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Existence of a positive ground state solution for a Kirchhoff type problem involving a critical exponent

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Abstract. We consider the following Kirchhoff type problem involving a critical nonlinearity:

$$\begin{cases} -\Big[a+b\Big(\int_\Omega |\nabla u|^2\,dx\Big)^m\Big]\Delta u = f(x,u)+|u|^{2^*-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^N$ ($N\ge 3$) is a smooth bounded domain with smooth boundary $\partial\Omega$, $a>0$, $b\ge 0$, and $0<m<2/(N-2)$. Under appropriate assumptions on f, we show the existence of a positive ground state solution via the variational method.

1. Introduction and main results. The purpose of this article is to investigate the existence of a ground state solution of the Kirchhoff type problem

$$\begin{cases} -(a+b\|u\|^{2m})\Delta u = f(x,u)+|u|^{2^*-2}u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ ($N\ge 3$) is a smooth bounded domain with smooth boundary $\partial\Omega$ and $0<m<2/(N-2)$. Here $2^*=2N/(N-2)$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for every $p\in[1,2^*]$, where $H_0^1(\Omega)$ denotes the usual Sobolev space endowed with the norm $\|u\|^2=\int_\Omega|\nabla u|^2\,dx$ and $L^p(\Omega)$ denotes the usual Lebesgue space with the norm $|u|_p=(\int_\Omega|u|^p\,dx)^{1/p}$; and $f:\overline{\Omega}\times\mathbb{R}^+\to\mathbb{R}$ is a continuous function.

The Kirchhoff equation which included the nonlocal term $M(\|u\|^2)$ was proposed by Kirchhoff [9] in the following problem:

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$$\begin{cases} u_{tt} - M(\|u\|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x); \end{cases}$$

the above equation as an extension of the classical d'Alembert wave equation for free vibrations of elastic strings. Those kinds of problems were also considered in nonlinear vibration theory [14, 15]. In mathematics, the Kirchhoff equation has also been extensively discussed, for example, in [2, 1, 12, 13, 6, 7, 10, 18, 19, 16].

In recent years, the Kirchhoff problem involving critical growth has attracted much attention:

$$\begin{cases} -M(\|u\|^2)\Delta u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda > 0$ is a parameter. Up to now, several existence results have been successfully obtained via the variational and topological methods. By letting the parameter λ be large enough, Alves et al. [1] have verified the existence of a positive solution for problem (1.2) with $N = 3$, and Hamydy et al. [8] have extended their result to the p-Kirchhoff problem. On the basis of [1], Figueiredo et al. [6, 7] have obtained some interesting results by using an appropriate truncation of M.

In the case $N = 3$ and $M(s) = a + bs$, (1.2) has the following form:

$$\begin{cases} -(a + b\|u\|^2)\Delta u = \lambda f(x,u) + u^5 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

By using the Brézis–Lieb Lemma [4], Xie et al. [19] have obtained a positive solution for problem (1.3) with $\lambda = 1$. D. Naimen has used the Second Concentration-Compactness Lemma of Lions [11] to obtain the following result:

THEOREM A (see [12]) *Let $a > 0$ and $b \geq 0$. Suppose that f satisfies the following assumptions:*

- (f_1) *f is continuous in $\overline{\Omega} \times \mathbb{R}$, $f(x,t) \geq 0$ for $t \geq 0$ and $f(x,t) = 0$ for $t \leq 0$, for all $x \in \overline{\Omega}$.*
- (f_2') *$\lim_{t\to 0^+} f(x,t)/t = 0$ and $\lim_{t\to\infty} f(x,t)/t^5 = 0$, uniformly for $x \in \overline{\Omega}$.*
- (f_3') *There exists a constant $\theta > 0$ such that $4 < \theta < 6$ and $f(x,t)t - \theta F(x,t) \geq 0$ for all $x \in \Omega$ and $t \geq 0$, where $F(x,t) = \int_0^t f(x,s)s\,ds$.*
- (f_4) *There exists a nonempty open set $\omega \subset \Omega$ such that $\lim_{t\to\infty} f(x,t)/t^3 = \infty$ uniformly for $x \in \omega$.*

Then problem (1.3) *has a positive solution for all $\lambda > 0$.*

In [13], D. Naimen attacks the Brézis–Nirenberg problem for a 4-dimensional Kirchhoff type problem with critical growth.

Motivated by the work mentioned above, in this paper we verify the existence of a ground state solution for problem (1.1). On the one hand, by giving a weaker assumption on f, we extend Theorem A. On the other hand, we encounter big problems in proving the local $(\mathrm{PS})_c$ condition and estimating the mountain pass value, and we use a new calculation method to overcome these problems.

To state our main results, we make the following assumptions on f.

- (f_1) f is continuous in $\overline{\Omega} \times \mathbb{R}$, $f(x,t) \geq 0$ for $t \geq 0$ and $f(x,t) = 0$ for $t \leq 0$, for all $x \in \overline{\Omega}$.
- (f_2) $\lim_{t\to 0^+} f(x,t)/t = 0$ and $\lim_{t\to\infty} f(x,t)/t^{2^*-1} = 0$, uniformly for $x \in \overline{\Omega}$.
- (f_3) $\frac{1}{2m+2} f(x,t)t - F(x,t) \geq 0$ for all $x \in \overline{\Omega}$ and $t \geq 0$.
- (f_4) There exists a nonempty open set $\omega \subset \Omega$ such that $\lim_{t\to\infty} f(x,t)/t^3 = \infty$ uniformly for $x \in \omega$.
- (f_5) There exist constants $\eta, \mu > 0$ such that $f(x,t) \geq \eta t$ for all $x \in \omega$ and $t \in [\mu, \infty)$, where ω is some nonempty open subset of Ω.
- (f_6) There exists a constant $\eta > 0$ such that $f(x,t) \geq \eta$ for all $x \in \omega$ and $t \in A$, where $A \subset (0, \infty)$ is a nonempty open interval and ω is a nonempty open subset of Ω.

The main results of this paper are the following theorems.

Theorem 1.1. *Suppose $N = 3$, $a > 0$, $b \geq 0$ and $0 < m < 2$. If* (f_1), (f_2'), (f_3) *and* (f_4) *hold, then problem* (1.1) *has a positive ground state solution.*

Corollary 1.1. *Let $a > 0$ and $b \geq 0$. Assume that assumptions* (f_1), (f_2'), (f_4) *are satisfied and*

(f_3'') $\frac{1}{4} f(x,t)t - F(x,t) \geq 0$ *for all* $x \in \overline{\Omega}$ *and* $t \geq 0$.

Then problem (1.3) *has a positive ground state solution.*

Remark 1.1. Corollary 1.1 essentialy extends Theorem A. To see this, it suffices to compare condition (f_3') with (f_3''): obviously, the latter is weaker. Moreover, there are functions covered by our Corollary 1.1, but not by Theorem A, for example,

$$f(x,t) = 4t^3 \ln(1+t^2) + \frac{2t^5}{1+t^2} \qquad \text{for } x \in \overline{\Omega} \text{ and } t \geq 0.$$

Theorem 1.2. *Suppose $N = 4$, $a > 0$, $b \geq 0$ and $0 < m < 1$. If* (f_1)–(f_3) *and* (f_5) *hold, then problem* (1.1) *has a positive ground state solution.*

Theorem 1.3. *Suppose $N \geq 5$, $a > 0$, $b \geq 0$ and $0 < m < 2/(N-2)$. If* (f_1)–(f_3) *and* (f_6) *hold, then problem* (1.1) *has a positive ground state solution.*

Remark 1.2. In this paper, we have to overcome various difficulties. The lack of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is the most difficult one. Moreover, we have to estimate the critical value. In addition, because of the parameter m, we also encounter some calculational problems which will be solved by a new method.

Remark 1.3. As far as we know, results similar to Theorems 1.3 and 1.4 for high-dimensional Kirchhoff problems are rare.

2. Proofs of theorems. We make use of the following notation.

- Let S be the best Sobolev constant, that is,
$$S := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\big(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\big)^{2/2^*}},$$
where $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \mid \partial u/\partial x_i \in L^2(\mathbb{R}^N),\ i = 1,\dots,N\}$.
- $\{u_n\}$ is called a $(\mathrm{PS})_c$ *sequence* for a functional I if $I(u_n) \to c$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$; and I satisfies the $(\mathrm{PS})_c$ *condition* if any $(\mathrm{PS})_c$ sequence has a convergent subsequence.
- C denotes various positive constants.
- $B(x,r) \subset \mathbb{R}^N$ denotes an open ball with center at x and radius r.

We know that finding a solution of problem (1.1) is equivalent to finding a critical point of the C^1 functional
$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}\|u\|^{2m+2} - \int_\Omega F(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx,$$
which implies that
$$\langle I'(u), v\rangle = (a + b\|u\|^{2m})\int_\Omega (\nabla u, \nabla v)\,dx - \int_\Omega f(x,u)v\,dx - \int_\Omega |u|^{2^*-1}v\,dx$$
for all $u, v \in H_0^1(\Omega)$.

The following lemma plays an important role in proving Lemmas 2.2 and 2.4.

Lemma 2.1. *Let $h(r) = a + bS^{mN/2}r^{2m} - r^{2^*-2}$ $(r > 0)$. Then:*

(1) *the equation $h(r) = 0$ has a unique positive solution r_0, which satisfies*
$$a + bS^{mN/2}r_0^{2m} = r_0^{2^*-2}; \tag{2.1}$$

(2) *the set of solutions of $h(r) \le 0$ is $\{r \mid r \ge r_0\}$.*

Proof. (1) Firstly, we show the monotonicity of $h(r)$ on $(0,\infty)$. We have
$$h'(r) = 2mbS^{mN/2}r^{2m-1} - (2^*-2)r^{2^*-3}.$$

The equation $h'(r) = 0$ has a unique positive solution

$$r_1 = \left(\frac{2mbS^{mN/2}}{2^* - 2}\right)^{\frac{1}{2^*-2m-2}}.$$

We easily see that $h(r)$ is increasing in $(0, r_1]$ and decreasing in $[r_1, \infty)$. For $h(0) = a > 0$, one has $h(r_1) > 0$. Since $h(r) \to -\infty$ as $r \to \infty$, we conclude that $h(r) = 0$ has a unique solution r_0 in $(0, \infty)$.

(2) Follows from (1) and the monotonicity of $h(r)$. ∎

The infimum in the definition of the Sobolev constant S is achieved by the function

$$U(x) = \frac{C}{(1 + |x|^2)^{(N-2)/2}},$$

and U satisfies

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^N,$$

which implies (see [17])

$$\int_{\mathbb{R}^N} |\nabla U|^2\, dx = \int_{\mathbb{R}^N} |U|^{2^*}\, dx = S^{N/2}. \tag{2.2}$$

Next, we consider the problem

$$\begin{cases} -\Big[a + b\Big(\int_{\mathbb{R}^N} |\nabla u|^2\, dx\Big)^m\Big]\Delta u = u^{2^*-1} & \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{2.3}$$

where $a, b > 0$ and $0 < m < 2/(N-2)$. Let $u = rU$, where $r \in (0, \infty)$, and insert it into (2.3); this yields

$$-\Big[a + b\Big(\int_{\mathbb{R}^N} |\nabla U|^2\, dx\Big)^m r^{2m}\Big] r\Delta U = r^{2^*-1}U^{2^*-1}.$$

According to (2.2), we have

$$a + bS^{mN/2}r^{2m} = r^{2^*-2}. \tag{2.4}$$

By Lemma 2.1(1), we conclude that r_0U is a positive solution of (2.3), which implies

$$aS^{N/2}r_0^2 + bS^{(m+1)N/2}r_0^{2m+2} = S^{N/2}r_0^{2^*}. \tag{2.5}$$

By taking full advantage of Lemma 2.1, we will verify that

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}\|u\|^{2m+2} - \int_{\Omega} F(x,u)\, dx - \frac{1}{2^*}\int_{\Omega} |u|^{2^*}\, dx$$

satisfies the local $(\mathrm{PS})_c$ condition.

LEMMA 2.2. *Let f satisfy* (f$_2$) *and* (f$_3$). *Suppose that $c < \Lambda$, where*

$$\Lambda = a/2r_0^2 S^{N/2} + \frac{b}{2m+2} r_0^{2m+2} S^{N(m+1)/2} - \frac{1}{2^*} r_0^{2^*} S^{N/2}.$$

Then I satisfies the $(\mathrm{PS})_c$ *condition.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence for I. We claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. In fact, since $I(u_n) \to c$ and $I'(u_n) \to 0$ in $H^{-1}(\Omega)$, by (f$_3$) we have

$$\begin{aligned} 1 + c + o(1)\|u_n\| &\ge I(u_n) - \frac{1}{2m+2}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2m+2}\right) a\|u_n\|^2 + \left(\frac{1}{2m+2} - \frac{1}{2^*}\right) \int_\Omega |u_n|^{2^*}\, dx \\ &\quad + \int_\Omega \left(\frac{1}{2m+2} f(x,u_n)u_n - F(x,u_n)\right) dx \\ &\ge \left(\frac{1}{2} - \frac{1}{2m+2}\right) a\|u_n\|^2. \end{aligned}$$

Since $a > 0$, we conclude $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence there exist a subsequence (still denoted by $\{u_n\}$) and $u \in H_0^1(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H_0^1(\Omega), \\ u_n \to u & \text{strongly in } L^p(\Omega) \text{ for all } 1 \le p < 2^*, \\ u_n(x) \to u(x) & \text{a.e. in } \Omega. \end{cases}$$

Set $w_n = u_n - u$. We claim that $\|w_n\| \to 0$. Otherwise, there exists a subsequence (still denoted by $\{w_n\}$) such that

$$\lim_{n\to\infty} \|w_n\| = l,$$

where l is a positive constant. Then

$$\|u_n\|^2 = \|w_n\|^2 + \|u\|^2 + o(1), \tag{2.6}$$

$$\|u_n\|^{2m+2} = (\|w_n\|^2 + \|u\|^2)^{m+1} + o(1). \tag{2.7}$$

Furthermore, from the Brézis–Lieb Lemma [4],

$$\int_\Omega |u_n|^{2^*}\, dx = \int_\Omega |w_n|^{2^*}\, dx + \int_\Omega |u|^{2^*}\, dx + o(1). \tag{2.8}$$

From $I'(u_n) \to 0$ in $H^{-1}(\Omega)$ and (2.7), we get

$$\begin{aligned} \lim_{n\to\infty} \langle I'(u_n), u\rangle &= a\|u\|^2 + b(l^2 + \|u\|^2)^m \|u\|^2 \\ &\quad - \int_\Omega f(x,u)u\, dx - \int_\Omega |u|^{2^*}\, dx = 0. \end{aligned} \tag{2.9}$$

From (f_3) and (2.9), we obtain

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}\|u\|^{2m+2} - \int_\Omega F(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx \tag{2.10}$$
$$= \left(\frac{1}{2} - \frac{1}{2m+2}\right)a\|u\|^2 + \left(\frac{1}{2m+2} - \frac{1}{2^*}\right)\int_\Omega |u|^{2^*}\,dx$$
$$+ \int_\Omega \left(\frac{1}{2m+2} f(x,u)u - F(x,u)\right)dx$$
$$- \frac{b}{2m+2}((l^2 + \|u\|^2)^m - \|u\|^{2m})\|u\|^2$$
$$\geq - \frac{b}{2m+2}((l^2 + \|u\|^2)^m - \|u\|^{2m})\|u\|^2 =: T.$$

On the other hand, since $I'(u_n) \to 0$ in $H^{-1}(\Omega)$, we get

$$\langle I'(u_n), u_n\rangle = a\|u_n\|^2 + b\|u_n\|^{2m+2} - \int_\Omega f(x,u_n)u_n\,dx - \int_\Omega |u_n|^{2^*}\,dx = o(1). \tag{2.11}$$

By (f_2), for any $\varepsilon > 0$, there exist constants $C, d(\varepsilon) > 0$ such that

$$|f(x,t)t| \leq \frac{\varepsilon}{2C}t^{2^*} + d(\varepsilon).$$

Let $\xi = \varepsilon/2d(\varepsilon) > 0$, and suppose $E \subseteq \Omega$ with $\operatorname{meas} E < \xi$. Then

$$\left|\int_E f(x,u_n)u_n\,dx\right| \leq \int_E |f(x,u_n)u_n|\,dx$$
$$\leq \int_E d(\varepsilon)\,dx + \frac{\varepsilon}{2C}\int_E |u_n|^{2^*}\,dx$$
$$\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,$$

where the last inequality follows from the Sobolev embedding and the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$. Therefore, $\{\int_\Omega f(x,u_n)u_n\,dx : n \in \mathbb{N}\}$ is equi-absolutely continuous. By Vitali's convergence theorem,

$$\int_\Omega f(x,u_n)u_n\,dx \to \int_\Omega f(x,u)u\,dx \quad \text{as } n \to \infty. \tag{2.12}$$

Applying the same method, we can also verify that

$$\int_\Omega F(x,u_n)\,dx \to \int_\Omega F(x,u)\,dx \quad \text{as } n \to \infty. \tag{2.13}$$

Combining (2.11) with (2.6)–(2.8) and (2.12) yields

$$a\|w_n\|^2 + a\|u\|^2 + b(\|w_n\|^2 + \|u\|^2)^{m+1} = \int_\Omega f(x,u)u\,dx + \int_\Omega |w_n|^{2^*}\,dx + \int_\Omega |u|^{2^*}\,dx + o(1). \tag{2.14}$$

By (2.9) and (2.14), we obtain

$$a\|w_n\|^2 + b[(\|w_n\|^2 + \|u\|^2)^{m+1} - (l^2 + \|u\|^2)^m\|u\|^2] = \int_\Omega |w_n|^{2^*}\,dx + o(1). \tag{2.15}$$

From (2.15) and $\int_\Omega |w_n|^{2^*}\,dx \le \|w_n\|^{2^*}/S^{2^*/2}$ we get

$$al^2 + bl^{2m+2} \le al^2 + b[(l^2 + \|u\|^2)^m l^2] \le l^{2^*}/S^{2^*/2},$$

which implies that

$$a + bS^{mN/2}(lS^{-N/4})^{2m} \le (lS^{-N/4})^{2^*-2}.$$

By Lemma 2.1(2), we obtain $lS^{-N/4} \ge r_0$, which implies that

$$l \ge r_0 S^{N/4}. \tag{2.16}$$

It follows from (2.6)–(2.8) and (2.13) that

$$\begin{aligned} I(u_n) &= \frac{a}{2}\|w_n\|^2 + \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}(\|w_n\|^2 + \|u\|^2)^{m+1} - \int_\Omega F(x,u_n)\,dx \\ &\quad - \frac{1}{2^*}\int_\Omega |w_n|^{2^*}\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx + o(1) \\ &= I(u) + \frac{a}{2}\|w_n\|^2 + \frac{b}{2m+2}[(\|w_n\|^2 + \|u\|^2)^{m+1} - \|u\|^{2m+2}] \\ &\quad + \frac{1}{2^*}\int_\Omega |w_n|^{2^*}\,dx + o(1). \end{aligned}$$

Therefore, by (2.15),

$$\begin{aligned} I(u) &= I(u_n) - \frac{a}{2}\|w_n\|^2 - \frac{b}{2m+2}[(\|w_n\|^2 + \|u\|^2)^{m+1} - \|u\|^{2m+2}] \\ &\quad - \frac{1}{2^*}\int_\Omega |w_n|^{2^*}\,dx + o(1) \\ &= I(u_n) - \left(\frac{1}{2} - \frac{1}{2^*}\right)a\|w_n\|^2 - \left(\frac{1}{2m+2} - \frac{1}{2^*}\right)b(\|w_n\|^2 + \|u\|^2)^{m+1} \\ &\quad + \frac{b}{2m+2}\|u\|^{2m+2} - \frac{b}{2^*}(l^2 + \|u\|^2)^m\|u\|^2 + o(1). \end{aligned}$$

Letting $n \to \infty$, by (2.5) and (2.16) we obtain

$$\begin{aligned}
I(u) &= c - \left(\frac{1}{2} - \frac{1}{2^*}\right)al^2 - \left(\frac{1}{2m+2} - \frac{1}{2^*}\right)b(l^2 + \|u\|^2)^{m+1} \\
&\quad + \frac{b}{2m+2}\|u\|^{2m+2} - \frac{b}{2^*}(l^2 + \|u\|^2)^m\|u\|^2 \\
&\le c - \left(\frac{1}{2} - \frac{1}{2^*}\right)al^2 - \left(\frac{1}{2m+2} - \frac{1}{2^*}\right)bl^{2m+2} + T \\
&\le c - \left(\frac{1}{2} - \frac{1}{2^*}\right)ar_0^2S^{N/2} - \left(\frac{1}{2m+2} - \frac{1}{2^*}\right)br_0^{2m+2}S^{N(m+1)/2} + T \\
&= c - \frac{a}{2}r_0^2S^{N/2} - \frac{b}{2m+2}r_0^{2m+2}S^{N(m+1)/2} + \frac{1}{2^*}r_0^{2^*}S^{N/2} + T \\
&= c - \Lambda + T < T,
\end{aligned}$$

which contradicts (2.10). Therefore, $u_n \to u$ in $H_0^1(\Omega)$. ∎

LEMMA 2.3. *Suppose that* (f_1) *and* (f_2) *hold. Then there exists* $\rho > 0$ *such that:*

(1) *there exists* $\alpha > 0$ *such that* $I(u) \ge \alpha > 0$ *whenever* $\|u\| = \rho$*;*
(2) *there exists* $e_0 \in H_0^1(\Omega)$ *such that* $\|e_0\| > \rho$ *and* $I(e_0) < 0$.

Proof. (1) By (f_2), for every ε there exists $C(\varepsilon) > 0$ such that

$$F(x,t) \le \varepsilon t^2 + C(\varepsilon)t^{2^*} \tag{2.17}$$

for all $t \ge 0$ and $x \in \overline{\Omega}$.

According to the Sobolev inequality and (2.17), we have

$$\begin{aligned}
I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}\|u\|^{2m+2} - \int_\Omega F(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx \\
&\ge \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}\|u\|^{2m+2} - \varepsilon|u|_2^2 - C(\varepsilon)|u|_{2^*}^{2^*} - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx \\
&\ge \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}\|u\|^{2m+2} - C\varepsilon\|u\|^2 - CC(\varepsilon)\|u\|^{2^*} - \frac{C}{2^*}\|u\|^{2^*}.
\end{aligned}$$

Hence there exist $\alpha > 0$ and $\rho > 0$ sufficiently small such that $I(u) \ge \alpha > 0$ for all $\|u\| = \rho$ whenever ε small enough.

(2) Fix $v \in H_0^1(\Omega)$ and $v \ne 0$. By (f_1) and (f_2), we have

$$\begin{aligned}
I(tv) &= \frac{a}{2}t^2\|v\|^2 + \frac{b}{2m+2}t^{2m+2}\|v\|^{2m+2} - \int_\Omega F(x,tv)\,dx - \frac{t^{2^*}}{2^*}\int_\Omega |v|^{2^*}\,dx \\
&\le \frac{a}{2}t^2\|v\|^2 + \frac{b}{2m+2}t^{2m+2}\|v\|^{2m+2} - \frac{t^{2^*}}{2^*}\int_\Omega |v|^{2^*}\,dx.
\end{aligned}$$

From the above, we see that $I(tv) \to -\infty$ as $t \to \infty$. Hence, we can choose $t_0 > 0$ large enough such that $\|t_0v\| > \rho$ and $I(t_0v) < 0$. Setting $e_0 = t_0v$ completes the proof. ∎

The Mountain Pass Lemma of [3] yields a sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying

$$I(u_n) \to c \geq \alpha > 0 \quad \text{and} \quad I'(u_n) \to 0,$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_0(\gamma(u)),$$
$$\Gamma = \{\gamma \in (C[0,1], H_0^1(\Omega)) \mid \gamma(0) = 0,\, \gamma(1) = e_0\}.$$

We know that S is also attained by the functions

$$y_\varepsilon(x) = \frac{C_\varepsilon}{(\varepsilon + |x|^2)^{(N-2)/2}}$$

for all $\varepsilon > 0$. Let

$$U_\varepsilon(x) = y_\varepsilon(x)/C_\varepsilon.$$

Without loss of generality, we may assume that $0 \in \omega$, where ω is some nonempty open set in Ω. Moreover, we choose a cut-off function $\phi \in C_0^\infty(\Omega)$ such that $0 \leq \phi \leq 1$ for all $x \in \Omega$ and

$$\phi(x) = \begin{cases} 1, & |x| \leq R, \\ 0, & |x| \geq 2R, \end{cases}$$

where $B_{2R}(0) \subset \Omega$. Set

$$u_\varepsilon(x) = \phi(x) U_\varepsilon(x), \tag{2.18}$$

$$v_\varepsilon(x) = \frac{u_\varepsilon(x)}{(\int_\Omega |u_\varepsilon|^{2^*}\, dx)^{1/2^*}}. \tag{2.19}$$

Then

$$\int_\Omega |v_\varepsilon|^{2^*}\, dx = 1, \tag{2.20}$$

$$\|v_\varepsilon\|^{2m+2} = S^{m+1} + O(\varepsilon^{N-2/2}), \tag{2.21}$$

$$\int_\Omega |v_\varepsilon|^q\, dx = \begin{cases} O(\varepsilon^{q(N-2)/4}), & 1 < q < N/(N-2), \\ O(\varepsilon^{q(N-2)/4}|\ln \varepsilon|), & q = N/(N-2), \\ O(\varepsilon^{2N-q(N-2)/4}), & N/(N-2) < q < 2^*. \end{cases} \tag{2.22}$$

Lemma 2.4. *Let f satisfy* (f$_1$) *and* (f$_2$). *Assume that there is a function $m(u)$ such that $f(x,u) \geq m(u) \geq 0$ for a.e. $x \in \omega$ and all $u \geq 0$, and the primitive $M(t) = \int_0^t m(s)\, ds$ satisfies*

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^{\varepsilon^{-1/2}} M\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(N-2)/2}\right] s^{N-1}\, ds = \infty. \tag{2.23}$$

Then there exists a constant $\varepsilon_0 > 0$ such that

$$\max_{t \geq 0} I(tv_\varepsilon) < \Lambda \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Proof. We define the functions

$$g(t) = I(tv_\varepsilon) = \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{2m+2}t^{2m+2}\|v_\varepsilon\|^{2m+2} - \frac{t^{2^*}}{2^*} - \int_\Omega F(x, tv_\varepsilon)\,dx,$$

$$\tilde{g}(t) = \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{2m+2}t^{2m+2}\|v_\varepsilon\|^{2m+2} - \frac{t^{2^*}}{2^*}.$$

Notice that $\lim_{t\to\infty}\tilde{g}(t) = -\infty$, $\tilde{g}(0) = 0$, and $\tilde{g}(t) > 0$ for $t > 0$ small enough. Hence there exists $t_\varepsilon \in (0,\infty)$ such that

$$\begin{aligned}
0 = \tilde{g}'(t_\varepsilon) &= t_\varepsilon(a\|v_\varepsilon\|^2 + b\|v_\varepsilon\|^{2m+2}t_\varepsilon^{2m} - t_\varepsilon^{2^*-2}) \\
&= t_\varepsilon[a(S + O(\varepsilon^{(N-2)/2})) + b(S^{m+1} + O(\varepsilon^{(N-2)/2})t_\varepsilon^{2m} - t_\varepsilon^{2^*-2}] \\
&= t_\varepsilon[aS + bS^{m+1}t_\varepsilon^{2m} - t_\varepsilon^{2^*-2} + O(\varepsilon^{(N-2)/2})(a + bt_\varepsilon^{2m})] \\
&= t_\varepsilon[aS + bS^{m+1}t_\varepsilon^{2m} - t_\varepsilon^{2^*-2} + O(\varepsilon^{(N-2)/2})],
\end{aligned}$$

which implies

$$aS + bS^{m+1}t_\varepsilon^{2m} - t_\varepsilon^{2^*-2} + O(\varepsilon^{(N-2)/2}) = 0. \tag{2.24}$$

Therefore,

$$a + bS^{Nm/2}\bigg(\frac{t_\varepsilon}{S^{(N-2)/2}}\bigg)^{2m} = \bigg(\frac{t_\varepsilon}{S^{(N-2)/2}}\bigg)^{2^*-2} + O(\varepsilon^{(N-2)/2}).$$

According to Lemma 2.1(1),

$$\frac{t_\varepsilon}{S^{(N-2)/2}} = r_0 + O(\varepsilon^{(N-2)/2}),$$

and so

$$t_\varepsilon = r_0 S^{(N-2)/2} + O(\varepsilon^{(N-2)/2}). \tag{2.25}$$

The function $\tilde{g}(t)$, actually, attains its maximum at t_ε and is increasing in the interval $[0, t_\varepsilon]$.

Since $\lim_{t\to\infty} g(t) = -\infty$, $g(0) = 0$, and $g(t) > 0$ as t small enough, it follows that $\sup_{t\ge 0} g(t)$ is attained for some $t_\varepsilon^0 > 0$, and

$$0 = g'(t_\varepsilon^0) = t_\varepsilon^0\bigg(a\|v_\varepsilon\|^2 + b\|v_\varepsilon\|^{2m+2}(t_\varepsilon^0)^{2m} - (t_\varepsilon^0)^{2^*-2} - \frac{1}{t_\varepsilon^0}\int_\Omega f(x, t_\varepsilon^0 v_\varepsilon)v_\varepsilon\,dx\bigg).$$

This yields

$$a\|v_\varepsilon\|^2 + b\|v_\varepsilon\|^{2m+2}(t_\varepsilon^0)^{2m} = (t_\varepsilon^0)^{2^*-2} + \frac{1}{t_\varepsilon^0}\int_\Omega f(x, t_\varepsilon^0 v_\varepsilon)v_\varepsilon\,dx. \tag{2.26}$$

By (f_2), for all δ, there exists $C > 0$ such that

$$|f(x,t)t| \le \delta t^{2^*} + Ct^2$$

for all $t \geq 0$ and $x \in \overline{\Omega}$. Therefore,

$$\left| \int_\Omega \frac{f(x, t_\varepsilon^0 v_\varepsilon) v_\varepsilon}{t_\varepsilon^0}\, dx \right| \leq \delta (t_\varepsilon^0)^{2^*-2} \int_\Omega v_\varepsilon^{2^*}\, dx + C \int_\Omega v_\varepsilon^2\, dx = \delta (t_\varepsilon^0)^{2^*-2} + C \int_\Omega v_\varepsilon^2\, dx$$

for all δ. Connecting this with (2.22), we obtain

$$\left| \int_\Omega \frac{f(x, t_\varepsilon^0 v_\varepsilon) v_\varepsilon}{t_\varepsilon^0}\, dx \right| \to 0 \quad \text{as } \varepsilon \to 0.$$

Combining (2.24)–(2.26) with (2.21), we get

$$t_\varepsilon^0 \to r_0 S^{(N-2)/2}. \tag{2.27}$$

By (2.21),

$$\begin{aligned}
g(t_\varepsilon^0) &\leq \tilde{g}(t_\varepsilon) - \int_\Omega F(x, t_\varepsilon^0 v_\varepsilon)\, dx \\
&= \frac{a}{2} t_\varepsilon^2 \|v_\varepsilon\|^2 + \frac{b}{2m+2} t_\varepsilon^{2m+2} \|v_\varepsilon\|^{2m+2} - \frac{t_\varepsilon^{2^*}}{2^*} \\
&\quad + \int_\Omega F(x, t_\varepsilon^0 v_\varepsilon)\, dx \\
&= \frac{a}{2} (r_0 S^{(N-2)/2})^2 (S + O(\varepsilon^{(N-2)/2})) - \frac{1}{2^*} (r_0 S^{(N-2)/2})^{2^*} \\
&\quad + \frac{b}{2m+2} (r_0 S^{(N-2)/2})^{2m+2} (S + O(\varepsilon^{(N-2)/2}))^{m+1} \\
&\quad - \int_\Omega F(x, t_\varepsilon^0 v_\varepsilon)\, dx \\
&= \frac{a}{2} r_0^2 S^{N/2} + \frac{b}{2m+2} r_0^{2m+2} S^{N(m+1)/2} - \frac{1}{2^*} r_0^{2^*} S^{N/2} \\
&\quad + O(\varepsilon^{(N-2)/2}) - \int_\Omega F(x, t_\varepsilon^0 v_\varepsilon)\, dx \\
&= \Lambda + O(\varepsilon^{(N-2)/2}) - \int_\Omega F(x, t_\varepsilon^0 v_\varepsilon)\, dx.
\end{aligned} \tag{2.28}$$

According to (2.18), (2.19), (2.27) and the assumption on f, we have

$$\int_\Omega F(x, t_\varepsilon v_\varepsilon)\, dx \geq \int_{|x|<\mathbb{R}} M\left(\frac{C\varepsilon^{(N-2)/4}}{(\varepsilon + |x|^2)^{(N-2)/4}} \right) dx$$

for $\varepsilon > 0$ small enough.

In the following, we will verify that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{(N-2)/2}} \int_{|x|<R} M\left(\frac{C\varepsilon^{(N-2)/4}}{(\varepsilon + |x|^2)^{(N-2)/4}} \right) dx = \infty. \tag{2.29}$$

In fact,

$$\frac{1}{\varepsilon^{(N-2)/2}}\int\limits_{|x|<R} M\biggl(\frac{C\varepsilon^{(N-2)/4}}{(\varepsilon+|x|^2)^{(N-2)/4}}\biggr)\,dx$$

$$= \frac{C}{\varepsilon^{(N-2)/2}}\int\limits_0^R M\biggl(\frac{C\varepsilon^{(N-2)/4}}{(\varepsilon+r^2)^{(N-2)/4}}\biggr) r^{N-1}dr$$

$$= C\varepsilon \int\limits_0^{R\varepsilon^{-1/2}} M\biggl[C\biggl(\frac{\varepsilon^{-1/2}}{1+s^2}\biggr)^{(N-2)/2}\biggr] s^{N-1}\,ds.$$

When $R \le 1$,

$$\varepsilon \int\limits_{R\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} M\biggl[C\biggl(\frac{\varepsilon^{-1/2}}{1+s^2}\biggr)^{(N-2)/2}\biggr] s^{N-1}\,ds \le C\varepsilon M(C\varepsilon^{(N-2)/4})\varepsilon^{-N/2},$$

which is bounded as $\varepsilon \to 0$. Combining this with (2.23), we get (2.29).

On the other hand, when $R \ge 1$, according to (2.23), we have (2.29) obviously. This implies that $\max_{t\ge 0} I(tv_\varepsilon) < \Lambda$ for ε small enough. ∎

The following lemma is based on [5, proof of Corollary 2.3].

LEMMA 2.5. *Suppose $N = 3$, and f satisfies* (f_1) *and* (f_4). *Then the assumption of Lemma* 2.4 *holds.*

Proof. We define $m(u) = \inf_{x\in\omega} f(x,u)$. According to ($f_4$), for all $Q > 0$, there exists a constant $G > 0$ such that $M(u) \ge Qu^4$ for all $u \ge G$. It follows that

$$\varepsilon \int\limits_0^{\varepsilon^{-1/2}} M\biggl[\biggl(\frac{\varepsilon^{-1/2}}{1+s^2}\biggr)^{1/2}\biggr] s^2\,ds \ge Q\varepsilon \int\limits_0^{C\varepsilon^{-1/4}} \frac{\varepsilon^{-1}}{(1+s^2)^2} s^2\,ds.$$

Therefore,

$$\liminf_{\varepsilon\to 0} \varepsilon \int\limits_0^{\varepsilon^{-1/2}} M\biggl[\biggl(\frac{\varepsilon^{-1/2}}{1+s^2}\biggr)^{1/2}\biggr] s^2\,ds \ge Q\int\limits_0^{\infty} \frac{s^2}{(1+s^2)^2}\,ds$$

for all $Q > 0$, which completes the proof. ∎

Proof of Theorem 1.1. Applying Lemma 2.3, we find that I has a mountain pass geometry. Then from the Mountain Pass Lemma, there is a sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying $I(u_n) \to c \ge \alpha > 0$ and $I'(u_n) \to 0$. Moreover, $c < \Lambda$ by Lemmas 2.4 and 2.5. It follows from Lemma 2.2 that $\{u_n\}$ has a convergent subsequence (still denoted by $\{u_n\}$). Suppose that $u_n \to u_0$ in $H_0^1(\Omega)$. By the continuity of I', u_0 is a solution of problem (1.1). Furthermore, $u_0 \ne 0$ for $c > 0$.

For the existence of a ground state solution, we define

$$E = \{I(u) \mid I'(u) = 0,\ u \neq 0\}.$$

Then $E \neq \emptyset$ since $u_0 \neq 0$ and $I'(u_0) = 0$. Now, we claim that E has an infimum. In fact, for any $u \in E$,

$$\langle I'(u), u\rangle = a\|u\|^2 + b\|u\|^{2m+2} - \int_\Omega f(x,u)u\,dx - \int_\Omega |u|^{2^*}\,dx = 0. \tag{2.30}$$

According to (2.30) and (f_3),

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{2m+2}\|u\|^{2m+2} - \int_\Omega F(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx \\ &= \left(\frac{1}{2} - \frac{1}{2m+2}\right)a\|u\|^2 + \left(\frac{1}{2m+2} - \frac{1}{2^*}\right)\int_\Omega |u|^{2^*}\,dx \\ &\quad + \int_\Omega \left(\frac{1}{2m+2}f(x,u)u - F(x,u)\right)dx \\ &\ge \left(\frac{1}{2} - \frac{1}{2m+2}\right)a\|u\|^2 \ge 0. \end{aligned}$$

Therefore, we can define

$$E_0 = \inf\{I(u) \mid I'(u) = 0,\ u \neq 0\}.$$

We get $\{v_n\}$ such that $I(v_n) \in E$ and $I(v_n) \to E_0$. Since we know $I'(u_0) = 0$ and $I(u_0) = c$, we have $E_0 \le c < \Lambda$. By Lemma 2.2, $\{v_n\}$ has a strongly convergent subsequence (still denoted by $\{v_n\}$). Hence, there exists $v_0 \in H_0^1(\Omega)$ such that $v_n \to v_0$ in $H_0^1(\Omega)$. Then $I(v_0) = E_0$ and $I'(v_0) = 0$.

Finally, we prove $v_0 \neq 0$. By (f_2), there exists a constant $C > 0$ such that

$$f(x,t)t \le \frac{a}{2}\lambda_1 t^2 + Ct^{2^*}$$

for all $t \ge 0$ and $x \in \overline{\Omega}$, where $\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \|u\|^2/|u|^2$. From $\langle I'(v_n), v_n\rangle = 0$ and the Sobolev inequality, it follows that

$$\begin{aligned} a\|v_n\|^2 \le a\|v_n\|^2 + b\|v_n\|^{2m+2} &= \int_\Omega f(x,v_n)v_n\,dx + \int_\Omega |v_n|^{2^*}\,dx \\ &\le \frac{a}{2}\lambda_1 \int_\Omega |v_n|^2\,dx + (C+1)\int_\Omega |v_n|^{2^*}\,dx \le \frac{a}{2}\|v_n\|^2 + C\|v_n\|^{2^*}. \end{aligned}$$

Therefore,

$$\frac{a}{2}\|v_n\|^2 \le C\|v_n\|^{2^*},$$

which implies $0 < C \le \|v_n\|$ for all n. Hence, $v_0 \neq 0$. Furthermore, $\langle I'(v_0), v_0^-\rangle = 0$, where $v_0^- = \max\{-v_0, 0\}$. Hence, $v_0 \ge 0$. According to the strong maximum principle, v_0 is a positive solution of problem (1.1), completing the proof of Theorem 1.1. ■

Proof of Corollary 1.1. Because (f_3) is (f_3'') in the case $m = 1$, Corollary 1.1 is a special case of Theorem 1.1. ■

The following lemma is based on [5, proof of Corollary 2.2].

LEMMA 2.6. *Suppose $N = 4$, and f satisfies* (f_1) *and* (f_5). *Then the assumption of Lemma* 2.4 *holds.*

Proof. By (f_1) and (f_5), we obtain

$$f(x,u) \geq \eta u \chi_{[\mu,\infty)}(u) = m(u)$$

for all $x \in \omega$, and $u \geq 0$, where $\chi_{[\mu,\infty)}$ is the characteristic function of $[\mu,\infty)$. Thus,

$$M(u) = \frac{1}{2}\eta(u^2 - \mu^2) \quad \text{for } u \geq \mu.$$

Therefore,

$$\varepsilon \int_0^{\varepsilon^{-1/2}} M\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)\right] s^3\, ds \geq \frac{1}{4}\eta\varepsilon \int_0^{C\varepsilon^{-1/4}} \frac{\varepsilon^{-1}}{(1+s^2)^2} s^3\, ds = C|\ln \varepsilon|.$$

Hence,

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^{\varepsilon^{-1/2}} M\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(N-2)/2}\right] s^{N-1}\, ds = \infty. \ ■$$

Proof of Theorem 1.2. By using Lemmas 2.1–2.4 and 2.6, much as in the proof of Theorem 1.1, we can easily show that problem (1.1) has a positive ground state solution. ■

The following lemma is based on [5, proof of Corollary 2.1].

LEMMA 2.7. *Suppose $N \geq 5$, and f satisfies* (f_1) *and* (f_6). *Then the assumption of Lemma* 2.4 *holds.*

Proof. By (f_1) and (f_6), we have

$$f(x,u) \geq \eta \chi_A(u) = m(u)$$

for all $x \in \omega$ and $u \geq 0$. Since A is nonempty, there exist constants $d \in A$ and $\xi > 0$ such that

$$M(u) \geq \xi > 0 \quad \text{for all } u \geq a.$$

If $\frac{\varepsilon^{-1/2}}{1+s^2} \geq a^{2/(N-2)}$, then

$$M\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(N-2)/2}\right] \geq \xi \quad \text{as } s \leq C\varepsilon^{-1/4}.$$

Therefore,

$$\varepsilon \int_0^{\varepsilon^{-1/2}} M\left[\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(N-2)/2}\right] s^{N-1}\,ds \geq \xi\varepsilon \int_0^{C\varepsilon^{-1/4}} s^{N-1}\,ds = C\varepsilon^{1-N/4}.$$

Since $N \geq 5$, we get $1 - N/4 < 0$. Hence $C\varepsilon^{1-N/4} \to \infty$ as $\varepsilon \to 0$. ■

Proof of Theorem 1.3. By using Lemmas 2.1–2.4 and 2.7, and reasoning as in the proof of Theorem 1.1, we can easily prove that problem (1.1) has a positive ground state solution. ■

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