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Strong Chang's Conjecture, Semi-Stationary Reflection, the Strong Tree Property and two-cardinal square principles

by

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Abstract. We prove that a strong version of Chang's Conjecture implies both the Strong Tree Property for ω_2 and the negation of the square principle $\square(\lambda, \omega)$ for every regular cardinal $\lambda \geq \omega_2$.

1. Introduction. In these notes we consider two equivalent principles: a strong version of Chang's Conjecture and the Semi-Stationary Reflection Principle. Given two sets x, y, we write $x \sqsubseteq y$ whenever $x \subseteq y$ and $x \cap \omega_1 = y \cap \omega_1$.

Definition 1.1. The principle CC^* asserts that for every regular cardinal $\kappa \geq \omega_2$, there are arbitrary large θ such that the following statement $\mathrm{CC}(\kappa, \theta)$ holds: For every countable $M \prec H_\theta$ and every $a \in [\kappa]^{\omega_1}$, there is a countable $M^* \prec H_\theta$ and $a^* \in M^* \cap [\kappa]^{\omega_1}$ such that $a^* \supseteq a$ and $M^* \sqsupseteq M$.

A first generalization of Chang's Conjecture of this kind was given by Shelah (see [21, Theorem 1.3, p. 398]). Similar general versions were studied in [25] and [5]. The Semi-Stationary Reflection Principle (SSR) was introduced by Shelah [22, Chapter XIII, Definition 1.5]. Given an ordinal λ and a set $X \subseteq [\lambda]^\omega$, we say X is *semi-stationary in* $[\lambda]^\omega$ if its $\sqsubseteq$-upward closure is stationary, i.e. the set $\{y \in [\lambda]^\omega : \exists x \in X \ (x \sqsubseteq y)\}$ is stationary. It is clear that every stationary set is semi-stationary.

Definition 1.2. The principle SSR asserts that the following statement $\mathrm{SSR}(\lambda)$ holds for every ordinal $\lambda \geq \omega_2$: for every semi-stationary subset $X \subseteq [\lambda]^\omega$, there is $W \in [\lambda]^{\omega_1}$ with $W \supseteq \omega_1$ such that $X \cap [W]^\omega$ is semi-stationary in $[W]^\omega$.

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Döbler and Schindler proved that CC^* and SSR are equivalent (see [5, Theorem 5.7]). Shelah showed that SSR is equivalent to the following statement:

(†) Every poset preserving stationary subsets of ω_1 is semiproper

(see [22, Chapter XIII, 1.7]). Although these principles are consequences of the Weak Reflection Principle (see for example [17]) or Rado's Conjecture [4], they have many important consequences of their own: In [9], it was already shown that (†) implies that the ideal NS_{ω_1} is precipitous. It was shown that under a weaker version of CC^*, the existence of a special $\aleph_2$-Aronszajn tree is equivalent to CH (see [26]), and that SSR implies the Singular Cardinal Hypothesis [19] and the negation of $\square(\lambda)$ for all regular cardinals $\lambda \geq \omega_2$. The present authors [28] showed recently that under a weak version of CC^*, the negation of CH entails the Tree Property for ω_2.

In Section 3, we discuss the relationship between CC^* and the Strong Tree Property. Looking for sufficient conditions for a tree to have a cofinal branch has led to many interesting combinatorial results. We recall that an infinite regular cardinal κ has the *Tree Property* ($TP(\kappa)$) if for every tree of height κ with levels of size $< \kappa$, there is a cofinal branch. König's Lemma states that $TP(\omega)$ holds [14], while Aronszajn showed that there is a tree of height ω_1 with each level at most countable and with no cofinal branches (see [15, Theorem 6, p. 96]). Baumgartner [1] proved that the Proper Forcing Axiom PFA implies $TP(\omega_2)$. However, $TP(\omega_2)$ turned out to be equiconsistent with the existence of a weakly compact cardinal ([16, Theorem 5.9] and [6]).

Jech introduced a strengthening of the Tree Property, now called the Strong Tree Property (see Section 3 for the definition). He noticed that an inaccessible cardinal κ has the Strong Tree Property if and only if κ is strongly compact (see [12, p. 174]). Weiß [31] showed that PFA implies $\aleph_2$ has the Strong Tree Property. Sakai and Veličković [19] proved that SSR, together with $MA_{\omega_1}(\text{Cohen})$, implies the Strong Tree Property at ω_2.

In this note, we show that it is enough to assume SSR and $\neg$CH for ω_2 to have the Strong Tree Property. We remark that SSR is consistent with both CH and $\neg$CH, and that CH implies $\neg TP(\omega_2)$. Therefore, our result is in certain sense optimal.

In Section 4, we study the relationship between SSR and the square principle $\square(\lambda, \omega)$ for every regular cardinal $\lambda \geq \omega_2$. The original square principle $\square_\lambda$ was introduced by Jensen [13]. He showed that $\square_\lambda$ holds in L for every uncountable cardinal λ. Schimmerling [20] generalized this square principle to weaker versions of the form $\square_{\kappa,\lambda}$. These two-cardinal versions have been extensively studied so far. For example, after the works of Cummings–Magidor and Baumgartner, we have a complete picture of the relationship between

MM and square principles of the form $\square_{\kappa,\lambda}$ ([2], [3]). Some partial results were also given on relations between Rado's Conjecture (RC) and $\square_{\kappa,\lambda}$ ([26], [27]). Sakai established in unpublished notes [18] a rather complete picture of relations between SSR and the square principles $\square_{\kappa,\lambda}$.

The square principle $\square(\lambda)$ (see Definition 4.1) has also been studied in several instances. Jensen showed that in L, if $\lambda > \omega$ is regular and $\square(\lambda)$ holds, then λ is not weakly compact (see [13, Theorem 6.1]). It has been proven that the negation of $\square(\lambda)$ for all regular cardinal $\lambda \geq \omega_2$ is implied by the Proper Forcing Axiom (Todorčević [24]), the Weak Reflection Principle (Veličković [29]), Rado's Conjecture (Todorčević [25]) and more recently by SSR (Sakai–Veličković [19]). Regarding a two-cardinal version $\square(\kappa, \lambda)$ (see Definition 4.1) and its relation to other combinatorial principles, some results have been already established, for example in [23] and [27]. In this paper, we prove that SSR is enough to have the negation of $\square(\lambda, \omega)$ for every regular cardinal $\lambda \geq \omega_2$.

2. Preliminaries. In these notes we will consider several types of stationary sets.

Given a limit ordinal γ, a subset $A \subseteq \gamma$ is *unbounded in* γ if $\sup(A) = \gamma$, and *closed in* γ if for every limit ordinal $\beta < \gamma$, if $A \cap \beta$ is unbounded in β, then $\beta \in A$. A set $A \subseteq \gamma$ is often called a *club set in* γ if it is closed and unbounded in γ. A set $S \subseteq \gamma$ is *stationary* if $S \cap A \neq \emptyset$ for every A club in γ.

The following result involving stationary sets is known as *Fodor's Lemma* or the *Pressing Down Lemma for ordinals.*

LEMMA 2.1 (Fodor [7]). *Let κ be a regular uncountable cardinal. Then for every stationary $S \subseteq \kappa$, and for every $f : S \to \kappa$ such that $f(\alpha) < \alpha$ for every $\alpha \in S$, there is $\xi < \kappa$ such that $f^{-1}(\{\xi\})$ is stationary.*

A general version of a stationary set was given originally by Jech. We will also use an equivalent version due to Kueker (see for example [10, Theorem 8.28]). Given an infinite set A and a regular cardinal μ, we denote by $[A]^{<\mu}$ the collection of subsets of A of size $< \mu$. Similarly, let $[A]^{\mu}$ denote the collection of all subsets of A of size μ. We say that a set $S \subseteq [A]^{\omega}$ is *stationary in* $[A]^{\omega}$ if for every function $F : [A]^{<\omega} \to A$, there is $X \in S$ such that $F(e) \in X$ for every $e \in [X]^{<\omega}$.

The following lemma is a generalized version of the Pressing Down Lemma (see [10, Theorem 8.24]).

LEMMA 2.2 (Jech). *For every stationary set $S \subseteq [A]^{\omega}$ and every $f : S \to A$ such that $f(X) \in X$ for every $X \in S$, there is $a \in A$ such that $f^{-1}(\{a\})$ is stationary.*

In general, we say that a set $S \subseteq [A]^\mu$ is *weakly stationary* if for every $F : [A]^{<\omega} \to A$, there is $X \in S$ such that $F(e) \in X$ for every $e \in [X]^{<\omega}$. Note that by Kueker's Theorem mentioned above, weakly stationary is the same as stationary when $\mu = \omega$. However, it is not so for $\mu > \omega$: see, for example, the discussion [11, end of Section 4.1]. This generalization of stationary set is due probably to Foreman, Magidor and Shelah [9], and used prominently by Woodin.

The Pressing Down Lemma for this kind of stationary sets is folklore, but we include a reference for completeness.

Lemma 2.3 (Folklore). *Given a set X and a regular cardinal μ, for every weakly stationary set $S \subseteq [X]^\mu$ and any regressive function $f : S \to X$, there is a weakly stationary set $S' \subseteq S$ such that $f\restriction_{S'}$ is constant.*

Proof. See for example [8, p. 912, Lemma 3.3]. ∎

All along these notes we only use the notion of weakly stationary. Since for $\mu = \omega$ weakly stationary and stationary coincide, we make an abuse of language and call both just stationary, even when $\mu \geq \omega_1$.

3. Semi-Stationary Reflection Principle and the Strong Tree Property for ω_2. We give the definitions regarding the Strong Tree Property.

Definition 3.1. Let $\lambda > \omega_1$ be a regular cardinal and let $\kappa \geq \lambda$. A (κ, λ)-*tree* is a system $\{\mathscr{F}_a \in P(2^a) : a \in [\kappa]^{<\lambda}\}$ such that

(1) for every a, $1 \leq |\mathscr{F}_a| < \lambda$,
(2) for $a, b \in [\kappa]^{<\lambda}$, $a \subseteq b \to \forall f \in \mathscr{F}_b\ \exists g \in \mathscr{F}_a\ (f\restriction_a = g)$.

Given a (κ, λ)-tree $\mathscr{F}$, we order its elements in the following way: for $f, g \in \mathscr{F}$, $f \leq_{\mathscr{F}} g$ if and only if $g\restriction_{\mathrm{dom}(f)} = f$. Observe that if $f \leq_{\mathscr{F}} g$, then in particular $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$. Note that $\leq_{\mathscr{F}}$ is transitive, but it is not necessarily a tree order. We say that $f, g \in \mathscr{F}$ are *compatible* if there is $h \in \mathscr{F}$ such that $h \geq_{\mathscr{F}} f, g$ (note that in a tree order, compatible is equivalent to comparable). For $A, B \subseteq \mathscr{F}$ we write $A \perp B$ if for every $f \in A$ and every $g \in B$, f and g are incompatible. Similarly, for $f, g \in \mathscr{F}$ and $A \subseteq \mathscr{F}$, we write $f \perp g$ and $f \perp A$ whenever $\{f\} \perp \{g\}$ and $\{f\} \perp A$ respectively. A *cofinal branch through* $\mathscr{F}$ is a function $B : \kappa \to 2$ such that $B\restriction_a \in \mathscr{F}$ for every $a \in [\kappa]^{<\lambda}$.

Definition 3.2. We say that λ has the *Strong Tree Property* if every (κ, λ)-tree has a cofinal branch for every $\kappa \geq \lambda$.

In this section, we prove that CC^* together with the negation of CH implies ω_2 has the Strong Tree Property. The proofs are based on the techniques of [26] and [28]: compare, for example, our Proposition 3.1, Lemma 3.1 and

Lemma 3.2 with Proposition 2.3, Lemma 2.4 and Lemma 2.5 in [26] and Proposition 3.1, Lemma 3.1 and Lemma 3.2 in [28] respectively.

Let $\kappa \geq \omega_2$ and fix a (κ, ω_2)-tree $\mathscr{F}$. Fix a level enumeration surjective function $e : [\kappa]^{\omega_1} \times \omega_1 \to \mathscr{F}$ such that $e(d, \xi) \in \mathscr{F}_d$.

We have the following:

PROPOSITION 3.1. *Given a (κ, ω_2)-tree $\mathscr{F}$, let $\langle A_d : d \in [\kappa]^{\omega_1} \rangle$ be a sequence of collections of nodes such that $A_d \in [\mathscr{F}_d]^\omega$ for every $d \in [\kappa]^{\omega_1}$. Then there are $b \in [\kappa]^{\omega_1}$ and E stationary in $[\kappa]^{\omega_1}$ such that for every $g \in \mathscr{F}_b$ and every $d \in E$, if g has an extension in $\mathscr{F}_d$, this extension is unique.*

Proof. Let θ be large enough such that $\{\mathscr{F}, e, \kappa, \ldots\}$ and all relevant parameters belong to H_θ.

We remark the following:

REMARK 3.1. *There are stationary many $M \prec H_\theta$ with $|M| = \aleph_1$ such that for every $A \in [M]^\omega$, there is $B \in M \cap [M]^\omega$ such that $B \supseteq A$.*

Proof. For any $g : H_\theta^{<\omega} \to H_\theta$, build a $\subseteq$-continuous chain $\langle M_\xi : \xi < \omega_1 \rangle$ of countable elementary submodels of H_θ such that for any $\xi \in \omega_1$, M_ξ is closed under g and $M_\xi \in M_{\xi+1}$. Let $M = \bigcup_{\xi<\omega_1} M_\xi$. It is easy to check that M is closed under g and $|M| = \omega_1$. Then if $A \in [M]^\omega$, there is $\xi \in \omega_1$ such that $A \subseteq M_\xi$, and so $M_\xi \in M_{\xi+1} \subseteq M$. ∎

Let $S \subseteq [H_\theta]^{\omega_1}$ be the stationary set of M's of Remark 3.1. For any $M \in S$, let $d_M = M \cap \kappa$. For $g, h \in \mathscr{F}_{d_M}$ with $g \neq h$, choose $\alpha_{g,h} \in d_M$ such that $g(\alpha_{g,h}) \neq h(\alpha_{g,h})$. Since A_{d_M} is countable, we can apply Remark 3.1 to the set $\{\alpha_{g,h} : g, h \in A_{d_M}\} \in [M]^\omega$ to find $B_M \supseteq \{\alpha_{g,h} : g, h \in A_{d_M}\}$ with $B_M \in M \cap [M]^\omega$. Using the Pressing Down Lemma, find $B \in H_\theta$ and $S' \subseteq S$ stationary such that $B_M = B$ for all $M \in S'$. By Menas' Lemma, the set $E = \{d_M : M \in S'\}$ is stationary in $[\kappa]^{\omega_1}$. Let $b = B \cap \kappa$. Then for every $f \in \mathscr{F}_b$, if f has an extension in $\mathscr{F}_d$ for $d \in E$, this extension is unique. ∎

PROPOSITION 3.2. *Let $\mathscr{F}$ be a (κ, ω_2)-tree with no cofinal branches. Let $\langle A_d : d \in [\kappa]^{\omega_1} \rangle$ be a sequence of collections of nodes such that $A_d \in [\mathscr{F}_d]^\omega$ for every $d \in [\kappa]^{\omega_1}$. Let θ be large enough such that $\{\mathscr{F}, e, \kappa, \ldots\}$ and all relevant parameters belong to H_θ, and let $N \prec H_\theta$ be such that $|N| = \aleph_1$ and $N \supseteq \omega_1$. Then for every $f \in \mathscr{F}$ with* $\mathrm{dom}(f) \supseteq N \cap \kappa$, *there is $d \in N \cap [\kappa]^{\omega_1}$ such that $f\restriction_d \notin A_d$.*

Proof. Suppose otherwise. Fix $f \in \mathscr{F}$ with $\mathrm{dom}(f) \supseteq N \cap \kappa$ such that $f\restriction_d \in A_d$ for every $d \in N \cap [\kappa]^{\omega_1}$. Apply Proposition 3.1 to find $b \in [\kappa]^{\omega_1}$ and a stationary set $E \subseteq [\kappa]^{\omega_1}$ such that one can define for every $g \in \mathscr{F}_b$ a function $F_g : E \to \mathscr{F}$, where $F_g(d)$ is the unique extension in $\mathscr{F}_d$ of g if the extension exists, or the empty set otherwise. Observe that by elementarity, we can take $E, b \in N$ such that F_g is defined in N for every $g \in \mathscr{F}_b \cap N$.

Furthermore, since N is closed under the level enumeration function, and $\omega_1 \cup \{b\} \subseteq N$, we get $\mathscr{F}_b \subseteq N$. In particular $f\restriction_b \in N$, and therefore $F_{f\restriction_b}$ is defined in N. To simplify notation, let $F = F_{f\restriction_b}$, and let $B = \bigcup_{d\in E} F(d)$ (which is also defined in N).

Observe that for any $d \in E$, we get $F(d) \neq \emptyset$, since $F(d) = f\restriction_d$. Also for $d, d' \in E$, $F(d)$ and $F(d')$ are $\leq_{\mathscr{F}}$-comparable since $F(d) = f\restriction_d$ and $F(d') = f\restriction_{d'}$, By our initial supposition of the proof, for every $d \in [\kappa]^{\omega_1} \cap N$, $B\restriction_d (= f\restriction_d) \in \mathscr{F}$. Therefore, by elementarity, B defines in N a cofinal branch in $\mathscr{F}$, a contradiction. ∎

Lemma 3.1. (CC*) *Let $\mathscr{F}$ be a (κ, ω_2)-tree with no cofinal branches. Then there are arbitrarily large θ such that for every countable $M \prec H_\theta$ there are $M_0, M_1 \prec H_\theta$ countable and $a_0 \in M_0 \cap [\kappa]^{\omega_1}$, $a_1 \in M_1 \cap [\kappa]^{\omega_1}$ with*

(1) *$M \cap \omega_1 = M_0 \cap \omega_1 = M_1 \cap \omega_1$,*
(2) *$\mathscr{F}_{a_0} \cap M_0 \perp \mathscr{F}_{a_1} \cap M_1$.*

Proof. Apply CC* to find θ sufficiently large such that all relevant parameters belong to H_θ and such that $\mathrm{CC}(\kappa, \theta)$ holds. Take $M \prec H_\theta$ countable. Our goal is to find a_0, a_1, M_0, M_1 such that (1) and (2) of the present lemma hold.

Let $\theta' > \theta$ be sufficiently large such that $M, \mathscr{F}, e, H_\theta$ and all relevant parameters are members of $H_{\theta'}$ and such that $\mathrm{CC}(\kappa, \theta)$ holds in $H_{\theta'}$. Take $N \prec H_{\theta'}$ of size $\aleph_1$ with $\omega_1 \subseteq N$ and containing all relevant parameters such as M and $\mathscr{F}$. Let $a = N \cap \kappa$. To build a_1 and M_1, simply apply CC* (outside N) to find $M_1 \prec H_\theta$ and $a_1 \in M_1 \cap [\kappa]^{\omega_1}$ such that $a_1 \supseteq a$ and $M_1 \sqsupseteq M$. We will show later that M_1, a_1 are the ones that we are looking for. To find a_0 and M_0 we need a little more. First we prove the following:

Claim 3.1. *Let K be a countable elementary submodel of H_θ with $K \in N$ and let $b \in [\kappa]^{\omega_1}$ with $b \supseteq a$. Then for every $f \in \mathscr{F}_b$, there is $K^* \sqsupseteq K$ with $K^* \in N$ and $c \in K^* \cap [\kappa]^{\omega_1}$ such that $f \perp K^* \cap \mathscr{F}_c$.*

Proof. Assume otherwise. Take $f \in \mathscr{F}_b$ such that for any $K^* \in N$ with $K^* \sqsupseteq K$ and for all $c \in K^* \cap [\kappa]^{\omega_1}$, there is $g_c \in K^* \cap \mathscr{F}_c$ compatible with f.

Remark 3.2. *For every $c \in K^* \cap [\kappa]^{\omega_1}$, f and g_c are not only compatible, but indeed $f \geq g_c$.*

Proof. This follows directly by showing that $c \subseteq a$ ($\subseteq b$) for every $c \in K^* \cap [\kappa]^{\omega_1}$. Since $K^* \in N$ and K^* is countable, we have $K^* \subseteq N$. So if $c \in K^* \cap [\kappa]^{\omega_1}$, in particular $c \in N$. Since $\omega_1 \subseteq N$, we also have $c \subseteq N$, and therefore $c \subseteq N \cap \kappa = a$. ∎

Working in N and using the fact that CC* holds in N, build a sequence $\langle (K_d, d') : d \in [\kappa]^{\omega_1} \rangle$ such that K_d is a countable submodel of H_θ, $K_d \sqsupseteq K$,

$d' \supseteq d$ and $d' \in K_d \cap [\kappa]^{\omega_1}$ for every $d \in [\kappa]^{\omega_1}$. For $d \in [\kappa]^{\omega_1}$, define

$$A_d = \{h \in \mathscr{F}_d : \exists g \in K_d \cap \mathscr{F}_{d'}\ (g \geq_{\mathscr{F}} h)\}.$$

Observe that whenever $d \subseteq d'$, if $h_0, h_1 \in \mathscr{F}_d$ and $g \in \mathscr{F}_{d'}$ with $h_0, h_1 \leq_{\mathscr{F}} g$, then $h_0 = h_1$ (since $h_0 = g\restriction_{\mathrm{dom}(h_0)} = g\restriction_d = g\restriction_{\mathrm{dom}(h_1)} = h_1$). Therefore the cardinality of A_d is at most the cardinality of K_d, which is countable. We can now apply Proposition 3.2 to N, f and $\langle A_d : d \in [\kappa]^{\omega_1}\rangle$ to find $d \in N \cap [\kappa]^{\omega_1}$ such that

$$f\restriction_d \notin A_d. \tag{1}$$

By our assumption at the beginning of the proof of this claim, and by Remark 3.2, there is $g \in K_d \cap \mathscr{F}_{d'}$ with $g \leq_{\mathscr{F}} f$. By definition of A_d, we have $g\restriction_d \in A_d$. But $g\restriction_d = (f\restriction_{d'})\restriction_d = f\restriction_d$, contradicting (1). ∎

We now continue with the proof of Lemma 3.1. Let $\{f_n : n \in \omega\}$ be an enumeration of $M_1 \cap \mathscr{F}_{a_1}$. Then, applying Claim 3.1, build a $\sqsubseteq$-increasing sequence $\langle M(n) : n \in \omega\rangle$ and a sequence $\langle c(n) : n \in \omega\rangle$ such that for every $n \in \omega$, we have $M(n) \sqsupseteq M$, $c(n) \in M(n) \cap [\kappa]^{\omega_1}$ and

$$f_n \perp M(n) \cap \mathscr{F}_{c(n)}. \tag{2}$$

Using CC*, find $M_0 \sqsupseteq \bigcup_{n\in\omega} M(n)$ with $M_0 \prec H_\theta$ and $a_0 \in M_0 \cap [\kappa]^{\omega_1}$ such that $a_0 \supseteq \bigcup_{n\in\omega} c_n$. We claim that (2) of Lemma 3.1 holds for a_0, a_1, M_0 and M_1, i.e. $\mathscr{F}_{a_0} \cap M_0 \perp \mathscr{F}_{a_1} \cap M_1$. To see that, take $n \in \omega$ and $g \in M_0 \cap \mathscr{F}_{a_0}$; we will show that $f_n \perp g$. Observe that

$$M(n) \cap \mathscr{F}_{c(n)} = M_0 \cap \mathscr{F}_{c(n)}, \tag{3}$$

since $c(n) \in M(n) \subseteq M_0$, $M_0 \cap \omega_1 = M(n) \cap \omega_1$ and the enumeration function e is in both M_0 and $M(n)$. Since $g \in M_0 \cap \mathscr{F}_{a_0}$ and $c(n) \in M(n) \subseteq M_0$, we have $g\restriction_{c(n)} \in M_0 \cap \mathscr{F}_{c(n)}$. Therefore, by (3), we get $g\restriction_{c(n)} \in M(n) \cap \mathscr{F}_{c(n)}$. Using (2), we obtain $f_n \perp g\restriction_{c(n)}$, and therefore $f_n \perp g$. ∎

We have the following:

LEMMA 3.2. (CC*) *Let $\mathscr{F}$ be a (κ, ω_2)-tree with no cofinal branches. For λ sufficiently large, if the set*

$$S_{\mathscr{F}} = \{M \in [H_\lambda]^\omega : \exists b \in [\kappa]^{\omega_1}\ \forall f \in \mathscr{F}_b\ \exists a \in M \cap [b]^{\omega_1}\ (f\restriction_a \notin M)\}$$

is nonstationary, then CH *holds.*

Proof. Suppose $S_{\mathscr{F}}$ is nonstationary, and let $F : [H_\lambda]^{<\omega} \to H_\lambda$ be a function such that if $M \in [H_\lambda]^\omega$ is closed under F, then $M \notin S_{\mathscr{F}}$. As before, let $e : [\kappa]^{\omega_1} \times \omega_1 \to \mathscr{F}$ be a surjective function such that $e(a, \xi) \in \mathscr{F}_a$ for every $\xi \in \omega_1$. Let θ be sufficiently large such that $\mathscr{F}, S_{\mathscr{F}}, F, e$ and all relevant parameters are in H_θ and the conclusion of Lemma 3.1 holds.

Using Lemma 3.1, build a binary tree $\langle M_\sigma\rangle_{\sigma\in 2^{<\omega}}$ of countable elementary submodels of H_θ with the property that for every $\sigma \in 2^{<\omega}$,

(1) $M_\sigma \cap \omega_1 = M_{\sigma\frown 0} \cap \omega_1 = M_{\sigma\frown 1} \cap \omega_1$, and
(2) there exist $a_0 \in M_{\sigma\frown 0} \cap [\kappa]^{\omega_1}$ and $a_1 \in M_{\sigma\frown 1} \cap [\kappa]^{\omega_1}$ such that $\mathscr{F}_{a_0} \cap M_{\sigma\frown 0} \perp \mathscr{F}_{a_1} \cap M_{\sigma\frown 1}$.

For every $r \in 2^\omega$, let $M_r = \bigcup_{n\in\omega} M_{r\restriction n}$. Let $b \in [\kappa]^{\omega_1}$ be such that $b \supseteq a$ for every $a \in M_\sigma \cap [\kappa]^{\omega_1}$ and every $\sigma \in 2^{<\omega}$. Since $M_r \prec H_\theta$ and $F \in M_r$, M_r is closed under F, we have $M_r \cap \kappa \notin S_{\mathscr{F}}$. So we can choose $f_r \in \mathscr{F}_b$ such that $f_r\restriction_a \in M_r$ for every $a \in M_r \cap [b]^{\omega_1}$.

Claim 3.2. *The map $r \mapsto f_r$ is an injection from 2^ω to $\mathscr{F}_b$ (and therefore* CH *holds).*

Proof. Let $r_0, r_1 \in 2^\omega$ with $r_0 \neq r_1$ and denote by f_i the node f_{r_i} for $i \in \{0,1\}$. We will find two predecessors of f_0 and f_1 that are incompatible. Let $n \in \omega$ be such that $r_0\restriction_n = r_1\restriction_n = \sigma$, and $r_0\restriction_{n+1} \neq r_1\restriction_{n+1}$. Without loss of generality, suppose $r_i(n) = i$ for $i \in \{0,1\}$. By the construction of our binary tree, we can take $a_0 \in M_{r_0\restriction_{n+1}}$ and $a_1 \in M_{r_1\restriction_{n+1}}$ such that $\mathscr{F}_{a_0} \cap M_{r_0\restriction_{n+1}} \perp \mathscr{F}_{a_1} \cap M_{r_1\restriction_{n+1}}$. However, observe that for $i \in \{0,1\}$, $a_i \in M_{r_i\restriction_{n+1}} \subseteq M_{r_i}$, and so $f_i\restriction_{a_i} \in M_{r_i\restriction_{n+1}}$. Therefore, $f_0\restriction_{a_0}$ and $f_1\restriction_{a_1}$ are incompatible, and so are f_0 and f_1. ∎

This finishes the proof of Lemma 3.2. ∎

We are ready to prove the main theorem of this section.

Theorem 3.1. (CC*) *If* CH *does not hold, then ω_2 has the Strong Tree Property.*

Proof. Assume CH does not hold, but there is a (κ, ω_2)-tree $\mathscr{F}$ with no cofinal branches. From Lemma 3.2, for λ sufficiently large, the set $S_{\mathscr{F}}$ is stationary in $[H_\lambda]^\omega$, and in particular it is semi-stationary. Without loss of generality, we can assume that every set in $S_{\mathscr{F}}$ is closed under e. Since CC* and SSR are equivalent [5, Theorem 5.7], we can apply SSR to obtain $X \in [H_\lambda]$ with $X \supseteq \omega_1$ such that $[X]^\omega \cap S_{\mathscr{F}}$ is semi-stationary. Let

$$S = \{x \in [X]^\omega : \exists M_x \in S_{\mathscr{F}} \cap [X]^\omega\ (x \sqsupseteq M_x)\},$$

which is stationary by definition of semi-stationary set. Take a stationary set $S' \subseteq S$ of size ω_1 ([1]). For $x \in S'$, using the definition of $S_{\mathscr{F}}$, choose $b_x \in [\kappa]^{\omega_1}$ such that for every $f \in \mathscr{F}_{b_x}$, there is $a \in M_x \cap [b_x]^{\omega_1}$ with $f\restriction_a \notin M_x$. Let $b = \bigcup_{x\in S'} b_x$ (and so $|b| = \omega_1$). Fix $f \in \mathscr{F}_b$. Then for $x \in S'$, we can choose $a_x \in M_x \cap [b_x]^{\omega_1}$ such that

$$(f\restriction_{b_x})\restriction_{a_x} = f\restriction_{a_x} \notin M_x. \tag{4}$$

Apply the Pressing Down Lemma to find $a \in [\kappa]^{\omega_1}$ and a stationary set $S'' \subseteq S'$ such that $a_x = a$ for every $x \in S''$. Observe that since S'' is

([1]) For example, let $h : X \to \omega_1$ be a bijection. So the set $\{h^{-1}[\alpha] : \alpha \in \omega_1 \setminus \omega\}$ is a club of size ω_1, and take its intersection with S.

stationary in $[X]^\omega$, it is in particular cofinal in $[X]^\omega$, and since $X \supseteq \omega_1$, we have $\bigcup_{x\in S''}(x \cap \omega_1) = \omega_1$. Therefore we can fix $x \in S''$ and $\xi \in x$ such that $e(a,\xi) = f{\restriction}_a$. However, M_x is closed under e, and $M_x \cap \omega_1 = x \cap \omega_1$ (since $x \sqsupseteq M_x$), and so $e(\xi, a) \in M_x$, contradicting (4). ∎

4. Square sequences. Given a set A of ordinals, we denote by $\mathrm{Lim}(A)$ the collection of limit points of A, i.e. $\alpha \in \mathrm{Lim}(A)$ if $\alpha > 0$ and $\sup(A\cap\alpha) = \alpha$ (so in particular, α is a limit ordinal). Observe also that if $A \subseteq B$, we have $\mathrm{Lim}(A) \subseteq \mathrm{Lim}(B)$.

We recall a two-cardinal version $\square(\lambda,\mu)$ of the square principle.

Definition 4.1. Given a regular cardinal λ and a cardinal $\mu \le \lambda$, $\langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda)\rangle$ is a (λ,μ)-*square sequence* or a $\square(\lambda,\mu)$-*sequence* if

(1) $1 \le |\mathscr{C}_\alpha| \le \mu$,
(2) for every $C \in \mathscr{C}_\alpha$, C is a closed and unbounded subset of α,
(3) for every $C \in \mathscr{C}_\beta$, if $\alpha \in \mathrm{Lim}(C)$, then $C \cap \alpha \in \mathscr{C}_\beta$.

Given a set $C \subseteq \lambda$, we say that C *trivializes* a (λ,μ)-square sequence $\langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda)\rangle$ if $C \cap \alpha \in \mathscr{C}_\alpha$ for every $\alpha \in \mathrm{Lim}(C)$.

We say that the *principle* $\square(\lambda,\mu)$ *holds* if there is a (λ,μ)-square sequence $\langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda)\rangle$ that is trivialized by no club.

We first give some lemmas which describe some properties of square sequences of the form $\square(\lambda,\mu)$.

Lemma 4.1. *For a (λ,μ)-square sequence $\langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda)\rangle$ the following are equivalent:*

(1) *There is a club $D \subseteq \lambda$ trivializing the sequence.*
(2) *There is $C \subseteq \lambda$ such that $\mathrm{Lim}(C)$ is unbounded in λ and a sequence $\langle C_\gamma : \gamma \in \mathrm{Lim}(C)\rangle$ such that for every $\gamma \in \mathrm{Lim}(C)$, $C_\gamma \in \mathscr{C}_\gamma$, and for $\alpha, \beta \in \mathrm{Lim}(C)$, if $\alpha < \beta$ then $C_\alpha = C_\beta \cap \alpha$.*

Proof. (1)⇒(2). Just set $C = D$ and $C_\gamma = D \cap \gamma$ for every $\gamma \in \mathrm{Lim}(D)$.

(2)⇒(1). Take C as in the assumption, and set $D = \bigcup_{\alpha\in\mathrm{Lim}(C)} C_\alpha$. We will show that $D \cap \alpha \in \mathscr{C}_\alpha$ for every $\alpha \in \mathrm{Lim}(D)$.

Take $\alpha \in \mathrm{Lim}(D)$ and $\beta \in \mathrm{Lim}(C)$ such that $\alpha \in C_\beta$. Using the properties of the sequence $\langle C_\gamma : \gamma \in \mathrm{Lim}(C)\rangle$, it is not difficult to show that α is also a limit point of C_β, and so $C_\beta \cap \alpha \in \mathscr{C}_\beta$. Therefore, it is enough to show that $D \cap \alpha = C_\beta \cap \alpha$. Note that already $C_\beta \subseteq D$. So $C_\beta \cap \alpha \subseteq D \cap \alpha$. Therefore, it remains to show that $D \cap \alpha \subseteq C_\beta \cap \alpha$. Observe that by the properties of $\langle C_\gamma : \gamma \in \mathrm{Lim}(C)\rangle$, we can easily verify $C_\gamma \cap \alpha \subseteq C_\beta \cap \alpha$ for every $\gamma \in \mathrm{Lim}(C)$, and therefore $D \cap \alpha = C_\beta \cap \alpha$.

We show that if $\mathrm{Lim}\,(C)$ is unbounded, then D is a club: To show that D is unbounded, take $\beta < \lambda$. Since $\mathrm{Lim}(C)$ is unbounded in λ, there is $\alpha > \beta$

with $\alpha \in \mathrm{Lim}(C)(\subseteq \mathrm{Lim}(\lambda))$ and C_α unbounded in α. To show that D is closed, take an increasing sequence $\langle \beta_\xi : \xi < \gamma \rangle$ of elements of D with $\gamma < \lambda$. Let $\beta = \sup\{\beta_\xi : \xi < \gamma\}$. We wish to show that $\beta \in D$. For every $\xi < \gamma$, there is $\alpha_\xi \in \mathrm{Lim}(C)$ such that $\beta_\xi \in C_{\alpha_\xi}$. Let $\alpha = \sup\{\alpha_\xi : \xi < \gamma\} < \lambda$. Since $\mathrm{Lim}(C)$ is unbounded in λ, let $\eta \in \mathrm{Lim}(C)$ with $\eta > \alpha$. By the properties of C, we have $C_\eta \cap \alpha_\xi = C_{\alpha_\xi}$ for every $\xi < \gamma$, and so $\{\beta_\xi : \xi < \gamma\} \subseteq C_\eta$. Since C_η is closed, $\sup\{\beta_\xi : \xi \in \gamma\} \in C_\eta \subseteq D$. ∎

Remark 4.1. Let λ be a regular uncountable cardinal, and let $\langle \mathscr{C}_\beta : \beta \in \mathrm{Lim}\,(\lambda)\rangle$ be a $\square(\lambda,\mu)$-sequence with $\lambda > \mathrm{cof}(\mu)^+$. For $\beta < \mu$, let $\mathscr{C}_\beta = \{C^\xi_\beta : \xi < \mu\}$. Then for every $\beta \in \lambda \cap \mathrm{Cof}(>\mu)$, there is $\alpha_\beta < \beta$ such that for every $C_\xi, C_\eta \in \mathscr{C}_\beta$, if $C_\xi \neq C_\eta$, then $C_\xi \cap \alpha_\beta \neq C_\eta \cap \alpha_\beta$.

Proof. For $C_\xi, C_\eta \in \mathscr{C}_\beta$ with $C_\xi \neq C_\eta$, choose $\alpha_{\{\xi,\eta\}} < \beta$ such that $C_\xi \cap \alpha_{\{\xi,\eta\}} \neq C_\eta \cap \alpha_{\{\xi,\eta\}}$. If $C_\xi = C_\eta$, let $\alpha_{\{\xi,\eta\}}$ be just any α below β. Let $\alpha_\beta = \sup\{\alpha_{\{\xi,\eta\}} : \{\xi,\eta\} \in [\mu]^2\}$. Since $\mathrm{cof}(\beta) > \mu$, we have $\alpha_\beta < \beta$, and therefore $C_\xi \cap \alpha_\beta \neq C_\eta \cap \alpha_\beta$ for every $\{\xi,\eta\} \in [\mu]^2$ with $C_\xi \neq C_\eta$. ∎

Lemma 4.2. *Let λ be a regular uncountable cardinal, and let $\langle \mathscr{C}_\beta : \beta \in \mathrm{Lim}\,(\lambda)\rangle$ be a $\square(\lambda,\mu)$-sequence with $\lambda > \mathrm{cof}(\mu)^+$ such that no club trivializes this sequence. For $\beta < \mu$, let $\mathscr{C}_\beta = \{C^\xi_\beta : \xi < \mu\}$. For any set $X \subseteq \lambda$ such that $X \cap \mathrm{Cof}(>\mu)$ is stationary, and for every $M \prec H_\theta$ with θ sufficiently large and $\{X, \langle \mathscr{C}_\beta : \beta \in \mathrm{Lim}(\lambda)\rangle\} \cup \mu \subseteq M$, if $\delta = \sup(M \cap \lambda)$, then for every $\xi \in \mu$, the set*

$$\{\alpha \in X \cap M : \alpha \notin \mathrm{Lim}(C^\xi_\delta)\}$$

is unbounded in δ.

Proof. Suppose it is not the case. Then there are $\xi^* \in \mu$ and $\gamma \in M \cap \lambda$ such that $X \cap M \setminus \gamma \subseteq \mathrm{Lim}(C^{\xi^*}_\delta)$. Let $X_0 = X \setminus \gamma$, so in particular $X_0 \in M$, and similarly $\mathrm{Lim}(X_0) \in M$. Observe also that $X_0 \cap \mathrm{Cof}(>\mu)$ is stationary. Applying Fodor's Lemma and Remark 4.1, there is $\alpha \in \lambda$ and a stationary subset $X_1 \subseteq X_0 \cap \mathrm{Cof}(>\mu)$ such that $\alpha_\beta = \alpha$ for every $\beta \in X_1$. Since $X_0 \in M$, by elementarity we can take $\alpha, X_1 \in M$.

Remark 4.2. $C^{\xi^*}_\delta \cap \alpha \in M$.

Proof. Pick any $\beta \in \mathrm{Lim}(X_1) \cap M \setminus \alpha$. As $\mathrm{Lim}(X_1) \cap M \subseteq X_0 \cap M \subseteq \mathrm{Lim}(C^{\xi^*}_\delta)$, there is $\xi_\beta \in \mu\,(\subseteq M)$ such that $C^{\xi^*}_\delta \cap \beta = C^{\xi_\beta}_\beta$. But then $C^{\xi^*}_\delta \cap \alpha = C^{\xi^*}_\delta \cap (\beta \cap \alpha) = (C^{\xi^*}_\delta \cap \beta) \cap \alpha = C^{\xi_\beta}_\beta \cap \alpha$. Since $\alpha, \beta, \xi_\beta \in M$, the set $C^{\xi_\beta}_\beta$ is defined in M, and so $C^{\xi^*}_\delta \cap \alpha \in M$. ∎

To simplify notation, write $C^* = C^{\xi^*}_\delta \cap \alpha$, so by Remark 4.2, $C^* \in M$.

Claim 4.1. *For every $\beta \in \mathrm{Lim}(X_1)$, there is a unique ξ_β such that $C^{\xi_\beta}_\beta \cap \alpha = C^*$.*

Proof. By the elementarity of M, it suffices to prove that Claim 4.1 holds in M. To show existence, using $\mathrm{Lim}(X_1)\cap M\subseteq \mathrm{Lim}(C^{\xi^*}_\delta)$, pick ξ_β such that $C^{\xi^*}_\delta\cap\beta=C^{\xi_\beta}_\beta$. Then $C^{\xi_\beta}_\beta\cap\alpha=(C^{\xi^*}_\delta\cap\beta)\cap\alpha=C^{\xi^*}_\delta\cap(\beta\cap\alpha)=C^{\xi^*}_\delta\cap\alpha=C^*$. To show uniqueness, take $\xi_\beta,\eta_\beta\in\mu$ such that $C^{\xi_\beta}_\beta\neq C^{\eta_\beta}_\beta$. Since $\beta\in X_1$, we have $C^{\xi_\beta}_\beta\cap\alpha\neq C^{\eta_\beta}_\beta\cap\alpha$, so both cannot be equal to C^*. ■

Define now $C_\beta=C^{\xi_\beta}_\beta$ for $\beta\in X_1$. Then the sequence $\langle C_\beta:\beta\in\mathrm{Lim}(X_1)\rangle$ is in M. Observe that for every $\gamma,\beta\in M\cap\mathrm{Lim}(X_1)$, if $\gamma<\beta$ we have $C_\beta\cap\gamma=(C^{\xi^*}_\delta\cap\beta)\cap\gamma=C^{\xi^*}_\delta\cap\gamma=C_\gamma$, contradicting Lemma 4.1. ■

In this section, we prove that assuming SSR, we can have the negation of $\square(\lambda,\omega)$ for every regular cardinal $\lambda\geq\omega_2$.

For a set A of ordinals, define $\sup^+(A)=\sup\{\alpha+1:\alpha\in A\}$. We will use the following useful implications of SSR given by Sakai–Veličković. Fix a regular cardinal $\lambda\geq\omega_2$. For countable sets of ordinals x and y, we write $x\sqsubseteq^* y$ if

- $x\sqsubseteq y$,
- $\sup^+(x)=\sup^+(y)$,
- $\sup^+(x\cap\gamma)=\sup^+(y\cap\gamma)$ for all $\gamma\in E^\lambda_{\omega_1}\cap x$.

Given $X\subset[\lambda]^\omega$ for some $\lambda\geq\omega_1$, we say that X is *weakly full* if X is upward closed under $\sqsubseteq^*$.

LEMMA 4.3 ([19, Lemma 2.2]). *Let $\lambda\geq\omega_2$. Suppose there is a weakly full stationary $X\subseteq[\lambda]^\omega$ such that for every $I\in[\lambda]^{\omega_1}$ with $\omega_1\subseteq I$, there is $J\subseteq\lambda$ such that $I\subseteq J$, $\sup^+(J)=\sup^+(I)$ and $X\cap[J]^\omega$ is nonstationary. Then* SSR(λ) *fails.*

Sakai and Veličković also present a game which will be used to construct a weakly full stationary set. Let λ be a regular cardinal $\geq\omega_2$. For a function $F:[\lambda]^{<\omega}\to\lambda$ let $G_1(\lambda,F)$ be the following game of length ω:

I	α_0 γ_0		α_1 γ_1		$\cdots$	α_n γ_n		$\cdots$
II	β_0		β_1		$\cdots$	β_n		$\cdots$

I and II in turn choose ordinals $<\lambda$. In the nth stage, first I chooses α_n, then II chooses β_n, and then I again chooses $\gamma_n>\beta_n$, with γ_n of cofinality ω_1. I wins if

$$\mathrm{cl}_F(\{\gamma_n:n\in\omega\})\cap[\alpha_m,\gamma_m)=\emptyset$$

for every $m\in\omega$, where $\mathrm{cl}_F(A)$ denotes the closure of the set A under F. Otherwise, II wins.

LEMMA 4.4 ([19, Lemma 2.3]). *Let λ be a regular cardinal $\geq\omega_2$ and let $F:[\lambda]^{<\omega}\to\lambda$. Then I has a winning strategy in the game $G_1(\lambda,F)$.*

Now we state our theorem.

THEOREM 4.1. *For every regular cardinal* $\lambda \geq \omega_2$, $\mathrm{SSR}(\lambda)$ *implies the negation of* $\square(\lambda, \omega)$.

Proof. Assuming that $\square(\lambda, \omega)$ holds, we will show that $\mathrm{SSR}(\lambda)$ fails.

Let $\langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda) \rangle$ be a (λ, ω)-square sequence that is trivialized by no club subset of λ. Without loss of generality, we can assume $|\mathscr{C}_\alpha| = \omega$ for every $\alpha \in \mathrm{Lim}(\lambda)$. Let $\langle C^n_\alpha : n < \omega \rangle$ enumerate $\mathscr{C}_\alpha$.

Let X be the set of all $x \in [\lambda]^\omega$ which have limit order type and there is a sequence $\langle \xi^x_n : n < \omega \rangle$ of ordinals below $\sup(x)$ such that for all $n \in \omega$,

(1) $\sup(x \cap C^n_{\sup^+(x)}) \leq \xi^x_n$,
(2) $\mathrm{cof}(\min(x \setminus \beta)) = \omega_1$ for all $\beta \in C^n_{\sup^+(x)} \setminus \xi^x_n$.

It is not hard to check that X is weakly full. We have the following.

LEMMA 4.5. *X is stationary in* $[\lambda]^\omega$.

Proof. Let $F : [\lambda]^{<\omega} \to \lambda$. We will find $x \in X$ closed under F. By Lemma 4.4, fix a winning strategy τ of I for $G_1(\lambda, F)$. Moreover let C be the set of all limit ordinals $< \lambda$ closed under τ and F. Note that C is club in λ.

Let θ be sufficiently large such that H_θ has all the relevant parameters. We are going to build inductively a sequence $\langle \mathfrak{M}_n : n \in \omega \rangle$ of structures of H_θ as follows: Fix a well-order $<$ of H_θ, let $\mathfrak{M}_0 = \langle H_\theta; \in, <, \langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda) \rangle, F, C, \ldots \rangle$, let $\langle M^0_\xi : \xi < \lambda \rangle$ be a strictly continuous $\subseteq$-increasing sequence of elementary submodels of $\mathfrak{M}_0$ of size $< \lambda$, and define $D_0 = \{\sup(M^0_\xi \cap \lambda) : \xi < \lambda\}$. Observe that D_0 is a club in λ and $D_0 \in H_\theta$. Suppose we have defined a structure $\mathfrak{M}_n$ of H_θ and a strictly continuous $\subseteq$-increasing sequence $\langle M^n_\xi : \xi < \lambda \rangle$ of elementary submodels of $\mathfrak{M}_n$ of size $< \lambda$. Define $D_n = \{\sup(M^n_\xi \cap \lambda) : \xi < \lambda\}$, so that D_n is a club in λ with $D_n \in H_\theta$. Let $\mathfrak{M}_{n+1} = \langle H_\theta; \in, <, \langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda) \rangle, F, C, D_0, \ldots, D_n, \ldots \rangle$.

Let

$$\mathfrak{M} = \langle H_\theta, \in, <, \langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda) \rangle, F, C, \{D_n : n \in \omega\}, \ldots \rangle.$$

Take again a strictly $\subseteq$-increasing continuous sequence $\langle M_\xi : \xi < \lambda \rangle$ of elementary submodels of $\mathfrak{M}$ such that $|M_\xi| < \lambda$ and $M_\xi \cap \lambda$ is transitive for every $\xi < \lambda$. Then the set $\{M_\xi \cap \lambda : \xi < \lambda\}$ is a club in λ, and since E^λ_ω is stationary in λ, we can fix $M \preceq \mathfrak{M}$, with $M \cap \lambda$ transitive and $M \cap \lambda \in E^\lambda_\omega$. Let $\delta = M \cap \lambda$. We have the following:

CLAIM 4.2. *There is an increasing sequence $\langle \delta_n : n < \omega \rangle$ of ordinals such that*

(1) $\delta_n \in C \setminus \bigcup_{i \leq n} \mathrm{Lim}(C^i_\delta)$,
(2) $\sup\{\delta_n : n \in \omega\} = \delta$.

Proof. Fix a strictly increasing sequence $\langle \epsilon_n : n < \omega \rangle \subseteq M$ of limit δ. We proceed by induction. To find δ_0, apply directly Lemma 4.2 to find $\delta_0 \in [\epsilon_0, \delta)$ with $\delta_0 \in C \setminus \mathrm{Lim}(C^0_\delta)$. Fix $n \in \omega$, and suppose we have already built δ_n above ϵ_n.

SUBCLAIM 4.1. *There is a sequence of intervals $[\beta^n_0, \delta^n_0) \supseteq \cdots \supseteq [\beta^n_i, \delta^n_i) \supseteq \cdots \supseteq [\beta^n_n, \delta^n_n)$ with $\beta^n_0 \geq \max\{\delta_n, \epsilon_{n+1}\}$ and $\delta^n_0 < \delta$, and there is a sequence $\langle M^n_i \preceq \mathfrak{M}_{n-i} : i \leq n \rangle$ of elementary submodels such that for every $i \leq n$,*

- $\delta^n_i = \sup(M^n_i \cap \lambda)$,
- $\beta^n_i \in M^n_i$,
- $[\beta^n_i, \delta^n_i) \cap C^j_\delta = \emptyset$ *for every* $j \leq i$.

Proof. Since $D_n \in M$, apply Lemma 4.2 to D_n, M, $\mathrm{Lim}(C^0_\delta)$ and $\max\{\delta_n, \epsilon_{n+1}\}$ to find $\delta^n_0 > \max\{\delta_n, \epsilon_{n+1}\}$ with $\delta^n_0 \in D_n \cap M \setminus \mathrm{Lim}(C^0_\delta)$.

Let $M^n_0 \preceq \mathfrak{M}_n$ be such that $\delta^n_0 = \sup(M^n_0 \cap \lambda)$. Take $\beta^n_0 \in M^n_0 \cap \lambda$ with $\beta^n_0 \geq \max\{\delta_n, \epsilon_{n+1}\}$ and such that $[\beta^n_0, \delta^n_0) \cap C^0_\delta = \emptyset$.

Observe that for $n = 0$ we are already done, so we can assume $n \geq 1$.

For $i < n$, suppose that we have found a sequence of intervals $[\beta^n_0, \delta^n_0) \supseteq \cdots \supseteq [\beta^n_i, \delta^n_i)$ with $\beta^n_0 \geq \max\{\delta_n, \epsilon_{n+1}\}$ and $\delta^n_0 < \delta$ and a sequence $\langle M^n_j \preceq \mathfrak{M}_{n-j} : j \leq i \rangle$ of elementary submodels such that for every $j \leq i$,

- $\delta^n_j = \sup(M^n_j \cap \lambda)$,
- $\beta^n_j \in M^n_j$,
- $[\beta^n_j, \delta^n_j) \cap C^k_\delta = \emptyset$ for every $k \leq j$.

Since $i < n$, the set D_{n-i-1} is well-defined, and since $M^n_i \preceq \mathfrak{M}_{n-i}$, we have $D_{n-i-1} \in M^n_i$.

CASE 1: $\delta^n_i \notin \mathrm{Lim}(C^{i+1}_\delta)$. Choose $\beta^n_{i+1} \in M^n_i \cap \lambda$ with $\beta^n_{i+1} \geq \beta^n_i$ and such that $[\beta^n_{i+1}, \delta^n_i) \cap C^{i+1}_\delta = \emptyset$. Since D_{n-i-1} is unbounded in λ, by elementarity we can find $\delta^n_{i+1} \in D_{n-i-1} \cap M^n_i$ with $\delta^n_{i+1} > \beta^n_{i+1}$, and thus $[\beta^n_{i+1}, \delta^n_{i+1}) \cap C^j_\delta = \emptyset$ for every $j \leq i+1$. Let $M^n_{i+1} \preceq M_{n-i-1}$ be such that $\delta^n_{i+1} = \sup(M^n_{i+1} \cap \lambda)$.

CASE 2: $\delta^n_i \in \mathrm{Lim}(C^{i+1}_\delta)$. Take $k \in \omega$ such that $C^{i+1}_\delta \cap \delta^n_i = C^k_{\delta^n_i}$. Apply Lemma 4.2 to M^n_i, $\mathrm{Lim}(C^k_{\delta^n_i})$, D_{n-i-1} and β^n_i to find $\delta^n_{i+1} \in M^n_i \cap \lambda$ with $\delta^n_{i+1} > \beta^n_i$ and $\delta^n_{i+1} \notin \mathrm{Lim}(C^k_{\delta^n_i})$. Let $M^n_{i+1} \preceq \mathfrak{M}_{n-i-1}$ be such that $\delta^n_{i+1} = \sup(M^n_{i+1} \cap \lambda)$. Take $\beta^n_{i+1} \in M^n_{i+1}$ such that $\beta^n_{i+1} \geq \beta^n_i$ and $[\beta^n_{i+1}, \delta^n_{i+1}) \cap C^k_{\delta^n_i} = \emptyset$. Then $[\beta^n_{i+1}, \delta^n_{i+1}) \cap C^j_\delta = \emptyset$ for every $j \leq i+1$. ∎

Observe that we have defined β^n_n, M^n_n and δ^n_n with $\delta^n_n = \sup(M^n_n \cap \lambda)$. To finish our construction, we again have two cases.

Case 1: $\delta_n^n \notin \mathrm{Lim}(C_\delta^{n+1})$. Choose $\beta_{n+1}^n \in M_n^n \cap \lambda$ with $\beta_{n+1}^n \geq \beta_n^n$ and such that $[\beta_{n+1}^n, \delta_n^n) \cap C_\delta^{n+1} = \emptyset$. Since $C \in M_n^n$, by elementarity we can choose $\delta_{n+1} \in [\beta_{n+1}^n, \delta_n^n) \cap C$, and so δ_{n+1} is as needed.

Case 2: $\delta_n^n \in \mathrm{Lim}(C_\delta^{n+1})$. Take $k \in \omega$ such that $C_\delta^{n+1} \cap \delta_n^n = C_{\delta_n^n}^k$. Apply Lemma 4.2 to M_n^n, $\mathrm{Lim}(C_{\delta_n^n}^k)$, β_n^n and C to find $\delta_{n+1} \in M_n^n \cap C \setminus \beta_n^n$ with $\delta_{n+1} \notin \mathrm{Lim}(C_{\delta_n^n}^k)$. ∎

Now let $\beta_n < \delta_n$ such that $[\beta_n, \delta_n) \cap \bigcup_{i \leq n} C_\delta^i = \emptyset$. Then let $\langle \alpha_n, \gamma_n : n \in \omega \rangle$ be a sequence of I's moves according to τ against $\langle \beta_n : n \in \omega \rangle$. Moreover let $x = \mathrm{cl}_F(\{\gamma_n : n \in \omega\})$. It suffices to prove that $x \in X$. To see this, first note that $\sup^+(x) = \delta$ because δ is closed under F. We are going to check that setting $\xi_n^x = \delta_n$ will witness $x \in X$. Fix $n \in \omega$. Observe that for $m \geq n$, we have $C_\delta^n \cap \delta_m \subseteq \beta_m \subseteq \gamma_m$ by the choice of β_m. Also note that $\alpha_{m+1} < \delta_m$, because $\beta_m \in \delta_m$ and δ_m is closed under τ (since $\delta_m \in C$). Hence $C_\delta^n \cap [\delta_m, \delta_{m+1}) \subseteq [\alpha_{m+1}, \gamma_{m+1})$ for every $m \geq n$. Note that $x \cap [\alpha_{k+1}, \gamma_{k+1}) = \emptyset$ for each $k \in \omega$ because I wins with the play $\langle \alpha_k, \beta_k, \gamma_k : k \in \omega \rangle$. Thus $x \cap C_\delta^n \subseteq \delta_n$. Moreover for $m \geq n$, $\min(x \setminus \beta) = \gamma_{m+1}$ for all $\beta \in C_\delta^n \cap [\delta_m, \delta_{m+1})$, and $\mathrm{cof}(\gamma_{m+1}) = \omega_1$ by the rule of $G_1(\lambda, F)$. Therefore, $\delta_n = \xi_n^x$ witnesses $x \in X$.

This finishes the proof of Subclaim 4.1. ∎

Claim 4.3. *The hypothesis of Lemma* 4.3 *holds for* X.

Proof. The proof is the same as in [19, proof of Claim 2], by fixing just one C_δ^i for some $i \in \omega$. ∎

This completes the proof of Lemma 4.5. ∎

5. Final remarks and open questions. Strong Chang's Conjecture is a consequence of the Weak Reflection Principle and Rado's Conjecture. Sakai and Veličković showed that WRP, together with MA_{ω_1}(Cohen), implies that $\aleph_2$ has the Super Tree Property. However, they also showed that SSR and MA_{ω_1}(Cohen) together do not imply ω_2 has the Super Tree Property (see [19, Theorem 3.5]). Some natural questions arise:

Question 5.1. *Is* WRP + ¬CH *enough to have the Super Tree Property for* ω_2*?*

Question 5.2. *Does Rado's Conjecture, together with* ¬CH*, imply* ω_2 *has the Super Tree Property?*

For example, it is known that if a strongly compact cardinal is Levy collapsed to ω_2, then Rado's Conjecture holds. If starting from a model with a strongly compact cardinal κ, we can force Rado's Conjecture together with the negation of CH by a proper forcing which is an iteration of length κ

of small forcings, then this would answer this question negatively by [30, Corollary 6.10]. We thank the referee for pointing this out.

The following question is also still open:

QUESTION 5.3. *Is* $\mathrm{WRP}(\omega_2)$ *enough to prove that the game* $G(\omega^2)$ *has a winning strategy, so we can get* $\mathrm{WRP}(\omega_2) + \neg\mathrm{CH} \to \mathrm{TP}(\omega_2)$*?*

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