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Topological algebras of random elements

by

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Abstract. Let $L_0(\Omega; A)$ be the Fréchet space of Bochner-measurable random variables with values in a unital complex Banach algebra A. We study $L_0(\Omega; A)$ as a topological algebra, investigating the notion of spectrum in $L_0(\Omega; A)$, the Jacobson radical, ideals, hulls and kernels. Several results on automatic continuity of homomorphisms are developed, including versions of well-known theorems of C. Rickart and B. E. Johnson.

1. Introduction. The study of random elements in various finite- and infinite-dimensional vector spaces has a long history, encompassing such areas as random-matrix theory [10], [13], the theory of random operators [2], and probability theory on Banach spaces [2], [21]–[30]. In the present paper, we focus on the case when the elements are chosen from a Banach algebra A, investigating how the theory of Banach algebras reflects on the space of random elements of A.

Inspiration for the present work stems from the earlier work of the authors. If T is a compact Hausdorff space, papers [4], [5], [32] can be viewed as studying the space $C(T; L_0(\mathbb{C}))$ of continuous functions with random values, whereas the current work deals with the space $L_0(\Omega; C(T))$ of random continuous functions. It is easy to see that

$$L_0(\Omega; C(T)) \subset C(T; L_0(\mathbb{C})).$$

On the other hand, automatic continuity of random derivations was consid-

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ered in detail in the work of the second author and A. R. Villena [33], [34], motivating our interest here in automatic continuity of homomorphisms.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and A be a complex Banach algebra. To avoid trivialities, we shall take μ to be nonatomic and complete. For simplicity, we will assume throughout the paper that A has an identity e, although much of what we say below can be modified to hold for the nonunital case by relating it to the unital case in the standard way by adjoining a unit. We wish to study the algebra $L_0(\Omega; A)$ of all equivalence classes of Bochner-measurable functions (sometimes called Effros measurable, as in [8]) from Ω to A, with the topology of convergence in measure.

After a short review in Section 2 of the needed properties of the space $L_0(\Omega; X)$ when X is a Banach space, we begin to study $L_0(\Omega; A)$ as a topological algebra. In particular, we observe that the set of invertible elements is not open (Prop. 2.1), in contrast to the case of a Banach algebra. This in turn explains why maximal (left/right/two-sided) ideals in $L_0(\Omega; A)$ are not closed but dense.

We examine the notion of spectrum for such algebras in Section 3. Easy examples (Examples 3.1) show that the classical notion of spectrum of an element $\boldsymbol{a} \in L_0(\Omega; A)$ as those complex numbers λ for which $\boldsymbol{a} - \lambda$ is not invertible in $L_0(\Omega; A)$ can lead to an empty or noncompact spectrum. This leads us to exploit the "stochastic spectrum"

$$\sigma_{\mathrm{s}}(\boldsymbol{a}) = \{\boldsymbol{\lambda} \in L_0(\Omega) \colon \boldsymbol{a}(\omega) - \boldsymbol{\lambda}(\omega) \notin \operatorname{Inv} A \text{ a.s.}\}$$

and the corresponding "stochastic spectral radius" and "stochastically quasinilpotent" elements. Both are defined in the obvious way, and the spectral radius is shown to exist as a random variable (Prop. 3.4). This leads to a study of the Jacobson radical. Since maximal ideals are not closed, the radical is somewhat mysterious. Nevertheless, in the important case when A is separable, we prove (Theorem 3.10) that the natural conjecture holds:

$$\operatorname{Rad} L_0(\Omega; A) = L_0(\Omega; \operatorname{Rad} A).$$

This has two interesting consequences: Although all maximal ideals are dense in $L_0(\Omega; A)$, the radical is closed. And $L_0(\Omega; A)$ is semisimple if and only if A is.

Section 4 is devoted to the ideal theory of $L_0(\Omega; A)$. When A is simple, we prove that the closed ideals in $L_0(\Omega; A)$ are the obvious ones, namely those determined by the characteristic functions of measurable sets (Theorem 4.3). When A is a commutative Banach algebra, there are appropriate notions of hulls and kernels in the theory of ideals of $L_0(\Omega; A)$. The section concludes by showing that if X is a separable compact Hausdorff space, then hulls determine unique closed ideals in $L_0(\Omega; C(X))$, and that spectral synthesis holds for a large class of ideals (Theorem 4.7).

Finally, in Section 5, we develop in the current context versions of the celebrated theorems on automatic continuity of homomorphisms due to C. Rickart (Theorem 5.2) and B. E. Johnson (Theorem 5.7).

For the general theory of Banach algebras, see, for example, [3], [9], or [19].

2. The spaces $L_0(\Omega;X)$ and $L_0(\Omega;A)$. For a Banach space X, and with respect to the Borel σ-algebra on X, the space $L_0(\Omega;X)$ consists of all (equivalence classes of) almost-sure limits of $\mathcal{F}$-measurable simple functions, or equivalently, all $\mathcal{F}$-measurable functions with separable range. Note that the classical Pettis Measurability Theorem [20], [14, Sec. 3.5], [11, Sec. II.1] asserts that $L_0(\Omega;X)$ consists of all (equivalence classes of) functions with separable range which are "weakly measurable", and it is well known that these two conditions imply $\mathcal{F}$-measurability.

The restriction to the class of Bochner-measurable functions, as opposed to all (equivalence classes of) Borel-measurable functions from Ω to X stems from the well-known fact that if X is not separable, then the sum of two Borel-measurable functions and the norm function need not be Borel-measurable. Elements of $L_0(\Omega;X)$ will be denoted $\boldsymbol{x},\boldsymbol{y}$, and so on. Recall that for $\boldsymbol{x}\in L_0(\Omega;X)$, the norm function $\|\boldsymbol{x}(\cdot)\|$ is measurable, and a metric on $L_0(\Omega;X)$ that induces its topology is given by

$$d(\boldsymbol{x}_1,\boldsymbol{x}_2)=\mathbb{E}[\min\{\|\boldsymbol{x}_1-\boldsymbol{x}_2\|,1\}]=\int_\Omega \min\{\|\boldsymbol{x}_1-\boldsymbol{x}_2\|,1\}\,d\mu.$$

This metric enjoys the following properties:

(i) (translation invariance) $d(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{y}+\boldsymbol{z})=d(\boldsymbol{x},\boldsymbol{y})$;
(ii) (subadditivity) $d(\boldsymbol{x}+\boldsymbol{y},0)\le d(\boldsymbol{x},0)+d(\boldsymbol{y},0)$;
(iii) (monotonicity) $d(\boldsymbol{\lambda x},\boldsymbol{\lambda y})\le d(\boldsymbol{x},\boldsymbol{y})$ if $\boldsymbol{\lambda}\in L_0(\Omega)$ with $0\le\boldsymbol{\lambda}\le 1$.

It is also well known that $L_0(\Omega;X)$ is a Fréchet topological vector space which is not locally convex, and if A is a Banach algebra with identity, then $L_0(\Omega;A)$ is a topological algebra with identity. If we denote $L_0(\Omega,\mathbb{C})$ by $L_0(\Omega)$, then $L_0(\Omega;X)=L_0(\Omega)\overline{\otimes}X$, the completion being in the topological vector space topology of $L_0(\Omega;X)$, and we may consider $L_0(\Omega;X)$ as a module over $L_0(\Omega)$. Note that if $(\Omega',\mathcal{F}',\nu)$ is a probability space, Y is a Banach space, $\psi\colon\Omega'\to\Omega$ is a measurable map, $\varphi\colon X\to Y$ is continuous, and $\boldsymbol{x}\in L_0(\Omega;X)$, then

$$\varphi\circ\boldsymbol{x}\circ\psi\in L_0(\Omega';Y).$$

Recall that a subset E of $L_0(\Omega;X)$ is bounded in the topological vector space sense if and only if for every $\varepsilon>0$ there exists $M_\varepsilon>0$ such that

$$\mu[\|\boldsymbol{x}\|\ge M_\varepsilon]<\varepsilon,\quad \boldsymbol{x}\in E.$$

Such sets are usually called *stochastically bounded.*

In the obvious way, we may embed X and $\mathbb{C}$ in $L_0(\Omega; X)$ as constant random variables, while if A is as above, we may embed $L_0(\Omega)$ in $L_0(\Omega; A)$ as multiples of the identity. This embeds X and $L_0(\Omega)$ as closed subspaces of $L_0(\Omega; X)$ and $L_0(\Omega; A)$, respectively, since it is clear that the relative topologies on X and $L_0(\Omega)$ are the given ones. We will denote (embedded) elements $L_0(\Omega)$ as $\boldsymbol{\lambda}, \boldsymbol{\eta}, \ldots$, while scalars and elements of X are in light face. In particular, the characteristic (indicator) function of a set $\Omega_0 \subset \Omega$ is denoted by $\boldsymbol{\chi}_{\Omega_0}$.

We shall need some terminology from the theory of multi-valued functions and their selections. If X is a topological space, a multifunction (or relation) $F\colon \Omega \to X$ is called *closed* if $F(\omega)$ is closed in X for all $\omega \in \Omega$. The *graph* of F, i.e. the relation itself, is the set $\mathrm{Gr}(F) = \{(\omega, x)\colon x \in F(\omega)\}$. By a (*measurable*) *selection* for F we mean an element $\boldsymbol{f} \in L_0(\Omega; X)$ such that $\boldsymbol{f}(\omega) \in F(\omega)$ for all $\omega \in \Omega$. We shall call F *measurable* if $\mathrm{Gr}(F)$ is $\mathcal{F} \times \mathcal{B}(X)$-measurable, where $\mathcal{B}(X)$ denotes the Borel σ-algebra of X.

If F is a multifunction as above, the *closure* $\overline{F}$ is the multifunction given by $\overline{F}(\omega) = \overline{F(\omega)}$, where bar denotes closure. If $\{F_\iota\}_{\iota \in I}$ is a family of such multifunctions, define the multifunction $\bigcup_\iota F_\iota$ in the obvious way: $(\bigcup_\iota F_\iota)(\omega) = (\bigcup_\iota F_\iota(\omega))$. A standard reference on measurable relations and selections is [1, Chapter 8].

Let B be a complex algebra with unit e, and let $\mathrm{Inv}\, B$ denote the set of invertible elements of B. In our Banach algebra A, $\mathrm{Inv}\, A$ is open and inversion is continuous on $\mathrm{Inv}\, A$. It is easy to see that an element $\boldsymbol{a} \in L_0(\Omega; A)$ is invertible in $L_0(\Omega; A)$ if and only if $\boldsymbol{a} \in \mathrm{Inv}\, A$ a.s. The relationship between invertibility and the topology on $L_0(\Omega; A)$ is given by the following proposition.

Recall that if $a \in \mathrm{Inv}\, A$ and $\|x - a\| < \|a^{-1}\|^{-1}$, then $x \in \mathrm{Inv}\, A$. And if $a, b \in \mathrm{Inv}\, A$ are such that $\|b - a\| \le \frac{1}{2}\|a^{-1}\|^{-1}$, then

$$\|b^{-1} - a^{-1}\| \le 2\|a^{-1}\|^2 \|b - a\|.$$

Proposition 2.1. *Let $\boldsymbol{a} \in \mathrm{Inv}\, L_0(\Omega; A)$ and $\varepsilon > 0$.*

(i) *There is a neighborhood U of $\boldsymbol{a}$ in $L_0(\Omega; A)$ such that*

$$\mu[\boldsymbol{b} \in \mathrm{Inv}\, A] > 1 - \varepsilon$$

for all $\boldsymbol{b} \in U$. On the other hand, no open set in $L_0(\Omega; A)$ consists entirely of invertible elements.

(ii) *There is a neighborhood V of $\boldsymbol{a}$ in $L_0(\Omega; A)$ such that if*

$$\boldsymbol{b} \in (\mathrm{Inv}\, L_0(\Omega; A)) \cap V,$$

then $\mu[\|\boldsymbol{b}^{-1} - \boldsymbol{a}^{-1}\| \ge \varepsilon] < \varepsilon$. Thus the inversion map on $\mathrm{Inv}\, L_0(\Omega; A)$ *is continuous.*

Proof. (i) Choose $M > 2/\varepsilon$ such that $\mu(\Omega_0) > 1 - \varepsilon/2$, where
$$\Omega_0 = \{\omega\colon \|a^{-1}(\omega)\| < M\}.$$
Let
$$U = \{\boldsymbol{b} \in L_0(\Omega; A)\colon \mu[\|\boldsymbol{b} - \boldsymbol{a}\| \ge 1/M] < 1/M\}.$$
Then for all $\boldsymbol{b} \in U$,
$$\begin{aligned}\mu[\boldsymbol{b} \in \operatorname{Inv} A] &\ge \mu(\Omega_0 \cap [\|\boldsymbol{b} - \boldsymbol{a}\| < 1/M]) \ge \mu(\Omega_0) - 1/M \\ &> 1 - \varepsilon/2 - 1/M > 1 - \varepsilon.\end{aligned}$$
The second assertion is clear, since any invertible element of $L_0(\Omega; A)$ can be modified on a set of arbitrarily small positive measure so as not to be invertible.

(ii) Let Ω_0 be as in (i) and
$$V = \{\boldsymbol{b} \in L_0(\Omega; A)\colon \mu[\|\boldsymbol{b} - \boldsymbol{a}\| \ge \varepsilon/2M^2] < \varepsilon/2\}.$$
If $\boldsymbol{b} \in (\operatorname{Inv} L_0(\Omega; A)) \cap V$ and $\omega \in \Omega_0$, then outside of a set of measure at most $\varepsilon/2$,
$$\|\boldsymbol{b}^{-1} - \boldsymbol{a}^{-1}\| \le 2\|\boldsymbol{a}^{-1}\|^2\|\boldsymbol{b} - \boldsymbol{a}\| < 2M^2\left(\frac{\varepsilon}{2M^2}\right) = \varepsilon.$$
Thus
$$\mu[\|\boldsymbol{b}^{-1} - \boldsymbol{a}^{-1}\| \ge \varepsilon] \le \mu(\Omega_0^c) + \varepsilon/2 < \varepsilon. \ ■$$

3. The spectrum and the radical. For a complex algebra B with unit e, recall that the *spectrum* of an element b in B is the set
$$\sigma_B(b) = \{\lambda \in \mathbb{C}\colon a - \lambda e \notin \operatorname{Inv} B\}.$$
When B is a Banach algebra, it is well known that for b in B, $\sigma_B(b)$ is a compact, nonempty set in $\mathbb{C}$. But for $\boldsymbol{a}$ in $L_0(\Omega; A)$ neither of these properties is guaranteed, as shown by the following examples.

Examples 3.1. (1) For many algebras A and every probability space we are considering, there exist $\{a_n\}_{n=1}^{\infty} \subset A$ and $\{\Omega_n\}_{n=1}^{\infty}$ disjoint subsets of Ω of positive measure such that
$$\bigcup_{n=1}^{\infty} \sigma_A(a_n) = \mathbb{C} \quad \text{and} \quad \bigcup_{n=1}^{\infty} \Omega_n = \Omega.$$
If we set
$$\boldsymbol{a}(\omega) = a_n, \qquad \omega \in \Omega_n,\ n = 1, 2, \ldots,$$
then for every $n \in \mathbb{N}$ and $\lambda \in \sigma_A(a_n)$, $\mu[\lambda \in \sigma_A(\boldsymbol{a})] > 0$, so $\sigma_{L_0(\Omega;A)}(\boldsymbol{a}) = \mathbb{C}$.

(2) Let $a \in A$ be such that $\sigma_A(a) = \{1\}$ (e.g., $a = e$). If $\boldsymbol{\varphi}$ is a measurable function which maps Ω onto $\mathbb{C}$ in such a way that
$$\mu[\boldsymbol{\varphi} = \lambda] = 0$$

for all $\lambda \in \mathbb{C}$, and if $\boldsymbol{a}(\omega) = \boldsymbol{\varphi}(\omega)a$, then $\boldsymbol{a} - \lambda \in \operatorname{Inv} L_0(\Omega; A)$ for all $\lambda \in \mathbb{C}$, so $\sigma_{L_0(\Omega;A)}(\boldsymbol{a}) = \emptyset$.

To overcome these difficulties, we introduce the following definition. The notation parallels that of [10] for random matrices.

DEFINITION 3.2. For $\boldsymbol{a} \in L_0(\Omega; A)$, the *stochastic spectrum* of $\boldsymbol{a}$ is the set $\sigma_{\mathrm{s}}(\boldsymbol{a})$ of "random spectral elements," i.e.,

$$\begin{aligned}\sigma_{\mathrm{s}}(\boldsymbol{a}) &= \{\boldsymbol{\lambda} \in L_0(\Omega) \colon \boldsymbol{a}(\omega) - \boldsymbol{\lambda}(\omega) \notin \operatorname{Inv} A \text{ a.s.}\}\\ &= \{\boldsymbol{\lambda} \in L_0(\Omega) \colon \boldsymbol{\lambda}(\omega) \in \sigma_A(\boldsymbol{a}(\omega)) \text{ a.s.}\}.\end{aligned}$$

It follows easily from Proposition 3.3 below that every element of $\sigma_{L_0(\Omega;A)}(\boldsymbol{a})$ agrees with an element of $\sigma_{\mathrm{s}}(\boldsymbol{a})$ on a set of positive probability. In particular, for $\boldsymbol{a}$ as in Example 3.1(1),

$$\sigma_{\mathrm{s}}(\boldsymbol{a}) = \{\boldsymbol{\lambda} \in L_0(\Omega) \colon \boldsymbol{\lambda}(\omega) \in \sigma_A(a_n),\ \omega \in \Omega_n,\ n = 1, 2, \ldots\},$$

while in Example 3.1(2), $\sigma_{\mathrm{s}}(\boldsymbol{a}) = \{\boldsymbol{\varphi}\}$.

Although $\sigma_{L_0(\Omega;A)}(\boldsymbol{a})$ may be empty, $\sigma_{\mathrm{s}}(\boldsymbol{a})$ cannot be empty, as we will point out. We would like the natural choice of spectrum to be nonempty, so it seems appropriate to use the stochastic spectrum as our working notion of spectrum in $L_0(\Omega; A)$.

If Λ is a set of $\mathcal{F}$-measurable, $\mathbb{C}$- or A-valued functions and $\omega \in \Omega$, let $\Lambda(\omega) = \{\boldsymbol{\lambda}(\omega) \colon \boldsymbol{\lambda} \in \Lambda\}$.

PROPOSITION 3.3. *For any $\boldsymbol{a} \in L_0(\Omega; A)$, we have $\sigma_{\mathrm{s}}(\boldsymbol{a}) \neq \emptyset$. In fact, there is a countable set $\Lambda \subset \sigma_{\mathrm{s}}(\boldsymbol{a})$ such that $\Lambda(\omega)$ is dense in $\sigma_{\mathrm{s}}(\boldsymbol{a}(\omega))$ a.s.*

Proof. Recall that the spectrum function is upper semicontinuous on A, meaning that for any open set U in $\mathbb{C}$, $\{a \colon \sigma_A(a) \subset U\}$ is open. So for any such U and $\boldsymbol{a} \in L_0(\Omega; A)$, we have $\{\omega \colon \sigma_A(\boldsymbol{a}(\omega)) \subset U\} \in \mathcal{F}$. Thus $\sigma_A(\boldsymbol{a}(\cdot))$ is a measurable multifunction in the sense of [18]. The classical theorem of Kuratowski and Ryll-Nardzewski [18] now says that there exists a measurable selection for this multifunction, so $\sigma_{\mathrm{s}}(\boldsymbol{a}) \neq \emptyset$. The second assertion follows from the generalization of the above theorem which appears in [15, Theorem 5.6]. ∎

REMARK 3.4. There is another natural notion of spectrum on $L_0(\Omega; A)$, namely the *full spectrum*

$$\begin{aligned}\sigma(\boldsymbol{a}) &= \{\boldsymbol{\lambda} \in L_0(\Omega) \colon \boldsymbol{a} - \boldsymbol{\lambda} \notin \operatorname{Inv} L_0(\Omega; A)\}\\ &= \{\boldsymbol{\lambda} \in L_0(\Omega) \colon \mu(\{\omega \colon \boldsymbol{\lambda}(\omega) \in \sigma(\boldsymbol{a}(\omega))\}) > 0\}.\end{aligned}$$

Clearly $\sigma(\boldsymbol{a})$ contains both $\sigma_{L_0(\Omega;A)}(\boldsymbol{a})$ and $\sigma_{\mathrm{s}}(\boldsymbol{a})$. Although it has a certain aesthetic appeal, it seems to be unwieldy and does not lead to the results we are seeking.

Definition 3.5. Let $r(a)$ denote the spectral radius of an element a in A, i.e.

$$r(a) = \sup\{|\lambda| \colon \lambda \in \sigma_A(a)\}.$$

For $\boldsymbol{a} \in L_0(\Omega; A)$ define the *stochastic spectral radius* of $\boldsymbol{a}$ to be the function

$$\boldsymbol{r}(\boldsymbol{a})(\omega) = r(\boldsymbol{a}(\omega)) \text{ a.s.}$$

Corollary 3.6. *For $\boldsymbol{a} \in L_0(\Omega; A)$, $\boldsymbol{r}(\boldsymbol{a})$ is a random variable, and as random variables,*

$$\boldsymbol{r}(\boldsymbol{a}) = \sup\{|\boldsymbol{\lambda}| \colon \boldsymbol{\lambda} \in \sigma_{\mathrm{s}}(\boldsymbol{a})\} = \lim_{n\to\infty} \|\boldsymbol{a}^n\|^{1/n} \ \textit{a.s.}$$

Proof. Proposition 2.3 implies that $\boldsymbol{r}(\boldsymbol{a})$ is the supremum of a countable family of random variables, hence is measurable. By the spectral radius formula, the corollary follows. ∎

The notion of spectrum leads naturally to a discussion of radicals and semisimplicity. Recall that for any algebra B with identity, its *Jacobson radical* $\operatorname{Rad} B$ is the intersection of all maximal left (or equivently right) ideals in B and is the largest ideal contained in the set $\mathrm{QN}(B)$ of quasinilpotent elements (or topologically nilpotent elements in the case of a Banach algebra), i.e. elements with spectrum $\{0\}$ or empty. Equivalently,

$$\begin{aligned} \operatorname{Rad} B &= \{x \in B \colon bx - e \in \operatorname{Inv}(B)\ \forall b \in B\} \\ &= \{x \in B \colon xb - e \in \operatorname{Inv}(B)\ \forall b \in B\}. \end{aligned}$$

(See [3, Prop. 26.16], and for each ideal J, note that $J \subset \text{q-Inv}(B)$ means $J \subset \mathrm{QN}(B)$.) In particular, when B is commutative, $\operatorname{Rad} B = \mathrm{QN}(B)$. In the case of $L_0(\Omega; A)$ we can say more.

Definition 3.7. Let

$$\begin{aligned} \mathrm{QN}_{\mathrm{s}}(L_0(\Omega; A)) &= \{\boldsymbol{a} \in L_0(\Omega; A) \colon \sigma_{\mathrm{s}}(\boldsymbol{a}) = \{0\}\} \\ &= \{\boldsymbol{a} \in L_0(\Omega; A) \colon \boldsymbol{r}(\boldsymbol{a}) = 0 \text{ a.s.}\}. \end{aligned}$$

Lemma 3.8. *Let X and Y be topological spaces and $\Phi \colon X \to Y$ be a closed, upper semicontinuous multifunction. If F is a closed set in Y, then the map $\Psi(x) = \Phi(x) \cap F$, $x \in X$, is upper semicontinuous.*

Proof. Let $x_0 \in X$ and U be a neighborhood of $\Psi(x_0)$. Then $U \cup F^c$ is a neighborhood of $\Phi(x_0)$. Hence there is a neighborhood V of x_0 such that for all $x \in V$, $\Phi(x) \subset U \cup F^c$, so $\Psi(x) = \Phi(x) \cap F \subset U$. ∎

Lemma 3.9. *Let $\boldsymbol{a} \in L_0(\Omega; A)$. Then $\boldsymbol{a} \in \mathrm{QN}_{\mathrm{s}}(L_0(\Omega; A))$ if and only if $\boldsymbol{a} - \boldsymbol{\lambda} \in \operatorname{Inv}(L_0(\Omega; A))$ for all $\boldsymbol{\lambda} \in \operatorname{Inv}(L_0(\Omega))$.*

Proof. If $\boldsymbol{a} \in \mathrm{QN}_{\mathrm{s}}(L_0(\Omega; A))$ and $\boldsymbol{\lambda} \in \operatorname{Inv}(L_0(\Omega))$, then clearly $\boldsymbol{\lambda} \notin \sigma_A(\boldsymbol{a})$ a.s., so $\boldsymbol{a} - \boldsymbol{\lambda} \in \operatorname{Inv}(L_0(\Omega; A))$. Conversely, if $\boldsymbol{a} \notin \mathrm{QN}_{\mathrm{s}}(L_0(\Omega; A))$, then for some $0 < \varepsilon < 1$ we have $\mu(\Omega_\varepsilon) > 0$, where $\Omega_\varepsilon = \{\omega \colon r(\boldsymbol{a}(\omega)) \ge \varepsilon\}$.

By Lemma 3.8 and the selection theorem [18], we can find $\boldsymbol{\lambda}_\varepsilon \in L_0(\Omega)$ such that $\boldsymbol{\lambda}_\varepsilon(\omega) \in \sigma_A(\boldsymbol{a}(\omega))$ and $|\boldsymbol{\lambda}_\varepsilon(\omega)| \geq \varepsilon$ for all $\omega \in \Omega_\varepsilon$. If

$$\boldsymbol{\lambda}(\omega) = \begin{cases} \boldsymbol{\lambda}_\varepsilon(\omega), & \omega \in \Omega_\varepsilon, \\ 1, & \omega \notin \Omega_\varepsilon, \end{cases}$$

then $\boldsymbol{\lambda} \in \mathrm{Inv}(L_0(\Omega))$ but $\boldsymbol{a} - \boldsymbol{\lambda} \notin \mathrm{Inv}(L_0(\Omega; A))$. ∎

THEOREM 3.10. *Let A be a Banach algebra with identity. Then*

$$\mathrm{Rad}\, A \subset L_0(\Omega; \mathrm{Rad}\, A) \subset \mathrm{Rad}\, L_0(\Omega; A) \subset \mathrm{QN_s}(L_0(\Omega; A)) \subset \mathrm{QN}(L_0(\Omega; A)).$$

If A is separable, then $\mathrm{Rad}\, L_0(\Omega; A) = L_0(\Omega; \mathrm{Rad}\, A)$.

Proof. Suppose $\boldsymbol{a} \in \mathrm{Rad}\, L_0(\Omega; A)$ and $\boldsymbol{a} - \boldsymbol{\lambda}$ is not left-invertible in $L_0(\Omega; A)$. Then $\boldsymbol{a}$ and $\boldsymbol{a} - \boldsymbol{\lambda}$ are in a maximal left ideal of $L_0(\Omega; A)$, so $\boldsymbol{\lambda} \notin \mathrm{Inv}(L_0(\Omega))$. By Lemma 3.9,

$$\mathrm{Rad}\, L_0(\Omega; A) \subset \mathrm{QN_s}(L_0(\Omega; A)),$$

so $\mathrm{Rad}\, L_0(\Omega; A)$ is the largest ideal in $L_0(\Omega; A)$ contained in $\mathrm{QN_s}(L_0(\Omega; A))$. Since clearly $L_0(\Omega; \mathrm{Rad}\, A)$ is an ideal of $L_0(\Omega; A)$ (see below) contained in $\mathrm{QN_s}(L_0(\Omega; A))$, it is contained in the radical. The remaining containments are obvious.

For each $a \in A$, let $E(a) = \{b \in A \colon ba - e \notin \mathrm{Inv}(A)\}$. Thus $E(a)$ is closed in A and empty if and only if $a \in \mathrm{Rad}\, A$. Let

$$E = \{(a, b) \in A \times A \colon b \in E(a)\}.$$

Then E is closed in $A \times A$, so if $\boldsymbol{a} \in L_0(\Omega; A)$, it follows that

$$F = \{(\boldsymbol{a}(\omega), b) \colon \omega \in \Omega,\ b \in E_{\boldsymbol{a}(\omega)}\}$$

is a product-measurable subset of $\Omega \times A$.

Now suppose $\boldsymbol{a} \notin \mathrm{Rad}\, A$ on a set of positive measure. By conditioning on such a set, let us assume that set is all of Ω. That means $E(\boldsymbol{a}(\omega)) \neq \emptyset$ for all $\omega \in \Omega$. Thus F becomes a closed multifunction which is product measurable and such that each $F(\omega)$ is nonempty. By [15, Theorem 5.7] (cf. [1, Theorem 8.1.4]), if A is separable this implies that there is a selection $\boldsymbol{b}$ for F, which means $\boldsymbol{b}\boldsymbol{a} - e \notin \mathrm{Inv}\, L_0(\Omega; A)$. ∎

COROLLARY 3.11. *If $L_0(\Omega; A)$ is semisimple, then A is semisimple; the converse holds whenever A is separable or commutative.*

As for Banach algebras, we have the following observation.

LEMMA 3.12. *Let B be a closed, unital subalgebra of A. Then*

$$\mathrm{Rad}\, L_0(\Omega; A) \cap L_0(\Omega; B) \subset \mathrm{Rad}\, L_0(\Omega; B).$$

Proof. The set $\operatorname{Rad} L_0(\Omega; A) \cap L_0(\Omega; B)$ is an ideal in $L_0(\Omega; B)$ (see below), and

$$\mathrm{QN_s}(L_0(\Omega; B)) = \mathrm{QN_s}(L_0(\Omega; A)) \cap L_0(\Omega; B).$$

Since $\operatorname{Rad} L_0(\Omega; B)$ is the largest ideal in $\mathrm{QN_s}(L_0(\Omega; B))$, the lemma follows. ∎

COROLLARY 3.13. *If A is a C^*-algebra, then $L_0(\Omega; A)$ is semisimple.*

Proof. It is well known that every C^*-algebra is semisimple (see e.g. [19, Prop. 2.3.17, 2.3.18], [9, Cor. 3.2.13]). If A is a C^*-algebra and

$$\boldsymbol{a} \in \operatorname{Rad} L_0(\Omega; A),$$

then $\boldsymbol{a}$ is a.s. separably valued, so there is a separable, unital C^*-subalgebra B of A such that $\boldsymbol{a} \in L_0(\Omega; B)$. But then $\boldsymbol{a} \in \operatorname{Rad} L_0(\Omega; B)$ by Lemma 3.12, so by Theorem 3.10(ii) $\boldsymbol{a} \in \operatorname{Rad} L_0(\Omega; B) = \{0\}$. ∎

DEFINITION 3.14. Let A be commutative. For φ in the Gelfand space Φ_A, denote also by φ the homomorphism $\varphi\colon L_0(\Omega; A) \to L_0(\Omega)$ given by

$$\varphi(\boldsymbol{a})(\omega) = \varphi(\boldsymbol{a}(\omega)) \text{ a.s.}$$

Call this φ a *stochastic character*. For $\boldsymbol{a} \in L_0(\Omega; A)$, set $\widehat{\boldsymbol{a}}(\varphi) = \varphi(\boldsymbol{a})$ for all $\varphi \in \Phi_A$. Thus $\boldsymbol{a} \mapsto \widehat{\boldsymbol{a}}$ is a continuous homomorphism from $L_0(\Omega; A)$ to $L_0(\Omega; C(\Phi_A))$.

COROLLARY 3.15. *Let A be commutative. The map $\boldsymbol{a} \mapsto \widehat{\boldsymbol{a}}$ is injective if and only if A is semisimple.*

4. Ideals, hulls, and kernels. If I is an ideal in $L_0(\Omega; A)$, then $I \cap A$ is a (perhaps trivial) ideal in A. Conversely, if J is an ideal in A, then $L_0(\Omega; J) = \{\boldsymbol{a}\colon \boldsymbol{a} \in J \text{ a.s.}\}$ is an ideal in $L_0(\Omega; A)$. For I and J as above, we have $L_0(\Omega; J) \cap A = J$; and if J is closed in A, then $L_0(\Omega; J)$ is closed in $L_0(\Omega; A)$, since every Cauchy sequence in $L_0(\Omega; A)$ has a subsequence converging a.s. On the other hand, if I is closed, then $I \cap A$ is closed.

In general, the ideal structure of $L_0(\Omega; A)$ is complicated. For instance, one can develop many variations on the theme of the following example.

EXAMPLE 4.1. Let $\Omega = [0, 1]$, μ be ordinary Lebesgue measure, and I_t, $0 < t < 1$, be closed ideals in A such that $I_s \subset I_t$ if $s < t$. Let

$$J = \bigcap_{0<t<1} \{\boldsymbol{a}\colon \boldsymbol{a}(\omega) \in I_t,\ 0 \le \omega \le t\}.$$

Then J is a closed ideal in $L_0(\Omega; A)$.

In one case, however, we have a complete answer, based on the following lemma.

Lemma 4.2. *Suppose that A is simple, and let I be a nontrivial closed ideal in $L_0(\Omega; A)$. Then I contains an element which is invertible on a set of positive probability.*

Proof. Let $0 \neq \boldsymbol{a} \in I$. Since $\boldsymbol{a}$ is measurable, all the values of $\boldsymbol{a}$ lie in a closed, separable subspace A_0 of A. Let $a_1, a_2, \ldots$ be a countable dense subset of A_0, and for $1 \leq i, n < \infty$ let $B(i,n)$ be the closed ball of radius $1/n$ about a_i. There exists $\varepsilon > 0$ such that

$$\mu(\{\omega\colon \|\boldsymbol{a}(\omega)\| \geq \varepsilon\}) > 0.$$

Set $F_0 = \{a\colon \|a\| \geq \varepsilon\}$ and $\Omega_0 = \{\omega\colon \boldsymbol{a}(\omega) \in F_0\}$. Suppose that

$$F_0 \supset F_1 \supset \cdots \supset F_n$$

have been defined so that $\operatorname{diam} F_j \leq 1/j$ and $\mu(\Omega_j) > 0$, where

$$\Omega_j = \{\omega\colon \boldsymbol{a}(\omega) \in F_j\}, \quad j = 1, \ldots, n.$$

Since the balls $B(i, n+1)$, $i = 1, 2, \ldots$, cover A_0, we can choose i_{n+1} such that if we set $F_{n+1} = F_n \cap B(i_{n+1}, n+1)$ and $\Omega_{n+1} = \{\omega\colon \boldsymbol{a}(\omega) \in F_{n+1}\}$, then $\mu(\Omega_{n+1}) > 0$. Thus we obtain a decreasing sequence $\{F_n\}$ of nonvoid closed subsets of F_0 and measurable sets $\Omega_n = \{\omega\colon \boldsymbol{a}(\omega) \in F_n\}$ of positive probability such that $\operatorname{diam} F_n \to 0$.

Let $a_0 \in \bigcap_{n=1}^{\infty} F_n$. Since $a_0 \neq 0$, there exist $b_1, \ldots, b_m$, $c_1, \ldots, c_m \in A$ such that $b_1 a_0 c_1 + \cdots + b_m a_0 c_m = e$. It follows that

$$x(a) = b_1 a c_1 + \cdots + b_m a c_m$$

is invertible for all a in some neighborhood of a_0. In particular, for some j, $x(a)$ is invertible for all $a \in F_j$, so $b_1 \boldsymbol{a} c_1 + \cdots + b_m \boldsymbol{a} c_m$ is the desired element of I. ∎

Theorem 4.3. *Let A be simple, and let I be a closed ideal in $L_0(\Omega; A)$. Then there exists $\Omega_I \in \mathcal{F}$ such that*

$$I = L_0(\Omega; A)\boldsymbol{\chi}_{\Omega_I} = \{\boldsymbol{a} \in L_0(\Omega; A)\colon \boldsymbol{a} = 0 \textit{ on } X \setminus \Omega_I\}.$$

Proof. By Lemma 4.2 there exist $0 \neq \boldsymbol{a} \in I$ and $\Omega_{\boldsymbol{a}} \in \mathcal{F}$ such that $\mu(\Omega_{\boldsymbol{a}}) > 0$ and $\boldsymbol{a}(\omega)$ is invertible for all $\omega \in \Omega_{\boldsymbol{a}}$. It follows from the continuity of inversion in A that $\boldsymbol{\chi}_{\Omega_{\boldsymbol{a}}} \in I$, so $L_0(\Omega; A)\boldsymbol{\chi}_{\Omega_{\boldsymbol{a}}} \subset I$.

Now, if $\Omega_1, \Omega_2 \in \mathcal{F}$ are such that $\boldsymbol{\chi}_{\Omega_1}, \boldsymbol{\chi}_{\Omega_2} \in I$, then clearly $\boldsymbol{\chi}_{\Omega_1 \cup \Omega_2} \in I$. And if $\Omega_1 \subset \Omega_2 \subset \cdots$ are such $\boldsymbol{\chi}_{\Omega_n} \in I$ for all n and $\Omega_0 = \bigcup_n \Omega_n$, then

$$\boldsymbol{\chi}_{\Omega_0} = \lim_{n \to \infty} \boldsymbol{\chi}_{\Omega_n} \in I.$$

It follows that there exists $\Omega_I \in \mathcal{F}$ such that $\boldsymbol{\chi}_{\Omega_I} \in I$ and

$$\mu(\Omega_I) = \sup\{\mu(E)\colon E \in \mathcal{F},\ \boldsymbol{\chi}_E \in I\}.$$

Since $\boldsymbol{\chi}_E \notin I$ for all $E \subset \Omega \setminus \Omega_I$ with $\mu(E) > 0$, the theorem follows from Lemma 4.2. ∎

For the remainder of this section, A will be assumed commutative. Recall that if E is a closed set in Φ_A, then the *kernel* of E is the closed ideal of elements $a \in A$ such that $\widehat{a}(\varphi) = 0$ for all $\varphi \in E$. And if I is a closed ideal in A, then the *hull* of I is the closed set

$$\{\varphi \in \Phi_A \colon \widehat{a}(\varphi) = 0 \text{ for all } a \in I\}.$$

Our goal is to introduce the corresponding notions for the algebras $L_0(\Omega; A)$.

DEFINITION 4.4. Let $E \colon \Omega \to \Phi_A$ be a closed multifunction. By the *kernel* of E we shall mean the set

$$I(E) = \{\boldsymbol{a} \in L_0(\Omega; A) \colon \widehat{\boldsymbol{a}}(\varphi)(\omega) = 0 \text{ for all } \varphi \in E(\omega) \text{ a.s.}\}.$$

On the other hand, if I is a closed ideal in $L_0(\Omega; A)$, then a *hull* for I is a closed multifunction $Z = Z(I) \colon \Omega \to \Phi_A$ satisfying:

(i) (vanishing) For all $\boldsymbol{a} \in I$, $\widehat{\boldsymbol{a}}(\varphi)(\omega) = 0$ for all $\varphi \in Z(\omega)$ a.s.
(ii) (maximality) If W is a closed multifunction in Φ satisfying (i), then $W \subset Z$ (i.e., $W(\omega) \subset Z(\omega)$) a.s.

PROPOSITION 4.5. *If $E \colon \Omega \to \Phi_A$ is a closed multifunction, then $I(E)$ is a closed ideal in $L_0(\Omega; A)$. If I is a closed ideal in $L_0(\Omega; A)$ such that a hull $Z(I)$ exists, then the hull is unique up to null sets. If J is a closed ideal in $L_0(\Omega; A)$ with hull $Z(J)$, then $I(Z(J)) \supset J$, while if E is as above and if $Z(I(E))$ exists, then $Z(I(E)) \supset E$.*

The proof is straightforward and left to the reader.

THEOREM 4.6. *Let A be a commutative Banach algebra.*

(i) *Suppose that A is (completely) regular and Φ_A is separable. If*

$$E \colon \Omega \to \Phi_A$$

is a proper, closed, measurable multifunction, then $I(E) \neq \{0\}$.
(ii) *If I is a (topologically) countably generated, closed ideal in $L_0(\Omega; A)$, then $Z(I)$ exists and is measurable.*

Proof. (i) By a *proper* multifunction E we mean here that

$$\mu[E \neq \Phi_A] > 0.$$

Without loss of generality, we may assume that $E \neq \Phi_A$ on all of Ω. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a countable dense set in Φ_A. For each n,

$$E_n = \{\omega \colon \varphi_n \notin E(\omega)\} \in \mathcal{F}$$

and $\bigcup_n E_n = \Omega$. Hence $\mu(E_{n_0}) > 0$ for some n_0. Since Φ_A is compact and metrizable, [15, Theorem 3.5] implies that the function

$$\boldsymbol{f}(\omega) = \operatorname{dist}(\varphi_{n_0}, E(\omega))$$

is measurable. It follows that there exist $E' \in \mathcal{F}$ and $\varepsilon > 0$ such that $\mu(E') > 0$ and $\boldsymbol{f}(\omega) \geq \varepsilon$ for all $\omega \in E'$. By hypothesis, we can find $a_0 \in A$

such that $\widehat{a}_0(\varphi_0) = 1$ and $\widehat{a}_0(\varphi) = 0$ if $d(\varphi, \varphi_0) \geq \varepsilon$, where d denotes a metric on Φ_A. So if

$$\boldsymbol{a}(\omega) = \begin{cases} a_0, & \omega \in E', \\ 0, & \omega \notin E', \end{cases}$$

then $0 \neq \boldsymbol{a} \in I(E)$.

(ii) Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots$ be generators of I. Then $\widehat{\boldsymbol{a}}_n \in L_0(\Omega; C(\Phi_A))$, so let $Z(\boldsymbol{a}_n) = \widehat{\boldsymbol{a}}_n^{-1}(0)$, $n = 1, 2, \ldots$, defined up to null sets. Since convergence of a sequence in $L_0(\Omega; A)$ to some $\boldsymbol{a}$ implies convergence a.s. of a subsequence to $\boldsymbol{a}$, it is easy to see that $Z(I) = \bigcap_{n \in \mathbb{N}} Z(\boldsymbol{a}_n)$. Now, up to a μ-null set, each $\widehat{\boldsymbol{a}}_n$ takes values in a separable subset of $C(\Phi_A)$, and the range of any function in $C(\Phi_A)$ is separable. Thus neglecting a null set, there is a separable topological quotient space Y of Φ_A with quotient map $\pi \colon \Phi_A \to Y$ and a sequence $\boldsymbol{f}_n \in L_0(\Omega; C(Y))$ such that $\widehat{\boldsymbol{a}}_n = \boldsymbol{f}_n \circ \pi$ a.s., for $n = 1, 2, \ldots$. And since Φ_A is compact, so is Y, hence Y is metrizable. An easy exercise shows that the map $f \mapsto Z(f) = f^{-1}(0)$ is upper semicontinuous on $C(Y)$. Thus the multifunctions $Z(\boldsymbol{f}_n(\cdot))$ are measurable in the sense of [18] and hence each $\mathrm{Gr}(Z(\boldsymbol{f}_n))$ is measurable by [15, Theorem 5.3]. But then for all n, $\mathrm{Gr}(Z(\boldsymbol{a}_n)) = (\mathrm{Id} \times \pi)^{-1}(\mathrm{Gr}(Z(\boldsymbol{f}_n))) \in \mathcal{F} \times \mathcal{B}(\Phi_A)$. Hence $Z(I)$ exists and is measurable. ∎

The classical example which led to the notions of hull and kernel in Banach algebras is the case $A = C(X)$ for X a compact Hausdorff space, which is, of course, regular. Here $\Phi_A = X$ in the natural way, so our theorem mimics the classical results. But in this case the situation is more complete.

Theorem 4.7. *Let X be a separable, compact Hausdorff space. If $E \colon \Omega \to X$ is a closed, measurable multifunction, then $Z(I(E)) = E$. If I is a countably generated closed ideal in $L_0(\Omega; C(X))$, then $I(Z(I)) = I$.*

Proof. Let $\boldsymbol{\varphi}(\omega)(x) = \mathrm{dist}(x, E(\omega))$. Then $\boldsymbol{\varphi}$ is continuous on X for each $\omega \in \Omega$ and measurable for each $x \in X$. By a well-known observation of Carathéodory, $\boldsymbol{\varphi}$ is jointly measurable on $\Omega \times X$, hence an element of $L_0(\Omega; C(X))$. Since $Z(I(E)) \subset Z(\boldsymbol{\varphi}) = E$, we obtain equality from Proposition 4.5.

Let I be the ideal in $L_0(\Omega; C(X))$ generated topologically by $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \ldots\}$. For each n, $\|\boldsymbol{f}_n\| \in L_0(\Omega)$, so set

$$\boldsymbol{f} = \sum_{n=1}^{\infty} \frac{\boldsymbol{f}_n}{2^n \|\boldsymbol{f}_n\|},$$

where we understand any term to be the constant function 0 whenever $\|\boldsymbol{f}_n\| = 0$. Then $\boldsymbol{f} \in I$, and $Z(I) = \boldsymbol{f}^{-1}(0)$.

Let $\boldsymbol{g} \in I(Z(I))$ and $\varepsilon > 0$. For each $j = 1, 2, \ldots$ and $\omega \in \Omega$, let

$$\boldsymbol{U}_j(\omega) = \{x \colon \mathrm{dist}(x, Z(I)(\omega)) < 1/j\}.$$

Since X is separable, it is easy to see that $\boldsymbol{U}_j$ is a measurable multifunction. (The modulus of continuity of any element of $L_0(\Omega; C(X))$ is easily seen to be measurable.) Hence the sets

$$\Omega_j = \{\omega\colon |\boldsymbol{g}(\omega)(x)| < \varepsilon/3 \text{ for all } x \in \boldsymbol{U}_j(\omega)\}$$

are measurable; $\Omega_j \subset \Omega_{j+1}$, $j = 1, 2, \ldots$; and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Thus we can choose j_0 such that $\mu(\Omega_{j_0}^c) < \varepsilon$.

Recall that if F and G are disjoint closed multifunctions in X, then the function

$$\boldsymbol{\Phi}(\omega)(x) = \frac{\operatorname{dist}(x, F(\omega))}{\operatorname{dist}(x, F(\omega)) + \operatorname{dist}(x, G(\omega))}$$

is an element of $L_0(\Omega; C(X))$ such that $\boldsymbol{\Phi}(x) = 0$ if $x \in F$ and $\boldsymbol{\Phi}(x) = 1$ if $x \in G$. Using this observation and the stochastic version of Tietze's Extension Theorem [4, Theorem 2.3], one can construct $\boldsymbol{h} \in L_0(\Omega; C(X))$ such that $\boldsymbol{f}\boldsymbol{h} = 1$ on $U_{j_0}^c$ and $\|\boldsymbol{f}\boldsymbol{h}\| \le 2$ a.s.

Set $\boldsymbol{q} = \boldsymbol{f}\boldsymbol{g}\boldsymbol{h}$. Then $\boldsymbol{q} \in I$. Let $\omega \in \Omega_{j_0}$. If $x \in Z(I)(\omega)$, then

$$\boldsymbol{g}(x) = 0 = \boldsymbol{q}(x).$$

If $x \notin \boldsymbol{U}_{j_0}(\omega)$, then $\boldsymbol{f}(x)\boldsymbol{h}(x) = 1$, so $\boldsymbol{q}(x) = \boldsymbol{g}(x)$. And if

$$x \in \boldsymbol{U}_{j_0}(\omega) \setminus Z(I)(\omega),$$

then

$$|\boldsymbol{g}(x) - \boldsymbol{q}(x)| = |\boldsymbol{g}(x)|\,|1 - \boldsymbol{f}(x)\boldsymbol{h}(x)| < 3\,\frac{\varepsilon}{3} = \varepsilon.$$

Thus $\mu[\|\boldsymbol{g} - \boldsymbol{q}\| \ge \varepsilon] < \varepsilon$. Since ε was arbitrary and I is closed, we conclude that $\boldsymbol{g} \in I$. And since $\boldsymbol{g}$ was chosen arbitrarily in $I(Z(I))$, Proposition 4.5 implies that $I(Z(I)) = I$. ∎

5. Homomorphisms and automatic continuity. In this section we prove analogues of two well-known theorems on the automatic continuity of homomorphisms between Banach algebras, in the context of L_0-algebras. If A and B are unital Banach algebras, a mapping $\theta\colon L_0(\Omega; A) \to L_0(\Omega; B)$ will be called a *modular unital homomorphism* if it is a unital homomorphism which is also a module homomorphism over $L_0(\Omega)$.

LEMMA 5.1. *Let A and B be a unital Banach algebra and*

$$\theta\colon L_0(\Omega; A) \to L_0(\Omega; B)$$

be a unital homomorphism. Then θ is modular if and only if

$$\sigma_{\mathrm{s}}(\theta(\boldsymbol{a})) \subset \sigma_{\mathrm{s}}(\boldsymbol{a}), \quad \boldsymbol{a} \in L_0(\Omega; A). \tag{1}$$

Proof. Suppose θ is modular and $\boldsymbol{a} \in L_0(\Omega; A)$. If $\boldsymbol{\lambda} \notin \sigma_{\mathrm{s}}(\boldsymbol{a})$, then there exists $\Omega_0 \subset \Omega$ with $\mu(\Omega_0) > 0$ such that

$$\boldsymbol{a}(\omega) - \boldsymbol{\lambda}(\omega) \in \operatorname{Inv}(A), \qquad \omega \in \Omega_0.$$

Let

$$\boldsymbol{c}(\omega) = \begin{cases} (\boldsymbol{a}(\omega) - \boldsymbol{\lambda}(\omega))^{-1}, & \omega \in \Omega_0, \\ 0, & \omega \notin \Omega_0. \end{cases}$$

Then $\boldsymbol{c} \in L_0(\Omega; A)$ and $\boldsymbol{c}(\boldsymbol{a} - \boldsymbol{\lambda}) = \boldsymbol{\chi}_{\Omega_0}$, so $\theta(\boldsymbol{c})(\theta(\boldsymbol{a}) - \boldsymbol{\lambda}) = \theta(\boldsymbol{\chi}_{\Omega_0}) = \boldsymbol{\chi}_{\Omega_0}$. Hence $\boldsymbol{\lambda} \notin \sigma_{\mathrm{s}}(\theta(\boldsymbol{a}))$, so (1) holds.

Conversely, suppose that (1) holds for all $\boldsymbol{a} \in L_0(\Omega; A)$. Then in particular, for all $\boldsymbol{\lambda} \in L_0(\Omega)$, we have $\emptyset \neq \sigma_{\mathrm{s}}(\theta(\boldsymbol{\lambda})) \subset \sigma_{\mathrm{s}}(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}\}$. Hence $\theta(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$, so $\theta(\boldsymbol{\lambda}\boldsymbol{a}) = \theta(\boldsymbol{\lambda})\theta(\boldsymbol{a}) = \boldsymbol{\lambda}\theta(\boldsymbol{a})$ for all $\boldsymbol{a} \in L_0(\Omega; A)$, i.e., θ is modular. ∎

THEOREM 5.2. *Let A and B be unital Banach algebras and*

$$\theta \colon L_0(\Omega; A) \to L_0(\Omega; B)$$

be a modular unital homomorphism. Suppose that either

(i) *B is simple and the range of θ is dense in $L_0(\Omega; B)$, or*
(ii) *B is commutative and semisimple.*

Then θ is continuous.

Proof. Let $S(\theta)$ be the *separating subspace* of θ, i.e., the set of all $\boldsymbol{b} \in L_0(\Omega; B)$ for which there exists a sequence $\{\boldsymbol{a}_n\}$ in $L_0(\Omega; A)$ such that $\boldsymbol{a}_n \to 0$ and $\theta(\boldsymbol{a}_n) \to \boldsymbol{b}$. Then $S(\theta)$ is a closed ideal in $L_0(\Omega; B)$, since θ has dense range, and θ is continuous if and only if $S(\theta) = \{0\}$ by the Closed Graph Theorem, since $L_0(\Omega; A)$ and $L_0(\Omega; B)$ are Fréchet spaces.

Assume first that B is simple, and suppose $S(\theta) \neq \{0\}$. By Theorem 4.3 there exists $\Omega_0 \subset \Omega$ such that $\mu(\Omega_0) > 0$ and $\boldsymbol{\chi}_{\Omega_0} \in S(\theta)$. Choose $\boldsymbol{a}_n \in L_0(\Omega; A)$, $n = 1, 2, \ldots$, so that $\boldsymbol{a}_n \to 0$ and $\theta(\boldsymbol{a}_n) \to \boldsymbol{\chi}_{\Omega_0}$. By Lemma 5.1,

$$\sigma_{\mathrm{s}}(\theta(\boldsymbol{a})) \subset \sigma_{\mathrm{s}}(\boldsymbol{a}), \qquad \boldsymbol{a} \in L_0(\Omega; A).$$

If $\boldsymbol{\lambda}_n \in \sigma_{\mathrm{s}}(\theta(\boldsymbol{a}_n))$, then almost surely on Ω_0 our hypotheses imply that

$$1 - \boldsymbol{\lambda}_n \in \sigma_{\mathrm{s}}(e - \theta(\boldsymbol{a}_n)) = \sigma_{\mathrm{s}}(\theta(e - \boldsymbol{a}_n)).$$

Thus passing to a subsequence, we have

$$1 = \boldsymbol{\lambda}_n + (1 - \boldsymbol{\lambda}_n) \leq |\boldsymbol{\lambda}_n| + |1 - \boldsymbol{\lambda}_n| \leq \|\boldsymbol{a}_n\| + |1 - \boldsymbol{\lambda}_n| \to 0$$

almost surely on Ω_0. Hence $S(\theta) = \{0\}$.

Now suppose that B is commutative and semisimple. For each stochastic character φ on $L_0(\Omega; B)$ and $\boldsymbol{b} \in S(\theta)$, the previous case implies that

$$\varphi(\boldsymbol{b}) \in S(\varphi\theta) = \{0\}.$$

Thus $S(\theta) \subset \mathrm{Rad}(L_0(\Omega; B)) = \{0\}$ by Corollary 3.11. ∎

The following celebrated theorem was proved by B. E. Johnson in 1967:

THEOREM 5.3 ([17]). *Let A and B be unital Banach algebras with B semisimple. Then every surjective homomorphism $\theta \colon A \to B$ is continuous.*

COROLLARY 5.4. *Every semisimple, unital Banach algebra has a unique Banach-algebra norm.*

This theorem has been generalized by several authors in various directions for locally convex Fréchet algebras (see [7], [12], [16]). The following theorem is a version of Theorem 5.3 valid for algebras of random elements.

Our proof is based on the short proof of Theorem 5.3 obtained by T. Ransford [31]. That proof is based on the following lemma.

LEMMA 5.5 ([31]). *Let A be a Banach algebra and let $p(z)$ be a polynomial with coefficients in A. Then for any $R > 0$,*

$$r(p(1))^2 \le \sup_{|z|=R} r(p(z)) \sup_{|z|=1/R} r(p(z)).$$

REMARK 5.6. It was pointed out to the authors by A. Rodríguez-Palacios that the proof of the above lemma in [31] uses implicitly the following variant of Dini's Theorem: *Let E be a compact Hausdorff space, and let $\{f_n\}$ be a decreasing sequence of nonnegative continuous functions on E.* Then

$$\lim_{n\to\infty} \sup_{t\in E} f_n(t) = \sup_{t\in E} \lim_{n\to\infty} f_n(t).$$

For a proof, see [6, Lemma 3.7.2].

THEOREM 5.7. *Let A and B be unital Banach algebras and*

$$\theta\colon L_0(\Omega; A) \to L_0(\Omega; B)$$

be a modular unital homomorphism. If θ is surjective then the separating ideal $S(\theta)$ is contained in $\operatorname{Rad} L_0(\Omega; B)$. *In particular, if $L_0(\Omega; B)$ is semisimple then θ is continuous.*

Proof. Let $\boldsymbol{b} \in S(\theta)$ and choose $\boldsymbol{a}, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \in L_0(\Omega; A)$ such that $\theta(\boldsymbol{a}) = \boldsymbol{b}$ and

$$\lim_{n\to\infty} \boldsymbol{a}_n = 0, \qquad \lim_{n\to\infty} \theta(\boldsymbol{a}_n) = \boldsymbol{b}. \tag{2}$$

By passing to a subsequence we may assume the sequences in (2) converge a.s. For each n let $\boldsymbol{p}_n(z) = \boldsymbol{p}_n(z, \omega)$ be the random linear polynomial with values in $L_0(\Omega; B)$ given by

$$\boldsymbol{p}_n(z) = z\theta(\boldsymbol{a}_n) + (\theta(\boldsymbol{a}) - \theta(\boldsymbol{a}_n)).$$

Then $\boldsymbol{p}_n(1) = \boldsymbol{b}$ and

$$\boldsymbol{r}(\boldsymbol{p}_n(z)) \le \|\boldsymbol{p}_n(z)\| \le |z|\|\theta(\boldsymbol{a}_n)\| + \|\theta(\boldsymbol{a}) - \theta(\boldsymbol{a}_n)\| \quad \text{a.s.} \tag{3}$$

Since θ is modular, it follows easily after choosing a sequence Λ in $\sigma_{\mathrm{s}}(\theta(\boldsymbol{x}))$ as in Proposition 3.3 and applying Lemma 5.1 that

$$\boldsymbol{r}(\theta(\boldsymbol{x})) \le \boldsymbol{r}(\boldsymbol{x}) \text{ a.s.}, \qquad \boldsymbol{x} \in L_0(\Omega; A).$$

Thus

$$\boldsymbol{r}(\boldsymbol{p}_n(z)) \le \boldsymbol{r}(z\boldsymbol{a}_n + (\boldsymbol{a} - \boldsymbol{a}_n)) \le |z|\|\boldsymbol{a}_n\| + \|\boldsymbol{a} - \boldsymbol{a}_n\| \quad \text{a.s.} \tag{4}$$

If we set $z = 1$, Lemma 5.5 and the estimates (3) and (4) give

$$\boldsymbol{r}(\boldsymbol{b})^2 \le (R\|\boldsymbol{a}_n\| + \|\boldsymbol{a} - \boldsymbol{a}_n\|)(R^{-1}\|\theta(\boldsymbol{a}_n)\| + \|\theta(\boldsymbol{a}) - \theta(\boldsymbol{a}_n)\|) \quad \text{a.s.}$$

Letting $n \to \infty$ and recalling the definitions of $\boldsymbol{a}_n$ and $\boldsymbol{a}$, we obtain

$$\boldsymbol{r}(\boldsymbol{b})^2 \le R^{-1}\|\boldsymbol{a}\|\,\|\boldsymbol{b}\| \quad \text{a.s.}$$

Now choosing $R_n \to \infty$ we see that $\boldsymbol{b} \in \mathrm{QN_s}(L_0(\Omega; B))$. It follows that the ideal $S(\theta)$ is contained in $\mathrm{Rad}\, L_0(\Omega; B)$. ■

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