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Supercongruences for the Almkvist–Zudilin numbers

by

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1. Introduction. The *Apéry numbers* $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ were valuable to R. Apéry in his celebrated proof [1] that $\zeta(3)$ is an irrational number. Since then these numbers have been a subject of much research. For example, they stand among a host of other sequences with the property

$$A(p^r n) \equiv_{p^{3r}} A(p^{r-1}n),$$

now known as *supercongruence*, a term dubbed by F. Beukers [2].

At the heart of many of these congruences sits the classical example $\binom{pb}{pc} \equiv_{p^3} \binom{b}{c}$ which is a stronger variant of the famous congruence $\binom{pb}{pc} \equiv_p \binom{b}{c}$ of Lucas. For a compendium of references on Apéry-type sequences, see [10].

Let us begin by fixing notational conventions. Denote the set of positive integers by $\mathbb{N}^+$. For $m \in \mathbb{N}^+$, let $\equiv_m$ represent congruence modulo m.

In this paper, we aim to investigate a similar type of supercongruences for the following family of sequences. For integers $i \geq 0$ and $n \geq 1$, define

$$a_i(n) := \sum_{k=0}^{\lfloor (n-i)/3 \rfloor} (-1)^{n-k} \binom{3k+i}{k} \binom{2k+i}{k} \binom{n}{3k+i} \binom{n+k}{k} 3^{n-3k-i}, \tag{1.1}$$

whose generating function is

$$\sum_{n=0}^{\infty} a_i(n) z^n = (-1)^i \sum_{k=0}^{\infty} \binom{4k+i}{k,k,k,k+i} \frac{z^{3k+i}}{(1+3z)^{4k+1+i}}.$$

In recent literature, $a_0(n)$ are referred to as the *Almkvist–Zudilin numbers*. Our motivation for the present work emanates from the following claim found in [6] (see also [3], [7]).

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Conjecture 1.1. *For a prime p and $n \in \mathbb{N}^+$, the Almkvist–Zudilin numbers satisfy*

$$a_0(pn) \equiv_{p^3} a_0(n).$$

Our main results can be summed up as:

If p is a prime and $n, i \in \mathbb{N}^+$, then $a_0(pn) \equiv_{p^3} a_0(n)$ and $a_i(pn) \equiv_{p^2} 0$.

The organization of the paper is as follows. Section 2 lays down some preparatory results to show the vanishing of $a_i(pn)$ modulo p^2 for $i > 0$. Section 3 sees the completion of the proof. Sections 4 and 5 exhibit its elaborate execution. The reduction brings in a *tighter* claim, and it also offers an advantage in allowing to work with a single sum instead of a double sum. In Section 6, we complete the proof for Conjecture 1.1. The paper concludes with Section 7 where we declare an improvement on the results from Section 3, which states a congruence for the family of sequences $a_i(pn)$ modulo p^3 when $i > 0$.

2. Preliminary results. *Fermat quotients* are numbers of the form

$$q_p(x) = \frac{x^{p-1} - 1}{p},$$

and they played a useful role in the study of cyclotomic fields and Fermat's Last Theorem (see [8]). The next three lemmas are known, but we give their proofs for completeness.

Lemma 2.1. *If p is a prime and $a \not\equiv_p 0$ then for $d \in \mathbb{Z}$,*

$$q_p(a^d) \equiv_{p^2} d\, q_p(a) + p\binom{d}{2} q_p(a)^2. \tag{2.1}$$

Proof. Since by Fermat's Little Theorem $a^{p-1} \equiv_p 1$, it follows that

$$(a^{p-1})^d = (1 + (a^{p-1} - 1))^d \equiv_{p^3} 1 + d(a^{p-1} - 1) + \binom{d}{2}(a^{p-1} - 1)^2. \ \blacksquare$$

Lemma 2.2. *Let $H_n = \sum_{j=1}^n 1/j$ be the nth harmonic number. Then, for $n \in \mathbb{N}^+$, we have*

$$\sum_{k=1}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\frac{1}{k} = -2H_n. \tag{2.2}$$

Proof. For an indeterminate y, a simple partial fraction decomposition shows the identity (see [5, Lemma 3.1])

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k}\frac{1}{k+y} = \frac{(-1)^n}{y}\prod_{j=1}^{n}\frac{y-j}{y+j}. \tag{2.3}$$

Now, subtract $1/y$ from both sides and take the limit as $y \to 0$. The right-hand side takes the form

$$\frac{1}{n!}\lim_{y\to 0}\frac{\prod_{j=1}^{n}(j-y)-\prod_{j=1}^{n}(j+y)}{y}=-2\sum_{k=1}^{n}\frac{1}{k}.$$

The conclusion is clear. ■

LEMMA 2.3. *Suppose p is a prime and $0 \le k < p/3$. Then*

$$(-1)^k\binom{\lfloor p/3\rfloor}{k}\binom{\lfloor p/3\rfloor+k}{k}\equiv_p\binom{3k}{k,k,k}3^{-3k}.$$

Proof. We observe that $\binom{n}{k}\binom{n+k}{k}=\binom{2k}{k}\binom{n+k}{2k}$. If $p\equiv_3 1$, then $\lfloor\frac{p}{3}\rfloor=\frac{p-1}{3}$ and hence

$$\begin{aligned}\binom{\frac{p-1}{3}+k}{2k}&=\frac{\frac{p-1}{3}\left(\frac{p-1}{3}+k\right)}{(2k)!}\prod_{j=1}^{k-1}\left(\frac{p-1}{3}\pm j\right)\\&\equiv_p\frac{(-1)^k(3k-1)}{3^{2k}(2k)!}\prod_{j=1}^{k-1}(3j\pm 1)=\frac{(-1)^k(3k)!}{3^{3k}(2k)!k!}.\end{aligned}$$

Therefore,

$$(-1)^k\binom{\frac{p-1}{3}}{k}\binom{\frac{p-1}{3}+k}{k}=(-1)^k\binom{2k}{k}\binom{\frac{p-1}{3}+k}{2k}\equiv_p\frac{(3k)!}{3^{3k}!k!^3}.$$

The case $p\equiv_3 -1$ runs analogously. ■

COROLLARY 2.4. *For a prime p and an integer $0 < i < p/3$, we have*

$$\sum_{k=1}^{p-1}\binom{3k}{k,k,k}\frac{3^{-3k}}{k}\equiv_p\sum_{k=1}^{\lfloor p/3\rfloor}\binom{3k}{k,k,k}\frac{3^{-3k}}{k}\equiv_p 3q_p(3),$$

$$\sum_{k=0}^{p-1}\binom{3k}{k,k,k}\frac{3^{-3k}}{k+i}\equiv_p\sum_{k=0}^{\lfloor p/3\rfloor}\binom{3k}{k,k,k}\frac{3^{-3k}}{k+i}\equiv_p 0.$$

Proof. For the first assertion, we combine (2.2), Lemma 2.3 and the congruence [4, p. 358]

$$H_{\lfloor p/3\rfloor}\equiv_p -3\sum_{r=1}^{\lfloor p/3\rfloor}\frac{1}{p-3r}\equiv_p -\frac{3q_p(3)}{2}.$$

The second congruence follows from (2.3) with $y=i$ and Lemma 2.3. ■

3. Main results on the sequences $a_i(n)$ for $i > 0$

THEOREM 3.1. *For a prime p and $n, i \in \mathbb{N}^+$ with $i < p/3$, we have*

$$a_i(pn)\equiv_{p^2} 0.$$

Proof. In (1.1), replace n by pn and $k = pm + r$ for $0 \le r \le p-1$ and some $m \in \mathbb{Z}$ (note: $3k + i = 3pm + 3r + i \le pn$) so that

$$a_i(pn) = \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{p-1} (-1)^{pn-pm-r} \binom{3pm+3r+i}{pm+r} \binom{2pm+2r+i}{pm+r} \cdot \binom{pn}{3pm+3r+i} \binom{pn+pm+r}{pm+r} 3^{pn-3pm-3r-i}.$$

If $t := 3r + i \ge p + 1$, it is easy to show that the following terms vanish modulo p^2:

$$\binom{3pm+t}{pm+r} \binom{2pm+2r+i}{pm+r} \binom{pn}{3pm+t} = \binom{3pm+t}{pm+r, pm+r, pm+r+i} \binom{pn}{3pm+t}.$$

Therefore, we may restrict to the remaining sum with $3r + i \le p$:

$$a_i(pn) = \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{\lfloor (p-i)/3 \rfloor} (-1)^{n-m-r} \binom{3pm+3r+i}{pm+r} \binom{2pm+2r+i}{pm+r} \cdot \binom{pn}{3pm+3r+i} \binom{pn+pm+r}{pm+r} 3^{pn-3pm-3r-i}.$$

We need Lucas's congruence $\binom{pb+c}{pd+e} \equiv_p \binom{b}{d}\binom{c}{e}$ to arrive at

$$a_i(pn) \equiv_p \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{\lfloor (p-i)/3 \rfloor} (-1)^{n-m-r} \binom{3m}{m} \binom{3r+i}{r} \binom{2m}{m} \binom{2r+i}{r} \cdot \binom{pn}{3pm+3r+i} \binom{n+m}{m} 3^{pn-3pm-3r-1}.$$

Again by Lucas's congruence and using $\binom{p-1}{j} \equiv_p (-1)^j$ for $0 \le j < p$, we get

$$\binom{pn}{3pm+3r+i} = \frac{pn}{3pm+3r+i} \binom{p(n-1)+p-1}{3pm+3r+i-1} \equiv_{p^2} \frac{pn}{3r+i} \binom{n-1}{3m} \binom{p-1}{3r+i-1} \equiv_{p^2} (-1)^{r+i-1} \frac{pn}{3r+i} \binom{n-1}{3m},$$

which leads to

$$a_i(pn) \equiv_{p^2} pn \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{\lfloor (p-i)/3 \rfloor} (-1)^{n-m-r} \binom{3m}{m} \binom{3r+i}{r} \binom{2m}{m} \binom{2r+i}{r} \cdot \frac{(-1)^{r+i-1}}{3r+i} \binom{n-1}{3m} \binom{n+m}{m} 3^{pn-3pm-3r-i}.$$

Next, we use Fermat's Little Theorem and *decouple* the double sum to obtain

$$a_i(pn) \equiv_{p^2} n \sum_{m=0}^{\lfloor n/3\rfloor} (-1)^{n-m+i-1} 3^{n-3m-i} \binom{3m}{m}\binom{2m}{m}\binom{n-1}{3m}\binom{n+m}{m} \cdot p \sum_{r=0}^{\lfloor (p-i)/3\rfloor} \binom{3r+i}{r}\binom{2r+i}{r}\frac{3^{-3r}}{3r+i}.$$

It suffices to verify that the *sum over* r vanishes modulo p. To achieve this, apply partial fraction decomposition and Corollary 2.4 (upgrading the sum to $\lfloor p/3\rfloor$ is *harmless* here). Thus,

$$\begin{aligned}\sum_{r=0}^{\lfloor p/3\rfloor} \binom{3r+i}{r}\binom{2r+i}{r}\frac{3^{-3r}}{3r+i} &= \sum_{r=0}^{\lfloor p/3\rfloor} \binom{3r}{r,r,r} 3^{-3r} \prod_{j=1}^{i-1}(3r+j) \prod_{j=1}^{i}(r+j)^{-1} \\ &= \sum_{j=1}^{i} \alpha_j(i) \sum_{r=0}^{\lfloor p/3\rfloor} \binom{3r}{r,r,r}\frac{3^{-3r}}{r+j} \equiv_p \sum_{j=1}^{i}\alpha_j(i)\cdot 0 = 0,\end{aligned}$$

where $\alpha_j(i) = (-1)^{i-j}\binom{3j-1}{i-1}\binom{i-1}{j-1} \in \mathbb{Z}$. We have enough reason to conclude the proof. ■

4. The reduction on the sequence $a_0(n)$. Our proof of Conjecture 1.1 requires a slightly more delicate analysis than what has been demonstrated in the previous sections for the sequences $a_i(n)$, where $i > 0$. As a first major step forward, we prove the following *somewhat* stronger result. This will be crucial in scaling down a double sum, which emerges (see proof below) as an expression for the sequence $a_0(pn)$, to a single sum.

THEOREM 4.1. *If p is a prime, then the congruence*

$$\sum_{r=1}^{p-1}(-1)^r \binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r}\binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}3^{-3r} \equiv_{p^3} p\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}q_p(3^{-(n-3m)}) \tag{4.1}$$

implies $a_0(pn) \equiv_{p^3} a_0(n)$.

Proof. In (1.1), replace n by pn and $k = pm + r$ for $0 \le r \le p-1$ and some $m \in \mathbb{Z}$. Using these new parameters, we write

$$a_0(pn) = \sum_{m=0}^{n-1} 3^{p(n-3m)}(-1)^{n-m} \sum_{r=0}^{p-1}(-1)^r \binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r} \cdot \binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}3^{-3r}.$$

Let us isolate the case $r = 0$; then, from $\binom{pb}{pc} \equiv_{p^3} \binom{b}{c}$ and the hypothesis we get

$$a_0(pn) \equiv_{p^3} \sum_{m=0}^{n-1} 3^{p(n-3m)}(-1)^{n-m}\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m} \cdot [1 + pq_p(3^{-(n-3m)})]$$
$$\equiv_{p^3} \sum_{m=0}^{n-1}(-1)^{n-m}\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}3^{(n-3m)} = a_0(n). \blacksquare$$

5. Further preliminary results. In this section, we build a few valuable results aiming at the proof of (4.1) and hence of Conjecture 1.1.

LEMMA 5.1. *Let $m, n \in \mathbb{N}^+$. For $p > 3$ a prime and an integer $0 \le r < p$, we have*

$$\binom{p(n+m)+r}{pm+r} \equiv_{p^2} \binom{n+m}{m}[1 + pnH_r], \tag{5.1}$$

and

$$\binom{pn}{3pm+3r} \equiv_{p^3} \left(\frac{p}{3r} - \frac{p^2m}{3r^2}\right)(-1)^r\binom{n}{3m}(n-3m)B_r(p,n,m) \tag{5.2}$$

where

$$B_r(p,n,m) = \begin{cases} -1 + pnH_{3r-1} & \text{if } 0 < r < p/3, \\ \dfrac{(n-3m-1)(1-pnH_{3r-1-p})}{3m+1} & \text{if } p/3 < r < 2p/3, \\ \dfrac{(n-3m-1)(n-3m-2)(-1+pnH_{3r-1-2p})}{(3m+1)(3m+2)} & \\ & \text{if } 2p/3 < r < p. \end{cases}$$

Proof. We revive a result found in [9, (27)], which is stated as follows. If $n = n_1p + n_0$ and $k = k_1p + k_0$ where $0 < n_0, k_0 < p$ then

$$\binom{n}{k} \equiv_{p^2} \binom{n_1}{k_1}\left[(n_1+1)\binom{n_0}{k_0} - (n_1+k_1)\binom{n_0-p}{k_0} - k_1\binom{n_0-p}{k_0+p}\right]. \tag{5.3}$$

For (5.1), apply (5.3) with $n_1 = n+m$, $n_0 = r = k_0$, $k_1 = m$. So,

$$\binom{p(n+m)+r}{pm+r} \equiv_{p^2} \binom{n+m}{m}\left[(1+m+n)\binom{r}{r} - (n+2m)\binom{r-p}{r} - m\binom{r-p}{r+p}\right].$$

Now apply (5.3) to $\binom{p+r}{r} \equiv_{p^2} 2 - \binom{r-p}{p}$ (with $n_1 = 1$, $n_0 = k_0 = r$, $k_1 = 0$), and to $\binom{r-p}{r+p} = \binom{-p+r}{-2p} \equiv_{p^2} -3 + 2\binom{r-p}{r}$ (with $n_1 = -1$, $n_0 = r$, $k_1 = -2$,

$k_0 = 0$). After substitution and simplifications, we obtain

$$\binom{p(n+m)+r}{pm+r} \equiv_{p^2} \binom{n+m}{m}\left(1+n\left(\binom{p+r}{r}-1\right)\right).$$

The desired result is reached as soon as we note that

$$\binom{p+r}{r} = \frac{1}{r!}\prod_{j=1}^{r}(p+j) \equiv_{p^2} 1+pH_r.$$

The congruence (5.2) demands a careful analysis. The setup begins by expressing $3r = \epsilon p + d$ where $0 < d < p$ and $\epsilon = \lfloor 3r/p \rfloor \in \{0,1,2\}$ which correspond to $0 < 3r < p$, $p < 3r < 2p$ and $2p < 3r < 3p$, respectively. Apply (5.3) with $n_1 = n-1$, $n_0 = p-1$, $k_1 = 3m+\epsilon$, $k_0 = d-1$. Follow this through using $\binom{-1}{j} = (-1)^j$. The outcome is

$$\begin{aligned}\binom{pn}{3pm+3r} &= \frac{pn}{3pm+3r}\binom{p(n-1)+p-1}{p(3m+\epsilon)+d-1} \\ &\equiv_{p^3} \frac{pn}{3pm+3r}\binom{n-1}{3m+\epsilon}\left[n\binom{p-1}{3r-1-\epsilon p} + (-1)^{r-\epsilon}(n-1)\right].\end{aligned}$$

Combining this step and the easy facts

$$\frac{1}{3pm+3r} \equiv_{p^2} \frac{1}{3r} - \frac{pm}{3r^2}, \qquad \binom{p-1}{j} \equiv_{p^2} (-1)^j[1-pH_j],$$

we reach the conclusion. ∎

Also the next congruence can be deduced from (5.3). However, here we offer a more direct approach.

Lemma 5.2. *Let $m \in \mathbb{N}^+$. For $p > 3$ a prime and an integer $0 \le r < p$, we have*

(5.4)
$$\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r} \equiv_{p^2} \binom{3m}{m,m,m}\binom{3r}{r,r,r}\left[1+3pm(H_{3r}-H_r)\right].$$

Proof. Since $(pm+k)^{-1} \equiv_{p^2} \frac{1}{k}\left(1-\frac{pm}{k}\right)$, we obtain

$$(pm+k)^{-3} \equiv_{p^2} \frac{1}{k^3}\left(1-\frac{pm}{k}\right)^3 \equiv_{p^2} \frac{1}{k^3}\left(1-\frac{3pm}{k}\right) = \frac{k-3pm}{k^4}.$$

For notational simplicity, denote $\binom{3j}{j,j,j} = \binom{3j}{j}\binom{2j}{j}$ by $\binom{3j}{j^3}$. We consider the expansion $\prod_{i=1}^{n}(\lambda_i+x) = \sum_{j=0}^{n} e_j(\lambda)x^{n-j}$ as our running theme, where e_j is the jth *elementary symmetric function* in the parameters $\lambda = (\lambda_1,\dots,\lambda_n)$. In particular, $e_n = 1$ and $e_{n-1}(1,\dots,n) = n!H_n$. The claim then follows from

$$\binom{3pm+3r}{(pm+r)^3} = \binom{3pm}{(pm)^3}\prod_{j=1}^{3r}(j+3pm)\prod_{k=1}^{r}(pm+k)^{-3}$$
$$\equiv_{p^2} \binom{3pm}{(pm)^3}\frac{1}{r!^4}\prod_{j=1}^{3r}(j+3pm)\prod_{k=1}^{r}(k-3pm)$$
$$\equiv_{p^2} \binom{3pm}{(pm)^3}\frac{1}{r!^4}(3r)!r![1+3pmH_{3r}-3pmH_r]. \blacksquare$$

REMARK 5.3. This fact is even more general as stated below but the proof is left to the interested reader. If $A, n \in \mathbb{N}^+$, $0 \le r < p$ are integers and $p > 3$ a prime, then

$$\frac{(Apm+Ar)!}{(pm+r)!^A} \equiv_{p^2} \binom{Am}{m,\ldots,m}\binom{Ar}{r,\ldots,r}[1+Apm(H_{Ar}-H_r)].$$

LEMMA 5.4. *If $p > 3$ is a prime then*

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r} \equiv_{p^2} -3q_p(1/3)+\frac{3p}{2}\,q_p(1/3)^2, \tag{5.5}$$

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r^2} \equiv_{p} -\frac{9}{2}\,q_p(1/3)^2, \tag{5.6}$$

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{(H_{3r}-H_r)3^{-3r}}{r} \equiv_{p} 0. \tag{5.7}$$

Proof. By (2.1), $q_p(1/27) \equiv_{p^2} 3\,q_p(1/3) + 3p\,q_p(1/3)^2$. Therefore, by (5) in [11, Theorem 4],

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r} = \sum_{r=1}^{p-1}\frac{(1/3)_r(2/3)_r}{(1)_r^2}\cdot\frac{1}{r} \equiv_{p^2} -q_p(1/27)+\frac{p}{2}\,q_p(1/27)^2$$
$$\equiv_{p^2} -3q_p(1/3)+\frac{3p}{2}\,q_p(1/3)^2.$$

In a similar way, by (6) in [11, Theorem 4],

$$\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r^2} = \sum_{r=1}^{p-1}\frac{(1/3)_r(2/3)_r}{(1)_r^2}\cdot\frac{1}{r^2} \equiv_p -\frac{1}{2}\,q_p(1/27)^2 \equiv_p -\frac{9}{2}\,q_p(1/3)^2.$$

By (1) in [11, Theorem 1],

$$\frac{(1/3)_r(2/3)_r}{(1)_r^2}\sum_{j=0}^{r-1}\left(\frac{1}{1/3+j}+\frac{1}{2/3+j}\right) = \sum_{k=0}^{r-1}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{1}{r-k}.$$

Hence (5.7) is implied by

$$\begin{aligned}
&\sum_{r=1}^{p-1}\binom{3r}{r,r,r}\frac{(3H_{3r}-H_r)3^{-3r}}{r}\\
&\quad=\sum_{r=1}^{p-1}\frac{(1/3)_r(2/3)_r}{(1)_r^2}\cdot\frac{1}{r}\cdot\sum_{j=0}^{r-1}\bigg(\frac{1}{1/3+j}+\frac{1}{2/3+j}\bigg)\\
&\quad=\sum_{r=1}^{p-1}\frac{1}{r}\sum_{k=0}^{r-1}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{1}{r-k}=\sum_{k=0}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\sum_{r=k+1}^{p-1}\frac{1}{r(r-k)}\\
&\quad=\sum_{r=1}^{p-1}\frac{1}{r^2}+\sum_{k=1}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\bigg(\frac{1}{k}\sum_{r=k+1}^{p-1}\bigg(\frac{1}{r-k}-\frac{1}{r}\bigg)\bigg)\\
&\quad\equiv_p\sum_{k=1}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{1}{k}(H_{p-1-k}-H_{p-1}+H_k)\\
&\quad\equiv_p\sum_{k=1}^{p-1}\binom{3k}{k,k,k}\frac{2H_k\,3^{-3k}}{k},
\end{aligned}$$

because $H_{p-1-k}\equiv_p H_k$ and $H_{p-1}\equiv_p\sum_{r=1}^{p-1}\frac{1}{r^2}\equiv_p\sum_{j=1}^{p-1}j\equiv_p 0$ as $p\neq 2$. ∎

6. Proof of Conjecture 1.1. In this section, we combine the results from the preceding sections to arrive at a proof of (4.1) (restated here for the reader's convenience) and therefore of Conjecture 1.1.

Theorem 6.1. *For a prime $p>3$ and $m,n\in\mathbb{N}^+$, we have*

$$\begin{aligned}
&\sum_{r=1}^{p-1}(-1)^r\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r}\binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}3^{-3r}\\
&\qquad\equiv_{p^3} p\binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}q_p(3^{-(n-3m)}).
\end{aligned}$$

Proof. Based on congruences (2.1), (5.1), (5.2), and (5.4), the assertion is equivalent to

(6.1)
$$\begin{aligned}
&\sum_{r=1}^{p-1}\binom{3r}{r,r,r}(1+3pm(H_{3r}-H_r))(1+pnH_r)\bigg(\frac{1}{3r}-\frac{pm}{3r^2}\bigg)B_r(p,n,m)3^{-3r}\\
&\qquad\equiv_{p^2} q_p(1/3)+\frac{p(n-3m-1)}{2}\,q_p(1/3)^2.
\end{aligned}$$

Now we split the sum on the left-hand side of (6.1) into three pieces,

$$S_1 = \sum_{r=1}^{\lfloor p/3 \rfloor} (\,\cdot\,), \qquad S_2 = \sum_{r=\lceil p/3 \rceil}^{\lfloor 2p/3 \rfloor} (\,\cdot\,), \quad \text{and} \quad S_3 = \sum_{r=\lceil 2p/3 \rceil}^{p-1} (\,\cdot\,).$$

As regards S_1,

$$S_1 \equiv_{p^2} \frac{1}{3} \sum_{r=1}^{\lfloor p/3 \rfloor} \binom{3r}{r,r,r} \left(-\frac{1}{r} - \frac{p(n-3m)}{3r^2} + \frac{p(n-3m)(H_{3r}-H_r)}{r} \right) 3^{-3r}.$$

If $p/3 < r < 2p/3$ then $\binom{3r}{r,r,r} \equiv_p 0$ and $1 + 3pm(H_{3r} - H_r) \equiv_p 1 + 3m$ with $B_r(p,n,m) \equiv_p (n-3m-1)/(3m+1)$. These imply that

$$\begin{aligned} S_2 &\equiv_{p^2} \sum_{r=\lceil p/3 \rceil}^{\lfloor 2p/3 \rfloor} \binom{3r}{r,r,r} (1 + 3pm(H_{3r}-H_r))(1 + pnH_r) \\ &\qquad\qquad \cdot \left(\frac{1}{3r} - \frac{pm}{3r^2} \right) B_r(p,n,m) 3^{-3r} \\ &\equiv_{p^2} \sum_{r=\lceil p/3 \rceil}^{\lfloor 2p/3 \rfloor} \binom{3r}{r,r,r} (1+3m) \left(\frac{1}{3r} \right) \frac{n-3m-1}{3m+1}\, 3^{-3r} \\ &\equiv_{p^2} \frac{n-3m-1}{3} \sum_{r=\lceil p/3 \rceil}^{\lfloor 2p/3 \rfloor} \binom{3r}{r,r,r} \frac{3^{-3r}}{r}. \end{aligned}$$

Finally, $S_3 \equiv_{p^2} 0$ because obviously $\binom{3r}{r,r,r} \equiv_{p^2} 0$ as long as $2p/3 < r < p$.

Again $\binom{3r}{r,r,r} \equiv_p 0$ if $p/3 < r < 2p/3$, and $\binom{3r}{r,r,r} \equiv_{p^2} 0$ if $2p/3 < r < p$. So, from (5.5) and (5.6) we know that

$$\sum_{r=1}^{\lfloor 2p/3 \rfloor} \binom{3r}{r,r,r} \frac{3^{-3r}}{r} \equiv_{p^2} \sum_{r=1}^{p-1} \binom{3r}{r,r,r} \frac{3^{-3r}}{r} \equiv_{p^2} -3q_p(1/3) + \frac{3p}{2}\, q_p(1/3)^2,$$

$$p \sum_{r=1}^{\lfloor p/3 \rfloor} \binom{3r}{r,r,r} \frac{3^{-3r}}{r^2} \equiv_{p^2} p \sum_{r=1}^{p-1} \binom{3r}{r,r,r} \frac{3^{-3r}}{r^2} \equiv_{p^2} -\frac{9p}{2}\, q_p(1/3)^2.$$

As before, $\binom{3r}{r,r,r} \equiv_{p^2} 0$ for $2p/3 < r < p$. Moreover, we have $\binom{3r}{r,r,r} \equiv_p 0$ and $pH_{3r} - pH_r \equiv_p 1$ for $p/3 < r < 2p/3$. Therefore, by (5.7),

$$\begin{aligned} 0 &\equiv_{p^2} p \sum_{r=1}^{p-1} \binom{3r}{r,r,r} \frac{(H_{3r}-H_r)3^{-3r}}{r} \equiv_{p^2} p \sum_{r=1}^{\lfloor 2p/3 \rfloor} \binom{3r}{r,r,r} \frac{(H_{3r}-H_r)3^{-3r}}{r} \\ &\equiv_{p^2} p \sum_{r=1}^{\lfloor p/3 \rfloor} \binom{3r}{r,r,r} \frac{(H_{3r}-H_r)3^{-3r}}{r} + \sum_{r=\lceil p/3 \rceil}^{\lfloor 2p/3 \rfloor} \binom{3r}{r,r,r} \frac{3^{-3r}}{r}. \end{aligned}$$

Putting all these together, we conclude that

$$
\begin{aligned}
&S_1+S_2+S_3\\
&\equiv_{p^2} \frac{1}{3}\sum_{r=1}^{\lfloor p/3\rfloor}\binom{3r}{r,r,r}\left(-\frac{1}{r}-\frac{p(n-3m)}{3r^2}+\frac{p(n-3m)(H_{3r}-H_r)}{r}\right)3^{-3r}\\
&\qquad+\frac{n-3m-1}{3}\sum_{r=\lceil p/3\rceil}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r}+0\\
&\equiv_{p^2} -\frac{1}{3}\sum_{r=1}^{\lfloor 2p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r}\\
&\qquad-\frac{n-3m}{9}\sum_{r=1}^{\lfloor p/3\rfloor}\binom{3r}{r,r,r}\frac{3^{-3r}}{r^2}+\frac{n-3m}{3}\cdot 0\\
&\equiv_{p^2} -\frac{1}{3}\left(-3q_p(1/3)+\frac{3p}{2}\,q_p(1/3)^2\right)-\frac{n-3m}{9}\left(-\frac{9p}{2}\,q_p(1/3)^2\right)\\
&\equiv_{p^2} q_p(1/3)+\frac{p(n-3m-1)}{2}\,q_p(1/3)^2,
\end{aligned}
$$

which is exactly what we expect. The proof is complete. ∎

7. Conclusions and remarks. In this final section, we extend the congruence on $a_i(n)$ (for $i>0$), discussed in the earlier sections, from modulo p^2 to modulo p^3. While stating our claim in its generality, we only exhibit proof outlines for $i=1$ as a prototypical example. For $i>1$, the details are similar and hence omitted. We believe the curious researcher would be able to account for these remaining cases.

Theorem 7.1. *For $n,i\in\mathbb{N}^+$ and a prime $p>2i$,*

$$
a_i(pn)\equiv_{p^3}(-1)^{i-1}\frac{a_1(pn)}{i^2\binom{2i-1}{i-1}}\equiv_{p^3}\frac{(-1)^{i-1}p^2\binom{n+2}{2}a_1(n)}{i^2\binom{2i-1}{i-1}}.
$$

Proof (*the case $i=1$; ingredients for $a_1(pn)\equiv_{p^3}p^2\binom{n+2}{2}a_1(n)$*).
(A) By partial fraction decomposition,

$$
\begin{aligned}
a_i(n)&=\frac{1}{3^i}\sum_{k=0}^{n-1}(-1)^{n-k}\binom{3k}{k}\binom{2k}{k}\binom{n}{3k}\binom{n+k}{k}\frac{\binom{n-3k}{i}3^{n-3k}}{\binom{k+i}{i}}\\
&=(-1)^i a_0(n)+\frac{i}{3^i}\sum_{j=1}^{i}(-1)^{j-1}\binom{i-1}{j-1}\binom{n+3j}{i}b_j(n),
\end{aligned}
$$

where for $j \in \mathbb{N}^+$,

$$b_j(n) := \sum_{k=0}^{n-1} (-1)^{n-k}(n-3k)\binom{3k}{k}\binom{2k}{k}\binom{n}{3k}\binom{n+k}{k}\frac{3^{n-3k}}{k+j}.$$

Thus, $a_0(np) \equiv_{p^3} a_0(n)$ implies

$$\begin{aligned} a_1(np) &= -a_0(np) + \frac{np+3}{3}\, b_1(np) \\ &\equiv_{p^3} -a_0(n) + \frac{np+3}{3}\, b_1(np) \\ &\equiv_{p^3} a_1(n) + \frac{np+3}{3}\, b_1(np) - \frac{p+3}{3}\, b_1(p). \end{aligned}$$

(B) Hence, it suffices to show that

$$b_1(np) \equiv_{p^3} \frac{3}{np+3}\left(p^2\binom{n+2}{2} - 1\right)a_1(n) + \frac{n+3}{np+3}\, b_1(n),$$

or, since $a_1(n) = -a_0(n) + (n+3)b_1(n)/3$,

$$b_1(np) \equiv_{p^3} p^2\binom{n+3}{3}b_1(n) + \left(1 - \frac{pn}{3} - \frac{p^2(n+3)(7n+6)}{18}\right)a_0(n). \tag{7.1}$$

(C) The above congruence follows from

(7.2)

$$\begin{aligned} \sum_{r=0}^{p-1} &\binom{3pm+3r}{pm+r}\binom{2pm+2r}{pm+r}\binom{pn}{3pm+3r}\binom{p(n+m)+r}{pm+r}\frac{(-1)^r 3^{-3r}}{pm+r+1} \\ &\equiv_{p^3} \left(\frac{p^2}{m+1}\binom{n+3}{3} + 1 - \frac{pn}{3} - \frac{p^2(n+3)(7n+6)}{18}\right) \\ &\qquad\qquad \cdot \binom{3m}{m}\binom{2m}{m}\binom{n}{3m}\binom{n+m}{m}3^{-(n-3m)(p-1)} \end{aligned}$$

By summing over m, it is immediate to recover (7.1).

(D) In order to prove (7.2), we have the old machinery,

$$\frac{1}{pm+r+1} \equiv_{p^2} \frac{1}{r+1} - \frac{mp}{(r+1)^2},$$

and

$$\sum_{r=0}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{r+1} = \frac{9p}{2}\binom{3p}{p,p,p}3^{-3p} \equiv_{p^2} p - 3p^2 q_p(1/3),$$

$$\sum_{r=0}^{p-1}\binom{3r}{r,r,r}\frac{3^{-3r}}{(r+1)^2} = \frac{9(9p+2)}{4}\binom{3p}{p,p,p}3^{-3p} - \frac{9}{2} \equiv_p -\frac{7}{2}.$$

(E) Finally, we can modify the previous proof as follows:

$$
\begin{aligned}
&\sum_{r=0}^{p-1}\binom{3r}{r,r,r}\frac{(3H_{3r}-H_r)3^{-3r}}{r+1}\\
&\qquad=\sum_{r=1}^{p-1}\frac{(1/3)_r(2/3)_r}{(1)_r^2}\cdot\frac{1}{r+1}\cdot\sum_{j=0}^{r-1}\biggl(\frac{1}{1/3+j}+\frac{1}{2/3+j}\biggr)\\
&\qquad=\sum_{r=1}^{p-1}\frac{1}{r+1}\sum_{k=0}^{r-1}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{1}{r-k}\\
&\qquad=\sum_{k=0}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\sum_{r=k+1}^{p-1}\frac{1}{(r+1)(r-k)}\\
&\qquad=\sum_{k=0}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\biggl(\frac{1}{k+1}\sum_{r=k+1}^{p-1}\biggl(\frac{1}{r-k}-\frac{1}{r+1}\biggr)\biggr)\\
&\qquad=\sum_{k=0}^{p-2}\frac{(1/3)_k(2/3)_k}{(1)_k^2}\cdot\frac{H_{p-1-k}-H_p+H_{k+1}}{k+1}\\
&\qquad\equiv_p\sum_{k=0}^{p-1}\binom{3k}{k,k,k}\frac{(H_k-H_p+H_{k+1})3^{-3k}}{k+1},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\sum_{r=0}^{p-1}\binom{3r}{r,r,r}\frac{(H_{3r}-H_r)3^{-3r}}{r+1}&\equiv_p\frac{1}{3}\sum_{k=0}^{p-1}\binom{3k}{k,k,k}\frac{(-1/p+1/(k+1))3^{-3k}}{k+1}\\
&\equiv_p\frac{1}{3}\biggl(-1-\frac{7}{2}\biggr)=-\frac{3}{2}. \qquad\blacksquare
\end{aligned}
$$

Remark 7.2. We showed that the conjecture $a_0(pn)\equiv_{p^3}a_0(n)$ holds true. If one combines the techniques established in this paper with the existing literature on supercongruences (see references below) for binomials of the type $\binom{p^rn+k}{p^tm+j}$, there is enough reliable verity to believe that $a_0(p^rn)\equiv_{p^{3r}}a_0(p^{r-1}n)$ might be approachable. However, at present we are unsure of the details.

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