) = o(\beta_n)$ (cf. [5, pp. 211–212]).

The following result from [14] is a refined substitute for Proposition 3.1 (see [14, Lemma 1.3b)] with $d_1 := d$, $d_2 := \mathrm{id}$ and $Y := \widetilde{X}$):

Proposition 4.1. *Let X and $\widetilde{X}$ be nuclear locally convex spaces such that $(d, X, \widetilde{X})$ has property* ($\underline{\mathrm{DN}}$). *Then*

$$\Delta(\widetilde{X}) \subset \widetilde{\Delta}(X).$$

The above setting will be used for

$$\widetilde{X} := \mathcal{H}_v(\mathbb{R})'_b \quad \text{and} \quad X := \mathcal{H}_v^\infty(\mathbb{R})'_b$$

where $\mathcal{H}_v^\infty(\mathbb{R})$ is an auxiliary weighted space of holomorphic germs given by

$$\mathcal{H}_v^\infty(\mathbb{R}) := \operatorname*{ind}_{n\to\infty} H_n$$

with

$$H_n := \Big\{ f \in \mathcal{H}(V_{1/n}) \ \Big|\ |f|_n := \sup_{z\in V_{1/n}} |f(z)| e^{-nv(nz)} < \infty \Big\},$$

which is much easier to handle than $\mathcal{H}_v(\mathbb{R})$.

Theorem 4.2. *Let $d : \mathcal{H}_v^\infty(\mathbb{R})'_b \to \mathcal{H}_v(\mathbb{R})'_b$ be the transpose of the natural inclusion $\mathcal{H}_v(\mathbb{R}) \hookrightarrow \mathcal{H}_v^\infty(\mathbb{R})$. Then $(d, \mathcal{H}_v^\infty(\mathbb{R})'_b, \mathcal{H}_v(\mathbb{R})'_b)$ has* ($\underline{\mathrm{DN}}$).

Proof. Since the spaces involved are reflexive, ${}^t d : \mathcal{H}_v(\mathbb{R}) \to \mathcal{H}_v^\infty(\mathbb{R})$ is the identity map. We therefore have to prove the following by (4.2): There is $\gamma > 0$ such that for any $n \in \mathbb{N}$ there are $k \in \mathbb{N}$ and $C_j, \delta > 0$ such that for any $f \in H_n$ with $|f|_n \le 1$ and any $r > 0$ there are $f_0 \in H_k$ and $f_2 \in H_\gamma(V_\gamma)$ such that $f = f_0 + f_2$ near $\mathbb{R}$ and

$$|f_0|_k \le C_1 e^{-\delta r} \quad \text{and} \quad \|f_2\|^v_{\gamma,\gamma} \le C_2 e^r. \tag{4.4}$$

This is proved by applying the Decomposition Theorem 2.2 twice:

(a) Given $f \in H_n$ with $|f|_n \le 1$ we set $F(z) := f(4z/(\widetilde{t}n))$ where $\widetilde{t}$ is taken from Theorem 2.2 for the weight v. Then $\|F\|^w_{-1,\widetilde{t}/4} \le 1$ for the weight function $w(z) := nv(z)$ (instead of v). Since the constants $\widetilde{t}$ for the weights v and w coincide (see the calculation of $\widetilde{t}$ in the proofs of [17, Theorem 2.2 and Lemma 2.4]) we can apply Theorem 2.2 (for the weight w and for $t_2 = \widetilde{t}/2$, $t = \widetilde{t}/4$, $t_0 = 1/j$ and $\tau = -1$, $\tau_0 = -j$, $\tau_2 = 1$ for sufficiently large j fixed according to the general estimates needed in Theorem 2.2). We thus get F_ℓ such that $F = F_0 + F_2$ near $\mathbb{R}$ and

$$\|F_0\|^w_{-K_0 j,1/j} \le C_1 e^{-Gr} \quad \text{and} \quad \|F_2\|^w_{1,\widetilde{t}/2} \le e^r.$$

We set $\widetilde{f}_\ell(z) := F_\ell(n\widetilde{t}z/4)$ and notice that $f = \widetilde{f}_0 + \widetilde{f}_2$ near $\mathbb{R}$, and for large n and $k \ge jn\max\{K_0, \widetilde{t}\}$,

$$|\widetilde{f}_0|_k \le \sup_{z\in V_{4/(jn\widetilde{t})}} |\widetilde{f}_0(z)|e^{-K_0 jnv(\mathrm{Re}(n\widetilde{t}z/4))} = \|F_0\|^w_{-K_0 j,1/j} \le C_1 e^{-Gr}$$

and

$$\|\widetilde{f}_2\|^v_{1,1/n} \le \sup_{z\in V_{2/n}} |\widetilde{f}_2(z)|e^{nv(\mathrm{Re}(n\widetilde{t}z/4)} = \|F_2\|^w_{1,\widetilde{t}/2} \le e^r. \tag{4.5}$$

(b) Since the domain of $\widetilde{f}_2$ is too small and depends on n, we apply Theorem 2.2 a second time, now for the weight v and the following choices. We take $f := \widetilde{f}_2 e^{-r}$ and notice that $\|f\|^v_{1,1/n} \le 1$ by (4.5). Set $\tau := 1$, $\tau_2 := 2$ and $t := 1/n$, $t_2 := \widetilde{t}/2$. For large n we can then choose $t_0, \tau_0 > 0$ sufficiently small to satisfy the general estimates in Theorem 2.2, so that we get holomorphic functions $\widetilde{f}_{2,l}$, $l = 0, 2$, such that $\widetilde{f}_2 e^{-r} = \widetilde{f}_{2,0} + \widetilde{f}_{2,2}$ near $\mathbb{R}$ and

$$\|\widetilde{f}_{2,0}\|^v_{K_0\tau_0,t_0} \le C_1 e^{-2r} \quad \text{and} \quad \|\widetilde{f}_{2,2}\|^v_{\widetilde{t}/2,\widetilde{t}/2} \le C_1 e^{2r/G}$$

since without loss of generality $\widetilde{t}/2 \le \tau_2 = 2$. Notice that we have applied Theorem 2.2 for $2r/G$ instead of r. We now set $f_{2,l} := \widetilde{f}_{2,l} e^r$ and get $\widetilde{f}_2 = f_{2,0} + f_{2,2}$ near $\mathbb{R}$ and also (if $k \ge 1/t_0$)

$$|f_{2,0}|_k \le \|f_{2,0}\|^v_{K_0\tau_0,t_0} \le C_1 e^{-r} \quad \text{and} \quad \|f_{2,2}\|^v_{\widetilde{t}/2,\widetilde{t}/2} \le C_1 e^{(1+2/G)r}.$$

The final decomposition of f is then given by $f_0 := \widetilde{f}_0 + f_{2,0}$ and $f_2 := f_{2,2}$, and the theorem follows by rescaling r. ∎

We now obtain an implicit estimate for (α_n):

Corollary 4.3. $\Lambda_0(\alpha) = \Delta(\mathcal{H}_v(\mathbb{R})'_b) \subset \widetilde{\Delta}(\mathcal{H}^\infty_v(\mathbb{R})'_b)$.

Proof. Apply Theorems 2.3 and 4.2, the estimate (2.4) and Propositions 3.1(b) and 4.1. ∎

Similarly to [5, Proposition 10.6.8] we have

$$\widetilde{\Delta}(E) \subset \widetilde{\Delta}(E_1) \tag{4.6}$$

if E_1 is isomorphic to a quotient of E.

To turn the estimate in Corollary 4.3 into an explicit one, we need a topological subspace of $\mathcal{H}^\infty_v(\mathbb{R})$ (and hence a quotient of $\mathcal{H}^\infty_v(\mathbb{R})'_b$) whose generalized diametral dimension can be calculated. Surprisingly, a suitable subspace is provided by the following weighted space of holomorphic functions on the unit disc $\mathbb{D}$:

$$\mathcal{H}_v^\infty(\mathbb{D}) := \Big\{ f \in \mathcal{H}(\mathbb{D}) \;\Big|\; \exists n \in \mathbb{N} : |||f|||_n := \sup_{z\in\mathbb{D}} |f(z)| e^{-nv(\frac{n}{1-|z|})} < \infty \Big\}.$$

To prove this, we use a composition operator defined by an analytic spiral (cf. [3, Proposition 5.3] where a similar idea was used to show that $\mathcal{H}(\mathbb{D})$ is a closed subspace of the space $\mathcal{A}(\mathbb{R})$ of real analytic functions on $\mathbb{R}$).

THEOREM 4.4. *$\mathcal{H}_v^\infty(\mathbb{D})$ is topologically isomorphic to a closed subspace of $\mathcal{H}_v^\infty(\mathbb{R})$.*

Proof. The embedding of $\mathcal{H}_v^\infty(\mathbb{D})$ into $\mathcal{H}_v^\infty(\mathbb{R})$ will be provided by the composition operator

$$T_\psi : \mathcal{H}_v^\infty(\mathbb{D}) \to \mathcal{H}_v^\infty(\mathbb{R}), \quad f \mapsto f \circ \psi,$$

defined by the function

$$\psi(z) := (z^2 + C)^{i/2} \bigg(1 + \frac{-1 + B \sin\big(\frac{1}{4} \ln(z^2 + C)\big)}{(z^2 + C)^{1/2}} \bigg)$$

where $C > 0$ is large and fixed, $B := (e^{2\pi} - 1)/(e^{2\pi} + 1) < 1$ and

$$w^d := \exp(d \ln(w)), \quad d \in \mathbb{C},$$

with $\ln(w) := \ln(|w|) + id \arg(w)$ for $|\arg(w)| < \pi$. Then $(z^2 + C)^d$ is defined for $z \in V_1$ if $C \geq 2$ since

$$\text{(4.7)} \quad \operatorname{Re}(z^2 + C) = x^2 - y^2 + C > x^2 + C - 1 \geq x^2 + 1 \quad \text{for } z = x + iy \in V_1.$$

(a) We first estimate $|\psi(z)|$ for $z = x + iy \in V_{1/C}$ as follows. Let $b := \arg(z^2 + C)$. Then

$$\text{(4.8)} \qquad |\psi(z)| \leq e^{|b|/2} \bigg(|1 - (z^2 + C)^{-1/2}| + \frac{B \big|\sin\big(\frac{1}{4} \ln(z^2 + C)\big)\big|}{|z^2 + C|^{1/2}} \bigg).$$

Notice that

$$\text{(4.9)} \qquad \frac{|b|}{2} \leq |\sin(b)| = \frac{2|xy|}{|z^2 + C|} \leq \frac{2}{C|z^2 + C|^{1/2}} \leq \frac{2}{C} \quad \text{for } z \in V_{1/C}$$

by (4.7) if $C \geq 2$. Set $\varepsilon := (1 - B)/4$. Then

$$\begin{aligned} \text{(4.10)} \quad |1 - (z^2 + C)^{-1/2}| &= \big|1 - |z^2 + C|^{-1/2} e^{-ib/2}\big| \\ &= [1 - 2\cos(b/2)|z^2 + C|^{-1/2} + |z^2 + C|^{-1}]^{1/2} \\ &\leq [1 - 2(1 - \varepsilon)|z^2 + C|^{-1/2}]^{1/2} \\ &\leq 1 - (1 - \varepsilon)|z^2 + C|^{-1/2} \quad \text{for } z \in V_{1/C} \end{aligned}$$

provided C is so large that by (4.9),

$$\cos(b/2) \geq \cos(2/C) \geq 1 - \varepsilon/2 \quad \text{and} \quad C \geq 1 + 1/\varepsilon.$$

Obviously,

$$|\sin(\xi+i\eta)| = (\sin^2(\xi)+\sinh^2(\eta))^{1/2} \le (1+\sinh^2(\eta))^{1/2} = \cosh(\eta) \le 1+\varepsilon/B \tag{4.11}$$

if $|\eta|$ is small. We apply this estimate to $\xi + i\eta := \frac{1}{4}\ln(z^2+C)$ (i.e. $\eta = b/4$) and combine it with (4.7)–(4.10). Set $B_1 := (1-B)/2 = 1/(e^{2\pi}+1)$ (and hence $1-B-2\varepsilon = B_1$). Thus by Taylor's theorem, for sufficiently large C,

$$\begin{aligned}|\psi(z)| &\le e^{2|z^2+C|^{-1/2}}(1-(1-B-2\varepsilon)|z^2+C|^{-1/2}) \\ &\le \left(1+\frac{3}{C}|z^2+C|^{-1/2}\right)(1-B_1|z^2+C|^{-1/2}) \\ &\le 1-\left(B_1-\frac{3}{C}\right)|z^2+C|^{-1/2} \\ &\le 1-\frac{B_1}{2}|z^2+C|^{-1/2} \quad \text{for } z\in V_{1/C}.\end{aligned} \tag{4.12}$$

Notice that $\psi(z)\in\mathbb{D}$ for $z\in V_{1/C}$ by (4.12). Also,

$$|\psi(z)| \le 1-\frac{B_1}{4|x|} \quad \text{for } z\in V_{1/C} \text{ and } |x|\ge C+1. \tag{4.13}$$

For $n\in\mathbb{N}$ we choose $k\ge n(1+4/B_1)+C$ and get, by (4.13),

$$\begin{aligned}\|T_\psi(f)\|_k &= \sup_{z\in V_{1/k}} |f(\psi(z))|e^{-kv(kx)} \\ &\le \sup_{w\in\mathbb{D}} |f(w)|e^{-nv(n/(1-|w|))} \sup_{z\in V_{1/k}} e^{nv(n/(1-|\psi(z)|))-kv(kx)} \\ &\le C_1|||f|||_n \sup_{x\in\mathbb{R}} e^{nv(4nx/B_1)-kv(kx)} \le C_1|||f|||_n.\end{aligned}$$

T_ψ is thus defined and continuous; it is injective by the identity theorem for analytic functions.

(b) We now show that the range of T_ψ is closed in $\mathcal{H}_v^\infty(\mathbb{R})$ (and explain the special choice of B). We define $x_k>0$ for $k\in\mathbb{N}$ by $\ln(x_k^2+C)=2\pi k$ and notice that

$$\psi(x_k) = (-1)^k[1+e^{-\pi k}(-1+B\sin(\pi k/2))]$$

and hence

$$\begin{aligned}\psi(x_{4k+1}) &= -1+e^{-(4k+1)\pi}(1-B) \\ &= -1+e^{-(4k+3)\pi}(B+1) = \psi(x_{4k+3})\end{aligned} \tag{4.14}$$

since $B=(e^{2\pi}-1)/(e^{2\pi}+1)$ by definition. For $x\in[x_{4k+1},x_{4k+3}]$ we have

$$|\psi(x)| \ge 1-e^{-(4k+1)\pi}(B+1) = 1-e^{-4k\pi}/\cosh(\pi). \tag{4.15}$$

Let $1-e^{-4k\pi}/\cosh(\pi) \ge |z| \ge 1-e^{-4\pi(k-1)}/\cosh(\pi)$. Then

$$x_{4k+3} \le C_2e^{4\pi(k-1)} \le C_3/(1-|z|),$$

and by the maximum principle and also (4.14) and (4.15) we get

$$\begin{aligned}|f(z)| &\le \sup_{x\in[x_{4k+1},x_{4k+3}]} |f(\psi(x))| \\ &\le e^{nv(nx_{4k+3})} \sup_{x\in[x_{4k+1},x_{4k+3}]} |f(\psi(x))|e^{-nv(nx)} \\ &\le e^{C_3 nv(C_3 n/(1-|z|))}\|T_\psi(f)\|_n.\end{aligned}$$

Hence range(T_ψ) is closed in $\mathcal{H}_v^\infty(\mathbb{R})$ and thus a (DFS)-space. So T_ψ is open by the open mapping theorem. The theorem is proved. ∎

PROPOSITION 4.5. *$\mathcal{H}_v^\infty(\mathbb{D})$ is isomorphic to $\Lambda_\infty(n/g(n))_b'$ where g is the inverse function of $f(t) := tv(t)$.*

Proof. (a) For $f \in \mathcal{H}_v^\infty(\mathbb{D})$ with $|||f|||_n \le 1$, by Cauchy's estimate for $|z| := 1 - R$ we get

$$\begin{aligned}\frac{|f^{(j)}(0)|}{j!} &\le \inf_{1>R>0} \frac{e^{nv(n/R)}}{(1-R)^j} \le \inf_{1/2\ge R>0} e^{nv(n/R)+2jR} \\ &\le e^{nv(g(j/n))+2n^2(j/n)/g(j/n)} = e^{(n+2n^2)v(g(j/n))} \\ &\le e^{(n+2n^2)v(g(j))} = e^{(n+2n^2)(j/g(j))} \quad \text{for large } j\end{aligned}$$

where we have chosen $1/R := g(j/n)/n$ to show the first inequality in the second line and used (3.1) several times.

(b) Conversely, if $(c_j) \in \Lambda_\infty(n/g(n))'$ then for $f(z) := \sum_{n=0}^\infty c_j z^j$, $|z| =: R < 1$ and $R' := (1+R)/2$ we get

$$\begin{aligned}|f(z)| &\le \sum_{n=0}^\infty e^{nj/g(j)} R^j \le \sum_{n=0}^\infty (R/R')^j \sup_{j\in\mathbb{N}_0} e^{nv(g(j))+j\ln(R')} \\ &\le \frac{2}{1-R} \sup_{j\in\mathbb{N}_0} e^{nv(g(j))-j(1-R')} = \frac{2}{1-R} \sup_{j\in\mathbb{N}_0} e^{nv(g(j))-j(1-R)/2}\end{aligned}$$

since $1 - R' = (1-R)/2$. We want to show that for large j,

$$nv(g(j)) - j(1-R)/2 \le 2nv(2n/(1-R)) \quad \text{for any } 0 < R < 1. \tag{4.16}$$

Set $t := 1 - R$ and $l := g(j)$, i.e. $j = g^{-1}(l) = lv(l)$. So (4.16) is equivalent to

$$nv(l) \le 2nv(2n/t) + ltv(l)/2 \quad \text{for any } 0 < t < 1 \tag{4.17}$$

for large l. Now, (4.17) clearly holds if $n \le lt/2$. On the other hand, if $n \ge lt/2$ and hence $l \le 2n/t$, (4.17) is also trivial since then $v(2n/t) \ge v(l)$. Summarizing, we have shown that

$$|f(z)| \le \frac{C_4}{1-|z|} e^{2nv(2n/(1-|z|))} \le C_5 e^{3nv(3n/(1-|z|))}$$

by (2.1). ∎

Finally, we can prove the main result of this paper:

THEOREM 4.6. *$\mathcal{H}_v(\mathbb{R})$ is (tamely) isomorphic to $\Lambda_0(n/g(n))'_b$ where $g(t)$ is the inverse function of $f(t) := tv(t)$.*

Proof. By Theorems 2.3 and 3.2(b) we need to show that there is $C > 0$ such that

$$\alpha_n \le Cn/g(n) \quad \text{for large } n \in \mathbb{N}. \tag{4.18}$$

By Corollary 4.3, (4.6), Theorem 4.4, Proposition 4.5 and (4.3) we have

$$\Lambda_0(\alpha_n) \subset \widetilde{\Delta}(\mathcal{H}_v^\infty(\mathbb{R})'_b) \subset \widetilde{\Delta}(\mathcal{H}_v^\infty(\mathbb{D})'_b) = \widetilde{\Delta}(\Lambda_\infty(n/g(n))) = \Lambda_\infty(n/g(n))'.$$

The inclusion $\Lambda_0(\alpha_n) \hookrightarrow \Lambda_\infty(n/g(n))'_b$ is continuous by the closed graph theorem. Grothendieck's factorization theorem (see [19, 24.33]) implies that $\Lambda_0(\alpha_n)$ is continuously embedded in a step space of $\Lambda_\infty(n/g(n))'_b$. This shows (4.18). ∎

COROLLARY 4.7. *$\mathcal{H}_v(\mathbb{R})$ is stable, i.e. $\mathcal{H}_v(\mathbb{R}) \times \mathcal{H}_v(\mathbb{R})$ is isomorphic to $\mathcal{H}_v(\mathbb{R})$.*

Proof. We have to prove by Theorem 4.6 that $\Lambda_0(n/g(n))$ is stable. This is obvious since $\beta_n := n/g(n)$ is increasing by (3.1) and

$$\beta_{2n}/2 = \frac{n}{g(2n)} \le \frac{n}{g(n)} = \beta_n$$

since g is increasing. ∎

5. Examples. We will discuss several examples of weighted spaces satisfying the assumptions of this paper. Specifically, we will calculate the crucial sequence $(n/g(n))$ up to equivalence.

We start with some canonical examples of weight functions (see [17, Section 5] for the easy calculations).

EXAMPLE 5.1. The following functions v are weight functions:

(i) $v(x) := v_{\alpha,\beta}(x) := (\ln(x))^\alpha(\ln(\ln(x)))^\beta$ for $x \ge x_0$ where $\alpha > 1$ and $\beta \in \mathbb{R}$ or $\alpha = 1$ and $\beta > 0$.
(ii) $v(x) := e^{v_{\alpha,\beta}(x)}$ for $x \ge x_0$ where $\alpha > 0$ and $\beta \in \mathbb{R}$.
(iii) $v(x) := v_{\alpha,\beta}(e^x) = x^\alpha(\ln(x))^\beta$ for $x \ge x_0$ where $\alpha > 0$ and $\beta \in \mathbb{R}$.
(iv) $v(x) := e^{ax^\alpha(\ln(x))^\beta}$ where $a > 0$ and $1 > \alpha > 0$ and $\beta \in \mathbb{R}$ or $\alpha = 1$ and $\beta \le 0$.

Of course, products of weight functions from Example 5.1 are also weight functions.

Two sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are said to be *equivalent* if there is $C > 1$ such that

$$\alpha_n/C \le \beta_n \le C\alpha_n \quad \text{for large } n.$$

Notice that $\Lambda_0(\alpha_n) = \Lambda_0(\beta_n)$ if $(\alpha_n)_{n\in\mathbb{N}}$ is equivalent to $(\beta_n)_{n\in\mathbb{N}}$. Hence we only need to calculate the sequence $(n/g(n))_{n\in\mathbb{N}}$ from Theorem 4.6 up to equivalence. We start with an easy observation:

LEMMA 5.2.

(a) *Let g_v be the inverse function of $f(t) := tv(t)$. Then*

$$g_v(t)g_{v^{-1}}(t) = t.$$

(b) *Let $v(x) := e^{w(\ln(x))}$ where $w \in C^1([0,\infty[)$ is positive, increasing and unbounded.*

(i) *$(n/g(n))_{n\in\mathbb{N}}$ is equivalent to $(v(n))_{n\in\mathbb{N}}$ if there is $C > 0$ such that*

$$w'(t)w(t) \le C \quad \text{for large } t. \tag{5.1}$$

(ii) *$(n/g(n))_{n\in\mathbb{N}}$ is equivalent to $(n/v^{-1}(n))_{n\in\mathbb{N}}$ if there is $C > 0$ such that*

$$t \le Cw'(t) \quad \text{for large } t. \tag{5.2}$$

Proof. (a) By (3.1) we have

$$t/g_v(t) = (\tau v^{-1}(\tau))^{-1}(t) = g_{v^{-1}}(t).$$

(b)(i) There is $C_1 \ge 1$ such that

$$\begin{aligned} v(t)/C_1 \le t/g(t) = t/(tv(t))^{-1} \le C_1 v(t) \text{ for large } t \\ \Leftrightarrow\ v(\tau v(\tau))/C_1 \le v(\tau) \le C_1 v(\tau v(\tau)) \text{ for large } \tau \\ \Leftrightarrow\ w(\xi + w(\xi)) \le \ln C_1 + w(\xi) \text{ for large } \xi. \end{aligned}$$

The latter estimate clearly holds since Taylor's theorem yields, for some $\eta \in [0, w(\xi)]$,

$$w(\xi + w(\xi)) \le w(\xi) + w(\xi)w'(\xi+\eta) \le w(\xi) + w(\xi+\eta)w'(\xi+\eta) \le w(\xi) + C$$

because w is positive and increasing and satisfies (5.1). This proves the claim.

(b)(ii) The function $v^{-1} = \exp \circ w^{-1} \circ \ln$ satisfies (5.1) (for w^{-1} instead of w) by (5.2). By a) we have $n/g_v(n) = n/(n/g_{v^{-1}}(n))$, which is equivalent to $n/v^{-1}(n)$ by (b)(i). ■

EXAMPLE 5.3. $(n/g(n))_{n\in\mathbb{N}}$ is equivalent to

(i) $(v(n))_{n\in\mathbb{N}}$ if $v(x) := v_{\alpha,\beta}(x) := (\ln(x))^\alpha(\ln(\ln(x)))^\beta$ for $x \ge x_0$ where $\alpha > 1$ and $\beta \in \mathbb{R}$ or $\alpha = 1$ and $\beta > 0$;
(ii) $(v(n))_{n\in\mathbb{N}}$ if $v(x) := e^{v_{\alpha,\beta}(x)}$ for $x \ge x_0$ where $1/2 > \alpha > 0$ and $\beta \in \mathbb{R}$ or $\alpha = 1/2$ and $\beta \le 0$;
(iii) $(ne^{-(\ln(n))^{1/\alpha}})_{n\in\mathbb{N}}$ if $v(x) := e^{(\ln(x))^\alpha}$ where $\alpha \ge 2$;
(iv) $(n^{\alpha/(\alpha+1)})_{n\in\mathbb{N}}$ if $v(x) := x^\alpha$ where $\alpha > 0$;
(v) $(n(\ln(n))^{-1/\alpha})_{n\in\mathbb{N}}$ if $v(x) := e^{ax^\alpha}$ where $1 \ge \alpha > 0$ and $a > 0$.

Proof. (a) We apply Lemma 5.2(b)(i) for $w(t) := \alpha \ln(t) + \beta \ln(\ln(t))$ in (i) and $w(t) := t^\alpha (\ln(t))^\beta$ in (ii), respectively. (5.1) is easily shown for α and β as specified.

(b) We apply Lemma 5.2(b)(ii) for $w(t) := t^\alpha$ in (iii) and $w(t) := ae^{\alpha t}$ in (v). (5.2) is easily shown for α as specified. Calculating v^{-1} in each case we get the desired formulas.

(c) We have $f(t) = tv(t) = t^{1+\alpha}$ in (iv) and therefore $g(t) = t^{1/(\alpha+1)}$ and $t/g(t) = t^{1-1/(\alpha+1)} = t^{\alpha/(\alpha+1)}$. ∎

To treat the weights $v(x) := e^{(\ln(x))^\alpha}$ for $1/2 < \alpha < 1$ we would have to take into account more and more terms in the Taylor expansion of $(\xi+\xi^\beta)^{1/\beta}$. The details are left to the reader.

The spaces S_α^1 of Gelfand–Shilov satisfy the assumptions of this paper. Recall that S_α^β is defined as follows (for $\alpha, \beta > 0$, see [4, Chap. IV]):

$$S_\alpha^\beta := \{f \in C^\infty(\mathbb{R}) \mid \exists A, B > 0\ \forall k, j \in \mathbb{N}_0 : |x^k f^{(j)}(x)| \le CA^k k^{k\alpha} B^j j^{j\beta}\}.$$

We thus get:

Example 5.4.

(a) S_α^1 is (tamely) isomorphic to $\Lambda_0(n^{1/(\alpha+1)})_b'$ for $\alpha > 0$.

(b) S_1^β is (tamely) isomorphic to $\Lambda_0(n^{1/(\beta+1)})_b'$ for $\beta > 0$.

Proof. (a) By [4, Chap. IV, Sect. 2], S_α^1 is (tamely) isomorphic to $\mathcal{H}_v(\mathbb{R})$ for the weight $v(x) := |x|^{1/\alpha}$. The claim thus follows from Example 5.3(iv) and Theorem 4.6.

(b) This follows from (a) since the Fourier transform is a tame isomorphism between S_1^β and S_β^1 by [4, Chap. IV, Sect. 6.2, formula (11)]. ∎

In particular, we recover the result proved in [16], that the space $P_*(\mathbb{R})_b'$ of Fourier hyperfunctions on $\mathbb{R}$ is (tamely) isomorphic to $\Lambda_0(n^{1/2})_b'$. Indeed, the space $P_*(\mathbb{R})$ of test functions for Fourier hyperfunctions is by definition (see [6])

$$P_*(\mathbb{R}) := \Big\{ f \;\Big|\; \exists n \in \mathbb{N} : f \in \mathcal{H}(V_{1/n}) \text{ and } \sup_{z \in V_{1/n}} |f(z)| e^{|\mathrm{Re}\, z|/n} < \infty \Big\},$$

that is, $P_*(\mathbb{R}) = S_1^1$ in the notation of Gelfand–Shilov. So this is the special case $\alpha = 1$ of Example 5.4(a).

The space $\widetilde{\mathcal{P}}(\mathbb{R})_b'$ of modified Fourier hyperfunctions is defined very similarly (see [7, 22]). We just use the conical neighborhoods

$$W_{1/n} := \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < (1 + |\mathrm{Re}\, z|)/n\}$$

of $\mathbb{R}$ instead of $V_{1/n}$ in the definition of the test function space $\widetilde{\mathcal{P}}(\mathbb{R})$, i.e.

$$\widetilde{\mathcal{P}}(\mathbb{R}) := \Big\{ f \;\Big|\; \exists n \in \mathbb{N} : f \in \mathcal{H}(W_{1/n}) \text{ and } \sup_{z \in W_{1/n}} |f(z)| e^{|\mathrm{Re}\, z|/n} < \infty \Big\}.$$

Notice that the weights are not changed. We now have the following unexpected result:

EXAMPLE 5.5. The space $\widetilde{\mathcal{P}}(\mathbb{R})'_b$ of modified Fourier hyperfunctions is isomorphic to $\Lambda_0(n/\ln(n))$.

Proof. $\widetilde{\mathcal{P}}(\mathbb{R})$ is isomorphic to $\mathcal{H}_v(\mathbb{R})$ for $v(x) := \exp(|x|)$ (see [17, Section 6]). The claim thus follows from Example 5.3(v) for $a = \alpha = 1$. ∎

Specifically, the spaces $P_*(\mathbb{R})'_b$ of Fourier hyperfunctions and $\widetilde{\mathcal{P}}(\mathbb{R})'_b$ of modified Fourier hyperfunctions are not isomorphic.

PROPOSITION 5.6. *$\widetilde{\mathcal{P}}(\mathbb{R})'_b$ is isomorphic to a closed subspace of any of the spaces $\mathcal{H}_v(\mathbb{R})'_b$.*

Proof. $\mathcal{H}_v(\mathbb{R})'_b$ isomorphic to $\Lambda_0(n/g(n))$ by Theorem 4.6. Moreover, $\Lambda_0(n/g(n))$ is nuclear by (2.4) and stable by Corollary 4.7. Hence we can apply the list of [23, p. 296], that is, we have to show that $\alpha_n = n/g(n) \le n/\ln(n)$, i.e. $\ln(n) \le Cg(n) = C(tv(t))^{-1}(n)$ for some C and large n. Taking inverses on both sides we have to show that $tv(t) \le e^{Ct}$ for some C and large t. The latter estimate now easily follows from (2.2). ∎

By [4, Chap. IV, Sect. 2.3], S_1^β can be identified for $0 < \beta < 1$ with the weighted space of entire functions

$$\mathcal{H}_{1,1/(1-\beta)} := \Big\{ f \in H(\mathbb{C}) \;\Big|\; \exists n \in \mathbb{N} : |f|_n := \sup_{z\in\mathbb{C}} |f(z)| e^{|\operatorname{Re} z|/n - n|\operatorname{Im} z|^{1/(1-\beta)}} < \infty \Big\}.$$

From Example 5.4(b) we thus get:

COROLLARY 5.7. *$\mathcal{H}_{1,1/(1-\beta)}$ is isomorphic to $\Lambda_0(n^{1/(\beta+1)})'_b$ for $1 > \beta > 0$.*

References

[1] C. Bessaga, A. Pełczyński and S. Rolewicz, *On diametral approximative dimension and linear homogeneity of F-spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 9 (1961), 677–683.

[2] P. Domański, *Notes on real analytic functions and classical operators*, in: Topics in Complex Analysis and Operator Theory (Valencia, 2010), O. Blasco et al. (eds.), Contemp. Math. 561, Amer. Math. Soc., 2012, 3–47.

[3] P. Domański and M. Langenbruch, *Composition operators on spaces of real analytic functions*, Math. Nachr. 254–255 (2003), 68–86.

[4] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, Vol. 2, Academic Press, New York, 1968.

[5] H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.

[6] A. Kaneko, *Introduction to Hyperfunctions*, Kluwer, Dordrecht, 1988.

[7] T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo Sect. IA 17 (1970), 467–517.

[8] A. N. Kolmogorov, *On the linear dimension of topological vector spaces*, Dokl. Akad. Nauk SSSR 120 (1958), 239–241 (in Russian).

[9] A. N. Kolmogorov and B. M. Tikhomirov, *ε-entropy and ε-capacity of sets in function spaces*, Amer. Math. Soc. Transl. (2) 17, Amer. Math. Soc., 1961, 227–364.

[10] M. Langenbruch, *On the functional dimension of solution spaces of hypoelliptic partial differential operators*, Math. Ann. 272 (1985), 201–223.

[11] M. Langenbruch, *Kolmogorov diameters in solution spaces of systems of partial differential equations*, Manuscripta Math. 53 (1985), 35–64.

[12] M. Langenbruch, *Sequence space representations for weighted solution spaces of hypoelliptic systems of partial differential operators*, Rend. Circ. Mat. Palermo 35 (1986), 169–202.

[13] M. Langenbruch, *Sequence space representations for solution spaces of partial differential equations*, Doğa Mat. 10 (1986), 167–175.

[14] M. Langenbruch, *Power series spaces and weighted solution spaces of partial differential operators*, Math. Z. 197 (1987), 71–88.

[15] M. Langenbruch, *Bases in solution sheaves of systems of partial dfferential equations*, J. Reine Angew. Math. 373 (1987), 1–36.

[16] M. Langenbruch, *Hermite functions and weighted spaces of generalized functions*, Manuscripta Math. 119 (2006), 269–285.

[17] M. Langenbruch, *Bases in spaces of analytic germs*, Ann. Polon. Math. 106 (2012), 223–242.

[18] G. G. Margarjan and V. N. Kazarjan, *On the functional dimension of the solution space of hypoelliptic equations*, Math. USSR-Sb. 43 (1982), 547–561.

[19] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.

[20] B. S. Mityagin, *Approximate dimension and bases in nuclear spaces*, Russian Math. Surveys 16 (1961), 59–127.

[21] B. S. Mityagin and G. M. Khenkin, *Linear problems in complex analysis*, Russian Math. Surveys 26 (1971), no. 4, 99–164.

[22] Y. Saburi, *Fundamental properties of modified Fourier hyperfunctions*, Tokyo J. Math. 8 (1985), 231–273.

[23] D. Vogt, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. 37 (1982), 269–301.

[24] D. Vogt, *Sequence space representations of spaces of test functions and distributions*, in: Functional Analysis, Holomorphy, and Approximation Theory (Rio de Janeiro, 1979), Lecture Notes in Pure Appl. Math. 83, Dekker, 1983, 405–443.

Michael Langenbruch
Institute of Mathematics
University of Oldenburg
D-26111 Oldenburg, Germany
E-mail: michael.langenbruch@uni-oldenburg.de