STUDIA MATHEMATICA 233 (2) (2016)

Bernstein type properties of two-sided hypersurfaces immersed in a Killing warped product

by

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Abstract. Our purpose is to apply suitable maximum principles in order to obtain Bernstein type properties for two-sided hypersurfaces immersed with constant mean curvature in a Killing warped product $M^n \times_\rho \mathbb{R}$, whose curvature of the base M^n satisfies certain constraints and whose warping function ρ is concave on M^n. For this, we study situations in which these hypersurfaces are supposed to be either parabolic, stochastically complete or, in a more general setting, L^1-Liouville. Rigidity results related to entire Killing graphs constructed over the base of the ambient space are also given.

1. Introduction. Killing vector fields are important objects which have been widely used in order to understand the geometry of submanifolds and, more particularly, of hypersurfaces immersed in Riemannian spaces. Alías, Dajczer and Ripoll [ADR] extended the classical Bernstein theorem [B] to the context of complete minimal surfaces in Riemannian spaces of nonnegative Ricci curvature carrying a Killing vector field. This was done under the assumption that the sign of the angle function between the global Gauss mapping and the Killing vector field remains unchanged along the surface. Afterwards, Dajczer, Hinojosa and de Lira [DHL] defined a notion of Killing graph in a class of Riemannian manifolds endowed with a Killing vector field and solved the corresponding Dirichlet problem for prescribed mean curvature under a hypothesis involving the domain data and the Ricci curvature of the ambient space. More recently, Dajczer and de Lira [DL] showed that an entire Killing graph of constant mean curvature lying inside a slab must

2010 *Mathematics Subject Classification*: Primary 53C42; Secondary 53B30, 53C50.
Key words and phrases: Killing warped product, parabolic hypersurfaces, stochastically complete hypersurfaces, L^1-Liouville hypersurfaces, entire Killing graphs.
Received 20 November 2015; revised 8 April 2016.
Published online 19 May 2016.

DOI: 10.4064/sm8464-4-2016

be a totally geodesic slice, under certain restrictions on the curvature of the ambient space. To prove their Bernstein type result, they used as the main ingredient the Omori–Yau maximum principle for the Laplacian in the sense of Pigola, Rigoli and Setti [PRS2].

In the particular case of a Riemannian product $M^n \times \mathbb{R}$, it was shown by Rosenberg, Schulze and Spruck [RSS] that if the Ricci curvature of the base M^n is nonnegative and its sectional curvature is bounded from below, then any entire minimal graph over M^n with nonnegative height function must be a slice. This extends the celebrated theorem by Bombieri, De Giorgi and Miranda [BGM] concerning entire minimal hypersurfaces in Euclidean space. In [LLP], the second and third authors jointly with Parente studied two-sided complete hypersurfaces immersed in $M^n \times \mathbb{R}$, whose base is also supposed to have sectional curvature bounded from below. In this setting, they extended a technique developed in [LP] obtaining sufficient conditions for a hypersurface to be a slice of the ambient space, provided that its angle function has suitable behavior.

We recall that a hypersurface is said to be *two-sided* if its normal bundle is trivial, that is, there exists on it a globally defined unit normal vector field.

Here, assuming constraints similar to [DL] on the curvature of M^n and supposing that the warping function ρ is concave on M^n, our purpose is to investigate Bernstein type properties of two-sided hypersurfaces immersed with constant mean curvature in a Killing warped product $M^n \times_\rho \mathbb{R}$.

In Section 2 we recall some basic facts concerning two-sided hypersurfaces immersed in $M^n \times_\rho \mathbb{R}$. In Section 3 we establish our first Bernstein type results concerning parabolic two-sided hypersurfaces (Theorems 3.1 and 3.3 and Corollaries 3.5 and 3.6). In particular, under some restrictions on the warping function and assuming that the base of the ambient space has a pole, we use a well known result due to Khas'minskii [K] to show that the immersed hypersurface is, in fact, parabolic (Theorem 3.7).

In Section 4 we consider a weaker case in which the immersed hypersurface is shown to be stochastically complete, and we apply the weak Omori–Yau maximum principle to obtain another Bernstein type theorem (Theorem 4.2). We also treat the more general setting where the hypersurface is just supposed to be L^1-Liouville (Theorem 4.5). Moreover, we discuss the case in which the two-sided hypersurface is an entire Killing graph constructed over the base of the ambient space (Corollaries 3.8 and 4.4).

Finally, we point out that most of our results do not require geodesic completeness of the base M^n of the ambient space. More precisely, these results are meaningful when M^n is either incomplete or n-dimensional with $n \geq 3$.

2. Killing warped products. Let $\overline{M}^{n+1}$ be an $(n+1)$-dimensional Riemannian manifold endowed with a Killing vector field Y. Suppose that the distribution orthogonal to Y is of constant rank and integrable. Given an integral leaf M^n of that distribution, let $\Psi : \mathbb{I} \times M^n \to \overline{M}^{n+1}$ be the flow generated by Y with initial values in $\overline{M}^{n+1}$, where $\mathbb{I}$ is the maximal interval of definition. Without loss of generality, in what follows we assume $\mathbb{I} = \mathbb{R}$.

In this setting, $\overline{M}^{n+1}$ can be regarded as the *Killing warped product* $M^n \times_\rho \mathbb{R}$, that is, the product manifold $M^n \times \mathbb{R}$ endowed with the warping metric

$$\langle\, , \rangle = \pi_M^*(\langle\, , \rangle_M) + (\rho \circ \pi_M)^2 \pi_{\mathbb{R}}^*(dt^2), \tag{2.1}$$

where π_M and $\pi_{\mathbb{R}}$ denote the canonical projections from $M^n \times \mathbb{R}$ onto each factor, $\langle\, , \rangle_M$ is the induced Riemannian metric on the base M^n, and the warping function $\rho \in C^\infty$ is $\rho = |Y| > 0$.

Throughout this work, we will deal with hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a Killing warped product $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ and which are *two-sided*. This means that there is a globally defined unit normal vector field N. In this setting, we will consider two particular smooth functions on Σ^n: the (vertical) *height function* $h = (\pi_{\mathbb{R}})|_\Sigma$ and the *angle function* $\Theta = \langle N, Y \rangle$.

For a two-sided hypersurface Σ^n with constant mean curvature in $M^n \times_\rho \mathbb{R}$, from [BCE, Proposition 2.12] (see also [FR, Proposition 1] and [ADR, Proposition 6]) we have

$$\Delta\Theta = -\big(\overline{\mathrm{Ric}}(N,N) + |A|^2\big)\Theta, \tag{2.2}$$

where $\overline{\mathrm{Ric}}$ denotes the Ricci tensor of $\overline{M}^{n+1}$ and $|A|$ stands for the Hilbert–Schmidt norm of the shape operator A of Σ^n with respect to N. According to [BCE, Proposition 2.9] (see also [MPR, Lemma 1]), formula (2.2) means that Θ is a *Jacobi function*, that is, it lies in the kernel of the *stability operator* $L = \Delta + \overline{\mathrm{Ric}}(N,N) + |A|^2$.

We close this section by observing that the Killing vector field Y determines in $M^n \times_\rho \mathbb{R}$ a codimension one foliation by totally geodesic slices $M^n \times \{t\}$, $t \in \mathbb{R}$. In general, the slices are not the only totally geodesic hypersurfaces in Killing warped products. For instance, if Γ is a geodesic of the hyperbolic plane $\mathbb{H}^2$, the cylinder $\Gamma \times \mathbb{R}$ is totally geodesic in $\mathbb{H}^2 \times \mathbb{R}$. So, a hypersurface being totally geodesic is strictly weaker than being a slice in general. Motivated by this fact, in the next section we establish some results which guarantee that (open pieces of) slices are the only totally geodesic hypersurfaces in $M^n \times_\rho \mathbb{R}$, under certain curvature constraints on M^n.

3. Parabolic hypersurfaces in $M^n \times_\rho \mathbb{R}$. Now, we present our first Bernstein type result concerning parabolic two-sided hypersurface immersed in a Killing warped product. For this, we recall that a Riemannian manifold without boundary Σ^n is said to be *parabolic* when the only superharmonic functions on Σ^n bounded from below are the constant ones.

THEOREM 3.1. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a Killing warped product with Ricci curvature of M^n satisfying $\mathrm{Ric}_M \geq -\kappa$ for some constant $\kappa > 0$, and with concave warping function ρ. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a parabolic two-sided hypersurface with constant mean curvature and with angle function Θ having strict sign* (*i.e. either $\Theta > 0$ everywhere or $\Theta < 0$ everywhere*). *If*

$$|\nabla h|^2 \leq \frac{\alpha}{\kappa(n-1)\rho^2}|A|^2 \tag{3.1}$$

for some constant $0 \leq \alpha < 1$, then Σ^n is contained in a slice. If in addition Σ^n is complete, then M^n is complete, Σ^n is a slice and $\overline{M}^{n+1} = M^n \times \mathbb{R}$ is a product space.

Proof. From [O, Corollary 7.43] we get

$$\begin{aligned} \overline{\mathrm{Ric}}(N,N) &= \overline{\mathrm{Ric}}(N^*,N^*) + \overline{\mathrm{Ric}}(N^\perp,N^\perp) \\ &= \mathrm{Ric}_M(N^*,N^*) - \frac{1}{\rho}\,\mathrm{Hess}_M\,\rho(N^*,N^*) - \langle N^\perp,N^\perp\rangle\frac{\Delta_M\rho}{\rho} \\ &= \mathrm{Ric}_M(N^*,N^*) - \frac{1}{\rho}\,\mathrm{Hess}_M\,\rho(N^*,N^*) - \Theta^2\frac{\Delta_M\rho}{\rho^3}, \end{aligned} \tag{3.2}$$

where N^* and $N^\perp$ are the orthogonal projections of N onto M^n and $\mathbb{R}$, respectively, and Hess_M and Δ_M are the Hessian and the Laplacian on M^n.

Consequently, from (3.2) and (2.2) we obtain

$$\Delta\Theta = -\left(\mathrm{Ric}_M(N^*,N^*) - \frac{1}{\rho}\,\mathrm{Hess}_M\,\rho(N^*,N^*) - \Theta^2\frac{\Delta_M\rho}{\rho^3} + |A|^2\right)\Theta. \tag{3.3}$$

On the other hand, denoting by $(\ )^\top$ the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n, we have

$$\nabla h = \frac{1}{\rho^2}Y^\top. \tag{3.4}$$

Moreover,

$$N^* = N - \frac{1}{\rho^2}\Theta Y. \tag{3.5}$$

Hence, from (3.4) and (3.5) it is not difficult to verify that

$$|\nabla h|^2 = \frac{1}{\rho^2}|N^*|^2_M. \tag{3.6}$$

We also note that since we are assuming that Θ has strict sign, for an appropriate choice of N we can suppose $\Theta > 0$ on Σ^n. Since ρ is assumed to be concave, and taking into account our constraint on Ric_M, from (3.3) and (3.6) we get

$$\Delta\Theta \le \big(\kappa(n-1)\rho^2|\nabla h|^2 - |A|^2\big)\Theta. \tag{3.7}$$

Using hypothesis (3.1), from (3.7) we obtain

$$\Delta\Theta \le (\alpha - 1)|A|^2\Theta. \tag{3.8}$$

Hence, Θ is a positive superharmonic function on Σ^n, and since we are assuming that Σ^n is parabolic, Θ must be constant on Σ^n. So, returning to (3.8), we see that Σ is totally geodesic. Therefore, hypothesis (3.1) ensures that h is constant on Σ^n, that is, Σ^n is contained in a slice of $\overline{M}^{n+1}$. If moreover Σ is complete, we infer that M^n is also complete, Σ is a slice of $\overline{M}^{n+1}$, and since in this case $\Theta = \rho$, we conclude that ρ is constant on M^n. ∎

Remark 3.2. We recall that a constant mean curvature hypersurface Σ^n is said to be *stable* when its stability operator L is nonpositive. As a consequence of [MPR, Corollary 1], if the angle function Θ of a stable complete parabolic hypersurface Σ^n with constant mean curvature in $M^n \times_\rho \mathbb{R}$ is bounded, then either Θ is identically zero, or it never vanishes on Σ^n. Hence, removing in Theorem 3.1 the hypothesis that Θ has strict sign and assuming that Σ^n is a stable complete parabolic hypersurface with Θ bounded, we will conclude that Σ^n is either a vertical cylinder, when Θ is identically zero, or a slice, when Θ does not vanish. This also applies to other results along this section.

Theorem 3.3. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a Killing warped product with* Ric_M *nonnegative and with ρ concave. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a parabolic two-sided hypersurface with constant mean curvature and with angle function Θ of strict sign. Then Σ^n is totally geodesic. If moreover* Ric_M *is strictly positive, then Σ^n is contained in a slice of $\overline{M}^{n+1}$. If in addition Σ^n is complete, then M^n is complete, Σ^n is a slice and $\overline{M}^{n+1} = M^n \times \mathbb{R}$ is a product space.*

Proof. Since ρ is concave and $\mathrm{Ric}_M \ge 0$, from (3.3) we find that (for an appropriate choice of N) Θ is a positive function on Σ^n such that

$$\Delta\Theta \le -\big(\mathrm{Ric}_M(N^*, N^*) + |A|^2\big)\Theta \le 0. \tag{3.9}$$

Thus, the parabolicity of Σ^n ensures that Θ is constant on Σ^n. So, returning to (3.9) we have $|A| \equiv 0$, that is, Σ^n is totally geodesic. Moreover, $\mathrm{Ric}_M(N^*, N^*) = 0$ on Σ^n. Assuming that $\mathrm{Ric}_M > 0$, we conclude that N^* vanishes identically on Σ^n, which means that Σ is contained in a slice of $\overline{M}^{n+1}$. If in addition Σ is complete, as in the last part of the proof of

Theorem 3.1 we find that M^n is complete, Σ is a slice of $\overline{M}^{n+1}$ and ρ is constant. ■

Remark 3.4. From the geometric point of view, parabolicity of a complete manifold is tightly related to the growth rate of the volume of geodesic balls. For instance, in [G2, Theorem 7.5], Grigor'yan showed that a sufficient criterion for Σ^n to be parabolic is that it is geodesically complete and, for some origin $o \in \Sigma^n$,

$$\mathrm{vol}(\partial B_r(o))^{-1} \notin L^1(\Sigma).$$

Here, $B_r(o)$ denotes the geodesic ball of Σ^n centered at o and of radius $r > 0$ and $L^1(\Sigma)$ stands for the space of Lebesgue integrable functions on Σ^n.

Furthermore, according to Cheng and Yau [CY] (see also [G2, Corollary 7.4] or [CM, Proposition 1.37]), the parabolicity of a complete manifold is also guaranteed if we assume the stronger condition that it has quadratic area growth, which means that

$$\mathrm{vol}(B_r) = \mathrm{O}(r^2) \quad \text{as } r \to \infty.$$

In particular, parabolicity holds if Σ^n has finite volume. Hence, the conclusions in Theorems 3.1 and 3.3 can be obtained if we assume that Σ^n has either finite volume or quadratic area growth.

On the other hand, Cao and Zhuo [CZ] proved that every n-dimensional gradient shrinking Ricci soliton has $\mathrm{vol}(B_r) \le Cr^n$ for some positive constant C. We recall that a Riemannian manifold (Σ^n, g) is called a *gradient shrinking Ricci soliton* if there exist $f \in C^\infty(\Sigma)$ and a positive constant λ satisfying

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g.$$

Hence, all 2-dimensional gradient shrinking Ricci solitons are parabolic.

Taking into account Remark 3.4, from Theorem 3.3 we obtain

Corollary 3.5. *Let $\overline{M}^3 = \mathbb{S}^2 \times_\rho \mathbb{R}$ be a Killing warped product with ρ concave. Let $\psi : \Sigma^2 \to \overline{M}^3$ be a complete two-sided surface with constant mean curvature and with angle function Θ of strict sign. If $(\Sigma^2, \nabla h)$ is a gradient shrinking Ricci soliton, then Σ^2 is isometric to $\mathbb{S}^2$ and $\overline{M}^3 = \mathbb{S}^2 \times \mathbb{R}$.*

From a classical result due to Ahlfors [A] and Blanc–Fiala–Huber [H], a complete surface of nonnegative Gaussian curvature is parabolic. So, taking into account this result jointly with [ADR, Proposition 8], it is not difficult to see that we can reason as in the proof of Theorem 3.3 to get

Corollary 3.6. *Let $\overline{M}^3 = M^2 \times_\rho \mathbb{R}$ be a Killing warped product with nonnegative Gaussian curvature K_M and with ρ concave. Let $\psi : \Sigma^2 \to \overline{M}^3$ be a complete two-sided surface with constant mean curvature, nonnegative Gaussian curvature and angle function Θ of strict sign. Then Σ^2 is totally* *geodesic. If in addition* $K_M(q) > 0$ *at some* $q \in \Sigma^2$*, then* M^2 *is complete,* Σ^2 *is a slice and* $\overline{M}^3 = M^2 \times \mathbb{R}$ *is a product space.*

In our next Bernstein type theorem, we will use a well known criterion due to Khas'minskii [K] to show that the immersed hypersurface is, in fact, parabolic.

THEOREM 3.7. *Let* $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ *be a Killing warped product with* ρ *concave and whose base* M^n *is complete noncompact with a pole and with nonnegative sectional curvature* K_M*. Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a two-sided hypersurface with constant mean curvature* H*. Suppose that the projection* π_M *is proper when restricted to* Σ^n *and that the angle function* Θ *has strict sign. If*

$$\frac{\partial \ln \rho}{\partial \varrho_M} \leq -\left(\frac{1}{\varrho_M} + |H|\right) \tag{3.10}$$

off a compact set, where ϱ_M *is the distance function on* M^n *measured from a fixed pole, then* Σ^n *is totally geodesic. If moreover* K_M *is strictly positive on* Σ^n*, then* Σ^n *is contained in a slice. If in addition* Σ^n *is complete, then* Σ^n *is a slice and* $\overline{M}^{n+1} = M^n \times \mathbb{R}$ *is a product space.*

Proof. We just need to prove that, under these conditions, the immersion is parabolic. For this, we will use a result due to Khas'minskii [K] (see also [G2, Corollary 5.4]): *if there exists on* Σ^n *a superharmonic function* ξ *outside a compact* $K \subset \Sigma^n$ *such that* $\xi(x) \to \infty$ *as* $x \to \infty$*, then* Σ^n *is parabolic.*

So, let the geodesic distance on M^n measured from a fixed pole $o \in M$ be denoted by $\varrho_M(x) = d_M(x, o)$ for $x \in M^n$. Let us also denote by $\overline{\nabla}$, ∇^M and ∇ the Levi-Civita connections of $\overline{M}^{n+1}$, M^n and Σ^n, respectively, and, slightly abusing the notation, let ϱ_M stand for the composition $\varrho_M \circ \pi_M \circ \psi$. In this setting, we have

$$\nabla \varrho_M = \overline{\nabla} \varrho_M - \langle \overline{\nabla} \varrho_M, N\rangle N. \tag{3.11}$$

For $v, w \in T\Sigma$, it follows that

$$\begin{aligned} \langle \nabla_v \nabla \varrho_M, w\rangle &= \langle \overline{\nabla}_v \nabla \varrho_M, w\rangle \\ &= \langle \overline{\nabla}_v (\overline{\nabla} \varrho_M - \langle \overline{\nabla} \varrho_M, N\rangle N), w\rangle \\ &= \langle \overline{\nabla}_v \overline{\nabla} \varrho_M, w\rangle - \langle \overline{\nabla}_v \langle \overline{\nabla} \varrho_M, N\rangle N, w\rangle \\ &= \langle \overline{\nabla}_v \overline{\nabla} \varrho_M, w\rangle - \langle \overline{\nabla}_v N, w\rangle \langle \overline{\nabla} \varrho_M, N\rangle \\ &= \langle \overline{\nabla}_v \overline{\nabla} \varrho_M, w\rangle + \langle Av, w\rangle \langle \overline{\nabla} \varrho_M, N\rangle. \end{aligned} \tag{3.12}$$

Considering polar coordinates (ϱ_M, θ) on M^n, we have

$$\overline{\nabla}\varrho_M = \sum_{\gamma,\eta} \overline{g}^{\gamma\eta} \frac{\partial \varrho_M}{\partial x^\gamma} \frac{\partial}{\partial x^\eta} = \partial_{\varrho_M},$$

that is, $\overline{\nabla}\varrho_M \in TM$.

Now, we note that

$$\begin{aligned}(3.13)\quad \langle \overline{\nabla}_v \overline{\nabla}\varrho_M, w\rangle = \langle \overline{\nabla}_{(\pi_M)_* v} \overline{\nabla}\varrho_M, (\pi_M)_* w\rangle + \langle \overline{\nabla}_{(\pi_M)_* v} \overline{\nabla}\varrho_M, (\pi_{\mathbb{R}})_* w\rangle \\ + \langle \overline{\nabla}_{(\pi_{\mathbb{R}})_* v} \overline{\nabla}\varrho_M, (\pi_M)_* w\rangle + \langle \overline{\nabla}_{(\pi_{\mathbb{R}})_* v} \overline{\nabla}\varrho_M, (\pi_{\mathbb{R}})_* w\rangle.\end{aligned}$$

Looking at each term above, we have

$$\begin{aligned}\langle \overline{\nabla}_{(\pi_M)_* v} \overline{\nabla}\varrho_M, (\pi_{\mathbb{R}})_* w\rangle = \langle \overline{\nabla}_{(\pi_{\mathbb{R}})_* v} \overline{\nabla}\varrho_M, (\pi_M)_* w\rangle = 0, \\ \langle \overline{\nabla}_{(\pi_{\mathbb{R}})_* v} \overline{\nabla}\varrho_M, (\pi_{\mathbb{R}})_* w\rangle = \rho \cdot \partial_{\varrho_M}(\rho) \langle (\pi_{\mathbb{R}})_* v, (\pi_{\mathbb{R}})_* w\rangle_{\mathbb{R}},\end{aligned}$$

and for the first term we get

$$\begin{aligned}\langle \overline{\nabla}_{(\pi_M)_* v} \overline{\nabla}\varrho_M, (\pi_M)_* w\rangle &= \langle (\pi_M)_* (\overline{\nabla}_{(\pi_M)_* v} \overline{\nabla}\varrho_M), (\pi_M)_* w\rangle_M \\ &= \langle \nabla^M_{(\pi_M)_* v} \partial_{\varrho_M}, (\pi_M)_* w\rangle_M \\ &= \langle \nabla^M_{(\pi_M)_* v^\perp} \partial_{\varrho_M}, (\pi_M)_* w^\perp\rangle_M \\ &\quad + \langle \nabla^M_{(\pi_M)_* v^\perp} \partial_{\varrho_M}, \langle (\pi_M)_* w, \partial_{\varrho_M}\rangle \partial_{\varrho_M}\rangle_M \\ &\quad + \langle \nabla^M_{\langle (\pi_M)_* v, \partial_{\varrho_M}\rangle \partial_{\varrho_M}} \partial_{\varrho_M}, (\pi_M)_* w^\perp\rangle_M \\ &\quad + \langle \nabla^M_{\langle (\pi_M)_* v, \partial_{\varrho_M}\rangle \partial_{\varrho_M}} \partial_{\varrho_M}, \langle (\pi_M)_* w, \partial_{\varrho_M}\rangle \partial_{\varrho_M}\rangle_M.\end{aligned}$$

From the above expression, using the fact that ∂_{ϱ_M} is a Killing vector field, so in particular $\nabla^M_{\partial_{\varrho_M}} \partial_{\varrho_M} = 0$, we obtain

$$\begin{aligned}\langle \nabla^M_{(\pi_M)_* v^\perp} \partial_{\varrho_M}, \langle (\pi_M)_* w, \partial_{\varrho_M}\rangle \partial_{\varrho_M}\rangle_M &= \langle \nabla^M_{\langle (\pi_M)_* v, \partial_{\varrho_M}\rangle \partial_{\varrho_M}} \partial_{\varrho_M}, (\pi_M)_* w^\perp\rangle_M \\ &= \langle \nabla^M_{\langle (\pi_M)_* v, \partial_{\varrho_M}\rangle \partial_{\varrho_M}} \partial_{\varrho_M}, \langle (\pi_M)_* w, \partial_{\varrho_M}\rangle \partial_{\varrho_M}\rangle_M = 0.\end{aligned}$$

Therefore (3.13) gives

$$\langle \overline{\nabla}_v \overline{\nabla}\varrho_M, w\rangle = \langle \nabla^M_{(\pi_M)_* v^\perp} \partial_{\varrho_M}, (\pi_M)_* w^\perp\rangle_M + \rho \frac{\partial \rho}{\partial \varrho_M} \langle (\pi_{\mathbb{R}})_* v, (\pi_{\mathbb{R}})_* w\rangle_{\mathbb{R}}.$$

Inserting this in (3.12) and taking the trace, we obtain

$$\begin{aligned}(3.14)\qquad \Delta\varrho_M = \sum_{j=1}^n \langle \nabla^M_{(\pi_M)_* e_j^\perp} \partial_{\varrho_M}, (\pi_M)_* e_j^\perp\rangle_M \\ + \rho \frac{\partial \rho}{\partial \varrho_M} \sum_{j=1}^n \langle (\pi_{\mathbb{R}})_* e_j, (\pi_{\mathbb{R}})_* e_j\rangle_{\mathbb{R}} + nH\langle \overline{\nabla}\varrho_M, N\rangle.\end{aligned}$$

So, applying the Hessian comparison theorem and taking into account that from (2.1) one has

$$\sum_{j=1}^{n}\langle(\pi_{\mathbb{R}})_*e_j,(\pi_{\mathbb{R}})_*e_j\rangle_{\mathbb{R}}\le\frac{n}{\rho^2},$$

from (3.14) we get

$$\Delta\varrho_M\le n\biggl(\frac{1}{\varrho_M}+\frac{\partial\ln\rho}{\partial\varrho_M}+|H|\biggr). \tag{3.15}$$

Thus, taking into account hypothesis (3.10), from (3.15) we conclude that ϱ_M is superharmonic outside a compact set, and since the projection π_M is proper when restricted to Σ^n, we deduce that $\varrho_M(x)\to\infty$ as $x\to\infty$. Hence, by Khas'minskii's result, Σ^n is parabolic. Therefore, since $K_M\ge 0$ and ρ is concave, the result follows from Theorem 3.3. ∎

Let $\overline{M}^{n+1}=M^n\times_\rho\mathbb{R}$ be a Killing warped product and let $\Psi:\mathbb{R}\times M^n\to\overline{M}^{n+1}$ be the flow generated by the Killing vector field Y. Fix an integral hypersurface M^n of the orthogonal distribution. According to [DL], the *entire Killing graph* $\Sigma(u)$ associated to a function $u\in C^\infty(M)$ is the hypersurface defined as

$$\Sigma(u)=\{\Psi(u(x),x):x\in M^n\}.$$

From the metric (2.1) on the ambient space, we see that $\Sigma(u)$ induces on the base M^n the metric

$$\langle\,,\rangle=\langle\,,\rangle_M+\rho^2du^2.$$

Thus, denoting by Du the gradient of u with respect to the metric of M^n, by a straightforward computation we verify that

$$N=\frac{1}{\rho\sqrt{1+\rho^2|Du|_M^2}}(Y-\rho^2\Psi_*(Du)) \tag{3.16}$$

gives an orientation on $\Sigma(u)$ such that

$$\Theta=\frac{\rho}{\sqrt{1+\rho^2|Du|_M^2}}>0. \tag{3.17}$$

Taking into account (3.17) and the fact that the projection π_M is proper when restricted to an entire Killing graph, from Theorem 3.7 we obtain

Corollary 3.8. *Let $\overline{M}^{n+1}=M^n\times_\rho\mathbb{R}$ be a Killing warped product with ρ concave and whose base M^n is complete noncompact with nonnegative sectional curvature. Let $\Sigma(u)$ be the entire Killing graph of a function $u\in C^\infty(M)$, with constant mean curvature. If inequality* (3.10) *holds, then $\Sigma(u)$ is a slice and $\overline{M}^{n+1}=M^n\times\mathbb{R}$ is a product space.*

4. Stochastic completeness and L^1-Liouville property in $M^n \times_\rho \mathbb{R}$. A Riemannian manifold Σ^n is said to be *stochastically complete* if, for some (and hence all) $(x,t) \in \Sigma \times (0,\infty)$, the heat kernel $p(x,y,t)$ of the Laplace–Beltrami operator Δ has the conservation property

$$\int_\Sigma p(x,y,t)\, d\mu(y) = 1. \tag{4.1}$$

From the probabilistic viewpoint, stochastic completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (4.1) means that the total probability of the particle to be found in the state space is constantly equal to one (cf. [E, G2, S]).

Any parabolic manifold is stochastically complete, but the converse is not true. For example, all Euclidean spaces $\mathbb{R}^n$ (with Euclidean measure) are stochastically complete, whereas $\mathbb{R}^n$ is parabolic if and only if $n \in \{1,2\}$. On the other hand, Pigola, Rigoli and Setti showed that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori–Yau maximum principle (see [PRS1, Theorem 1.1] or [PRS2, Theorem 3.1]):

LEMMA 4.1. *A Riemannian manifold Σ^n is stochastically complete if and only if for every $u \in C^2(\Sigma)$ satisfying $\inf_\Sigma u > -\infty$, there exists a sequence $\{p_j\} \subset \Sigma^n$ such that*

$$\lim_j u(p_j) = \inf_\Sigma u \quad \textit{and} \quad \liminf_j \Delta u(p_j) \geq 0.$$

Returning to our study of Bernstein type properties of hypersurfaces immersed in a Killing warped product, we get the following

THEOREM 4.2. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a Killing warped product with ρ concave and M^n complete noncompact with a pole and having nonnegative sectional curvature. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a two-sided hypersurface with constant mean curvature H. Suppose that the projection π_M is proper when restricted to Σ^n and the angle function Θ is bounded away from zero. If, for some positive constant λ,*

$$\left|\frac{\partial \ln \rho}{\partial \varrho_M}\right| \leq \lambda \varrho_M \tag{4.2}$$

off a compact set, where ϱ_M is the distance function on M^n measured from a fixed pole, then Σ^n is minimal. If in addition $n = 2$ and Σ^2 is complete, then Σ^2 is totally geodesic. If moreover the Gaussian curvature of Σ^2 is positive at some point of it, then M^2 is complete and Σ^2 is a slice of $\overline{M}^3$.

Proof. After an appropriate choice of N, we have $\Theta \geq C > 0$ for some positive constant C. Moreover, using hypothesis (4.2) jointly with the Hessian comparison theorem, it follows from (3.15) that $\Delta \varrho_M \leq \tilde{\lambda} \varrho_M$ for some positive constant $\tilde{\lambda}$, where (as in the proof of Theorem 3.7) ϱ_M stands for the composition $\varrho_M \circ \pi_M \circ \psi$. Thus, we can apply another Khas'minskii criterion [K] (see also [PRS2, Proposition 3.2]) to conclude that Σ^n is stochastically complete. Consequently, Lemma 4.1 guarantees the existence of a sequence $\{p_k\} \subset \Sigma^n$ such that

$$\lim_j \Theta(p_j) = \inf_\Sigma \Theta \quad \text{and} \quad \liminf_j \Delta\Theta(p_j) \geq 0.$$

On the other hand, since ρ is a concave function, from (3.3) we have

$$\Delta\Theta \leq -(\mathrm{Ric}_M(N^*, N^*) + |A|^2)\Theta.$$

Hence, since $|A|^2 = nH^2 + n(n-1)(H^2 - H_2)$, we get

$$\begin{aligned} 0 \leq \liminf_j \Delta\Theta(p_j) &\leq -\lim_j(\mathrm{Ric}_M(N^*, N^*) + |A|^2)\Theta(p_j) \\ &\leq -\lim_j(\mathrm{Ric}_M(N^*, N^*) + nH^2)\Theta(p_j) \leq 0. \end{aligned}$$

Consequently, since $\Theta \geq C > 0$ and $\mathrm{Ric}_M \geq 0$, we conclude that $H = 0$, that is, Σ^n is minimal. Thus, if $n = 2$ and Σ^2 is complete, we can apply [ADR, Proposition 8] to finish the proof. ∎

Remark 4.3. Considering the entire vertical graph

$$\Sigma^2(u) = \{(x, y, a \ln y) : y > 0\} \subset \mathbb{H}^2 \times \mathbb{R}, \qquad a \in \mathbb{R} \setminus \{0\},$$

where the 2-dimensional hyperbolic space $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is endowed with the complete metric $\langle\, ,\rangle_{\mathbb{H}^2} = y^{-2}(dx^2 + dy^2)$ and the smooth function $u : \mathbb{H}^2 \to \mathbb{R}$ is given by $u(x, y) = a \ln y$, we see that $\Theta = 1/\sqrt{1 + a^2}$ and $H = a/(2\sqrt{1 + a^2})$ (for more details, see [LLP, Example 10]). Hence, Theorems 3.3 and 4.2 do not hold when the sectional curvature of the base M^n is negative.

From Theorem 4.2 jointly with the half-space type theorem due to Rosenberg, Schulze and Spruck [RSS], already mentioned in our introduction, we obtain the following

Corollary 4.4. *Let $\overline{M}^{n+1} = M^n \times \mathbb{R}$ where M^n is complete noncompact with nonnegative sectional curvature. Let $\Sigma(u)$ be the entire Killing graph of a nonnegative function $u \in C^\infty(M)$, with constant mean curvature. If $|u|_{C^1(M)} \leq \alpha$ for some positive constant α, then $\Sigma(u)$ is a slice of $\overline{M}^{n+1}$.*

According to the terminology due to Bessa, Pigola and Setti [BPS], a smooth Riemannian manifold (Σ^n, g) is said to have the *L^1-Liouville property* (briefly, Σ^n is L^1-Liouville) if every nonnegative superharmonic function $u \in L^1(\Sigma)$ is constant. As observed by these authors, a stochastically complete manifold is L^1-Liouville. However, in general, an L^1-Liouville manifold may be stochastically incomplete (for details, see [BPS, Section 2]).

Assuming that the immersed hypersurface is L^1-Liouville, we get

THEOREM 4.5. *Let $\overline{M}^{n+1} = M^n \times_\rho \mathbb{R}$ be a Killing warped product with $\mathrm{Ric}_M \geq -\kappa$ for some constant κ and with ρ concave. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be an L^1-Liouville hypersurface with constant mean curvature and with angle function Θ of strict sign and $\Theta \in L^1(\Sigma)$.*

(a) *If $\kappa = 0$, then Σ^n is totally geodesic. If furthermore $|\nabla h| \leq \alpha |A|^\beta$ for some positive constants α and β, and Σ^n is complete, then Σ^n is a slice and $\overline{M}^{n+1} = M^n \times \mathbb{R}$ is a product space with M^n compact.*

(b) *If $\kappa > 0$ and, for some constant $0 < \alpha < 1$,*

$$|\nabla h|^2 \leq \frac{\alpha}{\kappa(n-1)\rho^2}|A|^2, \tag{4.3}$$

then Σ^n is contained in a slice of $\overline{M}^{n+1}$. If in addition Σ^n is complete, then Σ^n is a slice and $\overline{M}^{n+1} = M^n \times \mathbb{R}$.

(c) *If $\kappa < 0$, then Σ^n is contained in a slice of $\overline{M}^{n+1}$. If in addition Σ^n is complete, then Σ^n is a slice and $\overline{M}^{n+1} = M^n \times \mathbb{R}$ with M^n compact.*

Proof. In the set-up of (a), we have

$$\Delta\Theta \leq -(\mathrm{Ric}_M(N^*, N^*) + |A|^2)\Theta \leq 0. \tag{4.4}$$

Since Σ is L^1-Liouville and (after an appropriate choice of N) $\Theta > 0$, we see that Θ is constant. Therefore, by (4.4),

$$0 = \Delta\Theta \leq -(\mathrm{Ric}_M(N^*, N^*) + |A|^2)\Theta \leq 0.$$

So, Σ^n is totally geodesic. Since $|\nabla h| \leq \alpha |A|^\beta$, we conclude that Σ^n is, in fact, contained in a slice of $\overline{M}^{n+1}$. If moreover Σ^n is complete, then Σ^n is a slice, and since Θ is constant, ρ is constant on M^n. Moreover, $\mathrm{vol}(\Sigma) < \infty$. Therefore, [Y, Theorem 7] ensures that Σ^n is compact, and hence so is M^n.

In order to prove (b), we observe that (4.3) implies

$$\begin{aligned} \Delta\Theta &\leq -(\mathrm{Ric}_M(N^*, N^*) + |A|^2)\Theta \\ &\leq (\kappa(n-1)\rho^2|\nabla h|^2 - |A|^2)\Theta \leq (\alpha - 1)|A|^2\Theta \leq 0. \end{aligned}$$

Hence, we can reason as in the proof of (a). The proof of (c) is similar. ∎

REMARK 4.6. As observed by Bessa, Pigola and Setti [BPS], a result due to Grigor'yan [G1] ensures that a Riemannian manifold Σ^n is L^1-Liouville if and only if for some (hence any) $x \in \Sigma^n$,

$$\int_\Sigma G(x, y)\, d\mu(y) = \infty. \tag{4.5}$$

Here $G(x, y)$ stands for the Green kernel of Σ^n, which is defined as being the minimal, positive, fundamental solution of $-\Delta$. When Σ^n is parabolic, we have $G \equiv \infty$, and (4.5) is trivially satisfied. However, in this case, one already knows that positive superharmonic functions (without any further restriction) must be constant.

Acknowledgements. The third author is partially supported by CNPq, Brazil, grant 303977/2015-9. The authors would like to thank the referee for reading the manuscript in great detail and for his/her valuable suggestions and useful comments which improved the paper.

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