G_δ and co-meager semifilters

by

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Abstract. The ultrafilters on the partial order $([\omega]^\omega, \subseteq^*)$ are the free ultrafilters on ω, which constitute the space ω^*, the Stone–Čech remainder of ω. If U is an upperset of this partial order (i.e., a *semifilter*), then ultrafilters on U correspond to closed subsets of ω^* via Stone duality.

If U is large enough, then it is possible to get combinatorially nice ultrafilters on U by generalizing the corresponding constructions for $[\omega]^\omega$. In particular, if U is co-meager then there are ultrafilters on U that are weak P-filters (extending a result of Kunen). If U is G_δ (and hence also co-meager) and $\mathfrak{d} = \mathfrak{c}$, then there are ultrafilters on U that are P-filters (extending a result of Ketonen).

For certain choices of U, these constructions have applications in dynamics, algebra, and combinatorics. Most notably, we give a new proof of the fact that there are minimal-maximal idempotents in $(\omega^*, +)$. This was an outstanding open problem solved only recently by Zelenyuk.

1. Introduction. The two main theorems in this paper are extensions of two well-established theorems about special points in ω^*: Ketonen's proof that P-points exist under $\mathfrak{d} = \mathfrak{c}$, and Kunen's proof that weak P-points exist under ZFC. We show that each of these proofs can be carried out in a much more general setting. We also show how the extended results can be used to give a relatively easy proof that the semigroup $(\omega^*, +)$ contains idempotents that are both minimal and maximal.

Our starting point is the simple observation that certain semifilters behave very similarly to $[\omega]^\omega$ with respect to the $\subseteq^*$ relation. Since the theorems of Ketonen and Kunen can be interpreted as theorems about the partial order $([\omega]^\omega, \subseteq^*)$, these theorems naturally extend to partial orders of

the form $(\mathcal{S}, \subseteq^*)$ for certain nice semifilters $\mathcal{S}$. Specifically, in Section 3, we show that if $\mathfrak{d} = \mathfrak{c}$ then every G_δ semifilter admits an ultrafilter that is also a P-filter, and in Section 4, we show that every co-meager semifilter admits an ultrafilter that is a weak P-filter.

In the final Section 5, we show how these results can be used to give an alternative proof of Zelenyuk's result from [16], which answered a long-standing open question of Hindman and Strauss, namely whether $(\omega^*, +)$ contains left-maximal idempotents. Actually, our proof gives a strengthening of his result: we show that $(\omega^*, +)$ contains a minimal left ideal that is also a weak P-set. Any such ideal is prime, and the idempotents it contains are both minimal and left-maximal.

2. Preliminaries. A *semifilter* on ω is a subset $\mathcal{S}$ of $\mathcal{P}(\omega)$ such that $\emptyset \neq \mathcal{S} \neq \mathcal{P}(\omega)$ and $\mathcal{S}$ is closed upwards in $\subseteq^*$ (as usual, $A \subseteq^* B$ means $A \setminus B$ is finite). We think of semifilters as partial orders, naturally ordered by $\subseteq^*$. The largest possible semifilter is $[\omega]^\omega$, the set of all infinite subsets of ω.

A partial order is *antisymmetric* if $a \leq b$ and $b \leq a$ implies $a = b$; some authors even include this in the definition of a partial order. We note that our partial orders do not enjoy this property. However, each one has an *antisymmetric quotient*, namely the set of equivalence classes of the form $[X] = \{Y \subseteq \omega : X \subseteq^* Y \subseteq^* X\}$. For example, the antisymmetric quotient of $[\omega]^\omega$ is the familiar order $\mathcal{P}(\omega)/\text{fin}$ (without the bottom element). In what follows, we have no need (and no desire) to work with equivalence classes, and will not need to use the antisymmetry axiom anywhere. Therefore we choose to work with subsets of ω rather than equivalence classes thereof. It is worth pointing out, though, that all of our proofs and constructions "factor through" the antisymmetric quotient, and can be interpreted as results about $\mathcal{P}(\omega)/\text{fin}$ and its uppersets.

If $\mathcal{S}$ is a semifilter, then a *filter* on $\mathcal{S}$ is a filter on the partial order $(\mathcal{S}, \subseteq^*)$. Specifically, $\mathcal{F} \subseteq \mathcal{S}$ is a filter on $\mathcal{S}$ whenever

- $\mathcal{F} \neq \emptyset$;
- $A \in \mathcal{F}$ and $A \subseteq^* B$ implies $B \in \mathcal{F}$;
- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

An *ultrafilter* on $\mathcal{S}$ is a maximal filter on $\mathcal{S}$. We say $\mathcal{B} \subseteq \mathcal{S}$ is a *filter base* on $\mathcal{S}$ if $\{A \subseteq \mathbb{N} : B \subseteq^* A \text{ for some } B \in \mathcal{B}\}$ is a filter on $\mathcal{S}$. A set is *centered* in $\mathcal{S}$ if it is contained in some filter base.

The collection of all ultrafilters on $[\omega]^\omega$ is denoted ω^*. This set has a natural topology as the Stone–Čech remainder of ω, with basic open sets of the form $A^* = \{\mathcal{F} \in \omega^* : A \in \mathcal{F}\}$. Every filter $\mathcal{F}$ on $[\omega]^\omega$ corresponds to a closed subset of ω^*, namely $\hat{\mathcal{F}} = \bigcap_{A \in \mathcal{F}} A^*$. The set $\hat{\mathcal{F}}$ is called the *Stone dual* of $\mathcal{F}$. For more on Stone duality and the topology of ω^*, we refer the reader to [12].

If $\mathcal{F}$ is a filter on some semifilter $\mathcal{S}$, then $\mathcal{F}$ is also a filter on $[\omega]^\omega$, although an ultrafilter on $\mathcal{S}$ may not be an ultrafilter on $[\omega]^\omega$. Thus (ultra)filters on a semifilter $\mathcal{S}$ correspond to closed subsets of ω^*. For certain choices of $\mathcal{S}$, these closed sets may have interesting algebraic/dynamical/combinatorial properties, and for certain choices of the ultrafilter they may also have interesting topological properties. The interplay between these two choices will give rise to our applications in Section 5.

A subset X of ω^* is a *P-set* if, whenever $\langle U_n : n < \omega \rangle$ is a sequence of open sets each of which contains X, X is in the interior of $\bigcap_{n<\omega} U_n$. We say X is a *weak P-set* if the closure of each countable $D \subseteq \omega^* \setminus X$ is disjoint from X. We call $\mathcal{F}$ a (*weak*) *P-filter* if $\hat{\mathcal{F}}$ is a (weak) *P*-set.

We will also consider semifilters as subsets of the topological space 2^ω (identifying sets with their characteristic functions). The basic open neighborhoods of 2^ω are of the form

$$[\![A \restriction F]\!] = \{X \in 2^\omega : X \cap F = A \cap F\}$$

for $A \subseteq \omega$ and finite $F \subseteq \omega$. If $s \subseteq [0, n]$, we will write $[\![s]\!]$ for $[\![s \restriction [0, n]]\!]$.

We mention here a special semifilter that will appear in several places throughout this paper. A set $A \subseteq \omega$ is *thick* if A contains arbitrarily long intervals, and we let Θ denote the semifilter of thick sets. The ultrafilters on Θ correspond (via Stone duality) precisely to the minimal left ideals of $(\omega^*, +)$ (see [4, Lemma 3.2]). It was this observation that first motivated the study of ultrafilters on Θ, and this in turn motivated our work here.

We end this section by mentioning some results on the descriptive complexity of semifilters considered as subsets of 2^ω. Recall that a set has the *Baire property* if it differs from an open set by a meager set. All Borel sets as well as analytic and co-analytic sets have the Baire property. The following proposition (stated for semifilters in [1]) shows that definable semifilters are either very small or very large:

PROPOSITION 2.1. *If a semifilter has the Baire property then it is either meager or co-meager.*

Proof. Because semifilters are closed under making finite modifications, this follows from the "topological 0-1 law" (see, e.g., [8, Theorem 8.47]). ∎

Meager (or co-meager) filters have a very convenient characterization due to Talagrand and, independently, Jalaili-Naini (see [14], [7]). It was noticed in [1] that it applies to semifilters as well. Given two semifilters $\mathcal{S}, \mathcal{G}$ we say that $\mathcal{S}$ is *Rudin–Blass above* $\mathcal{G}$ (and write $\mathcal{S} \geq_{\mathrm{RB}} \mathcal{G}$) if there is a finite-to-one function $f : \omega \to \omega$ such that $A \in \mathcal{G}$ if and only if $f^{-1}[A] \in \mathcal{S}$ (this is the standard Rudin–Blass ordering extended to semifilters).

PROPOSITION 2.2. *A semifilter is co-meager iff it is Rudin–Blass above $[\omega]^\omega$. It is meager iff it is Rudin–Blass above the Fréchet filter.*

In what follows, the semifilters we work with will be either co-meager or G_δ. As one might expect, the latter property is strictly stronger than the former:

COROLLARY 2.3. *A semifilter $\mathcal{S}$ is co-meager if and only if it contains a G_δ semifilter.*

Proof. Because semifilters are closed under making finite modifications, every semifilter is dense in 2^ω. The "if" direction follows. For the "only if" direction, let $\mathcal{S}$ be a co-meager semifilter and, using the first part of Proposition 2.2, let $f : \omega \to \omega$ be a finite-to-one function such that $f^{-1}[A] \in \mathcal{S}$ for any infinite $A \subseteq \omega$.

For each n, let

$$U_n = \big\{ X \in 2^\omega : \exists \text{ distinct } m_1, \ldots, m_n \text{ with } \bigcup_{1 \le k \le n} f^{-1}(m_k) \subseteq X \big\}.$$

Then U_n is open and upwards closed with respect to $\subseteq$. Hence $\mathcal{G} = \bigcap_{n \in \omega} U_n$ is G_δ, and is easily seen to be upwards closed with respect to $\subseteq^*$. In other words, $\mathcal{G}$ is a G_δ semifilter, and $\mathcal{G} \subseteq \mathcal{S}$ by construction. ∎

3. P-filters from $\mathfrak{d} = \mathfrak{c}$. If $f, g \in \omega^\omega$, we say g *dominates* f, and write $f \le^* g$, whenever $\{ n \in \omega : f(n) \ge g(n) \}$ is finite. The *dominating number* $\mathfrak{d}$ is the smallest size of some $D \subseteq \omega^\omega$ such that every $f \in \omega^\omega$ is dominated by some $g \in D$.

In this section we show that if $\mathfrak{d} = \mathfrak{c}$ then every G_δ semifilter admits a P-ultrafilter. This extends a result of Ketonen, who first proved that $\mathfrak{d} = \mathfrak{c}$ implies the existence of P-points in ω^*.

LEMMA 3.1. *Let $\mathcal{S}$ be a semifilter, and let U be an open subset of 2^ω with $\mathcal{S} \subseteq U$. For each $X \in \mathcal{S}$, there is a function $f_X : \omega \to \omega$ such that, for every $m \in \omega$, $[\![X {\restriction} [m, f_X(m))]\!] \subseteq U$.*

Proof. Fix $m \in \omega$ and $X \in \mathcal{S}$. Let $\{M_i : i < 2^m\}$ enumerate the subsets of m. For each $i < 2^m$, let $X_i = M_i \cup (X \setminus m)$ and let n_i be the least natural number satisfying $[\![X_i {\restriction} n_i]\!] \subseteq U$. Then X_i is a finite modification of $X \in \mathcal{S}$, so $X_i \in \mathcal{S} \subseteq U$; therefore some such n_i must exist. Let $f_X(m) = \max\{n_i : i < 2^m\}$. If $Y \in [\![X {\restriction} [m, f_X(m))]\!]$, then for some i we have $Y \cap m = M_i$, which gives $Y \in [\![X_i {\restriction} f_X(m)]\!] \subseteq [\![X_i {\restriction} n_i]\!] \subseteq U$. ∎

The following lemma generalizes [9, Proposition 1.3]; see also [3, Proposition 6.24].

LEMMA 3.2. *Let $\mathcal{G}$ be a G_δ semifilter. Suppose $\{A_n : n \in \omega\}$ is a decreasing sequence in $\mathcal{G}$; also, suppose $\mathcal{B} \subseteq \mathcal{G}$, $|\mathcal{B}| < \mathfrak{d}$, and for each $B \in \mathcal{B}$, $\{B\} \cup \{A_n : n \in \omega\}$ is centered in $\mathcal{G}$. Then $\{A_n : n \in \omega\}$ has a bound $A \in \mathcal{G}$ such that, for any $B \in \mathcal{B}$, A and B have a common bound in $\mathcal{G}$.*

Proof. Replacing A_n with $\bigcap_{m \le n} A_m$ if necessary, we may assume that the A_n are decreasing. This only changes A_n finitely, and does not affect the other hypotheses or the conclusion of the lemma. Let $\langle U_n : n < \omega \rangle$ be a sequence of open subsets of 2^ω with $\mathcal{G} = \bigcap_{n \in \omega} U_n$. Replacing U_n with $\bigcap_{m \le n} U_m$ if necessary, we may assume that the U_n are also decreasing.

For each $B \in \mathcal{B}$, let $g_B(n) = f_{A_n \cap B}(n)$, where $f_{A_n \cap B}$ is the function described in Lemma 3.1. This is well-defined since $A_n \cap B \in \mathcal{G}$ (because $\{A_n : n \in \omega\} \cup \{B\}$ is centered in $\mathcal{G}$). As $|\mathcal{B}| < \mathfrak{d}$, there is some $h \in \omega^\omega$ that is not dominated by any g_B.

Let $A = \bigcup_{n \in \omega}(A_n \cap h(n))$. We claim that this A satisfies the conclusions of the lemma. There are two things to check: that A is a bound for $\{A_n : n \in \omega\}$, and that A and B have a common bound in $\mathcal{G}$ for any $B \in \mathcal{B}$.

Because the A_n are decreasing, $A \setminus A_n \subseteq \bigcup_{m < n}(A_m \cap h(m))$, which is a finite set. Thus $A \subseteq^* A_n$ for every n, and A is a bound for $\{A_n : n \in \omega\}$.

It remains to show that for any $B \in \mathcal{B}$, A and B have a common lower bound in $\mathcal{G}$.

Since h is not dominated by g_B, there is an infinite $C \subseteq \omega$ such that $g_B(n) < h(n)$ for all $n \in C$. By induction, find an infinite increasing sequence $\langle n_i : i \in \omega \rangle$ of elements of C such that, for each i, $h(n_i) < n_{i+1}$. Then set $\tilde{A} = \bigcup_{i \in \omega}([n_i, h(n_i)) \cap B \cap A_{n_i})$. By our requirements on the n_i, the intervals $[n_i, h(n_i))$ are disjoint. Clearly, $\tilde{A} \subseteq A \cap B$, and it remains to show $\tilde{A} \in \mathcal{G}$. For any i, we have $f_{A_{n_i} \cap B}(n_i) = g_B(n_i) < h(n_i)$, so that $\tilde{A} \cap [n_i, f_{A_{n_i} \cap B}(n_i)) = A_{n_i} \cap B \cap [n_i, f_{A_{n_i} \cap B}(n_i))$. By the definition of the function $f_{A_{n_i} \cap B}$, this means $\tilde{A} \subseteq U_{n_i}$. This is true for all the n_i, so $\tilde{A}$ is in infinitely many of the U_m. As the U_m are decreasing, $\tilde{A} \in \bigcap_{m \in \omega} U_m = \mathcal{G}$. ∎

The following theorem extends to G_δ semifilters the classical result of Ketonen about $[\omega]^\omega$ (see [9, Theorem 1.2 and Proposition 1.4] or [3, Theorem 9.25]).

THEOREM 3.3. *Let $\mathcal{G}$ be a G_δ semifilter. If $\mathfrak{d} = \mathfrak{c}$, then there is a P-ultrafilter on $\mathcal{G}$. In fact,*

(1) *If $\mathfrak{d} = \mathfrak{c}$, then every filter on $\mathcal{G}$ that is generated by fewer than $\mathfrak{c}$ sets is included in some P-ultrafilter on $\mathcal{G}$.*
(2) *Every ultrafilter on $\mathcal{G}$ that is generated by fewer than $\mathfrak{d}$ sets is a P-ultrafilter.*

Proof. For (1), let $\mathcal{F}_0$ be a basis for a filter on $\mathcal{G}$ with $|\mathcal{F}_0| < \mathfrak{d}$, and let $\{\langle X_n^\alpha : n < \omega \rangle : \alpha \in \mathfrak{c}\}$ be an enumeration of all countable decreasing sequences in $\mathcal{G}$. To avoid trivialities, we assume $\mathcal{F}_0$ contains the Fréchet filter. We construct by recursion an increasing sequence $\langle \mathcal{F}_\alpha : \alpha < \mathfrak{c} \rangle$ of filter bases such that $|\mathcal{F}_\alpha| = \aleph_0 \cdot |\alpha|$.

For limit α, we set $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$. Given that $\mathcal{F}_\alpha$ has already been constructed, we obtain $\mathcal{F}_{\alpha+1}$ as follows. If there is some $n < \omega$ such that $\mathcal{F}_\alpha \cup \{X_n^\alpha\}$ is not centered in $\mathcal{G}$, then we set $\mathcal{F}_{\alpha+1} = \mathcal{F}_\alpha$. If this is not the case, $\mathcal{F}_\alpha \cup \{X_n^\alpha : n \in \omega\}$ is centered in $\mathcal{G}$. Since $|\mathcal{F}_\alpha| < \mathfrak{d}$, we may apply Lemma 3.2 to find a lower bound X_ω^α for $\langle X_n^\alpha : n < \omega \rangle$ such that, for any $B \in \mathcal{F}_\alpha$, we have $X_\omega^\alpha \cap B \in \mathcal{G}$. In particular, $\mathcal{F}_\alpha \cup \{X_\omega^\alpha \cap B : B \in \mathcal{F}_\alpha\}$ is a filter base, and we define this to be $\mathcal{F}_{\alpha+1}$. Clearly $|\mathcal{F}_{\alpha+1}| = \aleph_0 \cdot |\mathcal{F}_\alpha|$, and this completes the recursive construction. Let $\mathcal{F}$ be the filter generated by $\bigcup_{\alpha < \mathfrak{c}} \mathcal{F}_\alpha$.

We must prove that $\mathcal{F}$ is a P-ultrafilter on $\mathcal{G}$. It is obvious that $\bigcup_{\alpha < \mathfrak{c}} \mathcal{F}_\alpha$ is a filter basis in $\mathcal{G}$, so $\mathcal{F}$ is a filter in $\mathcal{G}$. To see that $\mathcal{F}$ is an ultrafilter in $\mathcal{G}$, let $A \in \mathcal{G}$. There is some $\alpha < \mathfrak{c}$ such that $\langle X_n^\alpha : n < \omega \rangle$ is the constant sequence $X_n^\alpha = A$. If $\{A\} \cup \mathcal{F}$ is centered, so is $\{A\} \cup \mathcal{F}_\alpha$, and at step α of our construction we found a set $X_\omega^\alpha \subseteq^* A$ and let $X_\omega^\alpha \in \mathcal{F}_{\alpha+1}$. This implies $A \in \mathcal{F}$ (recall that $\mathcal{F}$ contains the Fréchet filter). Thus, for every $A \in \mathcal{G}$, either $A \in \mathcal{F}$ or $\{A\} \cup \mathcal{F}$ is not centered in $\mathcal{G}$. Hence $\mathcal{F}$ is an ultrafilter on $\mathcal{G}$.

To see that $\mathcal{F}$ is a P-filter, let $\langle A_n : n < \omega \rangle$ be a decreasing sequence of elements of $\mathcal{F}$. For some α, we have $X_n^\alpha = A_n$ for all n. At stage α of our construction, we added some set X_ω^α to $\mathcal{F}$ that is a lower bound for this sequence.

For (2), let $\mathcal{F}$ be an ultrafilter on $\mathcal{G}$, and let $\mathcal{B}$ be a basis for $\mathcal{F}$ such that $|\mathcal{B}| < \mathfrak{d}$. If $\langle A_n : n < \omega \rangle$ is a decreasing sequence in $\mathcal{F}$, then $\{A_n : n \in \omega\} \cup \mathcal{B}$ is centered. By Lemma 3.2, there is some $A_\omega \in \mathcal{G}$ that is a lower bound for $\{A_n : n < \omega\}$ and that, for every $B \in \mathcal{B}$, satisfies $A_\omega \cap B \in \mathcal{G}$. Then $\{A_\omega\} \cup \mathcal{B}$ is centered in $\mathcal{G}$. Since $\mathcal{B}$ is a basis for the ultrafilter $\mathcal{F}$ on $\mathcal{G}$, this implies $A_\omega \in \mathcal{F}$. Thus an arbitrary decreasing sequence of elements of $\mathcal{F}$ has a lower bound in $\mathcal{F}$, that is, $\mathcal{F}$ is a P-filter. ∎

4. Weak P-filters from ZFC. In this section we show that there are weak P-ultrafilters on any co-meager semifilter, in particular on the semifilter of thick sets. This will then be used in Section 5.

To construct an ultrafilter with nice combinatorial properties, one usually uses recursion. We first divide the combinatorial property into a large set of requirements, and then at step α of the construction the filter constructed so far is extended in such a way that the αth requirement is met. There are two main problems.

The first is that the recursive construction might stop before all of the requirements are met. To avoid this problem, one can use an idea going back to Pospíšil [13]: start with an independent system and make sure that at each step a large enough part of this system remains independent modulo the filter constructed so far. This guarantees, in particular, that the filter is

not an ultrafilter. Since there are independent systems of size $\mathfrak{c}$, this typically allows one to take care of $\mathfrak{c}$-many requirements.

The second problem is coming up with the requirements. For example, to get a weak P-point, the natural requirement would be given by a single countable sequence of ultrafilters and it would require that the constructed ultrafilter is not in the closure of this sequence. Unfortunately, there are far too many countable sequences of ultrafilters—we would need to meet $2^{\mathfrak{c}}$-many requirements, but our construction only has $\mathfrak{c}$-many steps. To overcome *this* problem Kunen [10] did something counterintuitive: he replaced the easier problem of constructing a weak P-point by a harder problem of constructing $\mathfrak{c}$-O.K. points. The clever part was that the combinatorial property of being a $\mathfrak{c}$-O.K. point, while decidedly uglier, can actually be divided into $\mathfrak{c}$-many requirements, thus leaving hope for our recursive construction.

We modify Kunen's proof and show that it allows us to construct weak P-ultrafilters on semifilters. First we need Kunen's definition (see [10]) of an O.K.-set: A (closed) subset $X \subseteq \omega^*$ is κ-*O.K.* if for each sequence $\langle U_n : n < \omega \rangle$ of open neighborhoods of X there is a family $\{V_\gamma : \gamma < \kappa\}$ of neighborhoods of X such that for each finite $\Gamma \in [\kappa]^{<\omega}$,

$$\bigcap_{\gamma \in \Gamma} V_\gamma \subseteq U_{|\Gamma|}.$$

A filter $\mathcal{F}$ is κ-O.K. if the corresponding closed set $\{p \in \omega^* : \mathcal{F} \subseteq p\} \subseteq \omega^*$ is κ-O.K.

Note that if $\kappa \leq \lambda$, then every λ-O.K. set is also κ-O.K. The following lemma, also due to Kunen, shows that κ-O.K. sets (with κ uncountable) are weak P-sets:

LEMMA 4.1. *A closed ω_1-O.K. set is a weak P-set.*

Proof. Let F be a closed ω_1-O.K. set and $D = \{p_n : n < \omega\}$ a countable set disjoint from F. Fix a descending sequence $\langle U_n : n < \omega \rangle$ of neighborhoods of F such that p_n is not contained in any U_m for $m > n$. For this sequence, choose open neighborhoods $\{V_\alpha : \alpha < \omega_1\}$ of F witnessing that it is ω_1-O.K. It is easy to see that p_n can only be an element of at most $n + 1$ V_α's, so we can fix $\alpha_n < \omega_1$ such that p_n is not contained in any V_β for $\beta \geq \alpha_n$. Let $\beta = \sup\{\alpha_n : n < \omega\} < \omega_1$. Then V_β is a neighborhood of F disjoint from D. ∎

REMARK 4.2. Later van Mill [11] generalized this lemma to show that a closed ω_1-O.K. subset of $X^* = \beta X \setminus X$ for any locally compact σ-compact X is actually disjoint even from the closure of any *ccc* subset of X^* disjoint from it.

LEMMA 4.3. *A filter $\mathcal{F}$ on ω is 2^ω-O.K. if for each sequence $\langle F_n : n < \omega \rangle$ of elements of $\mathcal{F}$ there are $\{V_\gamma : \gamma < 2^\omega\} \subseteq \mathcal{F}$ such that for each $n < \omega$ and $\gamma_1, \ldots, \gamma_n < 2^\omega$ the set*

$$V_{\gamma_1} \cap \cdots \cap V_{\gamma_n} \setminus F_n$$

is finite.

THEOREM 4.4. *If $\mathcal{S}$ is a co-meager semifilter then there is an ultrafilter on $\mathcal{S}$ which is a 2^ω-O.K. set (and hence a weak P-set).*

The proof uses large independent linked systems. For our purposes we will slightly modify the relevant defition.

Given a filter $\mathcal{F}$ and a semifilter $\mathcal{S}$ we say that a family $\mathcal{A}(C, R) = \{X^\beta_{\alpha,n} : \alpha \in C, n < \omega, \beta \in R\}$ of sets is a $C \times R$-*independent linked matrix modulo* $(\mathcal{F}, \mathcal{S})$ if

(1) for each $\alpha \in C$, $\beta \in R$ the sequence $\langle X^\beta_{\alpha,n} : n < \omega \rangle$ is increasing in $\subseteq$;

(2) for each $F \in \mathcal{F}$, each finite set $R_0 \in [R]^{<\omega}$ of rows, each choice $N : R_0 \to \omega$ of natural numbers and each choice $C_0 : R_0 \to [C]^{<\omega}$ of sets of columns such that $|C_0(\beta)| \leq N(\beta)$ for each $\beta \in R_0$ the intersection

$$F \cap \bigcap_{\beta \in R_0} \bigcap_{\alpha \in C_0(\beta)} X^\beta_{\alpha, N(\beta)}$$

is in $\mathcal{S}$;

(3) for each row $\beta \in R$ and any $n+1$ columns $C_0 \in [C]^{n+1}$ the intersection

$$\bigcap_{\alpha \in C_0} X^\beta_{\alpha,n}$$

is finite.

If $\mathcal{A}(C, R)$ is an independent matrix and $R_0 \subseteq R$ are some rows, we will use $\mathcal{A}(C, R \setminus R_0)$ to denote the matrix constructed from $\mathcal{A}(C, R)$ by deleting the rows (with indices) in R_0.

LEMMA 4.5 (Kunen). *There is a $2^\omega \times 2^\omega$-independent linked matrix modulo $(\mathcal{F}r, [\omega]^\omega)$.*

Kunen employed an elaborate recursive construction using trees. The following simple proof is due to P. Simon.

Proof of Lemma 4.5. We shall construct such a family consisting of subsets of the countable set $S = \{(k, f) : k \in \omega, f \in {}^{\mathcal{P}(k)}\mathcal{P}\mathcal{P}(k)\}$. Given $A, B \subseteq \omega$ and $n < \omega$ let

$$X^B_{A,n} = \{(k, f) \in S : |f(B \cap k)| \leq n \ \& \ A \cap k \in f(B \cap k)\}.$$

It is routine, if perhaps somewhat involved, to check that $\{X^B_{A,n} : n < \omega, A, B \subseteq \omega\}$ is a $2^\omega \times 2^\omega$-independent linked family modulo $(\mathcal{F}r, [\omega]^\omega)$. ∎

COROLLARY 4.6. *If $\mathcal{S}$ co-meager, then there is a $2^\omega \times 2^\omega$-independent linked matrix modulo $(\mathcal{F}r, \mathcal{S})$.*

Proof. By Proposition 2.2, there is a finite-to-one function $f : \omega \to \omega$ such that for each $X \subseteq \omega$ we have $X \in [\omega]^\omega \Leftrightarrow f^{-1}[X] \in \mathcal{S}$. Let $\mathcal{X} = \{X_{\alpha,n}^\beta : \alpha, \beta \in 2^\omega, n < \omega\}$ be the matrix given by Lemma 4.5. Let $Y_{\alpha,n}^\beta = f^{-1}[X_{\alpha,n}^\beta]$. It is easy to check that the Y's form the required independent matrix $\mathcal{Y}$. First notice that if $A \subseteq B$ then $f^{-1}[A] \subseteq f^{-1}[B]$. Condition (1) for $\mathcal{Y}$ now immediately follows from condition (1) for $\mathcal{X}$. Moreover, if A is infinite then $f^{-1}[A]$ is in $\mathcal{S}$ and condition (2) for $\mathcal{Y}$ now immediately follows from condition (2) for $\mathcal{X}$. Finally, if A is finite, then so is $f^{-1}[A]$ and condition (3) for $\mathcal{Y}$ follows from condition (3) for $\mathcal{X}$. ∎

We are now ready to prove the main Theorem 4.4. The proof constructs the maximal filter by a recursion which is kept going by a large independent linked matrix.

There will be two kinds of requirements that we will need to meet. One type takes care of a single countable sequence of neighborhoods that potentially comes into play in the definition of 2^ω-O.K. sets, the other type will take care of a single set in our semifilter to guarantee that the resulting filter is maximal.

The following two lemmas say that the requirements can be met at the cost of sacrificing at most countably many rows from our matrix. In both of these lemmas, $\mathcal{S}$ is some co-meager semifilter and, if $\mathcal{H}$ is a family of sets, $\langle \mathcal{H} \rangle$ is the filter generated by $\mathcal{H}$.

LEMMA 4.7. *Let $\mathcal{A}(C, R)$ be an independent linked matrix modulo $(\mathcal{F}, \mathcal{S})$ and $\langle Y_n : n < \omega \rangle$ a sequence of elements of $\mathcal{F}$. Fix any row $\beta \in R$. Then there are $\{V_\gamma : \gamma < 2^\omega\}$ such that $\mathcal{A}(C, R \setminus \{\beta\})$ is independent linked modulo $(\langle \mathcal{F} \cup \{V_\gamma : \gamma < 2^\omega\} \rangle, \mathcal{S})$, and for any finite set $\Gamma \in [2^\omega]^{<\omega}$ of indices the set*

$$\bigcap_{\gamma \in \Gamma} V_\gamma \setminus Y_{|\Gamma|}$$

is finite.

Proof. Write $Y_n' = \bigcap_{i \leq n} Y_i$ and let

$$V_\gamma = \bigcup_{n < \omega} Y_n' \cap X_{\gamma,n}^\beta.$$

We first check that the system remains independent modulo the larger filter. Conditions (1) and (3) are clearly satisfied. We check condition (2): Fix an element $F \in \mathcal{F}$, $\Gamma \in [2^\omega]^{<\omega}$ a finite set $R_0 \in [R \setminus \{\beta\}]^{<\omega}$ of rows, a sequence $N : R_0 \to \omega$ of natural numbers and a sequence $C_0 : R_0 \to [C]^{<\omega}$ of sets of columns satisfying $|C_0(\delta)| \leq N(\delta)$ for each $\delta \in R_0$. We need to verify that the intersection

$$S = F \cap \bigcap_{\gamma \in \Gamma} V_\gamma \cap \bigcap_{\delta \in R_0} \bigcap_{\alpha \in C_0(\delta)} X_{\alpha,N(\delta)}^\delta$$

is in $\mathcal{S}$. Let $R_0' = R_0 \cup \{\beta\}$, and extend N and C_0 to R_0' as follows: $N(\beta) = |\Gamma|$ and $C_0(\beta) = \Gamma$. Let $H = F \cap \bigcap_{i \leq |\Gamma|} Y_i$. Then

$$H \cap \bigcap_{\delta \in R_0'} \bigcap_{\alpha \in C_0(\delta)} X_{\alpha, N(\delta)}^{\delta}$$

is in $\mathcal{S}$ by assumption and is contained in S. Since $\mathcal{S}$ is upwards closed, we have $S \in \mathcal{S}$.

We now verify the second conclusion of the lemma. Let $\Gamma \in [2^\omega]^{<\omega}$. Since $Y_n' \supseteq Y_{n+1}'$ and $X_{\gamma,n}^\beta \subseteq X_{\gamma,n+1}^\beta$, it follows that

$$V_\gamma \setminus Y_{|\Gamma|} \subseteq \bigcup_{i < |\Gamma|} Y_i' \cap X_{\gamma,i}^\beta \subseteq X_{\gamma,|\Gamma|-1}^\beta.$$

In particular

$$\bigcap_{\gamma \in \Gamma} V_\gamma \setminus Y_{|\Gamma|} \subseteq \bigcap_{\gamma \in \Gamma} X_{\gamma,|\Gamma|-1}^\beta$$

where the second intersection is finite by the assumption on $\mathcal{A}(C, R)$. ∎

LEMMA 4.8. *Let $\mathcal{A}(C, R)$ be an independent linked matrix modulo $(\mathcal{F}, \mathcal{S})$ and $X \in \mathcal{S}$. Then there is a finite set R' of rows and an extension $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{A}(C, R \setminus R')$ is an independent linked matrix modulo $(\mathcal{F}', \mathcal{S})$, and either X or $\omega \setminus X$ is in $\mathcal{F}'$, or there is $F \in \mathcal{F}'$ such that neither $F \cap X$ nor $F \setminus X$ is in $\mathcal{S}$.*

Proof. If $\mathcal{A}(C, R)$ is not independent modulo $(\mathcal{F} \cup \{X\}, \mathcal{S})$ then there are an $F_0 \in \mathcal{F}$, a finite set R_0 of rows, sizes $N_0 : R_0 \to \omega$, and finite sets $C_0 : R_0 \to 2^\omega$ of columns each of size given by N_0 such that

$$X \cap F_0 \cap \bigcap_{\beta \in R_0} \bigcap_{\alpha \in C_0(\beta)} X_{\alpha, N_0(\beta)}^\beta$$

is not in $\mathcal{S}$. If $\mathcal{A}(C, R \setminus R_0)$ is not independent modulo $(\mathcal{F} \cup \{\omega \setminus X\}, \mathcal{S})$ then there are an $F_1 \in \mathcal{F}$, a finite set $R_1 \subseteq R \setminus R_0$ of rows, sizes $N_1 : R_1 \to \omega$, and finite sets $C_1 : R_1 \to 2^\omega$ of columns each of size given by N_1 such that

$$(\omega \setminus X) \cap F_1 \cap \bigcap_{\beta \in R_1} \bigcap_{\alpha \in C_1(\beta)} X_{\alpha, N_1(\beta)}^\beta$$

is not in $\mathcal{S}$. Then let

$$Z = F_0 \cap \bigcap_{\beta \in R_0} \bigcap_{\alpha \in C_0(\beta)} X_{\alpha, N_0(\beta)}^\beta \cap F_1 \cap \bigcap_{\beta \in R_1} \bigcap_{\alpha \in C_1(\beta)} X_{\alpha, N_1(\beta)}^\beta$$

and $R' = R_0 \cup R_1$. By construction $Z \cap X$ and $Z \setminus X$ are not in $\mathcal{S}$. We check that $\mathcal{A}(C, R \setminus R')$ is an independent linked matrix modulo $(\langle \mathcal{F} \cup \{Z\} \rangle, \mathcal{S})$. Conditions (1) and (3) are again clearly satisfied. To verify (2) fix $F_2 \in \mathcal{F}$, a finite set $R_2 \in [R \setminus R']^{<\omega}$ of rows, a sequence $N_2 : R_2 \to \omega$ of natural

numbers and a sequence $C_2 : R_2 \to [C]^{<\omega}$ of finite sets of columns satisfying $|C_2(\beta)| \leq N(\beta)$ for each $\beta \in R_2$. We need to show that the intersection

$$F \cap Z \cap \bigcap_{\beta \in R_2} \bigcap_{\alpha \in C_2(\beta)} X^{\beta}_{\alpha, N_2(\beta)}$$

is in $\mathcal{S}$. Since R_0, R_1 and R_3 are disjoint, we can let $R_3 = R_2 \cup R_1 \cup R_0$, $N_3 = N_0 \cup N_1 \cup N_2$ and $C_3 = C_0 \cup C_1 \cup C_2$. The above intersection is then equal to

$$F_0 \cap F_1 \cap F_2 \cap \bigcap_{\beta \in R_3} \bigcap_{\alpha \in C_3(\beta)} X^{\beta}_{\alpha, N_3(\beta)},$$

which is in $\mathcal{S}$ by the assumption on $\mathcal{A}(C, R)$. ∎

The proof of the theorem is now a routine recursive construction.

Proof of Theorem 4.4. Let $\mathcal{A}(2^\omega, 2^\omega)$ be an independent linked matrix modulo $(\mathcal{F}r, \mathcal{S})$ given by Corollary 4.6.

Enumerate $\mathcal{S}$ as $\{X_\alpha : \alpha \in 2^\omega\}$, and all countable sequences of sets from $\mathcal{S}$ as $\{\langle Y_{\alpha,n} : n < \omega \rangle : \alpha < 2^\omega\}$. Using the two lemmas we shall recursively construct a sequence $\langle \mathcal{F}_\alpha : \alpha < 2^\omega \rangle$ of filters putting the used rows into R_α along the way so that the following conditions are satisfied:

(1) $|R_\alpha| \leq \omega \cdot |\alpha|$ for each $\alpha < 2^\omega$;
(2) $\mathcal{F}_\alpha \subseteq F_\beta$ and $R_\alpha \subseteq R_\beta$ for each $\alpha < \beta < 2^\omega$;
(3) $\mathcal{A}(2^\omega, 2^\omega \setminus R_\alpha)$ is an independent linked matrix modulo $(\mathcal{F}_\alpha, \mathcal{S})$ for each $\alpha < 2^\omega$;
(4) if the sequence $\langle Y_{\alpha,n} : n < \omega \rangle$ is contained in $\mathcal{F}_\alpha$ then there are $\{V_\gamma : \gamma < 2^\omega\} \subseteq \mathcal{F}_{\alpha+1}$ such that for each finite set $\Gamma \in [2^\omega]^{<\omega}$ of indices the intersection $\bigcap_{\gamma \in \Gamma} V_\gamma \setminus Y_{\alpha,|\Gamma|}$ is finite;
(5) either $X_\alpha \in \mathcal{F}_{\alpha+1}$ or $\omega \setminus X_\alpha \in \mathcal{F}_{\alpha+1}$ or there is a set $F \in \mathcal{F}_{\alpha+1}$ such that both $F \cap X_\alpha \notin \mathcal{S}$ and $F \cap (\omega \setminus X_\alpha) \notin \mathcal{S}$.

We start by letting $\mathcal{F}_0 = \mathcal{F}r$ and $R_0 = \emptyset$. At limit stages we take unions, and at successor stages we use the previous lemmas to guarantee conditions (3) and (4). Finally, we let

$$\mathcal{F} = \bigcup_{\alpha < 2^\omega} \mathcal{F}_\alpha.$$

Condition (5) ensures that $\mathcal{F}$ is an ultrafilter on $\mathcal{S}$, while condition (4) guarantees that $\mathcal{F}$ is an 2^ω-O.K. set. ∎

5. Applications to algebra and combinatorics. Prior to this section, we have focused our attention on building certain kinds of ultrafilters in large semifilters. We now turn to applications in algebra and dynamics.

For $p \in \omega^*$, recall that $\sigma(p)$ is the unique ultrafilter generated by $\{A+1 : A \in p\}$. Equivalently, σ is the restriction to ω^* of the unique continuous

extension to $\beta\omega$ of the successor map on ω. This function σ, called the *shift map*, provides the canonical dynamical structure for ω^*.

Related to the shift map on ω^* is the standard additive semigroup structure on ω^* (or, at least, it is standard up to a left-right switch; we follow the conventions of [5]). Addition in ω^* is defined by setting $p + q = p\text{-}\lim_{n\in\omega}\sigma^n(q)$ for every $p, q \in \omega^*$. Here, "$p\text{-}\lim_{n\in\omega}\sigma^n(q)$" has its usual meaning: $r = p\text{-}\lim_{n\in\omega}\sigma^n(q)$ if and only if, for every neighborhood U of r, $\{n \in \omega : \sigma^n(q) \in U\} \in p$.

Recall that, if (X, f) is a dynamical system, then $Y \subseteq X$ is a *minimal subsystem* if Y is closed under f and closed topologically, and no proper nonempty subset of Y has these properties. If $(X, +)$ is a semigroup, then $Y \subseteq X$ is a *minimal left ideal* if $X + Y = Y$, and no proper nonempty subset of Y has this property.

These core notions from dynamics and algebra are, for ω^*, related to each other and to the notion of ultrafilters on semifilters:

LEMMA 5.1. *Let $\mathcal{F}$ be any filter on $[\omega]^\omega$. The following are equivalent:*

(1) $\mathcal{F}$ *is an ultrafilter on Θ.*
(2) $\hat{\mathcal{F}}$ *is a minimal dynamical subsystem of (ω^*, σ).*
(3) $\hat{\mathcal{F}}$ *is a minimal left ideal of $(\omega^*, +)$.*

Proof. The equivalence of (2) and (3) is well-known, and a proof can be found, e.g., in [2]. For the equivalence of these and (1), see [4, Lemma 3.2]. ∎

Recall that an *idempotent ultrafilter* is any $p \in \omega^*$ such that $p + p = p$. The idempotents of ω^* (indeed, any semigroup) admit a natural partial order as follows: if p and q are idempotent, then $p \leq q$ if and only if $p + q = q + p = p$. If $q + p = p$ (but not necessarily $p + q = p$), we write $p \leq_L q$.

It is known (see, e.g., [5, Theorem 1.38]) that an idempotent p is minimal with respect to $\leq$ if and only if p belongs to some minimal left ideal. Such ultrafilters are called *minimal idempotents*. It is also known that if q is any idempotent then there is a minimal idempotent p with $p \leq q$.

For many years it was a stubborn open question whether ω^* contains any $\leq_L$-maximal idempotents, and a good deal of work was done on this question (see [5, Questions 9.25 and 9.26], [6, Questions 5.5(2), (3)], [4, Problems 4.6 and 4.7], and [15]). In [16], Zelenyuk finally answered this question in the affirmative. The following application of Theorem 4.4 provides an alternative proof of Zelenyuk's result, and strengthens the result by showing that some minimal left ideal is a weak P-set.

THEOREM 5.2. *There is a minimal left ideal of ω^* that is also a weak P-set. It follows that:*

(1) *There is an idempotent ultrafilter that is both minimal and $\leq_L$-maximal.*

(2) *There is a minimal left ideal $L \subseteq \omega^*$ such that, for any $p, q \in \omega^*$, $p + q \in L$ if and only if $q \in L$.*

(3) *The minimal left ideals are not homeomorphically embedded in ω^*.*

For (3), recall that $Y, Z \subseteq X$ are *homeomorphically embedded* in X if there is some homeomorphism $h : X \to X$ such that $h(Y) = Z$. It is well-known that the minimal left ideals of ω^* are all homeomorphic, and in fact the homeomorphisms between them arise naturally from the algebraic structure (they are shifts of each other). This result says that the minimal left ideals are nonetheless topologically distinguishable, and the natural homeomorphisms between them cannot be extended to homeomorphisms of ω^*.

Proof of Theorem 5.2. To prove the main assertion of the theorem, first note that, by Lemma 5.1, it suffices to find an ultrafilter on Θ that is also a weak P-filter. This follows directly from Theorem 4.4 and the fact that Θ is G_δ. To see that Θ is G_δ, let

$$U_n = \{A \in 2^\omega : A \text{ contains an interval of length } n\};$$

then note that each U_n is open and $\Theta = \bigcap_{n \in \mathbb{N}} U_n$.

For (2), let L be a weak P-set and a minimal left ideal. If $q \in I$, then $p + q \in L$ because L is a left ideal. Since L is closed under σ and σ^{-1} by Lemma 5.1, we have $\{\sigma^n(q) : n \in \mathbb{N}\} \cap L = \emptyset$. As L is a weak P-set it follows that $\overline{\{\sigma^n(q) : n \in \mathbb{N}\}} \cap L = \emptyset$. If $q \notin L$, then $p + q = p\text{-}\lim_{n \in \omega} \sigma^n(q)$ is an element of $\overline{\{\sigma^n(q) : n \in \mathbb{N}\}}$, so $p + q \notin L$.

For (1), let L be a minimal left ideal that is a weak P-set, and let $q \in L$ be idempotent. Since $q \in L$ and L is a minimal left ideal, q is minimal. Let p be any idempotent other than q. If $p \in L$ then p is $\leq$-minimal, hence $\leq_L$-minimal (see [5, Proposition 1.36]), so $q \not\leq_L p$. If $p \notin L$ then $p + q \notin L$ by (1), in which case $q \not\leq_L p$. Thus q is $\leq_L$-maximal.

For (3), it suffices to note that some minimal left ideal is not a weak P-set. This is well-known and easy to prove: simply take some $p \in \omega^*$ that is not in any minimal left ideal, and note that $\omega^* + p = \overline{\omega + p}$ contains a minimal left ideal. ∎

Further applications of Section 4 and related ideas to the theory of semigroups would take us too far afield here, but these will be explored in a forthcoming sequel to this paper.

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