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On the Bishop–Phelps–Bollobás theorem for operators and numerical radius

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Abstract. We study the Bishop–Phelps–Bollobás property for numerical radius (for short, BPBp-nu) of operators on ℓ_1-sums and ℓ_∞-sums of Banach spaces. More precisely, we introduce a property of Banach spaces, which we call strongly lush. We find that if X is strongly lush and $X \oplus_1 Y$ has the weak BPBp-nu, then (X, Y) has the Bishop–Phelps–Bollobás property (BPBp). On the other hand, if Y is strongly lush and $X \oplus_\infty Y$ has the weak BPBp-nu, then (X, Y) has the BPBp. Examples of strongly lush spaces are $C(K)$ spaces, $L_1(\mu)$ spaces, and finite-codimensional subspaces of $C[0, 1]$.

1. Introduction. Let X be a (real or complex) Banach space and X^* be its dual space. The unit sphere of X will be denoted by S_X and the closed unit ball by B_X. We write $\mathcal{L}(X)$ for the space of all bounded linear operators on X. The *numerical radius* of $T \in \mathcal{L}(X)$ is defined by

$$v(T) = \sup\{|x^*(Tx)| : (x, x^*) \in \Pi(X)\},$$

where $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} : x^*(x) = 1\}$. It is clear that v is a seminorm on $\mathcal{L}(X)$ with $v(T) \leq \|T\|$ for every $T \in \mathcal{L}(X)$. We refer the reader to the classical monographs [10, 11] for background. An operator $T \in \mathcal{L}(X)$ *attains its numerical radius* if there exists $(x_0, x_0^*) \in \Pi(X)$ such that $v(T) = |x_0^*(Tx_0)|$. A lot of attention has been paid to the study of the denseness of numerical radius attaining operators [1, 3, 6, 8, 14, 15, 16, 27].

Motivated by the work [4] of M. Acosta, R. Aron, D. García and M. Maestre on the Bishop–Phelps–Bollobás property for operators, A. Guirao and O. Kozhushkina [19] introduced the notion of Bishop–Phelps–Bollobás property for numerical radius, which is a quantitative way to study the denseness of numerical radius attaining operators.

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Definition 1.1 ([22]). A Banach space X is said to have the *weak Bishop–Phelps–Bollobás property for numerical radius* (for short, *weak BPBp-nu*) if for every $0<\varepsilon<1$, there exists $\eta(\varepsilon)>0$ such that whenever $T\in\mathcal{L}(X)$ and $(x,x^*)\in\Pi(X)$ satisfy $v(T)=1$ and $|x^*(Tx)|>1-\eta(\varepsilon)$, there exist $S\in\mathcal{L}(X)$ and $(y,y^*)\in\Pi(X)$ such that

$$v(S)=|y^*(Sy)|,\quad \|T-S\|<\varepsilon,\quad \|x-y\|<\varepsilon,\quad \|x^*-y^*\|<\varepsilon.$$

A pair (X,Y) of Banach spaces is said to have the *Bishop–Phelps–Bollobas property for numerical radius* (for short, *BPBP-nu*) if together with all requirements of Definition 1.1, also $v(S)=|y^*(Sy)|=1$. From the definitions, it is clear that the BPBp-nu implies the weak BPBp-nu, while in [22] some conditions are given ensuring that the converse also holds.

Let X, Y be Banach spaces and denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X to Y. We recall that $T\in\mathcal{L}(X,Y)$ is said to be *norm attaining* if there is $x\in B_X$ such that $\|T\|=\|Tx\|$. A pair (X,Y) is said to have the *Bishop–Phelps–Bollobás property for operators* (for short, *BPBp*) [4] if, given $\varepsilon>0$, there exists $\eta(\varepsilon)>0$ such that whenever $T\in\mathcal{L}(X,Y)$ with $\|T\|=1$ and $x\in S_X$ satisfy $\|Tx\|>1-\eta(\varepsilon)$, there exist $S\in\mathcal{L}(X,Y)$ with $\|S\|=1$ and $y\in S_X$ such that

$$\|Sy\|=1,\quad \|T-S\|<\varepsilon,\quad \|x-y\|<\varepsilon.$$

It is shown in [19] that the real or complex spaces c_0 and ℓ_1 have the BPBp-nu. The result on ℓ_1 has been extended to the real space $L_1(\mathbb{R})$ by J. Falcó [18]. For the result on c_0, A. Avilés, A. J. Guirao and J. Rodríguez [7] gave sufficient conditions on a compact space K for the real space $C(K)$ to have the BPBp-nu, which, in particular, include all metrizable compact spaces. In [22] the BPBp-nu is studied for more general spaces. For instance, it is shown that finite-dimensional spaces and general $L_1(\mu)$ spaces have the BPBp-nu. It is also shown that $L_p(\mu)$ has the BPBp-nu for every measure μ when $1<p<\infty$, $p\neq 2$. It has been shown very recently [23] that every real Hilbert space has the BPBp-nu. As for negative results, it is shown in [22] that every separable infinite-dimensional Banach space can be equivalently renormed to fail the BPBp-nu, even though for reflexive spaces (actually for spaces with the Radon–Nikodým property [6]) the set of numerical radius attaining operators is always dense. To get this result, it is shown in [22] that there is a relation between the BPBp-nu and the Bishop–Phelps–Bollobás property for operators. More precisely, if $L_1(\mu)\oplus_1 Y$ has the BPBp-nu, then $(L_1(\mu),Y)$ has the BPBp [22, Theorem 15].

In this paper, we generalize this fact as follows. Let X,Y be Banach spaces. If X is strongly lush (see the definition below) and $X\oplus_1 Y$ has the weak BPBp-nu, then (X,Y) has the BPBp. On the other hand, if Y is strongly lush and $X\oplus_\infty Y$ has the weak BPBp-nu, then (X,Y) has the BPBp. It is also shown that none of the converses of these results holds. More precisely, there exist strongly lush spaces X and Y such that (X, Y) has the BPBp, but the set of numerical radius attaining operators is dense in neither $\mathcal{L}(X \oplus_1 Y)$ nor $\mathcal{L}(X \oplus_\infty Y)$.

We need some notation. Given a subset F of a Banach space X, we denote the absolutely closed convex hull of F by $\overline{\mathrm{aconv}}(F)$. For $C \subset X^*$, $\overline{\mathrm{aconv}}^{w^*}(C)$ denotes the absolutely weak-$*$ closed convex hull of C. We write $\mathrm{NA}(X)$ to denote the subset of those elements in X^* which attain their norm. Note that this set is dense by the classical Bishop–Phelps theorem [9]. Given $x^* \in \mathrm{NA}(X) \cap S_{X^*}$, we write $F(x^*)$ to denote the (non-empty) *face* generated by x^*, i.e. $F(x^*) = \{x \in B_X : x^*(x) = 1\}$.

DEFINITION 1.2. We say that a Banach space X is *strongly lush* if there is a subset C of S_{X^*} such that $B_{X^*} = \overline{\mathrm{aconv}}^{w^*}(C)$ and $B_X = \overline{\mathrm{aconv}}(F(x^*))$ for every $x^* \in C$.

This definition appeared, without name, in some papers, including [21, Corollary 4.5] or [24, Proposition 2.1]. There are many examples of spaces with this property, the easiest ones being the almost-CL-spaces [26, §2]. We recall that a Banach space is an *almost-CL-space* if its unit ball is the absolutely closed convex hull of every maximal face. $L_1(\mu)$ spaces and their isometric preduals (in particular, $C(K)$ spaces), the disk algebra etc. are examples of almost-CL-spaces (see [17, 21, 26] and references therein for background).

Moreover, separable lush spaces are strongly lush. We recall that a Banach space X is *lush* [13] if given $x, y \in S_X$ and $\varepsilon > 0$, there is $y^* \in S_{X^*}$ such that $\operatorname{Re} y^*(y) > 1 - \varepsilon$ and the distance from x to

$$\overline{\mathrm{aconv}}(\{z \in B_X : \operatorname{Re} y^*(z) > 1 - \varepsilon\})$$

is less than ε. We refer to [12, 13, 21, 24] and references therein for background. Almost-CL-spaces are lush, but the converse is not true [13]. As commented before, separable lush spaces are strongly lush ([21, Corollary 4.5] for the real case, [24, Proposition 2.1] for the complex case). This implies, in particular, that finite-codimensional subspaces of $C[0, 1]$ are strongly lush.

Let us also mention that there is a reformulation of strong lushness in terms of extreme points of the bidual ball: A Banach space X is strongly lush if and only if there exists a subset $C \subset S_{X^*}$ with $B_{X^*} = \overline{\mathrm{aconv}}^{w^*}(C)$ such that $|x^{**}(x^*)| = 1$ for every $x^* \in C$ and every extreme point x^{**} of $B_{X^{**}}$. Indeed, Milman's theorem shows that the necessity holds. The converse is shown by [5, Corollary 3.5].

2. The results. Let us present first the result for ℓ_1-sums, which generalizes [22, Theorem 15].

THEOREM 2.1. *Let X and Y be Banach spaces and suppose that X is strongly lush. If $X \oplus_1 Y$ has the weak BPBp-nu, then (X, Y) has the BPBp.*

Proof. Suppose that $X \oplus_1 Y$ has the weak BPBp-nu with a function η; we will show that (X, Y) has the BPBp with the function $\varepsilon \mapsto \eta\big(\frac{\varepsilon}{2+\varepsilon}\big)$. Fix $0 < \varepsilon < 1$ and let $\tilde{\varepsilon} = \varepsilon/(\varepsilon + 2)$. Let $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\tilde{\varepsilon})$. We pick $y_0^* \in S_{Y^*}$ such that

$$|y_0^*(Tx_0)| = \|Tx_0\| > 1 - \eta(\tilde{\varepsilon}).$$

We consider the extension $\tilde{T}$ of T from $X \oplus_1 Y$ to $X \oplus_1 Y$ given by

$$\tilde{T}(x, y) = (0, Tx) \quad ((x, y) \in X \oplus_1 Y).$$

We claim that $v(\tilde{T}) = \|\tilde{T}\| = 1$. Indeed, $v(\tilde{T}) \leq \|\tilde{T}\| = \|T\| = 1$. On the other hand,

$$\begin{aligned} v(\tilde{T}) &= \sup\big\{|(x^*, y^*)\tilde{T}(x, y)| : \max\{\|x^*\|, \|y^*\|\} = 1,\ \|x\| + \|y\| = 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^*(x) + y^*(y) = 1\big\} \\ &= \sup\big\{|(x^*, y^*)\tilde{T}(x, y)| : (x^*, y^*) \in B_{X^*} \times B_{Y^*},\ \|x\| + \|y\| = 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x^*(x) = \|x\|,\ y^*(y) = \|y\|\big\} \\ &\geq \sup\{|(x^*, y^*)\tilde{T}(x, 0)| : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1,\ y^* \in S_{Y^*}\} \\ &= \sup\{|y^*(Tx)| : y^* \in S_{Y^*},\ x \in S_X\} = \|T\|. \end{aligned}$$

Now, pick any $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$ and observe that

$$((x_0, 0), (x_0^*, y_0^*)) \in \Pi(X \oplus_1 Y)$$

and

$$|(x_0^*, y_0^*)\tilde{T}(x_0, 0)| = |y_0^*(Tx_0)| = \|Tx_0\| > 1 - \eta(\tilde{\varepsilon}).$$

Since $X \oplus_1 Y$ has the weak BPBp-nu with the function η, there exist $(x_1, y_1) \in S_{X \oplus_1 Y}$, $(x_1^*, y_1^*) \in S_{X^* \oplus_\infty Y^*}$ and $S' \in \mathcal{L}(X \oplus_1 Y)$ satisfying

$$x_1^*(x_1) + y_1^*(y_1) = 1, \qquad v(S') = |(x_1^*, y_1^*)S'(x_1, y_1)|$$

and

$$\|S' - \tilde{T}\| < \tilde{\varepsilon}, \qquad \|x_1 - x_0\| + \|y_1\| < \varepsilon, \qquad \max\{\|x_1^* - x_0^*\|, \|y_1^* - y_0^*\|\} < \varepsilon.$$

So $|v(S') - 1| < \tilde{\varepsilon}$ and $\big|\|S'\| - 1\big| < \tilde{\varepsilon}$. Hence

$$\begin{aligned} \left\|\frac{S'}{v(S')} - \tilde{T}\right\| &\leq \left\|\frac{S'}{v(S')} - S'\right\| + \|S' - \tilde{T}\| \\ &< \frac{\|S'\| \cdot |v(S') - 1|}{v(S')} + \tilde{\varepsilon} \leq \frac{(1 + \tilde{\varepsilon})\tilde{\varepsilon}}{1 - \tilde{\varepsilon}} + \tilde{\varepsilon} = \varepsilon. \end{aligned}$$

Write $\tilde{S} = S'/v(S')$ and observe that

$$v(\tilde{S}) = 1 = \big|(x_1^*, y_1^*)\tilde{S}(x_1, y_1)\big|, \qquad \|\tilde{S} - \tilde{T}\| < \varepsilon.$$

It follows that $x_1^*(x_1) = \|x_1\|$ and $y_1^*(y_1) = \|y_1\|$.

We claim that $y_1 = 0$. Otherwise,

$$\left\|\tilde{S}\left(0, \frac{y_1}{\|y_1\|}\right) - \tilde{T}\left(0, \frac{y_1}{\|y_1\|}\right)\right\| = \left\|\tilde{S}\left(0, \frac{y_1}{\|y_1\|}\right)\right\| < \varepsilon.$$

If $x_1 \neq 0$, then

$$\begin{aligned}1 &= |(x_1^*, y_1^*)\tilde{S}(x_1, y_1)| \\ &\leq \left|(x_1^*, y_1^*)\tilde{S}\left(\frac{x_1}{\|x_1\|}, 0\right)\right| \|x_1\| + \left|(x_1^*, y_1^*)\tilde{S}\left(0, \frac{y_1}{\|y_1\|}\right)\right| \|y_1\| \\ &\leq \|x_1\| + \varepsilon\|y_1\| < \|x_1\| + \|y_1\| = 1,\end{aligned}$$

a contradiction. The case $x_1 = 0$ is even easier.

By the claim, we have $x_1^*(x_1) = \|x_1\| = 1$. Next, write $\tilde{S}(x, y) = (\tilde{S}_1(x, y), \tilde{S}_2(x, y))$ and define $S_1 \in \mathcal{L}(X, X)$ and $S_2 \in \mathcal{L}(X, Y)$ by

$$S_1 x = \tilde{S}_1(x, 0), \quad S_2 x = \tilde{S}_2(x, 0) \quad (x \in X).$$

Observe that

$$\begin{aligned}1 = v(\tilde{S}) = |(x_1^*, y_1^*)\tilde{S}(x_1, 0)| &= |x_1^*(S_1 x_1) + y_1^*(S_2 x_1)| \\ &\leq \|S_1 x_1\| + \|S_2 x_1\| \leq \sup\{\|S_1 x\| + \|S_2 x\| : x \in B_X\} \\ &= \sup\{|x^*(S_1 x)| + |y^*(S_2 x)| : x \in B_X,\ x^* \in C,\ y^* \in S_{Y^*}\} \\ &= \sup\{|x^*(S_1 x) + y^*(S_2 x)| : x \in B_X,\ x^* \in C,\ y^* \in S_{Y^*}\}\end{aligned}$$

where we have used the fact that $\overline{\text{aconv}}^{w^*}(C) = B_{X^*}$. For $x^* \in C$, we have $B_X = \overline{\text{aconv}}(F(x^*))$ and the function $x \mapsto |x^*(S_1 x) + y^*(S_2 x)|$ is convex, so we may continue the previous chain of inequalities as follows:

$$\begin{aligned}&= \sup\{|x^*(S_1 x) + y^*(S_2 x)| : x^* \in C,\ x \in F(x^*),\ y^* \in S_{Y^*}\} \\ &= \sup\{|(x^*, y^*)\tilde{S}(x, 0)| : x^* \in C,\ x \in F(x^*),\ y^* \in S_{Y^*}\} \\ &\leq \sup\{|(x^*, y^*)\tilde{S}(x, y)| : ((x, y), (x^*, y^*)) \in \Pi(X \oplus_1 Y)\} = v(\tilde{S}) = 1.\end{aligned}$$

We conclude that

$$\begin{aligned}\sup\{\|S_1 x\| + \|S_2 x\| : x \in B_X\} &= \|S_1 x_1\| + \|S_2 x_1\| \\ &= |x_1^*(S_1 x_1) + y_1^*(S_2 x_1)| = 1,\end{aligned}$$

and it follows in particular that there exists $\omega \in S_{\mathbb{K}}$ such that

$$\|S_1 x_1\| = \omega x_1^*(S_1 x_1) \quad \text{and} \quad \|S_2 x_1\| = \omega y_1^*(S_2 x_1).$$

We now claim that $S_2 x_1 \neq 0$. Indeed, for all $x \in S_X$,

$$\varepsilon > \|\tilde{S} - \tilde{T}\| \geq \|S_1 x\| + \|S_2 x - Tx\|.$$

So $\|S_1\| \leq \varepsilon$ and $\|S_2 - T\| < \varepsilon$. If $S_2 x_1 = 0$, then

$$1 = \|S_1 x_1\| + \|S_2 x_1\| = \|S_1 x_1\| \leq \|S_1\| < \varepsilon,$$

a contradiction.

Finally, define $R \in \mathcal{L}(X,Y)$ by

$$Rx = S_2x + \omega \frac{S_2x_1}{\|S_2x_1\|} x_1^*(S_1x) \qquad \big(x \in X\big).$$

Observe that

$$\begin{aligned}|y_1^*(Rx_1)| &= \left| y_1^*(S_2x_1) + \omega\, \frac{y_1^*(S_2x_1)}{\|S_2x_1\|} x_1^*(S_1x_1) \right| \\ &= |x_1^*(S_1x_1) + y_1^*(S_2x_1)| = 1\end{aligned}$$

and $\|Rx\| \le \|S_2x\| + \|S_1x\| \le 1$. Therefore, $\|R\| = 1 = \|Rx_1\|$ and

$$\|R - T\| \le \|S_2 - T\| + \|S_1\| \le \|\tilde{S} - \tilde{T}\| < \varepsilon.$$

Notice also that $\|x_1 - x_0\| < \varepsilon$. This completes the proof. ∎

As mentioned in the introduction, almost-CL-spaces and separable lush spaces are strongly lush. Therefore, we have the following corollary.

Corollary 2.2. *Let X be an almost-CL-space or a separable lush space and let Y be a Banach space. If $X \oplus_1 Y$ has the weak BPBp-nu, then (X,Y) has the BPBp.*

Concerning ℓ_∞-sums, we have the following result in which a condition has to be imposed on the range space instead of on the domain space.

Theorem 2.3. *Let X and Y be Banach spaces and suppose that Y is strongly lush. If $X \oplus_\infty Y$ has the weak BPBp-nu, then (X,Y) has the BPBp.*

Proof. Suppose that $X \oplus_\infty Y$ has the BPBp-nu with a function η; we will show that (X,Y) has the BPBp with the function $\varepsilon \mapsto \eta\big(\frac{\varepsilon}{4+\varepsilon}\big)$. Fix $0 < \varepsilon < 1$ and let $\tilde{\varepsilon} = \varepsilon/(4+\varepsilon)$. Let $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\tilde{\varepsilon})$. Then, by the Bishop–Phelps theorem, there exists $y_0^* \in S_{Y^*} \cap \mathrm{NA}(Y)$ such that

$$|y_0^*(Tx_0)| = \|Tx_0\| > 1 - \eta(\tilde{\varepsilon}).$$

We pick $y_0 \in S_Y$ such that $y_0^*(y_0) = 1$. Now, we consider the extension $\tilde{T}$ of T from $X \oplus_\infty Y$ to $X \oplus_\infty Y$ defined by $\tilde{T}(x,y) = (0, Tx)$ for every $(x,y) \in X \oplus_\infty Y$. Clearly, $v(\tilde{T}) \le \|\tilde{T}\| = \|T\| = 1$ and, on the other hand,

$$\begin{aligned}v(\tilde{T}) &\ge \sup\big\{|(x^*,y^*)\tilde{T}(x,y)| : (x,y) \in S_X \times S_Y,\ \|x^*\| + \|y^*\| = 1, \\ &\qquad\qquad x^*(x) = \|x^*\|,\ y^*(y) = \|y^*\|\big\} \\ &\ge \sup\{|y^*(Tx)| : y^* \in S_{Y^*} \cap \mathrm{NA}(Y),\ x \in S_X\} = \|T\| = 1.\end{aligned}$$

So $v(\tilde{T}) = \|\tilde{T}\| = 1$. As $|(0,y_0^*)\tilde{T}(x_0,y_0)| = |y_0^*(Tx_0)| > 1 - \eta(\tilde{\varepsilon})$ and $X \oplus_\infty Y$ has the weak BPBp-nu with the function η, there exist $S' \in \mathcal{L}(X \oplus_\infty Y)$, $(x_1,y_1) \in S_{X \oplus_\infty Y}$ and $(x_1^*, y_1^*) \in S_{X^* \oplus_1 Y^*}$ such that

$$x_1^*(x_1) + y_1^*(y_1) = 1, \qquad v(S') = |(x_1^*, y_1^*)S'(x_1,y_1)|$$

and

$$\|\tilde{T}-S'\| < \tilde{\varepsilon}, \quad \max\{\|x_1-x_0\|, \|y_0-y_1\|\} < \varepsilon/2, \quad \|x_1^*\|+\|y_0^*-y_1^*\| < \varepsilon/2.$$

So $|v(S')-1| < \tilde{\varepsilon}$ and $\big|\|S'\|-1\big| < \tilde{\varepsilon}$. Hence

$$\left\|\frac{S'}{v(S')}-\tilde{T}\right\| \le \left\|\frac{S'}{v(S')}-S'\right\| + \|S'-\tilde{T}\| < \frac{\|S'\|\cdot|v(S')-1|}{v(S')}+\tilde{\varepsilon}$$
$$\le \frac{(1+\tilde{\varepsilon})\tilde{\varepsilon}}{1-\tilde{\varepsilon}}+\tilde{\varepsilon} = \varepsilon/2.$$

Now, for $\tilde{S} = S'/v(S')$ we have

$$v(\tilde{S}) = 1 = |(x_1^*, y_1^*)\tilde{S}(x_1, y_1)|, \quad \|\tilde{S}-\tilde{T}\| < \varepsilon/2.$$

Observe that

$$x_1^*(x_1) = \|x_1^*\|, \quad y_1^*(y_1) = \|y_1^*\|, \quad \|x_1^*\|+\|y_1^*\| = 1.$$

We claim that $x_1^* = 0$. Otherwise,

$$\left\|\left(\frac{x_1^*}{\|x_1^*\|}, 0\right)\tilde{S}(x_1, y_1)\right\| = \left\|\left(\frac{x_1^*}{\|x_1^*\|}, 0\right)\tilde{S}(x_1, y_1) - \left(\frac{x_1^*}{\|x_1^*\|}, 0\right)\tilde{T}(x_1, y_1)\right\|$$
$$\le \|\tilde{S}-\tilde{T}\| < \varepsilon.$$

Hence, if $y_1^* \neq 0$, then

$$1 = |(x_1^*, y_1^*)\tilde{S}(x_1, y_1)|$$
$$\le \left|\left(\frac{x_1^*}{\|x_1^*\|}, 0\right)\tilde{S}(x_1, y_1)\right|\|x_1^*\| + \left|\left(0, \frac{y_1^*}{\|y_1^*\|}\right)\tilde{S}(x_1, y_1)\right|\|y_1^*\|$$
$$\le \varepsilon\|x_1^*\| + \|y_1^*\| < \|x_1^*\| + \|y_1^*\| = 1,$$

a contradiction. The case $y_1^* = 0$ is similar.

By the claim, we get $y_1^*(y_1) = \|y_1\| = 1$. Write $\tilde{T}(x, y) = (0, \tilde{T}_2(x, y))$ and $\tilde{S}(x, y) = (\tilde{S}_1(x, y), \tilde{S}_2(x, y))$ for every $(x, y) \in X \oplus_\infty Y$.

We claim that $\|\tilde{S}_2\| = 1 = \|\tilde{S}_2(x_1, y_1)\|$. Indeed,

$$1 = v(\tilde{S}) = |(0, y_1^*)\tilde{S}(x_1, y_1)| = |y_1^*(\tilde{S}_2(x_1, y_1))| \le \|\tilde{S}_2(x_1, y_1)\|$$
$$\le \sup\{\|\tilde{S}_2(x, y)\| : x \in B_X,\, y \in B_Y\}$$
$$= \sup\{|y^*(\tilde{S}_2(x, y))| : x \in B_X,\, y \in B_Y,\, y^* \in C\}$$

where we have used the fact that $\overline{\text{aconv}}^{w^*}(C) = B_{X^*}$. For $y^* \in C$, the function $y \mapsto |y^*(\tilde{S}_2(x, y))|$ is convex and $B_Y = \overline{\text{aconv}}(F(y^*))$, so we may continue the previous chain of inequalities as follows:

$$= \sup\{|y^*(\tilde{S}_2(x, y))| : x \in B_X,\, y^* \in C,\, y \in F(y^*)\}$$
$$= \sup\{|(0, y^*)\tilde{S}(x, y)| : x \in B_X,\, y^* \in C,\, y \in F(y^*)\}$$
$$\le v(\tilde{S}) = 1,$$

which proves the claim.

As $\|x_0\| = 1$ and $\|x_0 - x_1\| < \varepsilon/2$, it follows that $\|x_1\| > 1 - \varepsilon/2$ (so, in particular, $x_1 \neq 0$), and $\bar{x}_1 = x_1/\|x_1\|$ satisfies

$$\|\bar{x}_1 - x_0\| < \varepsilon.$$

Next, we claim that $\|\tilde{S}_2(\bar{x}_1, y_1)\| = 1$. Otherwise,

$$\begin{aligned}\|\tilde{S}_2(x_1, y_1)\| &\leq \|x_1\|\,\|S_2(\bar{x}_1, y_1)\| + (1 - \|x_1\|)\|S_2(0, y_1)\| \\ &< \|x_1\| + (1 - \|x_1\|) = 1,\end{aligned}$$

a contradiction.

Finally, choose $x_2^* \in S_{X^*}$ with $x_2^*(\bar{x}_1) = 1$ and define $R \in \mathcal{L}(X, Y)$ by

$$Rx = \tilde{S}_2(x, x_2^*(x)y_1) \quad (x \in X).$$

We clearly have $\|R\| \leq \|\tilde{S}_2\| \leq 1$ and

$$\|R\bar{x}_1\| = \|\tilde{S}_2(\bar{x}_1, y_1)\| = 1.$$

So it is enough to show that $\|R - T\| < \varepsilon$. Note that for $x \in B_X$ and $y \in B_Y$,

$$\|\tilde{S}_2(x, y) - Tx\| = \|\tilde{S}_2(x, y) - \tilde{T}_2(x, y)\| \leq \|\tilde{S}_2 - \tilde{T}_2\| \leq \|\tilde{S} - \tilde{T}\| < \varepsilon/2.$$

In particular, for all $x \in B_X$,

$$\|Rx - Tx\| = \|\tilde{S}_2(x, x_2(x)y_1) - Tx\| < \varepsilon/2.$$

This completes the proof. ∎

As for the ℓ_1-sum, we obtain the following consequence.

Corollary 2.4. *Let Y be an almost-CL-space or a separable lush space and let X be a Banach space. If $X \oplus_\infty Y$ has the weak BPBp-nu, then (X, Y) has the BPBp.*

The proofs of Theorems 2.1 and 2.3 can be easily adapted to get analogous results for norm and numerical radius attaining operators:

Remark 2.5. *Let X and Y be Banach spaces.*

(a) *Suppose that X is strongly lush and the set of numerical radius attaining operators is dense in $\mathcal{L}(X \oplus_1 Y)$. Then the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$.*
(b) *Suppose that Y is strongly lush and that the set of numerical radius attaining operators is dense in $\mathcal{L}(X \oplus_\infty Y)$. Then the set of norm attaining operators is dense in $\mathcal{L}(X, Y)$.*

R. Payá [27] showed that there exists a strictly convex space X isomorphic to c_0 such that the set of numerical radius attaining operators is not dense in $\mathcal{L}(X \oplus_\infty c_0)$. Remark 2.5 allows us to give a similar example, with an easy proof.

Example 2.6. Let Y be any strictly convex space containing a copy of c_0. Then the set of numerical radius attaining operators is not dense in $\mathcal{L}(c_0 \oplus_1 Y)$. Indeed, otherwise Remark 2.5 implies that the set of norm attaining operators is dense in $\mathcal{L}(c_0, Y)$ (since c_0 is an almost-CL-space). However, this is not the case, as was shown by J. Lindenstrauss [25, Proposition 4].

As a final remark, we show that none of the converses to Theorem 2.1 and Theorem 2.3 (or even Corollaries 2.2 and 2.4) holds.

REMARK 2.7. *There exist almost-CL-spaces X and Y such that (X, Y) has the BPBp, but the set of numerical radius attaining operators is dense in neither $\mathcal{L}(X \oplus_1 Y)$ nor $\mathcal{L}(X \oplus_\infty Y)$.*

Indeed, J. Johnson and J. Wolfe [20] proved in 1982 that there is a compact metric space S such that the set of norm attaining operators is not dense in $\mathcal{L}(L_1[0,1], C(S))$. The proof was given for real spaces, but it is not difficult to check that it is also valid in the complex case. Now, let X and Y be the complex spaces $C(S)$ and $L_1[0,1]$, respectively. Then X and Y are almost-CL-spaces, and M. Acosta has recently shown [2] that (X, Y) has the BPBp. However, the set of numerical radius attaining operators is dense in neither $\mathcal{L}(X \oplus_1 Y)$ nor $\mathcal{L}(X \oplus_\infty Y)$. Otherwise, Remark 2.5 would imply that the set of norm attaining operators is dense in $\mathcal{L}(Y, X)$, which is not the case due to the above mentioned result of J. Johnson and J. Wolfe.

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