STUDIA MATHEMATICA 233 (2) (2016)

Non-universal families of separable Banach spaces

by

Ondřej Kurka (Praha)

Abstract. We prove that if $\mathcal{C}$ is a family of separable Banach spaces which is analytic with respect to the Effros Borel structure and no $X \in \mathcal{C}$ is isometrically universal for all separable Banach spaces, then there exists a separable Banach space with a monotone Schauder basis which is isometrically universal for $\mathcal{C}$ but not for all separable Banach spaces. We also establish an analogous result for the class of strictly convex spaces.

1. Introduction and main results. Let $\mathcal{C}$ be a class of Banach spaces. We say that a Banach space X is *isomorphically* [*isometrically*] *universal for* $\mathcal{C}$ if it contains an isomorphic [isometric] copy of every member of $\mathcal{C}$.

The present paper, together with the author's two recent papers [18, 19], establishes isometric counterparts of results concerning universality questions in separable Banach space theory and their natural connection with descriptive set theory (see [5, 1, 3, 4, 2, 10, 7, 8, 15, 17]). The three papers give a solution of the problem posed by G. Godefroy [14] whether there exists any isometric version of the amalgamation theory of S. A. Argyros and P. Dodos [2] which would provide isometrically universal spaces for small, or regular, isometric classes of Banach spaces.

For a class $\mathcal{C}$ of separable Banach spaces, it is a natural question whether $\mathcal{C}$ is "generic" in the sense that every separable Banach space which is isomorphically [isometrically] universal for $\mathcal{C}$ is also isomorphically [isometrically] universal for all separable Banach spaces.

Employing methods from descriptive set theory, J. Bourgain [5] strengthened a well-known result of W. Szlenk [21] and proved that the answer is positive for the class of separable reflexive spaces (in the isomorphic setting). The result was revisited by B. Bossard [4] who proved that any analytic set of separable Banach spaces (defined below) that contains every separable

2010 *Mathematics Subject Classification*: Primary 46B04, 54H05; Secondary 46B15, 46B20, 46B25.

Key words and phrases: isometrically universal Banach space, Effros Borel structure, analytic set, monotone basis, strict convexity.
Received 27 August 20115.
Published online 20 May 2016.

DOI: 10.4064/sm8380-4-2016

reflexive space up to isomorphism must also contain an element which is isomorphically universal for all separable Banach spaces.

These two results motivated the authors of [2] to introduce two concepts of Bourgain genericity and Bossard genericity. It is easy to show that a Bossard generic class is Bourgain generic, because the set of all spaces which can be isomorphically embedded into a separable Banach space X is analytic. The opposite implication, conjectured in [2], was proven only for classes of spaces with a basis.

To drop the reliance on a basis, P. Dodos [7] developed a parameterized version of a construction of $\mathcal{L}_\infty$-spaces due to J. Bourgain and G. Pisier [6]. This enabled him to finally prove the equivalence of Bourgain genericity and Bossard genericity.

In the present work, we find an isometric counterpart of a result from [7]. We prove the following theorem.

Theorem 1.1. *Let $\mathcal{C}$ be an analytic set of Banach spaces none of which is isometrically universal for all separable Banach spaces. Then there exists a Banach space E with a monotone basis which is isometrically universal for $\mathcal{C}$ but not for all separable Banach spaces.*

It follows from this theorem and from [13, Lemma 7(ii)] that the two genericities considered coincide in the isometric setting as well.

Corollary 1.2. *For a class $\mathcal{P}$ of separable Banach spaces, the following assertions are equivalent:*

(a) *A separable Banach space which is isometrically universal for $\mathcal{P}$ is also isometrically universal for all separable Banach spaces.*
(b) *Every analytic set $\mathcal{C}$ of separable Banach spaces containing all members of $\mathcal{P}$ up to isometry must also contain an element which is isometrically universal for all separable Banach spaces.*

We do not use the Bourgain–Pisier construction, as P. Dodos did in [7], which is not surprising, simply because isomorphic universality and isometric universality are quite different notions. Nevertheless, there are still analogies between our methods and methods from [7]. The main analogy is that the result has already been proven for classes of spaces with a monotone basis (see [18, Theorem 1.2]) and so our task is to find an embedding of a general separable space into a space with a monotone basis. The embedding must preserve non-universality and must be simple from the descriptive-set-theoretic viewpoint. One may notice that there are also analogies with the parameterized version of Zippin's embedding theorem [22] due to B. Bossard [3] (see also [9, Chapter 5]).

Let us remark that a more general version of Theorem 1.1 holds (cf. [18, remark (IV)]). Let H be a separable Banach space for which there are $a \in H$ and a subset $D \subset H$ whose closed linear span contains an isometric copy of H itself and such that $\|a \pm d\| = \|a\|$ for every $d \in D$. Then the theorem holds for the class of spaces not containing an isometric copy of H. Within the universal space $H = C(2^{\mathbb{N}})$, the property is fulfilled e.g. by the spaces $H = c_0$ and $H = \ell_1$.

The basic property of our embedding is that it creates no new line segments in the unit sphere. For this reason, the method works at the same time for the class of strictly convex spaces.

THEOREM 1.3. *Let $\mathcal{C}$ be an analytic set of separable strictly convex Banach spaces. Then there exists a strictly convex Banach space E with a monotone basis which is isometrically universal for $\mathcal{C}$.*

It was proven by E. Odell and Th. Schlumprecht [20] that there exists a separable reflexive space which is isomorphically universal for separable uniformly convex spaces (actually, there exists an isometrically universal space, see [19]). Since the set of all separable uniformly convex spaces is Borel (see [10, Corollary 5]), we obtain the following result.

COROLLARY 1.4. *There exists a separable strictly convex Banach space which is isometrically universal for all separable uniformly convex Banach spaces.*

2. Preliminaries. By a *basis* we mean a Schauder basis. A basis $x_1, x_2, \dots$ is said to be *monotone* if the associated partial sum operators $P_n : \sum_{k=1}^{\infty} a_k x_k \mapsto \sum_{k=1}^{n} a_k x_k$ satisfy $\|P_n\| \leq 1$.

A *Polish space* [*topology*] means a separable completely metrizable space [topology]. A set P equipped with a σ-algebra is called a *standard Borel space* if the σ-algebra is generated by a Polish topology on P. A subset of a standard Borel space is called *analytic* if it is a Borel image of a Polish space.

The following result can be found e.g. in [16, p. 297].

THEOREM 2.1 (Arsenin, Kunugui). *Let X be a standard Borel space, Y a Polish space and $R \subset X \times Y$ a Borel set such that all its sections $R_x = \{y \in Y : (x, y) \in R\}$, $x \in X$, are σ-compact. Then the projection $\pi_X(R)$ of R is Borel and there exists a Borel mapping $f : \pi_X(R) \to Y$ with $f(x) \in R_x$ for every $x \in \pi_X(R)$.*

For a topological space X, the set $\mathcal{F}(X)$ of all closed subsets of X is equipped with the *Effros Borel structure*, defined as the σ-algebra generated by the sets

$$\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$$

where U varies over open subsets of X. If X is Polish, then, equipped with this σ-algebra, $\mathcal{F}(X)$ forms a standard Borel space.

We will need the following basic fact (see e.g. [16, p. 76]).

Theorem 2.2 (Kuratowski, Ryll-Nardzewski). *Let X be a Polish space. Then there exists a sequence $d_1, d_2, \ldots : \mathcal{F}(X) \to X$ of Borel mappings such that $\{d_1(F), d_2(F), \ldots\}$ is dense in F for each non-empty $F \in \mathcal{F}(X)$.*

The *standard Borel space of subspaces* of a separable Banach space A is defined by

$$\mathcal{SE}(A) = \{F \in \mathcal{F}(A) : F \text{ is a linear subspace}\}$$

and considered as a subspace of $\mathcal{F}(A)$.

By an *analytic set of separable Banach spaces* we mean an analytic subset of $\mathcal{SE}(C([0,1]))$. It is well known that the spaces $C([0,1])$ and $C(2^{\mathbb{N}})$ are isometrically universal for all separable Banach spaces.

To end this short section, we recall a classical result from Banach space theory (see e.g. [11, p. 125]).

Theorem 2.3 (Banach, Dieudonné). *Let X be a Banach space and M be a convex subset of X^*. If $M \cap nB_{X^*}$ is w^*-closed for every $n \in \mathbb{N}$, then M is w^*-closed.*

3. First lemma

Lemma 3.1. *Let A be a separable Banach space. Then there exist an isometry $I : A \to C([0,1])$ and a sequence $p_1, p_2, \ldots$ of Borel functions $p_n : \mathcal{SE}(A) \to [0,1]$ such that, for every $X \in \mathcal{SE}(A)$:*

(i) *$p_1(X) = 0$ and $p_2(X) = 1$,*
(ii) *$p_n(X) \neq p_m(X)$ for $n \neq m$,*
(iii) *the sequence $p_1(X), p_2(X), \ldots$ is dense in $[0,1]$,*
(iv) *the subspace IX of $C([0,1])$ is closed in the topology generated by the points $p_1(X), p_2(X), \ldots$.*

To provide an isometry I, we follow a standard method. Let $\varphi : [0,1] \to (B_{A^*}, w^*)$ be a continuous surjection and let

$$(Ia)(t) = \varphi(t)(a), \qquad a \in A,\ t \in [0,1].$$

In several steps, we show that the choice of I works and a suitable sequence $p_1, p_2, \ldots$ of Borel functions exists.

Claim 3.2. *IA is closed in the pointwise topology.*

Proof. Assume that $f \in C([0,1])$ belongs to the closure of IA in the pointwise topology.

We show first that there is a function $h : B_{A^*} \to [-\|f\|, \|f\|]$ such that

$$f = h \circ \varphi.$$

We just need to check that $\varphi(u) = \varphi(v) \Rightarrow f(u) = f(v)$. Given $u, v \in [0,1]$ with $\varphi(u) = \varphi(v)$, we obtain $(Ia)(u) = \varphi(u)(a) = \varphi(v)(a) = (Ia)(v)$ for every $a \in A$. Since f belongs to the closure of IA in the pointwise topology, we get $f(u) = f(v)$.

We have $h(0) = 0$. Indeed, choosing $u \in [0,1]$ with $\varphi(u) = 0$, we obtain $(Ia)(u) = \varphi(u)(a) = 0$ for $a \in A$, and so $0 = f(u) = h(\varphi(u)) = h(0)$.

Further, h is affine. Indeed, let $a^*, b^* \in B_{A^*}$ and $\alpha, \beta \in [0,1]$ satisfy $\alpha + \beta = 1$. Choose $u, v, w \in [0,1]$ with $\varphi(u) = a^*$, $\varphi(v) = b^*$ and $\varphi(w) = \alpha a^* + \beta b^*$. We have

$$\begin{aligned}(Ia)(w) &= \varphi(w)(a) = (\alpha a^* + \beta b^*)(a) = \alpha a^*(a) + \beta b^*(a) \\ &= \alpha\varphi(u)(a) + \beta\varphi(v)(a) = \alpha(Ia)(u) + \beta(Ia)(v)\end{aligned}$$

for $a \in A$, and so

$$\begin{aligned}f(w) &= \alpha f(u) + \beta f(v), \\ h(\varphi(w)) &= \alpha h(\varphi(u)) + \beta h(\varphi(v)), \\ h(\alpha a^* + \beta b^*) &= \alpha h(a^*) + \beta h(b^*).\end{aligned}$$

Finally, we prove that h is w^*-continuous. Let $a_1^*, a_2^*, \ldots$ be a sequence in B_{A^*} converging to some a^* in the w^*-topology. We need to check that $h(a_n^*) \to h(a^*)$. Assume the opposite. Then there is a subsequence $a_{n_k}^*$ such that $h(a_{n_k}^*) \to c \neq h(a^*)$. For all $k \in \mathbb{N}$, pick $u_k \in [0,1]$ such that $\varphi(u_k) = a_{n_k}^*$. There is a subsequence u_{k_l} which converges to some u. Since φ is continuous, we have $\varphi(u_{k_l}) \to \varphi(u)$, and using $\varphi(u_k) = a_{n_k}^* \to a^*$, we obtain $\varphi(u) = a^*$. Since f is continuous, we have $f(u_{k_l}) \to f(u)$, and using $f(u_{k_l}) = h(\varphi(u_{k_l})) = h(a_{n_{k_l}}^*) \to c$, we obtain $f(u) = c$. Consequently, $f(u) = c \neq h(a^*) = h(\varphi(u)) = f(u)$, a contradiction.

We have shown that h is w^*-continuous, affine and $h(0) = 0$. So, h can be extended to a linear functional on A^* which is w^*-continuous by Theorem 2.3. Consequently, there is $a \in A$ such that $h(a^*) = a^*(a)$ for all $a^* \in B_{A^*}$. Then $f = Ia$, and the proof is finished. ∎

CLAIM 3.3. *There is a sequence of numbers $v_1, v_2, \ldots$ that is dense in $[0,1]$ and generates a topology in which IA is closed.*

Proof. By Claim 3.2, every $f \in C([0,1]) \setminus IA$ has a neighborhood $U_f \subset C([0,1]) \setminus IA$ of the form

$$U_f = \{g \in C([0,1]) : |g(u_k) - f(u_k)| < \varepsilon,\ 1 \le k \le n\}$$

for some ε, n and $u_1, \ldots, u_n$. Since $C([0,1]) \setminus IA$ can be covered by countably many such neighborhoods, the desired sequence is obtained by collecting all rational numbers and all numbers u_k associated to the members of this covering. ∎

CLAIM 3.4. *For every open ball $U \subset A$, the set*

$$R = \{(X, a^*) \in \mathcal{SE}(A) \times B_{A^*} : a^*(x) = 0,\ x \in X,\ a^*(x) > 0,\ x \in U\}$$

is Borel in $\mathcal{SE}(A) \times (B_{A^*}, w^*)$ *and all its sections* R_X, $X \in \mathcal{SE}(A)$, *are* σ*-compact.*

Proof. Let a be the center of U. Since every $a^* \neq 0$ maps open balls onto open intervals, we have

$$(X, a^*) \in R \iff a^*(x) = 0,\ x \in X,\ a^*(x) \geq 0,\ x \in U,\ a^*(a) > 0.$$

The dual unit ball B_{A^*} is compact in the w^*-topology. As $a^*(a) > 0$ if and only if $a^*(a) \geq 1/j$ for some $j \in \mathbb{N}$, it follows that the sections R_X, $X \in \mathcal{SE}(A)$, are σ-compact.

Let $x_1, x_2, \ldots$ be a dense sequence in U and let $d_1, d_2, \ldots : \mathcal{F}(A) \to A$ be Borel selectors provided by Theorem 2.2. Using the equivalence

$$(X, a^*) \in R \iff a^*(d_n(X)) = 0,\ a^*(x_n) \geq 0,\ n \in \mathbb{N},\ a^*(a) > 0$$

and the continuity of $(a^*, x) \in (B_{A^*}, w^*) \times A \mapsto a^*(x)$, we see that R is Borel. ∎

Claim 3.5. *There is a sequence* $s_1, s_2, \ldots$ *of Borel mappings* $s_n : \mathcal{SE}(A) \to (B_{A^*}, w^*)$ *such that, for every* $X \in \mathcal{SE}(A)$ *and* $a \in A \setminus X$, *there is* $n \in \mathbb{N}$ *with* $s_n(X)(a) \neq 0$ *and* $s_n(X)(x) = 0$ *for all* $x \in X$.

Proof. Let $U_1, U_2, \ldots$ be a countable basis of the norm topology of A consisting of open balls. For each $n \in \mathbb{N}$, set

$$R_n = \{(X, a^*) \in \mathcal{SE}(A) \times B_{A^*} : a^*(x) = 0,\ x \in X,\ a^*(x) > 0,\ x \in U_n\}.$$

By Claim 3.4 and Theorem 2.1, there exists a Borel mapping $s_n : \mathcal{SE}(A) \to (B_{A^*}, w^*)$ such that $(R_n)_X \neq \emptyset \Rightarrow s_n(X) \in (R_n)_X$ (we consider a Borel extension of the mapping provided by Theorem 2.1). Let us check that the mappings s_n work.

Let $X \in \mathcal{SE}(A)$ and $a \in A \setminus X$. For some $n \in \mathbb{N}$, we have $a \in U_n \subset A \setminus X$. By the Hahn–Banach theorem, there exists $a^* \in B_{A^*}$ such that $a^*(x) = 0$ for $x \in X$ and $a^*(x) > 0$ for $x \in U_n$. This means that $(R_n)_X$ is non-empty, and so $s_n(X) \in (R_n)_X$. Therefore, $s_n(X)$ has the desired property. ∎

Claim 3.6. *There is a sequence* $p_1, p_2, \ldots$ *of Borel functions* $p_n : \mathcal{SE}(A) \to [0,1]$ *such that* (iii) *and* (iv) *of Lemma* 3.1 *are valid for every* $X \in \mathcal{SE}(A)$.

Proof. Let numbers $v_1, v_2, \ldots$ be given by Claim 3.3 and Borel mappings $s_1, s_2, \ldots$ be given by Claim 3.5. Let

$$q_n(X) = \min \varphi^{-1}(s_n(X)), \qquad n \in \mathbb{N},\ X \in \mathcal{SE}(A).$$

Let us show that the function q_n is Borel for every $n \in \mathbb{N}$. It is sufficient to check that the mapping $a^* \mapsto \min \varphi^{-1}(a^*)$ is Borel from (B_{A^*}, w^*) into $[0,1]$. For $u \in [0,1]$, the set $\{a^* \in B_{A^*} : \min \varphi^{-1}(a^*) \leq u\} = \varphi([0,u])$ is compact, and thus Borel.

Further, let us show that, for every $X \in \mathcal{SE}(A)$, the subspace IX is closed in the topology generated by the points $q_1(X), q_2(X), \ldots$ and $v_1, v_2, \ldots$.

Given $f \in C([0,1]) \setminus IX$, there are two possibilities. If $f \notin IA$, then the properties of $v_1, v_2, \ldots$ guarantee that f does not belong to the closure of IA (and of IX in particular) in the relevant topology. If $f \in IA$, then we choose $a \in A$ with $Ia = f$. Necessarily, $a \notin X$, and thus there is $n \in \mathbb{N}$ with $s_n(X)(a) \neq 0$ and $s_n(X)(x) = 0$ for all $x \in X$. We have

$$f(q_n(X)) = (Ia)(q_n(X)) = \varphi(q_n(X))(a) = s_n(X)(a) \neq 0,$$

while

$$(Ix)(q_n(X)) = \varphi(q_n(X))(x) = s_n(X)(x) = 0, \quad x \in X.$$

Hence, f does not belong to the closure of IX in the topology considered.

Now, if we define

$$p_{2n-1}(X) = q_n(X), \quad p_{2n}(X) = v_n, \quad n \in \mathbb{N}, X \in \mathcal{SE}(A),$$

then, for every $X \in \mathcal{SE}(A)$, conditions (iii) and (iv) are valid. ∎

To finish the proof of Lemma 3.1, it remains to show that the sequence from Claim 3.6 can be modified to satisfy (i) and (ii) as well. If we define $p'_1(X) = 0, p'_2(X) = 1$ and $p'_n(X) = p_m(X)$ where m is the least natural number such that $|\{0, 1, p_1(X), \ldots, p_m(X)\}| = n$, it is easy to check that these functions are Borel and satisfy (i)–(iv).

4. Second lemma

Lemma 4.1. *Let A be a separable Banach space. Then there exist a separable Banach space Z, an isometry $J : A \to Z$, a collection $\{\|\cdot\|^X : X \in \mathcal{SE}(A)\}$ of norms on Z and a system $\{Q_n^X : X \in \mathcal{SE}(A), n \in \mathbb{N}\}$ of projections on Z such that, for every $X \in \mathcal{SE}(A)$:*

- (I) $\|z\| \leq \|z\|^X \leq 2\|z\|$ *for* $z \in Z$,
- (II) $\|z\| = \|z\|^X$ *if and only if* $z \in JX$,
- (III) $Q_1^X, Q_2^X, \ldots$ *is the sequence of partial sum operators associated with a basis of Z which is monotone in the sense that* $\|Q_n^X\| \leq 1$ *and* $\|Q_n^X\|^X \leq 1$.

Moreover,

- (IV) *the mapping $(X, z) \mapsto Q_n^X z$ is Borel from $\mathcal{SE}(A) \times Z$ into Z for every $n \in \mathbb{N}$,*
- (V) *the function $(X, z) \mapsto \|z\|^X$ is Borel from $\mathcal{SE}(A) \times Z$ into $\mathbb{R}$.*

The proof of this lemma consists of four parts. There are some analogies with a construction by Ghoussoub, Maurey and Schachermayer [12] and its parameterized version by Bossard [3] (see also [9, Chapter 5]).

❶ Fix an isometry $I : A \to C([0,1])$ and a sequence $p_1, p_2, \ldots$ of Borel functions $p_n : \mathcal{SE}(A) \to [0,1]$ satisfying properties (i)–(iv) from Lemma 3.1.

For every $X \in \mathcal{SE}(A)$, we define a sequence of projections $P_n^X : C([0,1]) \to C([0,1])$. Let $(P_1^X f)(t) = f(0)$ for every $f \in C([0,1])$ and $t \in [0,1]$. Given $n \geq 2$ and $f \in C([0,1])$, let $P_n^X f$ be the piecewise linear function which has the same values at $p_1(X), \dots, p_n(X)$ as f and is linear elsewhere.

Using (i)–(iii), we can easily verify that:

- $\|P_n^X f\| \leq \|f\|$,
- $P_n^X P_m^X = P_m^X P_n^X = P_{\min\{m,n\}}^X$,
- $P_n^X C([0,1])$ has dimension n,
- $P_n^X f \to f$ as $n \to \infty$.

Due to these properties, $P_1^X, P_2^X, \dots$ is the sequence of partial sum operators associated with a monotone basis of $C([0,1])$.

CLAIM 4.2. *For every $n \in \mathbb{N}$, the mapping $(X, f) \mapsto P_n^X f$ is Borel from $\mathcal{SE}(A) \times C([0,1])$ into $C([0,1])$.*

Proof. For $n \leq 2$, the assertion is clear, as P_n^X does not depend on X. For $n \geq 3$, the mapping is the composition of the Borel mapping $(X, f) \mapsto (p_3(X), \dots, p_n(X), f)$ and the continuous mapping which maps $(x_3, \dots, x_n, f)$ to the piecewise linear function which has the same values at $0, 1, x_3, \dots, x_n$ as f and is linear elsewhere. ∎

CLAIM 4.3. *Let $X \in \mathcal{SE}(A)$ and $f \in C([0,1])$. If $f \notin IX$, then there is $n \in \mathbb{N}$ such that $P_n^X f \notin P_n^X IX$.*

Proof. Property (iv) provides a neighborhood $W \subset C([0,1]) \setminus IX$ of f of the form

$$W = \{g \in C([0,1]) : |g(p_k(X)) - f(p_k(X))| < \varepsilon,\ 1 \leq k \leq n\}$$

for a large enough $n \in \mathbb{N}$ and a small enough $\varepsilon > 0$. This means that every $g \in IX$ satisfies $|g(p_k(X)) - f(p_k(X))| \geq \varepsilon$ for some $1 \leq k \leq n$. Since $(P_n^X g)(p_k(X)) = g(p_k(X))$ and $(P_n^X f)(p_k(X)) = f(p_k(X))$, it follows that $P_n^X g \neq P_n^X f$. ∎

❷ For every $X \in \mathcal{SE}(A)$, we define a sequence of projections $Q_i^X : \ell_2(C([0,1])) \to \ell_2(C([0,1]))$. For $\mathbf{f} = (f_1, f_2, \dots) \in \ell_2(C([0,1]))$, let

$$\begin{aligned}
Q_1^X \mathbf{f} &= (P_1^X f_1, 0, 0, 0, \dots),\\
Q_2^X \mathbf{f} &= (P_2^X f_1, 0, 0, 0, \dots),\\
Q_3^X \mathbf{f} &= (P_2^X f_1, P_1^X f_2, 0, 0, \dots),\\
Q_4^X \mathbf{f} &= (P_3^X f_1, P_1^X f_2, 0, 0, \dots),\\
Q_5^X \mathbf{f} &= (P_3^X f_1, P_2^X f_2, 0, 0, \dots),\\
Q_6^X \mathbf{f} &= (P_3^X f_1, P_2^X f_2, P_1^X f_3, 0, \dots),\\
Q_7^X \mathbf{f} &= (P_4^X f_1, P_2^X f_2, P_1^X f_3, 0, \dots),
\end{aligned}$$

and so on. In this way, $Q_1^X, Q_2^X, \dots$ is the sequence of partial sum operators associated with a monotone basis of $\ell_2(C([0,1]))$.

Further, define

$$U : C([0,1]) \to \ell_2(C([0,1])), \quad f \mapsto \frac{\sqrt{3}}{2}\Big(f, \frac{1}{2}f, \frac{1}{4}f, \dots\Big).$$

This is an isometry since

$$\|Uf\|^2 = \frac{3}{4}\Big(\|f\|^2 + \frac{1}{4}\|f\|^2 + \dots\Big) = \|f\|^2, \quad f \in C([0,1]).$$

Claim 4.4. *For every $i \in \mathbb{N}$, the mapping $(X, \mathbf{f}) \mapsto Q_i^X \mathbf{f}$ is Borel from $\mathcal{SE}(A) \times \ell_2(C([0,1]))$ into $\ell_2(C([0,1]))$.*

Proof. This follows from Claim 4.2 and the definition of Q_i^X. ∎

Claim 4.5. *Let $X \in \mathcal{SE}(A)$ and $\mathbf{f} \in \ell_2(C([0,1]))$. If $\mathbf{f} \notin UIX$, then there is $i \in \mathbb{N}$ such that $Q_i^X \mathbf{f} \notin Q_i^X UIX$.*

Proof. For a general $\mathbf{g} \in \ell_2(C([0,1]))$, we will denote its coordinates by g_k or $(\mathbf{g})_k$. There are two possibilities.

(1) Assume that $f_k \neq 2f_{k+1}$ for some $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $P_n^X f_k \neq 2P_n^X f_{k+1}$ and let $i \in \mathbb{N}$ be large enough that $n_k \geq n$ and $n_{k+1} \geq n$ in the expression $Q_i^X \mathbf{g} = (P_{n_j}^X g_j)_{j=1}^\infty$ (we consider $P_0^X = 0$). Then

$$\begin{aligned} P_n^X[(Q_i^X \mathbf{f})_k] &= P_n^X P_{n_k}^X f_k = P_n^X f_k \\ &\neq 2P_n^X f_{k+1} = 2P_n^X P_{n_{k+1}}^X f_{k+1} = 2P_n^X[(Q_i^X \mathbf{f})_{k+1}], \end{aligned}$$

while, for every $g \in C([0,1])$ (and in particular for every $g \in IX$),

$$\begin{aligned} P_n^X[(Q_i^X Ug)_k] &= P_n^X P_{n_k}^X[(Ug)_k] = P_n^X[(Ug)_k] = \frac{\sqrt{3}}{2^k} P_n^X g \\ &= 2P_n^X[(Ug)_{k+1}] = 2P_n^X P_{n_{k+1}}^X[(Ug)_{k+1}] = 2P_n^X[(Q_i^X Ug)_{k+1}], \end{aligned}$$

and so $Q_i^X Ug \neq Q_i^X \mathbf{f}$.

(2) Assume that $f_k = 2f_{k+1}$ for all $k \in \mathbb{N}$. Then $\mathbf{f} = Uf$ for $f = (2/\sqrt{3})f_1$. By our assumption, $f \notin IX$, and Claim 4.3 provides $n \in \mathbb{N}$ such that $P_n^X f \notin P_n^X IX$. Let $i \in \mathbb{N}$ be large enough that $n_1 \geq n$ in the expression $Q_i^X \mathbf{g} = (P_{n_j}^X g_j)_{j=1}^\infty$ (we consider $P_0^X = 0$). Then, for every $g \in IX$,

$$\begin{aligned} P_n^X[(Q_i^X Ug)_1] &= P_n^X P_{n_1}^X[(Ug)_1] = P_n^X[(Ug)_1] = \frac{\sqrt{3}}{2} P_n^X g \\ &\neq \frac{\sqrt{3}}{2} P_n^X f = P_n^X f_1 = P_n^X P_{n_1}^X f_1 = P_n^X[(Q_i^X \mathbf{f})_1], \end{aligned}$$

and so $Q_i^X Ug \neq Q_i^X \mathbf{f}$. ∎

Claim 4.6. *For $X \in \mathcal{SE}(A)$ and $i \in \mathbb{N}$, we have $\|Q_i^X U\| < 1$.*

Proof. There is $k \in \mathbb{N}$ such that points from the range of Q_i^X are supported by the first k coordinates. Given $f \in C([0,1])$, we have

$$\begin{aligned}\|Q_i^X U f\|^2 &= \left(\frac{\sqrt{3}}{2}\right)^2 \left\| \left(P_{n_1} f, \frac{1}{2} P_{n_2} f, \ldots, \frac{1}{2^{k-1}} P_{n_k} f, 0, 0, \ldots\right)\right\|^2 \\ &= \frac{3}{4}\left(\|P_{n_1} f\|^2 + \frac{1}{4}\|P_{n_2} f\|^2 + \cdots + \frac{1}{4^{k-1}}\|P_{n_k} f\|^2\right) \\ &\le \frac{3}{4}\left(\|f\|^2 + \frac{1}{4}\|f\|^2 + \cdots + \frac{1}{4^{k-1}}\|f\|^2\right) = \left(1 - \frac{1}{4^k}\right)\|f\|^2\end{aligned}$$

for some $n_1, \ldots, n_k \in \mathbb{N} \cup \{0\}$. It follows that $\|Q_i^X U\|^2 \le 1 - 1/4^k$. ■

❸ For every $X \in \mathcal{SE}(A)$, we define

$$\Omega^X = \overline{\mathrm{co}}\left(\frac{1}{2} B_{\ell_2(C([0,1]))} \cup \bigcup_{i=1}^{\infty} Q_i^X U I B_X\right).$$

Notice that $UIB_X \subset \Omega^X$ and $Q_i^X \Omega^X \subset \Omega^X$ for every $i \in \mathbb{N}$.

Claim 4.7. *The set*

$$\{(X, \mathbf{f}) \in \mathcal{SE}(A) \times \ell_2(C([0,1])) : \mathbf{f} \in \Omega^X\}$$

is Borel.

Proof. Let $\{\mathbf{f}_1, \mathbf{f}_2, \ldots\}$ be dense in $B_{\ell_2(C([0,1]))}$. Let $x_1, x_2, \ldots : \mathcal{SE}(A) \to B_A$ be Borel mappings such that $\{x_1(X), x_2(X), \ldots\}$ is dense in B_X for every $X \in \mathcal{SE}(A)$ (it is easy to find such a sequence using Theorem 2.2). We have

$$\begin{aligned}\mathbf{f} \in \Omega^X \ \Leftrightarrow\ & \forall l \in \mathbb{N}\ \exists m \in \mathbb{N}\ \exists k, n_1, \ldots, n_m \in \mathbb{N} \\ & \exists \gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{Q} \cap [0,1],\ \sum_{i=0}^{m} \gamma_i = 1 : \\ & \left\| \mathbf{f} - \left[\frac{1}{2}\gamma_0 \mathbf{f}_k + \sum_{i=1}^{m} \gamma_i Q_i^X U I x_{n_i}(X)\right]\right\| < \frac{1}{l}.\end{aligned}$$

It remains to note that, by Claim 4.4, the mapping $X \mapsto Q_i^X U I x_n(X)$ is Borel for all $i, n \in \mathbb{N}$. ■

Claim 4.8. *For $X \in \mathcal{SE}(A)$ and $\mathbf{f} \in \ell_2(C([0,1])) \setminus UIX$ with $\|\mathbf{f}\| = 1$, we have $\mathbf{f} \notin \Omega^X$.*

Proof. Claim 4.5 provides $i \in \mathbb{N}$ such that $Q_i^X \mathbf{f} \notin Q_i^X UIX$. Let $\mathbf{f}^* \in \ell_2(C([0,1]))^*$ be such that $\|\mathbf{f}^*\| = 1 = \mathbf{f}^*(\mathbf{f})$ and let z^* be a functional on $Q_i^X \ell_2(C([0,1]))$ such that $\|z^*\| = 1$, $z^*(Q_i^X \mathbf{f}) > 0$ and $z^*(Q_i^X \mathbf{g}) = 0$ for $\mathbf{g} \in UIX$. By Claim 4.6, there is $\varepsilon \in (0,1]$ such that $\|Q_j^X U\| \le 1 - \varepsilon$ for $1 \le j < i$. Define

$$\mathbf{g}^* = \mathbf{f}^* + \varepsilon \cdot z^* \circ Q_i^X.$$

Then

$$\mathbf{g}^*(\mathbf{f}) = \mathbf{f}^*(\mathbf{f}) + \varepsilon \cdot z^*(Q_i^X \mathbf{f}) > 1.$$

We claim that $\mathbf{g}^*$ separates $\mathbf{f}$ from Ω^X, showing that

$$\mathbf{g}^*(\mathbf{u}) \le 1, \quad \mathbf{u} \in \Omega^X.$$

If $\mathbf{u} \in \frac{1}{2}B_{\ell_2(C([0,1]))}$, then $\mathbf{g}^*(\mathbf{u}) \le \|\mathbf{g}^*\| \, \|\mathbf{u}\| \le (1+\varepsilon)\cdot \frac{1}{2} \le 1$. So, it remains to show that $\mathbf{g}^*(\mathbf{u}) \le 1$ for $\mathbf{u} = Q_j^X U g$ where $j \in \mathbb{N}$ and $g \in IB_X$. If $1 \le j < i$, then $\mathbf{g}^*(\mathbf{u}) \le \|\mathbf{g}^*\| \, \|Q_j^X U\| \, \|g\| \le (1+\varepsilon)(1-\varepsilon) \le 1$. If $j \ge i$, then $z^*(Q_i^X \mathbf{u}) = z^*(Q_i^X Q_j^X U g) = z^*(Q_i^X U g) = 0$ and

$$\mathbf{g}^*(\mathbf{u}) = \mathbf{f}^*(\mathbf{u}) + \varepsilon \cdot z^*(Q_i^X \mathbf{u}) = \mathbf{f}^*(\mathbf{u}) \le \|\mathbf{f}^*\| \, \|Q_j^X\| \, \|Ug\| \le 1,$$

which completes the verification of $\mathbf{f} \notin \Omega^X$. ∎

❹ Now, we are ready to finish the proof of Lemma 4.1. For every $X \in \mathcal{SE}(A)$, we define $\|\cdot\|^X$ as the norm on $\ell_2(C([0,1]))$ which has Ω^X for its unit ball. Let us check that properties (I)–(V) are valid for the choice $Z = \ell_2(C([0,1]))$ and $J = UI$.

(I) follows from $\frac{1}{2}B_{\ell_2(C([0,1]))} \subset \Omega^X \subset B_{\ell_2(C([0,1]))}$.

(II) We know that $UIB_X \subset \Omega^X \subset B_{\ell_2(C([0,1]))}$, which implies that $\|\mathbf{f}\| = \|\mathbf{f}\|^X$ for every $\mathbf{f} \in UIX$. Assume that $\mathbf{f} \in \ell_2(C([0,1])) \setminus UIX$. Assume moreover without loss of generality that $\|\mathbf{f}\| = 1$. By Claim 4.8, we have $\mathbf{f} \notin \Omega^X$, which means that $\|\mathbf{f}\|^X > 1 = \|\mathbf{f}\|$.

(III) It follows from $Q_i^X \Omega^X \subset \Omega^X$ that $\|Q_i^X\|^X \le 1$.

(IV) is already provided by Claim 4.4.

(V) By Claim 4.7, the pre-image of $[0,1]$ is Borel. Clearly, the pre-image of $[0,r]$ is also Borel for every $r > 0$, which gives (V).

5. Proof of main results. Let $A = C([0,1])$. Let a separable Banach space Z, an isometry $J : C([0,1]) \to Z$, a collection $\{\|\cdot\|^X : X \in \mathcal{SE}(C([0,1]))\}$ of norms on Z and a system $\{Q_n^X : X \in \mathcal{SE}(C([0,1])),\, n \in \mathbb{N}\}$ of projections on Z satisfy properties (I)–(V) from Lemma 4.1.

We are going to apply the same technique as in [18, Section 8] to obtain a new collection $\{|||\cdot|||^X : X \in \mathcal{SE}(C([0,1]))\}$ of norms on Z with the same properties and with the additional property that all line segments contained in the unit sphere of $(Z, |||\cdot|||^X)$ are contained in JX.

Let ϱ be a norm on $\mathbb{R}^3$ such that

- $\frac{1}{2}(|r| + |s|) \le \varrho(r,s,t) \le \max\{|r|,|s|,|t|\}$ and, in particular, the unit sphere contains the line segment $[(1,1,-1),(1,1,1)]$,
- $\varrho(r',s',t') \ge \varrho(r,s,t)$ for $0 \le r \le r', 0 \le s \le s', 0 \le t \le t'$,
- $\varrho(r,s,t') > \varrho(r,s,t)$ for $0 < r < s$ and $0 < t < t'$.

An example provided in [18] is the norm given by

$$B_{(\mathbb{R}^3,\varrho)} = \mathrm{co}\big(\{(\pm 1, \pm 1, \pm 1)\} \cup \sqrt{2}\, B\big),$$

where B stands for the Euclidean unit ball of $\mathbb{R}^3$.

For all $X \in \mathcal{SE}(C([0,1]))$, we define (considering $Q_0^X = 0$)

$$\sigma^X(z) = \Big(\sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \|Q_n^X z - Q_{n-1}^X z\|^2\Big)^{1/2}, \qquad z \in Z,$$

and

$$|||z|||^X = \varrho(\|z\|, \|z\|^X, \sigma^X(z)), \qquad z \in Z.$$

Claim 5.1.

(i) *σ^X is a strictly convex seminorm on Z,*
(ii) *$|||\cdot|||^X$ is a norm on Z,*
(iii) *$\sigma^X(z) \le \|z\|$,*
(iv) *$\|z\| \le |||z|||^X \le 2\|z\|$,*
(v) *$\sigma^X(Q_n^X z) \le \sigma^X(z)$,*
(vi) *$|||Q_n^X z|||^X \le |||z|||^X$,*
(vii) *the function $(X, z) \mapsto \sigma^X(z)$ is Borel,*
(viii) *the function $(X, z) \mapsto |||z|||^X$ is Borel.*

Proof. (i) As the range of $Q_n^X - Q_{n-1}^X$ is one-dimensional, there is $z_n^* \in Z^*$ such that $\|z_n^*\| \le 2$ and $\|Q_n^X z - Q_{n-1}^X z\| = |z_n^*(z)|$ for every $z \in Z$. Let

$$T : Z \to \ell_2, \quad z \mapsto \left(\frac{1}{2^{(n+2)/2}} z_n^*(z)\right)_{n=1}^{\infty}.$$

Then

$$\sigma^X(z) = \|Tz\|, \quad z \in Z.$$

At the same time, T is injective (if $Tz = 0$, then $Q_n^X z - Q_{n-1}^X z = 0$ for all n, and so $Q_n^X z = 0$ for all n). Therefore, (i) follows from strict convexity of ℓ_2.

(ii) Using (i) and the properties of ϱ, we get

$$\begin{aligned} |||u+v|||^X &= \varrho\big(\|u+v\|, \|u+v\|^X, \sigma^X(u+v)\big) \\ &\le \varrho\big(\|u\| + \|v\|, \|u\|^X + \|v\|^X, \sigma^X(u) + \sigma^X(v)\big) \\ &\le \varrho\big(\|u\|, \|u\|^X, \sigma^X(u)\big) + \varrho\big(\|v\|, \|v\|^X, \sigma^X(v)\big) \\ &= |||u|||^X + |||v|||^X. \end{aligned}$$

The verification of $|||\lambda z|||^X = |\lambda|\, |||z|||^X$ is similar.

(iii) We have

$$\sigma^X(z)^2 = \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \big\|Q_n^X z - Q_{n-1}^X z\big\|^2 \le \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} \cdot (2\|z\|)^2 = \|z\|^2.$$

(iv) Using (I), (iii) and the properties of ϱ, we obtain

$$\begin{aligned}\|z\| &\le \frac{1}{2}(\|z\| + \|z\|^X) \le \varrho\big(\|z\|, \|z\|^X, \sigma^X(z)\big)\\ &\le \max\{\|z\|, \|z\|^X, \sigma^X(z)\} = \|z\|^X \le 2\|z\|.\end{aligned}$$

(v) We have

$$\sigma^X(Q_m^X z) = \Big(\sum_{n=1}^{m} \frac{1}{2^{n+2}} \|Q_n^X z - Q_{n-1}^X z\|^2\Big)^{1/2} \le \sigma^X(z).$$

(vi) Using (v) and the properties of ϱ, we obtain

$$\begin{aligned}|||Q_n^X z|||^X &= \varrho\big(\|Q_n^X z\|, \|Q_n^X z\|^X, \sigma^X(Q_n^X z)\big)\\ &\le \varrho\big(\|z\|, \|z\|^X, \sigma^X(z)\big) = |||z|||^X.\end{aligned}$$

(vii) follows from (IV) and the definition of σ^X.

(viii) follows from (V), (vii) and the definition of $|||\cdot|||^X$. ∎

CLAIM 5.2. *We have* $|||Jx|||^X = \|x\|$ *for* $x \in X \in \mathcal{SE}(C([0,1]))$. *In particular,* $(Z, |||\cdot|||^X)$ *contains an isometric copy of* X.

Proof. Assume that $\|x\| = 1$. Note that $\|Jx\|^X = \|Jx\| = \|x\| = 1$ due to (II). At the same time, $\sigma^X(Jx) \le \|Jx\| = 1$ by Claim 5.1(iii). Since the unit sphere $S_{(\mathbb{R}^3,\varrho)}$ contains the line segment $[(1,1,-1),(1,1,1)]$, we obtain $|||Jx|||^X = 1 = \|x\|$. ∎

CLAIM 5.3. *Let* $X \in \mathcal{SE}(C([0,1]))$ *and let* $[u,v]$ *be a non-degenerate line segment in* Z *such that* $|||\cdot|||^X$ *is constant on* $[u,v]$. *Then* $[u,v] \subset JX$.

Proof. It is enough to show that $w = \frac{1}{2}(u+v) \in JX$ (the argument can be repeated for any subsegment of $[u,v]$). Assume $w \notin JX$. By Claim 5.1(i),

$$\sigma^X(w) < \frac{1}{2}(\sigma^X(u) + \sigma^X(v)).$$

At the same time, $\|w\| < \|w\|^X$ by (I) and (II), and the third property of ϱ yields

$$\varrho\Big(\|w\|, \|w\|^X, \frac{1}{2}(\sigma^X(u) + \sigma^X(v))\Big) > \varrho\big(\|w\|, \|w\|^X, \sigma^X(w)\big) = |||w|||^X.$$

The computation

$$\begin{aligned}\frac{1}{2}(|||u|||^X + |||v|||^X) &= \frac{1}{2}\Big(\varrho\big(\|u\|, \|u\|^X, \sigma^X(u)\big) + \varrho\big(\|v\|, \|v\|^X, \sigma^X(v)\big)\Big)\\ &\ge \varrho\Big(\frac{1}{2}(\|u\| + \|v\|), \frac{1}{2}(\|u\|^X + \|v\|^X), \frac{1}{2}(\sigma^X(u) + \sigma^X(v))\Big)\\ &\ge \varrho\Big(\|w\|, \|w\|^X, \frac{1}{2}(\sigma^X(u) + \sigma^X(v))\Big) > |||w|||^X\end{aligned}$$

concludes the proof. ∎

Claim 5.4.

(1) $(Z, |||\cdot|||^X)$ *is isometrically universal for all separable Banach spaces if and only if* X *has the same property.*
(2) $(Z, |||\cdot|||^X)$ *is strictly convex if and only if* X *is strictly convex.*

Proof. We check only the implication "⇒" in (1), since the other implications follow from Claims 5.2 and 5.3. Set

$$\begin{aligned} &\varDelta = \{0,1\}^{\mathbb{N}}, && \varDelta(i) = \{\gamma \in \varDelta : \gamma(1) = i\}, && i = 0,1,\\ &H = C(\varDelta), && H(i) = \{h \in H : \gamma \notin \varDelta(i) \Rightarrow h(\gamma) = 0\}, && i = 0,1. \end{aligned}$$

Assume that there is an isometry $I : H \to (Z, |||\cdot|||^X)$ and denote

$$z = I(\mathbf{1}_{\varDelta(0)}).$$

We claim that the space JX (and therefore X by Claim 5.2) is universal, showing that I maps $H(1)$ into JX.

Given $h \in H(1)$ such that $\|h\| \le 1$, observe that $\|\mathbf{1}_{\varDelta(0)}\| = \|\mathbf{1}_{\varDelta(0)} \pm h\| = 1$, and so $|||z|||^X = |||z \pm Ih|||^X = 1$. By Claim 5.3, we have $Ih \in JX$. ∎

Our last claim is similar to [7, Theorem 17] and [9, Theorem 5.19].

Claim 5.5. *The set*

$$\mathcal{R} = \{(X,Y) \in \mathcal{SE}(C([0,1]))^2 : Y \text{ is isometric to } (Z, |||\cdot|||^X)\}$$

is analytic.

Proof. Let $s_1, s_2, \dots$ be a dense sequence in Z. Recall that the function $(X,z) \mapsto |||z|||^X$ is Borel by Claim 5.1(viii). Therefore, $\mathcal{R}$ is a projection of a Borel set in $\mathcal{SE}(C([0,1]))^2 \times Z^{\mathbb{N}}$, as

$$\begin{aligned} (X,Y) \in \mathcal{R} \ \Leftrightarrow\ & \exists (z_1, z_2, \dots) \in Z^{\mathbb{N}} :\\ & \Big[(\forall k \in \mathbb{N}\ \forall l \in \mathbb{N}\ \exists n \in \mathbb{N} : \|s_k - z_n\| < 1/l)\\ & \& \Big(\forall m \in \mathbb{N}\ \forall \gamma_1, \dots, \gamma_m \in \mathbb{Q} :\\ & \Big|\Big|\Big| \sum_{n=1}^{m} \gamma_n z_n \Big|\Big|\Big|^X = \Big\| \sum_{n=1}^{m} \gamma_n d_n(Y) \Big\| \Big) \Big] \end{aligned}$$

where $d_1, d_2, \dots : \mathcal{F}(C([0,1])) \to C([0,1])$ are provided by Theorem 2.2. ∎

Let us finish the proof of Theorems 1.1 and 1.3. Depending on the theorem we want to prove, let P denote the property of being not isometrically universal for all separable Banach spaces or the property of being strictly convex.

By [18, Theorem 1.2], the theorems have already been proven under the assumption that the members of $\mathcal{C}$ have a monotone basis. Therefore, it is sufficient to show the following.

Let $\mathcal{C}$ be an analytic set of separable Banach spaces which satisfy P. Then there exists an analytic set $\mathcal{C}'$ of Banach spaces which satisfy P such that every member of $\mathcal{C}'$ has a monotone basis and an isometric copy of every member of $\mathcal{C}$ is contained in a member of $\mathcal{C}'$.

Given such a $\mathcal{C}$, the set

$$\mathcal{C}' = \{Y \in \mathcal{SE}(C([0,1])) : Y \text{ is isometric to } (Z, \|\!|\cdot|\!\|^X) \text{ for some } X \in \mathcal{C}\}$$

is analytic by Claim 5.5, since it is a projection of the analytic set $\mathcal{R} \cap (\mathcal{C} \times \mathcal{SE}(C([0,1])))$.

Let us check that $\mathcal{C}'$ works. By Claim 5.4, every $Y \in \mathcal{C}'$ satisfies P. By Claim 5.1(vi), every $Y \in \mathcal{C}'$ has a monotone basis. Finally, every $X \in \mathcal{C}$ is contained in some $Y \in \mathcal{C}'$ by Claim 5.2.

Acknowledgments. This research was supported by the grant GAČR 14-04892P. The author is a junior researcher at the University Centre for Mathematical Modelling, Applied Analysis and Computational Mathematics (MathMAC). He is also a member of the Nečas Center for Mathematical Modeling.

References

[1] S. A. Argyros, *A universal property of reflexive hereditarily indecomposable Banach spaces*, Proc. Amer. Math. Soc. 129 (2001), 3231–3239.

[2] S. A. Argyros and P. Dodos, *Genericity and amalgamation of classes of Banach spaces*, Adv. Math. 209 (2007), 666–748.

[3] B. Bossard, *An ordinal version of some applications of the classical interpolation theorem*, Fund. Math. 152 (1997), 55–74.

[4] B. Bossard, *A coding of separable Banach spaces. Analytic and coanalytic families of Banach spaces*, Fund. Math. 172 (2002), 117–152.

[5] J. Bourgain, *On separable Banach spaces, universal for all separable reflexive spaces*, Proc. Amer. Math. Soc. 79 (1980), 241–246.

[6] J. Bourgain and G. Pisier, *A construction of $\mathcal{L}_\infty$-spaces and related Banach spaces*, Bol. Soc. Brasil. Mat. 14 (1983), 109–123.

[7] P. Dodos, *On classes of Banach spaces admitting "small" universal spaces*, Trans. Amer. Math. Soc. 361 (2009), 6407–6428.

[8] P. Dodos, *Quotients of Banach spaces and surjectively universal spaces*, Studia Math. 197 (2010), 171–194.

[9] P. Dodos, *Banach Spaces and Descriptive Set Theory: Selected Topics*, Lecture Notes in Math. 1993, Springer, 2010.

[10] P. Dodos and V. Ferenczi, *Some strongly bounded classes of Banach spaces*, Fund. Math. 193 (2007), 171–179.

[11] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books Math. 8, Springer, 2001.

[12] N. Ghoussoub, B. Maurey and W. Schachermayer, *Slicings, selections and their applications*, Canad. J. Math. 44 (1992), 483–504.

[13] G. Godefroy, *Universal spaces for strictly convex Banach spaces*, Rev. R. Acad. Cien. Ser. A Mat. 100 (2006), 137–146.
[14] G. Godefroy, *Analytic sets of Banach spaces*, Rev. R. Acad. Cien. Ser. A Mat. 104 (2010), 365–374.
[15] G. Godefroy and N. J. Kalton, *Isometric embeddings and universal spaces*, Extracta Math. 22 (2007), 179–189.
[16] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, 1995.
[17] O. Kurka, *Genericity of Fréchet smooth spaces*, Rev. R. Acad. Cien. Ser. A Mat. 106 (2012), 371–406.
[18] O. Kurka, *Amalgamations of classes of Banach spaces with a monotone basis*, arXiv: 1504.06862 (2015).
[19] O. Kurka, *Zippin's embedding theorem and amalgamations of classes of Banach spaces*, Proc. Amer. Math. Soc., to appear.
[20] E. Odell and Th. Schlumprecht, *A universal reflexive space for the class of uniformly convex Banach spaces*, Math. Ann. 335 (2006), 901–916.
[21] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. 30 (1968), 53–61.
[22] M. Zippin, *Banach spaces with separable duals*, Trans. Amer. Math. Soc. 310 (1988), 371–379.

Ondřej Kurka
Department of Mathematical Analysis
Charles University
Sokolovská 83
186 75 Praha 8, Czech Republic
E-mail: kurka.ondrej@seznam.cz