Commutators with fractional integral operators

by

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Abstract. We investigate weighted norm inequalities for the commutator of a fractional integral operator and multiplication by a function. In particular, we show that, for $\mu, \lambda \in A_{p,q}$ and $\alpha/n + 1/q = 1/p$, the norm $\|[b, I_\alpha] : L^p(\mu^p) \to L^q(\lambda^q)\|$ is equivalent to the norm of b in the weighted BMO space $\mathrm{BMO}(\nu)$, where $\nu = \mu\lambda^{-1}$. This work extends some of the results on this topic existing in the literature, and continues a line of investigation which was initiated by Bloom in 1985 and was recently developed further by the first author, Lacey, and Wick.

1. Introduction and statement of main results. Recall the classical fractional integral operator, or Riesz potential, on $\mathbb{R}^n$: let $0 < \alpha < n$ be fixed and, for a Schwartz function f define the fractional integral operator (or Riesz potential) I_α by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy.$$

These operators have been studied since 1949, when they were introduced by Marcel Riesz, and have since found many applications in analysis—such as Sobolev embedding theorems and PDEs. Also recall the Calderón–Zygmund operators:

$$Tf(x) := \int_{\mathbb{R}^n} K(x,y) f(y) \, dy, \quad x \notin \operatorname{supp} f,$$

where the kernel satisfies the standard size and smoothness estimates:

$$|K(x, y)| \leq \frac{C}{|x - y|^n},$$

$$|K(x + h, y) - K(x, y)| + |K(x, y + h) - K(x, y)| \leq C \frac{|h|^\delta}{|x - y|^{n+\delta}},$$

for all $|x - y| > 2|h| > 0$ and a fixed $\delta \in (0, 1]$.

To contrast the two, note for example that fractional integral operators are positive, which in many cases makes them easier to work with (as one example of this, it is almost trivial to dominate the fractional integral operators by sparse operators, though this is not important to us in the present setting). On the other hand, the fractional integral operators do not commute with dilations and therefore can never boundedly map $L^p(dx)$ to itself. Additionally, the kernel of the fractional integral operator does not satisfy the standard estimates above. Therefore, the theory of fractional integral operators is not just a subset of the theory of Calderón–Zygmund operators. Because of this, results which are known for Calderón–Zygmund operators also need to be proved for the fractional integral operators.

In this paper we will characterize the triples (b, μ, λ), where b is a function and μ and λ are $A_{p,q}$ weights (to be defined shortly), such that the commutator $[b, I_\alpha]$ is bounded from $L^p(\mu^p)$ to $L^q(\lambda^q)$. Commutators with Riesz potentials were first studied in [Chan].

Our characterization will be in terms of the norm of b in a certain weighted BMO space, built from the weights μ and λ. This is an adaptation to the fractional integral setting of a viewpoint introduced by Bloom [B] in 1985, and recently investigated by the first author, Lacey and Wick [HLW1, HLW2]. Specifically, Bloom characterized $\|[b, H] : L^p(\mu) \to L^p(\lambda)\|$, where H is the Hilbert transform and μ, λ are A_p weights, in terms of $\|b\|_{\mathrm{BMO}(\nu)}$, where $\mathrm{BMO}(\nu)$ is the weighted BMO space associated with the weight $\nu := \mu^{1/p} \lambda^{-1/p}$. Recall that the Hilbert transform is the one-dimensional prototype for Calderón–Zygmund operators, a role played by the Riesz transforms in $\mathbb{R}^n$.

A modern dyadic proof of Bloom's result was recently given in [HLW1], and the techniques developed were then used to extend the result to all Calderón–Zygmund operators in [HLW2]. In particular, it was proved that

$$(1.1) \qquad \|[b, T] : L^p(\mu) \to L^p(\lambda)\| \leq c\|b\|_{\mathrm{BMO}(\nu)},$$

for all A_p weights μ, λ, and all Calderón–Zygmund operators T on $\mathbb{R}^n$, for some constant c depending on n, T, μ, λ and p. Specializing to the Riesz transforms, a lower bound was also proved. The center of the proof of (1.1) was the Hytönen Representation Theorem, which allows one to recover T from averaging over some dyadic operators, called dyadic shifts. Then the upper bound reduced to these dyadic operators.

We take a similar approach in this paper, where the role of the dyadic shifts will be played by the dyadic version of the fractional integral operator I_α, given by

$$(1.2) \qquad I_\alpha^{\mathcal{D}} f := \sum_{Q \in \mathcal{D}} |Q|^{\alpha/n} \langle f \rangle_Q \mathbb{1}_Q.$$

Our main result is:

THEOREM 1.1. *Suppose that $\alpha/n + 1/q = 1/p$ and $\mu, \lambda \in A_{p,q}$. Let $\nu := \mu\lambda^{-1}$. Then*

$$\|[b, I_\alpha] : L^p(\mu^p) \to L^q(\lambda^q)\| \simeq \|b\|_{\mathrm{BMO}(\nu)}.$$

It is important to observe that we require that each weight belong to a certain $A_{p,q}$ class and this will imply that $\mu\lambda^{-1}$ is an A_2 weight and, in particular, an A_∞ weight. Standard properties of these weight classes will be used throughout the paper, without tracking dependencies on the particular weight characteristics. The liberal use of these properties indicates the subtleties involved in the general two-weight setting. For an excellent account of this and other topics related to fractional integral operators, see [Cr].

The paper is organized as follows. In Section 2, we will give the requisite background material and definitions. Note, however, that most of the material not relating strictly to fractional integral operators (such as the Haar system, A_p weights, and weighted BMO) is standard and was also needed in [HLW2] where it is discussed in more detail. In Section 3 we will briefly discuss how the fractional integral operator can be recovered as an average of dyadic operators. In Section 4 we will prove $\|[b, I_\alpha] : L^p(\mu^p) \to L^q(\lambda^q)\| \lesssim \|b\|_{\mathrm{BMO}(\nu)}$, and in Section 5 we will prove the reverse inequality: $\|b\|_{\mathrm{BMO}(\nu)} \lesssim \|[b, I_\alpha] : L^p(\mu^p) \to L^q(\lambda^q)\|$.

2. Background and notation

2.1. The Haar system. Let $\mathcal{D}$ be a dyadic grid on $\mathbb{R}^n$ and let $Q \in \mathcal{D}$. For every $\epsilon \in \{0, 1\}^n$, let h_Q^ϵ be the usual Haar function defined on Q. For convenience, we write $\epsilon = 1$ if $\epsilon = (1, \ldots, 1)$. Note that, in this case, $\int h_Q^1 = 1$. Otherwise, if $\epsilon \neq 1$, then $\int h_Q^\epsilon = 0$. Moreover, recall that $\{h_Q^\epsilon\}_{Q \in \mathcal{D}, \epsilon \neq 1}$ forms an orthonormal basis for $L^2(\mathbb{R}^n)$. For a function f, a cube $Q \in \mathcal{D}$ and $\epsilon \neq 1$, we denote

$$\widehat{f}(Q, \epsilon) := \langle f, h_Q^\epsilon \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2(\mathbb{R}^n)$.

2.2. A_p classes and weighted BMO. Let w be a *weight* on $\mathbb{R}^n$, that is, a locally integrable, almost everywhere positive function. For a subset

$Q \subset \mathbb{R}^n$ we denote

$$w(Q) := \int_Q w \, dx \quad \text{and} \quad \langle w \rangle_Q := \frac{w(Q)}{|Q|}.$$

Given $1 < p < \infty$, a weight w is said to belong to the *Muckenhoupt A_p class* provided that

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{p-1} < \infty,$$

where p' denotes the Hölder conjugate of p, and the supremum is over all cubes $Q \subset \mathbb{R}^n$. Moreover, $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$ and, in this case, $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{p'-1}$. Furthermore, if $1 < p < q < \infty$, then $A_p \subset A_q$, with $[w]_{A_q} \le [w]_{A_p}$ for all $w \in A_p$.

For a dyadic lattice $\mathcal{D}$, recall the dyadic square function:

$$(S_{\mathcal{D}} f)^2 = \sum_{P \in \mathcal{D}, \epsilon \neq 1} |\widehat{f}(Q, \epsilon)|^2 \frac{\mathbb{1}_Q}{|Q|}.$$

Another property of A_p weights which will be useful for us is the following well-known *weighted Littlewood–Paley Theorem*:

THEOREM 2.1. *Let $w \in A_p$. Then*

$$\|S_{\mathcal{D}} : L^p(w) \to L^p(w)\| \simeq c(n, p, [w]_{A_p}).$$

For a weight w on $\mathbb{R}^n$, the *weighted BMO space* $\mathrm{BMO}(w)$ is defined to be the space of all locally integrable functions b that satisfy

$$(2.1) \qquad \|b\|_{\mathrm{BMO}(w)} := \sup_Q \frac{1}{w(Q)} \int_Q |b - \langle b \rangle_Q| \, dx < \infty,$$

where the supremum is over all cubes Q in $\mathbb{R}^n$. For a general weight, the definition of the BMO norm is highly dependent on its L^1 average. But, if the weight is A_∞, one is free to replace the L^1-norm by larger averages. Namely, for $w \in A_p$, define

$$(2.2) \qquad \|b\|_{\mathrm{BMO}^{p'}(w)} := \sup_Q \left(\frac{1}{w(Q)} \int_Q |b - \langle b \rangle_Q|^{p'} \, dw' \right)^{1/p'},$$

where w' denotes the conjugate weight $w^{1-p'}$. Then

$$(2.3) \qquad \|b\|_{\mathrm{BMO}(w)} \le \|b\|_{\mathrm{BMO}^{p'}(w)} \le C(n, p, [w]_{A_\infty}) \|b\|_{\mathrm{BMO}(w)}.$$

The proof is similar to the proof in the unweighted case. In particular, the first inequality is a straightforward application of Hölder's inequality, and the second inequality follows from a suitable John–Nirenberg property (which requires a suitable Calderón–Zygmund decomposition). The details are in [MW2].

For a dyadic grid $\mathcal{D}$ on $\mathbb{R}^n$, we define the dyadic versions of the norms above by taking supremum over $Q \in \mathcal{D}$ instead of over all cubes Q in $\mathbb{R}^n$, and denote these spaces by $\mathrm{BMO}_{\mathcal{D}}(w)$ and $\mathrm{BMO}_{\mathcal{D}}^{p'}(w)$. Clearly $\mathrm{BMO}(w) \subset \mathrm{BMO}_{\mathcal{D}}(w)$ for any choice of $\mathcal{D}$, and the equivalence in (2.3) also holds for the dyadic versions of these spaces.

A fact which will be crucial to our proof is the following:

LEMMA 2.2. *If $w \in A_2$, then*

$$(2.4) \qquad |\langle b, \Phi \rangle| \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{D}}^2(w)} \|S_{\mathcal{D}} \Phi\|_{L^1(w)}.$$

This comes from a duality relationship between dyadic weighted BMO spaces and dyadic weighted Hardy spaces. For a more detailed discussion and a proof of this fact, see [HLW2, Section 2.6]. We remark here that Lemma 2.2 was also fundamental for the proof of the upper bound (1.1) in [HLW2], essentially for the following reason: if μ, λ are A_p weights, then $\nu := \mu^{1/p} \lambda^{-1/p}$ is an A_2 weight. Thus the duality statement above applied to ν eventually yields, through Hölder's inequality, some bounds in terms of $L^p(\mu)$ and $L^{p'}(\lambda)$ norms. This is also the strategy we will adapt to the fractional integral case, which makes use of $A_{p,q}$ classes instead. We discuss these next.

2.3. $A_{p,q}$ classes. Throughout this section, α, n, p, q are fixed and satisfy $1/p - 1/q = \alpha/n$. We recall first the fractional maximal operator,

$$M_\alpha f := \sup_Q |Q|^{\alpha/n} \langle |f| \rangle_Q \mathbb{1}_Q,$$

with the supremum being over all cubes Q. This was first introduced in [MW1], where it was used to prove weighted inequalities for I_α, a result analogous to the classical result [CF] of Coifman and Fefferman, relating the Hardy–Littlewood maximal operator and singular integrals. We will be working with the dyadic version of this operator, $M_\alpha^{\mathcal{D}}$, defined for a dyadic grid $\mathcal{D}$ just as above, but only taking supremum over $Q \in \mathcal{D}$.

Also in [MW1] was introduced a generalization of A_p classes for the fractional integral setting: we say that a weight w belongs to the $A_{p,q}$ *class* provided that

$$[w]_{A_{p,q}} := \sup_Q \langle w^q \rangle_Q \langle w^{-p'} \rangle_Q^{q/p'} < \infty.$$

See [R, RS, CM2, CM1, Cr] for other generalizations.

We will use the following important result concerning $A_{p,q}$ weights due to, for example, Sawyer and Muckenhoupt and Wheeden [S1, S2, MW1]:

THEOREM 2.3. *Let w be a weight. Then the following are equivalent:*

  (i) $w \in A_{p,q}$;
  (ii) $\|M_\alpha^{\mathcal{D}} : L^p(w^p) \to L^q(w^q)\| \simeq C(n, \alpha, p, [w]_{A_{p,q}})$;
  (iii) $\|I_\alpha^{\mathcal{D}} : L^p(w^p) \to L^q(w^q)\| \simeq C(n, \alpha, p, [w]_{A_{p,q}})$.

We now make two observations about $A_{p,q}$ weights which will be particularly useful to us. First, we note that:

(2.5)   If $w \in A_{p,q}$, then $w^p \in A_p$, $w^{-p'} \in A_{p'}$, $w^q \in A_q$, and $w^{-q'} \in A_{q'}$,

where all weights above have Muckenhoupt characteristics bounded by powers of $[w]_{A_{p,q}}$. To see that $w^p \in A_p$, first notice $w \in A_{p,q}$ if and only if $w^q \in A_{q_0}$, with $[w^q]_{A_{q_0}} = [w]_{A_{p,q}}$, where

$$q_0 := 1 + q/p' = q(1 - \alpha/n).$$

Since the A_p classes are increasing and $q_0 < q$, we see that $w^q \in A_q$. In turn, this gives $w^{-q'} = (w^q)^{1-q'} \in A_{q'}$. The other two statements in (2.5) follow in a similar fashion from the fact that $w \in A_{p,q}$ if and only if $w^{-1} \in A_{q',p'}$.

Second, suppose that $\mu, \lambda \in A_{p,q}$ and let $\nu := \mu\lambda^{-1}$. Since $\mu^p, \lambda^p \in A_p$, Hölder's inequality implies $\nu \in A_2$ (with $[\nu]_{A_2}^p \leq [\mu^p]_{A_p}[\lambda^p]_{A_p}$), a fact which will be used in proving the upper bound. Moreover, we claim that for any cube Q,

(2.6) $$\mu^p(Q)^{1/p}\lambda^{-q'}(Q)^{1/q'} \lesssim \nu(Q)|Q|^{\alpha/n},$$

a fact which will be useful in proving the lower bound. To see this, note first that

$$\langle \mu^p \rangle_Q^{1/p} \langle \mu^{-p'} \rangle_Q^{1/p'} \lesssim 1 \quad \text{and} \quad \langle \lambda^{-q'} \rangle_Q^{1/q'} \langle \lambda^q \rangle_Q^{1/q} \lesssim 1,$$

which simply come from $\mu^p \in A_p$ and $\lambda^q \in A_q$. Since $p' > q'$, Hölder implies

$$\left(\frac{1}{|Q|} \int_Q \mu^{-q'}\, dx \right)^{1/q'} \leq \left(\frac{1}{|Q|} \left(\int_Q \mu^{-p'}\, dx \right)^{q'/p'} \left(\int_Q dx \right)^{1-q'/p'} \right)^{1/q'}$$

$$= \left(\frac{1}{|Q|} \int_Q \mu^{-p'}\, dx \right)^{1/p'},$$

and hence $\langle \mu^{-q'} \rangle_Q^{1/q'} \leq \langle \mu^{-p'} \rangle_Q^{1/p'}$. Combining these estimates gives

$$\langle \mu^p \rangle_Q^{1/p} \langle \lambda^{-q'} \rangle_Q^{1/q'} \lesssim \frac{1}{\langle \mu^{-p'} \rangle_Q^{1/p'}} \frac{1}{\langle \lambda^q \rangle_Q^{1/q}} \lesssim \frac{1}{\langle \mu^{-q'} \rangle_Q^{1/q'} \langle \lambda^q \rangle_Q^{1/q}} \leq \frac{1}{\langle \nu^{-1} \rangle_Q} \leq \langle \nu \rangle_Q.$$

The last two inequalities are applications of Hölder's inequality and the fact that $\nu^{-1} = \mu^{-1}\lambda$. This proves (2.6).

3. Averaging over dyadic fractional integral operators. In this section, we show that I_α can be recovered from (1.2) by averaging over dyadic lattices. The proof here is modified (and abridged) from the proof in [PTV], but it is possible to modify any of the proofs in, for example, [P, H, L]. For the sake of clarity, we only give the proof for the one-dimensional case.

Given an interval $[a, b)$ (it is not too important that the interval be closed on the left and open on the right) of length r, we can create a dyadic lattice $\mathcal{D}_{a,r}$ in a standard way. In particular, $\mathcal{D}_{a,r}$ is the dyadic lattice on $\mathbb{R}$ with intervals of length $r2^{-k}$, $k \in \mathbb{Z}$, and the point a is not in the interior of any of the intervals in $\mathcal{D}_{a,r}$. For example, $\mathcal{D}_{0,1}$ is the standard dyadic lattice on $\mathbb{R}$. For a given lattice $\mathcal{D}_{a,r}$, we let $\mathcal{D}_{a,r}^k$ denote the intervals in $\mathcal{D}_{a,r}$ with length $r2^{-k}$. In this section we slightly abuse notation and let $h_I^1 = |I|^{-1/2}\mathbb{1}_I$.

Define

$$\mathbb{P}_{(a,r)}^0 f(x) := \sum_{I \in \mathcal{D}_{a,r}^0} |I|^\alpha \langle f, h_I^1 \rangle h_I^1(x).$$

With r and x fixed, we can parameterize the dyadic grids by the set $(-r, 0]$ and we can give this set the probability measure da/r. For a fixed $x \in \mathbb{R}$, we want to compute

$$\mathbb{E}(\mathbb{P}_{(a,r)}^0 f(x)) = \int_{-r}^0 \mathbb{P}_{(a,r)}^0 f(x) \, \frac{da}{r}.$$

Let $\tau_t f(x) := f(x + t)$ be the translation operator and note that $\mathbb{P}_{a-t}\tau_t = \tau_t \mathbb{P}_a$. From this it easily follows that $\mathbb{E}\mathbb{P}_{(a,r)}^0 \tau_t = \tau_t \mathbb{P}_{(a,r)}^0$. That is, $\mathbb{E}\mathbb{P}_{(a,r)}^0$ is given by convolution. Let

$$\mathbb{E}\mathbb{P}_{(a,r)}^0 f(x) = F_{0,r} * f(x).$$

We want to compute $F_{0,r}$. First, note that $\mathbb{P}_{a,r}^0$ is convolution with the function $(r^\alpha/r)\mathbb{1}_{[-r/2,r/2]}$. Therefore,

$$F_{0,r} * f(x) = \mathbb{E}\mathbb{P}_{(a,r)}^0 f(x)$$

$$= \mathbb{E}\mathbb{P}_{(a/2,r)}^0 f(x) \int_{x-r/2}^{x+r/2} \int_{\mathbb{R}} f(s) \frac{r^\alpha}{r} \mathbb{1}_{-r/2,r/2}(t - s) \, ds \, \frac{dt}{r}.$$

Using Fubini, we see that

$$F_{0,r}(x) = \int_{x-r/2}^{x+r/2} \frac{r^\alpha}{r} \mathbb{1}_{[-r/2,r/2]}(t) \, \frac{dt}{r} = \frac{r^\alpha}{r} \mathbb{1}_{[-r/2,r/2]}(x) \left(1 - \left|\frac{x}{r}\right|\right)$$

$$= \frac{r^\alpha}{r} F_{0,1}(x/r).$$

Now, fix an $r \in [1, 2)$ and define

$$F_r = \sum_{n \in \mathbb{Z}} F_{0,2^n r}.$$

The grids $\mathcal{D}_{a,r}^k$, $k \in \mathbb{Z}$, can be unioned to form a dyadic lattice (here a is fixed). Call r the *calibre* of the dyadic lattice. Convolution with F_r is

averaging over all the dyadic lattices $\mathcal{D}_{a,r}$ with fixed calibre r, that is,

$$F_r * f = \mathbb{E}\mathbb{P}_{\mathcal{D}_{a,r}} f.$$

Finally, we need to average over $r \in [1,2)$. Set $F(x) := \int_1^2 F_r(x)\frac{dr}{r}$. Now, we want to compute $F(x)$. We have

$$F(x) = \int\limits_1^2 F_r(x)\frac{dr}{r} = \int\limits_1^2 \sum_{n \in \mathbb{Z}} F_{0,2^n r}(x)\frac{dr}{r} = \int\limits_0^\infty F_{0,\rho}(x)\frac{d\rho}{\rho}$$

$$= \int\limits_0^\infty F_{0,1}\left(\frac{x}{\rho}\right)\frac{\rho^\alpha}{\rho^2}\,d\rho = \int\limits_0^\infty \mathbb{1}_{-1/2,1/2}\left(\frac{x}{\rho}\right)\left(1 - \left|\frac{x}{\rho}\right|\right)\frac{\rho^\alpha}{\rho^2}\,dr.$$

Now, if $x > 0$, making the change of variable $t = x/\rho$, we see that

$$F(x) = \frac{x^\alpha}{x}\int\limits_0^\infty F_{0,1}(y)\frac{dy}{y^\alpha} = c_\alpha 1/x^{1-\alpha}.$$

Doing a similar computation for when $x < 0$, we see that $F(x) = c_\alpha/|x|^{1-\alpha}$.

4. Upper bound. The decomposition in Section 3 means that the upper bound in Theorem 1.1 follows from the following, where the implied constants are independent of the dyadic lattice:

LEMMA 4.1. *Suppose that* $\alpha/n + 1/q = 1/p$ *and* $\mu, \lambda \in A_{p,q}$. *Let* $\nu := \mu\lambda^{-1}$. *Then*

$$\|[b, I_\alpha^\mathcal{D}] : L^p(\mu^p) \to L^q(\lambda^q)\| \lesssim \|b\|_{\mathrm{BMO}(\nu)}.$$

Proof. We show that $[b, I_\alpha^\mathcal{D}]$ can be decomposed as the sum of four operators which will be fairly easy to bound. First note that for $\epsilon \neq 1$,

$$I_\alpha^\mathcal{D} h_Q^\epsilon = \sum_{P \in \mathcal{D}:\, P \subsetneq Q} |P|^{\alpha/n} h_Q^\epsilon(P)\mathbb{1}_P = \Big(\sum_{P \in \mathcal{D}:\, P \subsetneq Q} |P|^{\alpha/n}\mathbb{1}_P\Big)h_Q^\epsilon = c_\alpha |Q|^{\alpha/n} h_Q^\epsilon.$$

Similarly,

$$I_\alpha^\mathcal{D}\mathbb{1}_Q = (1 + c_\alpha)|Q|^{\alpha/n}\mathbb{1}_Q + |Q|\sum_{R \in \mathcal{D}:\, Q \subsetneq R} |R|^{\alpha/n}\frac{\mathbb{1}_R}{|R|}.$$

Using these computations we obtain

$$I_\alpha^\mathcal{D}(h_P^\epsilon h_Q^\eta) = \begin{cases} c_\alpha |P \cap Q|^{\alpha/n} h_P^\epsilon h_Q^\eta & \text{if } P \neq Q, \\ & \text{or if } P = Q \text{ and } \epsilon \neq \eta, \\ (1 + c_\alpha)|Q|^{\alpha/n}\frac{\mathbb{1}_Q}{|Q|} + \sum_{R \supsetneq Q} |R|^{\alpha/n}\frac{\mathbb{1}_R}{|R|} & \text{if } P = Q \text{ and } \epsilon = \eta. \end{cases}$$

Thus

$$
[h_P^\epsilon, I_\alpha^\mathcal{D}]h_Q^\eta =
\begin{cases}
c_\alpha h_Q^\eta(P)h_P^\epsilon(|Q|^{\alpha/n} - |P|^{\alpha/n}) & \text{if } P \subsetneq Q, \\
-|Q|^{\alpha/n}\frac{\mathbb{1}_Q}{|Q|} - \sum_{R\supsetneq Q}|R|^{\alpha/n}\frac{\mathbb{1}_R}{|R|} & \text{if } P = Q \text{ and } \epsilon = \eta, \\
0 & \text{if } Q \subsetneq P, \\
& \text{or if } Q = P \text{ and } \epsilon \neq \eta.
\end{cases}
$$

Expressing b and f in terms of their Haar coefficients, we obtain

$$
[b, I_\alpha^\mathcal{D}]f = \sum_{P,Q\in\mathcal{D}} \sum_{\epsilon,\eta\neq 1} \widehat{b}(P,\epsilon)\widehat{f}(Q,\eta)[h_P^\epsilon, I_\alpha^\mathcal{D}]h_Q^\eta.
$$

Hence

(4.1)
$$
[b, I_\alpha^\mathcal{D}]f = c_\alpha T_1 f - c_\alpha \Pi_{b,\alpha}^{(0,1,0)}f - \Pi_{b,\alpha}^{(0,0,1)}f - T_2 f,
$$

where

$$
\Pi_{b,\alpha}^{(0,1,0)}f := \sum_{Q\in\mathcal{D},\,\epsilon\neq 1} \widehat{b}(Q,\epsilon)\langle f\rangle_Q |Q|^{\alpha/n}h_Q^\epsilon,
$$

$$
\Pi_{b,\alpha}^{(0,0,1)}f := \sum_{Q\in\mathcal{D},\,\epsilon\neq 1} \widehat{b}(Q,\epsilon)\widehat{f}(Q,\epsilon)|Q|^{\alpha/n}\frac{\mathbb{1}_Q}{|Q|},
$$

$$
T_1 f := \sum_{P\in\mathcal{D},\,\epsilon\neq 1} \widehat{b}(P,\epsilon)\Big(\sum_{Q\supsetneq P,\,\eta\neq 1} \widehat{f}(Q,\eta)h_Q^\eta(P)|Q|^{\alpha/n}\Big)h_P^\epsilon,
$$

$$
T_2 f := \sum_{P\in\mathcal{D},\,\epsilon\neq 1} \widehat{b}(P,\epsilon)\widehat{f}(P,\epsilon)\Big(\sum_{Q\supsetneq P}|Q|^{\alpha/n}\frac{\mathbb{1}_Q}{|Q|}\Big).
$$

We will show that all of these operators are bounded $L^p(\mu^p) \to L^q(\lambda^q)$. Below, all implied constants are allowed to depend on n, α, p, $[\mu]_{A_{p,q}}$, and $[\lambda]_{A_{p,q}}$. Also all inner products below are taken with respect to dx, and therefore it is enough to show

$$
|\langle Tf, g\rangle| \lesssim \|b\|_{\mathrm{BMO}(\nu)}\|f\|_{L^p(\mu^p)}\|g\|_{L^{q'}(\lambda^{-q'})}
$$

for each of the four operators above (this is because the dual of $L^q(\lambda^q)$ with respect to the unweighted inner product is $L^{q'}(\lambda^{-q'})$). The idea, which is taken from [HLW1, HLW2], is to write the bilinear form $\langle Tf, g\rangle$ as $\langle b, \Phi\rangle$ and then show that $\|S_\mathcal{D}\Phi\|_{L^1(\nu)}$ is controlled by $\|f\|_{L^p(\mu^p)}\|g\|_{L^{q'}(\lambda^{-q'})}$; by the weighted H^1-BMO duality, this is enough to prove the claim.

The estimates for the two paraproducts are almost identical, and we only give the proof for $\Pi_{b,\alpha}^{(0,1,0)}$. First with

$$
\Phi := \sum_{Q\in\mathcal{D},\,\epsilon\neq 1} \langle f\rangle_Q |Q|^{\alpha/n}\widehat{g}(Q,\epsilon)h_Q^\epsilon,
$$

we have

$$\langle \Pi_{b,\alpha}^{(0,1,0)} f, g \rangle = \langle b, \Phi \rangle.$$

Then

$$(S_{\mathcal{D}}\Phi)^2 = \sum_{Q \in \mathcal{D}, \epsilon \neq 1} |\langle f \rangle_Q|^2 |Q|^{2\alpha/n} |\widehat{g}(Q, \epsilon)|^2 \frac{\mathbb{1}_Q}{|Q|} \leq (M_\alpha f)^2 (S_{\mathcal{D}} g)^2.$$

Therefore,

$$\|S_{\mathcal{D}}\Phi\|_{L^1(\nu)} \leq \|M_\alpha f\|_{L^q(\mu^q)} \|S_{\mathcal{D}} g\|_{L^{q'}(\lambda^{-q'})} \lesssim \|f\|_{L^p(\mu^p)} \|g\|_{L^{q'}(\lambda^{-q'})},$$

where the last inequality follows from Theorem 2.3 for the fractional maximal function, and from Theorem 2.1 and the fact that $\lambda^{-q'} \in A_{q'}$ for the dyadic square function. The proof for $\Pi_{b,\alpha}^{(0,0,1)}$ is very similar, and we omit the details.

Now let us look at T_1. As above, we have $\langle T_1 f, g \rangle = \langle b, \Phi \rangle$ with

$$\Phi := \sum_{P \in \mathcal{D}, \epsilon \neq 1} \widehat{g}(P, \epsilon) \Big(\sum_{Q \supsetneq P, \eta \neq 1} \widehat{f}(Q, \eta) h_Q^\eta(P) |Q|^{\alpha/n} \Big) h_P^\epsilon,$$

Then

$$(S_{\mathcal{D}}\Phi)^2 \leq \sum_{P \in \mathcal{D}, \epsilon \neq 1} |\widehat{g}(P, \epsilon)|^2 \Big(\sum_{Q \supsetneq P, \eta \neq 1} \langle |f| \rangle_Q |Q|^{\alpha/n} \Big)^2 \frac{\mathbb{1}_P}{|P|}$$
$$\leq (I_\alpha^{\mathcal{D}} |f|)^2 (S_{\mathcal{D}} g)^2.$$

From Theorems 2.3 and 2.1, it follows that

$$\|S_{\mathcal{D}}\Phi\|_{L^1(\nu)} \leq \|I_\alpha^{\mathcal{D}} |f|\|_{L^q(\mu^q)} \|S_{\mathcal{D}} g\|_{L^{q'}(\lambda^{-q'})} \lesssim \|f\|_{L^p(\mu^p)} \|g\|_{L^{q'}(\lambda^{-q'})}.$$

The estimates for T_2 are similar and we omit the details. ∎

5. Lower bound. In this section, we prove the lower bound in Theorem 1.1, which follows immediately from the lemma below. In particular, we will show the following:

LEMMA 5.1. *For all cubes Q,*

$$\frac{1}{\nu(Q)} \int\limits_Q |b(x) - \langle b \rangle_Q| \, dx \lesssim \|[b, I_\alpha] : L^p(\mu^p) \to L^q(\lambda^q)\|.$$

Proof. The proof follows the lines of the proof in [Chaf]. We first make some reductions. As with unweighted BMO, we can replace the $\langle b \rangle_Q$ with

any constant. Indeed,

$$\frac{1}{\nu(Q)} \int\limits_Q |b(x) - \langle b \rangle_Q| \, dx \leq \frac{1}{\nu(Q)} \int\limits_Q |b(x) - C_Q| \, dx + \frac{|Q|}{\nu(Q)} |C_Q - \langle b \rangle_Q|$$

$$\leq \frac{2}{\nu(Q)} \int\limits_Q |b(x) - C_Q| \, dx.$$

Second, let P be the cube with $l(P) = 4l(Q)$, where $l(Q)$ is the side length of Q, and with the same "bottom left corner" as Q. By the doubling property of A_∞ weights, we have $\nu(P) \simeq \nu(Q)$, and therefore it is enough to prove

$$\frac{1}{\nu(P)} \int\limits_Q |b(x) - C_Q| \, dx \lesssim \|[b, I_\alpha] : L^p(\mu^p) \to L^q(\lambda^q)\|.$$

Finally, let P_R be the "upper right half" of P. Below, we will use $C_Q = \langle b \rangle_{P_R}$.

Now, for $x \in Q$ and $y \in P_R$,

$$\frac{|x - y|}{2\sqrt{n}\,|P|^{1/n}} \geq \frac{\sqrt{n}\,|Q|^{1/n}}{2\sqrt{n}\,|P|^{1/n}} = \frac{1}{8} \quad \text{and} \quad \frac{|x - y|}{2\sqrt{n}\,|P|^{1/n}} \leq \frac{\sqrt{n}\,|P|^{1/n}}{2\sqrt{n}\,|P|^{1/n}} \leq \frac{1}{2}.$$

The point is that there is a function, $K(x)$, that is smooth on $[-1, 1]^n$, has a smooth periodic extension to $\mathbb{R}^n$, and is equal to $|x|^{n-\alpha}$ for $1/8 \leq |x| \leq 1/2$. Therefore, for $x \in Q$ and $y \in P_R$,

$$\left(\frac{|x - y|}{2\sqrt{n}\,|P|^{1/n}} \right)^{n-\alpha} = K\left(\frac{x - y}{2\sqrt{n}\,|P|^{1/n}} \right).$$

Important for us is the fact that K has a Fourier expansion with summable coefficients.

We are now ready to prove the main estimate. First, we denote $\sigma(x) = \operatorname{sgn}(b(x) - \langle b \rangle_{P_R})$. Then

$$\int\limits_Q |b(x) - \langle b \rangle_{P_R}| \, dx = \frac{1}{|P_R|} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} (b(x) - b(y))\sigma(x)\mathbb{1}_Q(x)\mathbb{1}_{P_R}(y) \, dy \, dx$$

$$= \frac{1}{|P_R|} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \frac{b(x) - b(y)}{\left(\frac{|x-y|}{2\sqrt{n}|P|^{1/n}} \right)^{n-\alpha}} \left(\frac{|x - y|}{2\sqrt{n}\,|P|^{1/n}} \right)^{n-\alpha} \sigma(x)\mathbb{1}_Q(x)\mathbb{1}_{P_R}(y) \, dy \, dx$$

$$\simeq |P|^{-\alpha/n} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \frac{b(x) - b(y)}{|x - y|^{n-\alpha}} K\left(\frac{x - y}{2\sqrt{n}\,|P|^{1/n}} \right) \sigma(x)\mathbb{1}_Q(x)\mathbb{1}_{P_R}(y) \, dy \, dx.$$

Observe that the integral above is positive, so the "$\simeq$" is not a problem. Expanding K in its Fourier series yields

$$K\left(\frac{x - y}{2\sqrt{n}|P|^{1/n}} \right) = \sum_k a_k e^{ikx/2\sqrt{n}|P|^{1/n}} e^{-iky/2\sqrt{n}|P|^{1/n}},$$

and inserting this into the integral, we continue:

$$|P|^{-\alpha/n}\sum_k a_k \int\limits_Q \int\limits_{P_R} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}}\sigma(x)e^{ikx/c|P|^{1/n}}e^{-iky/c|P|^{1/n}}\,dy\,dx$$

$$= |P|^{-\alpha/n}\sum_k a_k \int\limits_{\mathbb{R}} h_k(x)[b,I_\alpha]f_k(x)\,dx,$$

where $h_k(x) = \sigma(x)e^{ikx/c|P|^{1/n}}\mathbb{1}_P(x)$ and $f_k(y) = e^{-iky/c|P|^{1/n}}\mathbb{1}_{P_R}(y)$. We control the integral by

$$\int\limits_{\mathbb{R}} h_k(x)[b,I_\alpha]f_k(x)\,dx \le \|[b,I_\alpha]:L^p(\mu^p)\to L^q(\lambda^q)\|\|f_k\|_{L^p(\mu^p)}\|h_k\|_{L^{q'}(\lambda^{-q'})}$$

$$= \|[b,I_\alpha]:L^p(\mu^p)\to L^q(\lambda^q)\|\mu^p(P_R)^{1/p}\lambda^{-q'}(P)^{1/q'}$$

$$\le \|[b,I_\alpha]:L^p(\mu^p)\to L^q(\lambda^q)\|\mu^p(P)^{1/p}\lambda^{-q'}(P)^{1/q'}.$$

By (2.6), this is dominated by

$$\|[b,I_\alpha]:L^p(\mu^p)\to L^q(\lambda^q)\|\,|P|^{\alpha/n}\nu(P).$$

This completes the proof. ∎

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