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On complemented copies of $c_0(\omega_1)$ in $C(K^n)$ spaces

by

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Abstract. Given a compact Hausdorff space K we consider the Banach space of real continuous functions $C(K^n)$ or equivalently the n-fold injective tensor product $\hat{\bigotimes}_{\varepsilon}^{n} C(K)$ or the Banach space of vector valued continuous functions $C(K, C(K, C(K \dots, C(K) \dots)$. We address the question of the existence of complemented copies of $c_0(\omega_1)$ in $\hat{\bigotimes}_{\varepsilon}^{n} C(K)$ under the hypothesis that $C(K)$ contains such a copy. This is related to the results of E. Saab and P. Saab that $X \hat{\otimes}_{\varepsilon} Y$ contains a complemented copy of c_0 if one of the infinite-dimensional Banach spaces X or Y contains a copy of c_0, and of E. M. Galego and J. Hagler that it follows from Martin's Maximum that if $C(K)$ has density ω_1 and contains a copy of $c_0(\omega_1)$, then $C(K \times K)$ contains a complemented copy of $c_0(\omega_1)$.

Our main result is that under the assumption of $\clubsuit$ for every $n \in \mathbb{N}$ there is a compact Hausdorff space K_n of weight ω_1 such that $C(K)$ is Lindelöf in the weak topology, $C(K_n)$ contains a copy of $c_0(\omega_1)$, $C(K_n^n)$ does not contain a complemented copy of $c_0(\omega_1)$, while $C(K_n^{n+1})$ does contain a complemented copy of $c_0(\omega_1)$. This shows that additional set-theoretic assumptions in Galego and Hagler's nonseparable version of Cembrano and Freniche's theorem are necessary, as well as clarifies in the negative direction the matter unsettled in a paper of Dow, Junnila and Pelant whether half-pcc Banach spaces must be weakly pcc.

1. Introduction. Given a compact Hausdorff space K, the geometry of the Banach space $C(K \times K)$, besides being interesting in its own right, is important because $C(K \times K)$ spaces form paradigmatic examples of Banach spaces of vector valued continuous functions $C(K, C(K))$ and of the injective tensor products $C(K) \hat{\otimes}_{\varepsilon} C(K)$, and hence they are relevant to the investigations of the properties of tensor products $X \hat{\otimes}_{\varepsilon} Y$ in terms of the properties of X and Y. It is well known, by a surprising and celebrated result of P. Cembranos [3] and F. Freniche [7], that if $C(K)$ contains a copy

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of c_0 (i.e., $C(K)$ is infinite-dimensional), then $C(K\times K)$ always contains a complemented copy of c_0. This result has been generalized by E. Saab and P. Saab [17] to any tensor product $X\hat{\otimes}_\varepsilon Y$ of infinite-dimensional Banach spaces X,Y one of which contains c_0 or even to spaces of compact operators [16].

A consistent nonseparable version of this result has recently been obtained by E. M. Galego and J. Hagler [8, Corollary 4.7] who in particular proved that it is relatively consistent with ZFC that if $C(K)$ has density ω_1 and $C(K)$ has a copy of $c_0(\omega_1)$, then $C(K\times K)$ has a complemented copy of $c_0(\omega_1)$. Their proof relies on Todorcevic's analysis of nonseparable Banach spaces [20, Corollary 6] under the assumption of an additional set-theoretic axiom known as Martin's Maximum [6], which in particular implies that $2^\omega>\omega_1$ [6, 20]. However A. Dow, H. Junnila and J. Pelant [4, Example 2.16] constructed in ZFC a Banach space (of the form $C(K)$) of density 2^ω which contains a copy of $c_0(\omega_1)$ but its injective tensor square has no complemented copy of $c_0(\omega_1)$ (e.g., by [4, Corollaries 4.2 and 2.16]). For more on complemented copies of $c_0(\omega_1)$ see [1] and [10].

In order to discuss our results, let us recall some terminology. A Banach space X is called *weakly pcc* if any point-finite family of open sets in the weak topology on X is countable, and is *half-pcc* if any point-finite family of half-spaces (i.e., sets of the form $\{x\in X : x^*(x)>a\}$ for some $a\in\mathbb{R}$ and $x^*\in X^*$) is countable. This kind of chain conditions were first considered by Rosenthal [15] and are fundamental in the work of Dow, Junnila and Pelant [4]. If X is a Banach space, then a sequence $(x^*_\xi)_{\xi<\omega_1}$ of elements of the unit sphere of X^* is called an *ω_1-Josefson–Nissenzweig sequence* if $(x^*_\xi(x))_{\xi<\omega_1}$ belongs to $c_0(\omega_1)$ for any $x\in X$. This notion plays a crucial role in the result of Galego and Hagler and of Todorcevic mentioned above [8, 20].

THEOREM 1.1 ([4, 20]). *Let X be a Banach space. The following conditions are equivalent:*

(1) *There is a bounded linear operator $T: X\to c_0(\omega_1)$ with nonseparable range.*
(2) *There is an ω_1-Josefson–Nissenzweig sequence in X.*
(3) *X is not half-pcc.*

Proof. (1)⇒(2). If $\phi_\alpha = T^*(\delta_\alpha)/\|T^*(\delta_\alpha)\|$, where $\delta_\alpha(f)=f(\alpha)$ for each $\alpha<\omega_1$ and $f\in c_0(\omega_1)$, then $\phi_\alpha(x)=T(x)(\alpha)/\|T^*(\delta_\alpha)\|$. But for αs from an uncountable set $A\subseteq\omega_1$ the numbers $\|T^*(\delta_\alpha)\|$ are uniformly separated from zero. Hence $(\phi_\alpha)_{\alpha\in A}$ is an ω_1-Josefson–Nissenzweig sequence in X.

(2)⇒(1). Given an ω_1-Josefson–Nissenzweig sequence $(\phi_\alpha)_{\alpha<\omega_1}$ in X define an operator $T: X\to c_0(\omega_1)$ by $T(x)=(\phi_\alpha(x))_{\alpha<\omega_1}$. By the Hahn–Banach theorem it has a nonseparable range.

The equivalence of (1) and (3) is proved in [4, Theorem 1.6]. ∎

Note that the conditions above are in general much weaker than containing a complemented copy of $c_0(\omega_1)$, but following the ideas of Cembranos and Freniche, Galego and Hagler showed that if $C(K)$ contains a copy of $c_0(\omega_1)$, then $C(K \times K)$ contains a complemented copy of $c_0(\omega_1)$ under one of the above conditions (1)–(3) on the space $X = C(K)$. Actually, Todorcevic proved that under Martin's Maximum any nonseparable Banach space of density ω_1 satisfies the above conditions, which gives the previously mentioned result of Galego and Hagler under Martin's Maximum. The separable version of Todorcevic's result is the classical theorem of Josefson and Nissenzweig [9, 12] which implies that on any infinite-dimensional Banach space there is a linear operator with range dense in c_0.

One of the goals of the research project leading to this paper was to decide whether Galego and Hagler's nonseparable version of Cembranos and Freniche's theorem, and consequently the Saabs' theorem, is indeed sensitive to its additional set-theoretic assumption, i.e., whether one can or cannot obtain the same result without that assumption. We prove that one cannot: it is consistent that there exists a compact Hausdorff space K such that the density of $C(K)$ is ω_1 and $C(K)$ contains a copy of $c_0(\omega_1)$, but $C(K \times K)$ does not contain a complemented copy of $c_0(\omega_1)$. In fact we have encountered two different kinds of examples of spaces satisfying the above statement. We found one of them existing already in the literature ([4, Example 2.17] discussed in Section 4 and denoted DJP_1 there) after we had constructed our original space. The analysis of the differences between these examples led us to a more delicate result:

THEOREM 1.2. *It is consistent that there are compact Hausdorff spaces K_n for all $1 \le n \le \omega$ such that each $C(K_n)$ contains a copy of $c_0(\omega_1)$, while $C(K_n^m)$ with $n \le \omega$ and $m < \omega$ contains a complemented copy of $c_0(\omega_1)$ if and only if $n < m$.*

Proof. K_n is constructed in Section 3; its properties are stated in Theorem 3.1. Since K_n is $(n+1)$-diverse (see Definition 1.5), by Proposition 2.3 the space $C(K_n^n)$ is half-pcc and so by Theorem 1.1, $C(K_n^n)$ does not contain a complemented copy of $c_0(\omega_1)$. On the other hand, by Theorem 3.1 there is an n-to-1 continuous map from $K_n \setminus \{\infty\}$ onto $[0, \omega_1)$ with the order topology, so $C(K_n^{n+1})$ contains a complemented copy of $c_0(\omega_1)$ by Proposition 2.1.

K_ω is the example from Proposition 3.5; if we are not interested in additional properties, which we state in Theorem 1.8, it is Example 2.17 of [4] which we call DJP_1 and analyze in Section 4. Since these are nonseparable scattered compact spaces, they contain uncountable subspaces of isolated points, and so $C(K_\omega)$ contains $c_0(\omega_1)$. The properties of K_ω follow from Proposition 3.5 and Theorem 1.6. The properties of DJP_1 follow from Corollary 4.3 and Theorem 1.1. ∎

For additional properties of the K_ns see Theorem 1.8. In fact our constructions are generalizations of the space from [10] which can serve as K_1 above. As we note in Corollary 4.2 that for K compact scattered, if $C(K)$ is weakly pcc, then $C(K^n)$ is weakly pcc for every n, our examples K_n for $n < \omega$ from Theorem 1.2 give the following:

THEOREM 1.3. *It is consistent that there are half-pcc spaces $C(K)$ of density ω_1 which are not weakly pcc.*

This seems to have been unknown until now (see [4, p. 1330]), but we still do not know if such examples can be constructed without any additional assumptions (see Proposition 4.5). Among the three equivalent conditions of Theorem 1.1 the topological notion of being half-pcc is the simplest. Moreover in the case of scattered compact spaces it admits a simple topological criterion extracted in [4] from a paper of Arhangel'skii and Tkačuk [2]. Our refinements of these notions are the following:

DEFINITION 1.4. Let K be a compact space, let $m \in \mathbb{N}$ and let $F_1, \ldots, F_k$ a partition of $\{1, \ldots, m\}$. A point $(x_1, \ldots, x_m) \in K^m$ is said to be $(F_1, \ldots, F_k)$-*diverse* if $\{x_j : j \in F_i\} \cap \{x_j : j \notin F_i\} = \emptyset$ for all $1 \le i \le k$.

DEFINITION 1.5. Let K be a Hausdorff compact space and $n \in \mathbb{N}$. We say that K is n-*diverse* if for any given $m \in \mathbb{N}$ and for any partition $F_1, \ldots, F_k$ of $\{1, \ldots, m\}$ with $k \le n$, any sequence $\{(x_1^\alpha, \ldots, x_m^\alpha)\}_{\alpha<\omega_1} \subseteq K^m$ of $(F_1, \ldots, F_k)$-diverse points admits a cluster point which is $(F_1, \ldots, F_k)$-diverse.

THEOREM 1.6. *Suppose that K is a scattered compact space. Then $C(K)$ is weakly pcc if and only if K is n-diverse for each $n \in \mathbb{N}$.*

Proof. By [4, Proposition 2.2 and Lemma 2.13] (cf. [2, Proposition 2.7]) we need to prove that $K^n \setminus \Delta_n$ is ω_1-compact for all $n \in \mathbb{N}$ if and only if K is n-diverse for all $n \in \mathbb{N}$ (for terminology see Section 4). For the forward implication, given a sequence $v^\xi = (x_1^\xi, \ldots, x_m^\xi)$ of $(F_1, \ldots F_k)$-diverse points in K^m one can assume that there is a partition of $\{1, \ldots, m\}$ into sets $(A_i)_{i\le l}$ for some $l \le m$ such that the coordinates of the points v^ξ in the same part of the partition are all equal. Form $w^\xi \in K^l \setminus \Delta_l$ from the coordinates of v^ξ in distinct A_ks. Use the ω_1-compactness to obtain an accumulation point of w^ξs in $K^l \setminus \Delta_l$ and apply it to find an $(F_1, \ldots, F_k)$-diverse accumulation point of v^ξs. The backward implication is clear. ∎

THEOREM 1.7. *Let K be a compact, totally disconnected space and let $n \in \mathbb{N}$. Each of the following conditions implies the next.*

(1) *K is $(n+1)$-diverse.*
(2) *$C(K^n)$ is half-pcc.*

(3) *$C(K^n)$ contains no complemented isomorphic copy of $c_0(\omega_1)$.*
(4) *There is no point $\infty \in K$ such that $K \setminus \{\infty\}$ can be mapped onto $[0, \omega_1)$ by an $(n-1)$-to-1 continuous map.*

Proof. (1)⇒(2) by Proposition 2.3; (2)⇒(3) by Theorem 1.1; and Proposition 2.1 yields (3)⇒(4). ∎

Our examples K_n for $n \leq \omega$, unlike [4, Example 2.17], have all these properties for a given $n \in \mathbb{N}$ and none of these properties for bigger numbers. Another feature of our examples is the Lindelöf property of the weak topology:

THEOREM 1.8. *It is consistent that there are totally disconnected compact Hausdorff spaces K_n as in Theorem 1.2 such that $C(K_n)$ is Lindelöf in the weak topology and K_n is $(n+1)$-diverse and there is a point $\infty \in K_n$ such that $K_n \setminus \{\infty\}$ can be mapped onto $[0, \omega_1)$ by an n-to-1 continuous map.*

Proof. Applying Theorems 3.1 and 1.7 we are left with proving the Lindelöf property of $C(K_n)$ with the weak topology. First note that the Lindelöf property of the weak topology in $C(K)$ for K scattered is equivalent to this property for the pointwise convergence topology. Now the Lindelöf property follows from the fact that $K_n^{(\omega_1)} = \emptyset$ by a theorem of G. Sokolov [19, Theorem 2.3] which says that for such scattered spaces the Lindelöf property of $C_p(K)$ is equivalent to $\aleph_0$-monolithicity of K, that is, the property that the closure of every countable set has a countable network. In our case the closures of countable sets are included in sets of the form $[0, \beta] \cup \{\omega_1\}$ for some countable ordinal β (see Theorem 3.1) and so are metrizable because they are countable and scattered. ∎

The Lindelöf property of the weak topology is relevant here because it is proved in [4, Proposition 1.16] that a nonseparable weakly Lindelöf determined $C(K)$ space cannot be half-pcc. Also, as noted in [4, beginning of Section 1], the Lindelöf property of the weak topology is equivalent to being paracompact. Thus our spaces $C(K_n)$ are paracompact in the weak topology in contrast to the space $C(DJP_1)$ from [4] (see Corollary 4.3 below) which is not even σ-metacompact in the weak topology since it is weakly pcc.

We obtain our consistent examples assuming the combinatorial principle ♣ of Ostaszewski (see [13]; it is explained at the beginning of Section 2 below) and we note that this principle is also sufficient to obtain [4, Example 2.17] originally deduced from $\diamondsuit$ of R. Jensen (see [11]). The advantage of ♣ over $\diamondsuit$ is that the former is compatible with both CH and its negation, while $\diamondsuit$ implies CH. This kind of example cannot be obtained without additional set-theoretic assumptions because under Martin's Maximum, Banach spaces of density ω_1 map continuously and linearly into $c_0(\omega_1)$ with nonseparable ranges [20] and under the P-ideal dichotomy any weakly Lindelöf $C(K)$ space with $K^{(\omega_1)} = \emptyset$ containing an isomorphic copy of $c_0(\omega_1)$ contains a complemented copy of $c_0(\omega_1)$ [10, Theorem 3.2]. In this context, because DJP_2 (see Section 4) is a subspace of a separable scattered space, one should also recall a result of R. Pol [14, Theorem 2] saying that if K is separable, nonmetrizable, scattered of countable height, then $C(K)$ is not Lindelöf in the weak topology (see also [5]). But we do not know if dropping the requirement of $C(K)$ being Lindelöf in the weak topology one can construct in ZFC a K of weight continuum such that $C(K^n)$ contains complemented copies of $c_0(\omega_1)$ for some n and not for others. Using the notion of n-diverse spaces in the context of the ZFC Example 2.16 of [4], called DJP_2 here, doees not seem to work as in the case of DJP_1, which is shown in Propositions 4.4 and 4.5.

We will denote by $\mathcal{L}(\omega_1)$ the set of all countable ordinals which are limit ordinals. The other notation is standard.

2. n-to-1 maps onto $[0, \omega_1)$, n-diverse spaces and the pcc

PROPOSITION 2.1. *Let K be a compact totally disconnected space and $\infty \in K$. If there exists a continuous surjective map $\phi : K \setminus \{\infty\} \to [0, \omega_1)$ such that $|\phi^{-1}[\{\alpha\}]| \le n$ for all $\alpha < \omega_1$ and some $n \in \mathbb{N}$, where $[0, \omega_1)$ is endowed with the order topology, then $C(K^{n+1})$ contains a complemented copy of $c_0(\omega_1)$. In particular $C(K^{n+1})$ is not half-pcc.*

Proof. Set $L = K\setminus\{\infty\}$. For each $\gamma \in \omega_1$ pick any $x_\gamma \in \phi^{-1}[\{\gamma\}]$. We will consider the sets of points $x_{\gamma+1}, \ldots, x_{\gamma+(n+2)}$ for $\gamma \in \mathcal{L}(\omega_1)$. All these points are isolated in L and hence in K because successors are isolated in $[0, \omega_1)$ and ϕ is continuous. Note that given a pairwise disjoint clopen partition of K into sets $U_1, \ldots, U_k$ for some $k \in \mathbb{N}$, the set of all $\gamma \in \mathcal{L}(\omega_1)$ such that the sets $U_1, \ldots, U_k$ separate all the points $x_{\gamma+1}, \ldots, x_{\gamma+(n+2)}$ is at most finite. Indeed, at most one of the clopen sets, say U_j for some $1 \le j \le k$, contains ∞. So, any accumulation point of an infinite sequence $(x_{\gamma_m+i})_{m\in\mathbb{N}}$ from outside U_j with $1 \le i \le n+2$ and with an increasing sequence $(\gamma_m)_{m\in\mathbb{N}}$ must be in $\phi^{-1}[\{\sup_{m\in\mathbb{N}} \gamma_m\}]$. This set has at most n elements, while there will be $n+1$ such distinct accumulation points, if $U_1, \ldots, U_k$ separate all the points $x_{\gamma_m+1}, \ldots, x_{\gamma_m+(n+2)}$ for infinitely many $m \in \mathbb{N}$—a contradiction.

Now we define measures on K^{n+1} which will serve to define a projection from $C(K^{n+1})$ onto a copy of $c_0(\omega_1)$. Let $S(k)$ denote the set of all permutations of $\{1, \ldots, k\}$, and for $\sigma \in S(k)$ let $\mathrm{sgn}(\sigma)$ denote the sign of the permutation σ. Let λ_γ be a Radon measure on K^{n+1} defined by

$$\lambda_\gamma = \sum_{\sigma\in S(n+2)} \mathrm{sgn}(\sigma) \cdot \delta_{\{(x_{\gamma+\sigma(1)},\ldots,x_{\gamma+\sigma(n+1)})\}}.$$

Note that σ above is a permutation of all $n + 2$ points but the coordinates of points of K^{n+1} use only the first $n + 1$ numbers.

Claim. *For any clopen $V_1, \dots, V_{n+1}$ in K the set $\{\gamma \in \mathcal{L}(\omega_1) : \lambda_\gamma(V_1 \times \dots \times V_{n+1}) \neq 0\}$ is finite.*

To see this, first note that if $U_1, \dots, U_k$ are all the Boolean components of the finite field of sets generated by $V_1, \dots, V_{n+1}$, then only for finitely many γs the sets $U_1, \dots, U_k$, and so $V_1, \dots, V_{n+1}$, separate all the points $x_{\gamma+1}, \dots, x_{\gamma+(n+2)}$.

For those γ such that none of $V_1, \dots, V_{n+1}$ separate say $x_{\gamma+i}$ from $x_{\gamma+j}$ and for $\sigma \in S(n+2)$ such that $\sigma(i), \sigma(j) < n+2$ we have

$$(x_{\gamma+\sigma(1)}, \dots, x_{\gamma+\sigma(i)}, \dots, x_{\gamma+\sigma(j)}, \dots, x_{\gamma+\sigma(n+1)}) \in V_1 \times \dots \times V_{n+1}$$

if and only if

$$(x_{\gamma+\sigma'(1)}, \dots, x_{\gamma+\sigma'(i)}, \dots, x_{\gamma+\sigma'(j)}, \dots, x_{\gamma+\sigma'(n+1)}) \in V_1 \times \dots \times V_{n+1}$$

where σ' is obtained by composing σ with the transposition of i and j. In this case $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\sigma')$ and so the contributions of $\delta_{\{(x_{\gamma+\sigma(1)}, \dots, x_{\gamma+\sigma(n+1)})\}}$ and $\delta_{\{(x_{\gamma+\sigma'(1)}, \dots, x_{\gamma+\sigma'(n+1)})\}}$ to λ_γ on $V_1 \times \dots \times V_{n+1}$ cancel each other.

For those γ such that none of $V_1, \dots, V_{n+1}$ separate say $x_{\gamma+i}$ from $x_{\gamma+j}$ and for $\sigma \in S(n+2)$ such that $\sigma(j) = n+2$ we have

$$(x_{\gamma+\sigma(1)}, \dots, x_{\gamma+\sigma(i)}, \dots, x_{\gamma+\sigma(n+1)}) \in V_1 \times \dots \times V_{n+1}$$

if and only if

$$(x_{\gamma+\sigma'(1)}, \dots, x_{\gamma+\sigma'(j)}, \dots, x_{\gamma+\sigma'(n+1)}) \in V_1 \times \dots \times V_{n+1}$$

where σ' is obtained by composing σ with the transposition of i and j, i.e., $\sigma'(i) = n+2$. In this case $\mathrm{sgn}(\sigma) = -\mathrm{sgn}(\sigma')$ and so the contributions of $\delta_{\{(x_{\gamma+\sigma(1)}, \dots, x_{\gamma+\sigma(n+1)})\}}$ and $\delta_{\{(x_{\gamma+\sigma'(1)}, \dots, x_{\gamma+\sigma'(n+1)})\}}$ on $V_1 \times \dots \times V_{n+1}$ cancel each other as well.

Having fixed distinct $1 \le i, j \le n+2$ such that the sets $V_1, \dots, V_{n+1}$ do not separate the points $x_{\gamma+i}$ and $x_{\gamma+j}$ for a fixed $\gamma < \omega_1$, it is clear that the set of all permutations can be partitioned into two sets such that the σs above belong to one of them and the σ's to the other, just by reversing the roles of i and j in the permutation. Hence the contribution of any point in the definition of λ_γ is canceled by the contribution of the corresponding point from the other group and so $\lambda_\gamma(V_1 \times \dots \times V_{n+1}) = 0$ when one pair of points in $\{x_{\gamma+1}, \dots, x_{\gamma+(n+2)}\}$ is not separated by any of the sets $V_1, \dots, V_{n+1}$. But as previously noted, this holds for all but finitely many γs. This completes the proof of the claim.

Now note that for each $f \in C(K^{n+1})$ we have $(\lambda_\gamma(f))_{\gamma \in \mathcal{L}(\omega_1)} \in c_0(\mathcal{L}(\omega_1))$. Indeed, consider the operator $T : C(K^{n+1}) \to \ell_\infty(\mathcal{L}(\omega_1))$ given by $T(f) = (\lambda_\gamma(f))_{\gamma \in \mathcal{L}(\omega_1)}$. It is a well defined bounded linear operator and the claim means that $T(f) \in c_0(\mathcal{L}(\omega_1))$ for f being the characteristic function of any clopen subset of K^{n+1}. So it is true for any $f \in C(K^{n+1})$ by the Weierstrass–Stone theorem and the fact that $c_0(\mathcal{L}(\omega_1))$ is a closed subspace of $\ell_\infty(\mathcal{L}(\omega_1))$.

Now we are in a position to construct the required projection onto a copy of $c_0(\omega_1)$. Since all points $x_{\gamma+i}$ for $\gamma \in \mathcal{L}(\omega_1)$ and $i \in \mathbb{N}$ are isolated, we may consider

$$Y = \overline{\operatorname{span}\{\chi_{\{(x_{\gamma+1},\dots,x_{\gamma+n+1})\}} : \gamma \in \mathcal{L}(\omega_1)\}}$$

where the closure is taken in $C(K^{n+1})$. It is clear that Y is isometric to $c_0(\omega_1)$. We define

$$P(f) = \sum_{\gamma \in \mathcal{L}(\omega_1)} \lambda_\gamma(f) \cdot \chi_{\{(x_{\gamma+1},\dots,x_{\gamma+n+1})\}}, \qquad f \in C(K^{n+1}).$$

For any $f \in C(K^{n+1})$ we see that $(\lambda_\gamma(f))_{\gamma \in \mathcal{L}(\omega_1)} \in c_0(\mathcal{L}(\omega_1))$ and it follows that P defines a bounded operator from $C(K^{n+1})$ to $C(K^{n+1})$. Moreover $P[C(K^{n+1})] \subseteq Y$.

To see that P is a projection, we only need to check that $P\restriction_Y = \mathrm{Id}_Y$, and for that it is enough to prove that $P(\chi_{\{(x_{\gamma+1},\dots,x_{\gamma+n+1})\}}) = \chi_{\{(x_{\gamma+1},\dots,x_{\gamma+n+1})\}}$ for each $\gamma \in \mathcal{L}(\omega_1)$. This is clear since for all $\gamma, \theta \in \mathcal{L}(\omega_1)$

$$\lambda_\theta(\chi_{\{(x_{\gamma+1},\dots,x_{\gamma+n+1})\}}) = \lambda_\theta(\{(x_{\gamma+1},\dots,x_{\gamma+n+1})\}) = \begin{cases} 0 & \text{if } \theta \neq \gamma, \\ 1 & \text{if } \theta = \gamma. \end{cases} \ \blacksquare$$

Corollary 2.2. *Suppose K is the space obtained under the assumption* ♣ *in* [10, *Section* 4]. *Then $C(K)$ is half-pcc and contains an isomorphic copy of $c_0(\omega_1)$. Moreover $C(K \times K)$ contains a complemented copy of $c_0(\omega_1)$, in particular $C(K)$ is not weakly pcc.*

Proposition 2.3. *If a compact scattered Hausdorff K is $(n+1)$-diverse for some $n \in \mathbb{N}$, then $C(K^n)$ is half-pcc.*

Proof. Consider the half-spaces $H_\alpha = \{f \in C(K^n) : \int f\,d\mu_\alpha > a_\alpha\}$ for $a_\alpha \in \mathbb{R}$ and some Radon measures μ_α on K^n and any $\alpha < \omega_1$. We will prove that this collection cannot be point-finite. It is enough to prove that its refinement is not point-finite, so we may assume that the a_αs are all equal to some $a \in \mathbb{R}$ larger than uncountably many a_αs. By going to an uncountable subset we may assume that there is an $\varepsilon > 0$ and finite sets $G_\alpha \subseteq K$ such that there is $y^\alpha = (y_1^\alpha, \dots, y_n^\alpha) \in G_\alpha^n$ with

$$|\mu_\alpha(\{y^\alpha\})| > 2\varepsilon \quad \text{and} \quad |\mu_\alpha(K^n \setminus G_\alpha^n)| < \varepsilon.$$

Here we have used the fact that all Radon measures on scattered compacta are atomic. Going further to a smaller uncountable subsequence we may assume that there is a partition $I_1 \cup \dots \cup I_k = \{1, \dots, n\}$ for some $k \leq n$ such that $y_i^\alpha = y_j^\alpha$ for all $\alpha < \omega_1$ if and only if $i, j \leq n$ are in the same set of the partition.

We may assume that all the sets $G_\alpha = \{x_1^\alpha, \ldots, x_m^\alpha\}$ have the same cardinality $m \in \mathbb{N}$ and that the first k elements of each G_α (in the above enumeration) yield all different coordinates of y^α, so that $x_i^\alpha = y_j^\alpha$ if and only if $j \in I_i$ for $i \leq k$ and $j \leq n$. Form points $(x_1^\alpha, \ldots, x_m^\alpha) \in K^m$. They are $(\{1\}, \ldots, \{k\}, \{k+1, \ldots, m\})$-diverse so let $(x_1, \ldots, x_m)$ be a $(\{1\}, \ldots, \{k\}, \{k+1, \ldots, m\})$-diverse cluster point. It follows that there are pairwise disjoint clopen neighborhoods $U_1, \ldots, U_k$ of $x_1, \ldots, x_k$ respectively such that

$$\{x_{k+1}, \ldots, x_m\} \cap (U_1 \cup \cdots \cup U_k) = \emptyset.$$

Now let $V_j = U_i$ if and only if $j \in I_i$ for $j \leq n$ and $i \leq k$. Consider $V_1 \times \cdots \times V_n$. We will prove now that for infinitely many αs we have $G_\alpha^n \cap V_1 \times \cdots \times V_n = \{y^\alpha\}$. Indeed $W = U_1 \times \cdots \times U_k \times [K \setminus (U_1 \cup \cdots \cup U_k)]^{m-k}$ is a clopen neighborhood of the cluster point $(x_1, \ldots, x_m)$, so the points $(x_1^\alpha, \ldots, x_m^\alpha)$ are in it for infinitely many αs. For the same αs we must have $y^\alpha \in V_1 \times \cdots \times V_n$. Now if $y = (y_1, \ldots, y_n) \in G_\alpha^n$ is in $V_1 \times \cdots \times V_n$ note that none of the coordinates of y may be among $\{x_{k+1}^\alpha, \ldots, x_m^\alpha\}$ because these must be in $K \setminus (U_1 \cup \cdots \cup U_k) = K \setminus (V_1 \cup \cdots \cup V_n)$. Only one of the remaining possible values $\{x_1^\alpha, \ldots, x_k^\alpha\}$ of the coordinates of y may belong to V_j for $j \leq n$, namely x_i^α such that $j \in I_i$, because V_js are pairwise disjoint, so we conclude that $y = y^\alpha$. Hence for infinitely many $\alpha < \omega_1$ we have

$$\Big|\int \chi_{V_1 \times \cdots \times V_n}\, d\mu_\alpha\Big| \geq |\mu_\alpha(\{y^\alpha\})| - \Big|\int_{K^n \setminus G_\alpha^n} \chi_{V_1 \times \cdots \times V_n}\, d\mu_\alpha\Big| > \varepsilon.$$

Now considering $f = \pm(a/\varepsilon)\chi_{V_1 \times \cdots \times V_n}$ we obtain $\int f\, d\mu_\alpha > a$ for infinitely many αs, which shows that the H_αs do not form a point-finite family and completes the proof of the proposition. ∎

3. A compact space from ♣.

Our main result of this section is as follows:

Theorem 3.1 (♣). *For each $n \in \mathbb{N}$ there is an $(n+1)$-diverse nonseparable Hausdorff compact scattered topology τ on $[0, \omega_1]$ of height $\omega + 1$ and weight ω_1 where the sets $[0, \alpha + (n-1)] \cup \{\omega_1\}$ are closed for all $\alpha < \omega_1$. Moreover, there is a finite-to-one surjective function $\phi : [0, \omega_1) \to [0, \omega_1)$ which is τ-to-order-topology continuous such that $|\phi^{-1}[\{\alpha\}]| \leq n$ for all $\alpha < \omega_1$.*

Ostaszewski's principle ♣ ([13]) is stated as follows:

Definition 3.2. ♣ is the following sentence: There is a sequence $(S_\alpha)_{\alpha \in \mathcal{L}(\omega_1)}$ such that for each $\alpha \in \mathcal{L}(\omega_1)$:

(1) $S_\alpha \subseteq \alpha$;
(2) S_α converges to α in the order topology;
(3) for every uncountable $X \subseteq \omega_1$ there is $\alpha \in \mathcal{L}(\omega_1)$ such that $S_\alpha \subseteq X$.

In order to establish Theorem 3.1, we need to enrich our terminology. Following the notation of [10], for an ordinal $\alpha \leq \omega_1$ we define $F_0(\alpha) = \alpha = \{\beta : \beta < \alpha\}$ and for $n > 0$ we let $F_{n+1}(\alpha)$ be the set of all finite sequences of elements of $F_n(\alpha)$. Define $F(\alpha) = \bigcup_{n\in\mathbb{N}} F_n(\alpha)$. For $A \in F(\alpha)$ such that $A \in F_n(\alpha)$, by induction on $n \in \mathbb{N}$ we define the *support* of A, denoted $\mathrm{supp}(A)$, as the union of all sets $\mathrm{supp}(B)$ where B is a term of the sequence A, with $\mathrm{supp}(A) = \{A\}$ for $A \in F_0(\alpha)$. If $A, B \in F(\alpha), \alpha < \omega_1$, then we write $A < B$ if $\beta < \gamma$ for all $\beta \in \mathrm{supp}(A)$ and $\gamma \in \mathrm{supp}(B)$.

For a collection $\mathcal{W} \subseteq F(\omega_1)$, we say that it is *consecutive* if $A < B$ or $B < A$ whenever A and B are two distinct elements of $\mathcal{W}$.

DEFINITION 3.3. A collection $\mathcal{W}$ of elements of $F(\omega_1)$ *converges to* $\gamma \in \mathcal{L}(\omega_1)$ if $\mathcal{W}$ is consecutive and for every $\beta < \gamma$ the set $\{A \in \mathcal{W} : \mathrm{supp}(A) \not\subseteq [\beta + 1, \gamma)\}$ is finite.

In fact, we will deal only with elements of $F_0(\alpha)$ or $F_1(\alpha)$ for $\alpha \leq \omega_1$. For our purpose, we need the following version, which is in fact equivalent to Ostaszewski's ♣ [10, Lemma 4.4]:

DEFINITION 3.4. ♣′ is the following sentence: There is a sequence $(\mathcal{S}'_\alpha)_{\alpha\in\mathcal{L}(\omega_1)}$ such that for each $\alpha \in \mathcal{L}(\omega_1)$:

(i) $\mathcal{S}'_\alpha \subseteq F(\alpha)$;
(ii) $\mathcal{S}'_\alpha$ converges to α in the sense of Definition 3.3;
(iii) for every uncountable consecutive $\mathcal{W} \subseteq F(\omega_1)$ there is $\alpha \in \mathcal{L}(\omega_1)$ such that $\mathcal{S}'_\alpha \subseteq \mathcal{W}$.

Proof of Theorem 3.1. Given $n \in \mathbb{N}$, we fix a sequence $(\mathcal{S}'_\gamma)_{\gamma\in\mathcal{L}(\omega_1)}$ satisfying the conditions of ♣′. This sequence will hereafter be referred to as the *♣′-sequence*. For each $\gamma \in \mathcal{L}(\omega_1)$, since $\mathcal{S}'_\gamma$ converges to γ in the sense of Definition 3.3, without losing generality we may assume that $\mathcal{S}'_\gamma = \{s_r(\gamma) : r \in \mathbb{N}\}$ satisfies $\max\{\mathrm{supp}(s_r(\gamma))\} + (n-1) < \min\{\mathrm{supp}(s_t(\gamma))\}$ for any integers $r < t$.

The set of points of our space will be the set $[0, \omega_1]$ where $[0, \omega_1)$ is a locally compact space which is one-point compactified by adding ω_1. To define the locally compact topology on $[0, \omega_1)$, for each $\gamma < \omega_1$ we will construct a countable neighborhood basis $\mathcal{B}_\gamma$ at γ. We use transfinite induction. We start by letting τ_0 be the topology on $[0, 0]$ generated by $\mathcal{B}_0 = \{\{0\}\}$. Given $\gamma < \omega_1$, we assume that we have constructed for each $\beta < \gamma$ a topology τ_β on $[0, \beta]$ satisfying:

(1) $[0, \beta]$ is a scattered locally compact Hausdorff space of height not greater than ω;
(2) each point $\delta \in [0, \beta]$ has a countable neighborhood basis $\mathcal{B}_\delta$ consisting only of compact clopen sets and such that $\delta = \max\{V\}$ for each $V \in \mathcal{B}_\delta$;

(3) if $\alpha < \beta$, then $\tau_\alpha \subseteq \tau_\beta$ and $\{U \cap [0,\alpha] : U \in \tau_\beta\} = \tau_\alpha$;
(4) if $\alpha + (n-1) < \beta$, then $[0, \alpha + (n-1)]$ is closed in $[0,\beta]$.

Then we consider the topology τ^*_γ on $[0,\gamma)$ whose basis is $\bigcup_{\beta<\gamma} \tau_\beta$.

By conditions (1)–(3) the space $[0,\gamma)$ endowed with the topology τ^*_γ is Hausdorff, locally compact scattered with height not greater than ω. We will find an appropriate countable local basis for γ, $\mathcal{B}_\gamma$, and then define on $[0,\gamma]$ the topology τ_γ generated by $\tau^*_\gamma \cup \mathcal{B}_\gamma$.

If $\gamma < \omega_1$ let γ' be the greatest limit ordinal such that $\gamma' \le \gamma$. In the construction of $\mathcal{B}_\gamma$ we shall consider the following three cases:

Case 1: $\gamma' + (n-1) < \gamma$. Then we define $\mathcal{B}_\gamma = \{\{\gamma\}\}$.

Case 2: $\gamma' \le \gamma \le \gamma' + (n-1)$ and the following condition fails to hold:

($*$) For every $r \in \mathbb{N}$ every member $s_r(\gamma)$ of $\mathcal{S}'_\gamma$ is a $(k+1)$-tuple $s_r(\gamma) = (G^1_r, \ldots, G^{k+1}_r) \in \omega_1^{n_1} \times \cdots \times \omega_1^{n_{k+1}}$ for $0 \le k \le n$ such that $\{\mathrm{supp}(G^i_r) : 1 \le i \le k+1,\ r \in \mathbb{N}\}$ is a pairwise disjoint family and the heights of the points of $\mathrm{supp}(G^i_r)$ for $1 \le i \le k+1$ considered within the space $([0,\gamma'), \tau^*_{\gamma'})$ for all $r \in \mathbb{N}$ are uniformly bounded by some $p \in \mathbb{N}$.

Then we define $\mathcal{B}_\gamma = \{\{\gamma\}\}$ as in the first case.

Case 3: $\gamma' \le \gamma \le \gamma' + (n-1)$ and ($*$) holds. In this case we assume $\gamma = \gamma'$ (and so γ is a limit ordinal), and we will define $\mathcal{B}_{\gamma+i}$ for all $0 \le i < n$ at once. Consider the set $\mathcal{S}'_\gamma = \{s_r(\gamma) : r \in \mathbb{N}\}$ from our fixed $\clubsuit'$-sequence and assume ($*$). Let $m = \sum_{1\le i\le k+1} n_i$ and define x^j_r for $1 \le j \le m$ so that the sequence $(x^1_r, \ldots, x^m_r)$ is the concatenation of the sequences $G^1_r, \ldots, G^{k+1}_r$. Set $F_1 = \{1, \ldots, n_1\}, \ldots, F_{k+1} = \{\sum_{1\le i\le n_k} n_i, \ldots, m\}$.

Now we use our inductive hypotheses for τ^*_γ. By applying (1)–(4) and the fact that S'_γ converges to γ ((ii) of Definition 3.4) we may find a collection $\{W^j_r(\gamma) : 1 \le j \le m,\ r \in \mathbb{N}\}$ of τ^*_γ-clopen compact sets such that:

- $x^j_r = \max\{W^j_r(\gamma)\}$;
- $\{W^j_r(\gamma) : 1 \le j \le m,\ r \in \mathbb{N}\}$ are pairwise disjoint and converge to γ;
- the heights in τ^*_γ of all points of $W^j_r(\gamma)$ are not greater than p.

For each $1 \le i \le k$ and each $r \in \mathbb{N}$ define

$$V_r(\gamma + (i-1)) = \{\gamma + (i-1)\} \cup \bigcup_{t\ge r} \bigcup_{l\in F_i} W^l_t(\gamma).$$

By the construction it follows that for every $r, s \in \mathbb{N}$ we have:

- $V_r(\gamma + (i-1)) \cap V_s(\gamma + (j-1)) = \emptyset$ whenever $i \ne j$;
- $V_r(\gamma + (i-1)) \cap (\mathrm{supp}(G^j_s) \cup \mathrm{supp}(G^{k+1}_s)) = \emptyset$ whenever $i \ne j$.

We define

$$\mathcal{B}_{\gamma+(i-1)} = \begin{cases} V_r(\gamma+(i-1)) : r \in \mathbb{N}\} & \text{for } 1 \le i \le k, \\ \{\{\gamma+(i-1)\}\} & \text{for } k+1 \le i \le n. \end{cases}$$

It is straightforward to check that $([0, \gamma+(i-1)], \tau_{\gamma+(i-1)})$ satisfies conditions (1)–(3) for each $1 \le i \le n$. Given $1 \le i \le n$ we fix $\beta = \gamma + (i-1)$ and we will show that $([0,\beta], \tau_\beta)$ also satisfies (4). Assume the opposite, that is, $\alpha + (n-1) < \beta$ and there is $\delta \in \overline{[0, \alpha+(n-1)]}^{\tau_\beta}$ such that $\alpha+(n-1) < \delta$. By the inductive hypothesis (3) we may assume that $\gamma \le \delta \le \gamma + (n-1)$ and so $\alpha + (n-1) < \delta$ implies that $\alpha < \gamma$, since γ is a limit ordinal. We consider $\mathcal{B}_\delta = \{V_r(\delta) : r \in \mathbb{N}\}$. Since δ is not isolated in $([0,\beta], \tau_\beta)$ we may assume that $\delta = \gamma + (i-1)$ for $1 \le i \le k$. Recalling the construction of our space, there must exist $r_0 \in \mathbb{N}$ such that $\delta \in V_{r_0}(\delta) \subseteq [\alpha + n, \delta]$. This is a contradiction and so $[0, \alpha + (n-1)]$ is closed in $([0,\beta], \tau_\beta)$.

Finally, we consider on $[0, \omega_1)$ the locally compact Hausdorff topology generated by the basis $\bigcup_{\gamma<\omega_1} \tau_\gamma$. This space will be denoted by L_n and its one-point compactification will be denoted by $K_n = L_n \dot\cup \{\omega_1\}$. This concludes the construction of the compact space.

Claim 1. *The space K_n has height not greater than $\omega + 1$.*

Indeed, for any $\gamma < \omega_1$, the heights of the points of $V_0(\gamma) \setminus \{\gamma\}$ are uniformly bounded by some $p \in \mathbb{N}$, so the height of γ cannot be greater than $p + 1$.

Claim 2. *There is a finite-to-one continuous function $\phi : L_n \to [0, \omega_1)$ such that $|\phi^{-1}[\{\alpha\}]| \le n$ for all $\alpha < \omega_1$.*

Let $L_n^{(1)}$ be the set of all accumulation points of L_n. We define $\phi : L_n \to [0, \omega_1)$ by setting first $\phi(\alpha) = \alpha$ if $\alpha \in L_n \setminus L_n^{(1)}$. If $\alpha \in L_n^{(1)}$, then there are $\gamma \in \mathcal{L}(\omega_1)$ and $0 \le i \le n-1$ such that $\alpha = \gamma + i$. In this case we define $\phi(\alpha) = \phi(\gamma + i) = \gamma$.

It is clear that $|\phi^{-1}[\{\alpha\}]| \le n$ for all $\alpha < \omega_1$ and that $\phi[L_n]$ is homeomorphic to the interval $[0, \omega_1)$ with the order topology. To see that ϕ is continuous, recall that the intervals $[0, \alpha]$ for $\alpha < \omega_1$ generate the Boolean algebra of all clopen subsets of $[0, \omega_1)$. So it is enough to note that for every $\alpha < \omega_1$ the preimage $\phi^{-1}[[0,\alpha]]$ is clopen in L_n. These preimages are intervals of the form $[0, \alpha']$ for some $\alpha' < \omega_1$, which are open in L_n by (2).

Claim 3. *The compact space K_n is $(n+1)$-diverse.*

Let $m \in \mathbb{N}$ and $F_1, \ldots, F_{k+1}$ be a partition of $\{1, \ldots, m\}$ such that $k \le n$. Let $\{(x_1^\alpha, \ldots, x_m^\alpha)\}_{\alpha<\omega_1} \subseteq K_n^m$ be a sequence of $(F_1, \ldots, F_{k+1})$-

diverse points. We will prove that this sequence has an $(F_1, \ldots, F_{k+1})$-diverse cluster point.

Using Claim 1, by passing to an uncountable subset we may assume that the collection $\{\{x_1^\alpha, \ldots, x_m^\alpha\} : \alpha < \omega_1\}$ constitutes a Δ-system of finite sets of points of bounded finite heights less than some $p \in \mathbb{N}$ with root $\Delta = \{x_j^\alpha : j \in D\}$ for some $D \subseteq \{1, \ldots, m\}$. We will denote the element $x_j^\alpha \in \Delta$ by a_j for $j \in D$. By passing to a further uncountable subset we may assume that $\delta < x_j^\alpha$ for every $\delta \in \Delta$ and every $j \in \{1, \ldots, m\} \setminus D$. By (2) a cluster point of a set cannot lie strictly below all elements of the set, so by taking a smaller $k \leq n$ we may assume that all the sets $F_i' = F_i \setminus D$ for $1 \leq i \leq k+1$ are nonempty. By changing the enumeration of F_i's we may assume that if $\omega_1 = a_j$ for some $1 \leq j \leq m$, then $j \in F_{k+1}$.

Define sequences $G_\alpha^i = (x_j^\alpha : j \in F_i')$ for each $1 \leq i \leq k+1$ and $\alpha < \omega_1$. For each $\alpha < \omega_1$ define $(y_1^\alpha, \ldots, y_m^\alpha)$ to be the concatenation of the sequences $G_\alpha^1, \ldots, G_\alpha^{k+1}$. It follows that $\{(y_1^\alpha, \ldots, y_m^\alpha)\}_{\alpha<\omega_1} \subseteq K_n^m$ is a sequence of $(F_1', \ldots, F_{k+1}')$-diverse points.

Note that $\{\mathrm{supp}(G_\alpha^1, \ldots, G_\alpha^{k+1}) : \alpha < \omega_1\}$ is pairwise disjoint since $\{\{x_1^\alpha, \ldots, x_m^\alpha\} : \alpha < \omega_1\}$ is a Δ-system, so by passing to a further subsequence we may assume that $\mathcal{W} = \{(G_\alpha^1, \ldots, G_\alpha^{k+1}) : \alpha < \omega_1\}$ is consecutive.

Recalling the use of the $\clubsuit'$-sequence in the construction of K, there is $\gamma \in \mathcal{L}(\omega_1)$ such that $\mathcal{S}_\gamma' \subseteq \mathcal{S}$ and $\mathcal{S}_\gamma'$ can be enumerated in increasing order as

$$\mathcal{S}_\gamma' = \big\{\{G_r^1(\gamma), \ldots, G_r^{k+1}(\gamma)\} : r \in \mathbb{N}\big\}.$$

We define $z = (z_1, \ldots, z_m)$ in the following way: for $1 \leq i \leq k$, if $j \in F_i' = F_i \setminus D$ we define $z_j = \gamma + (i-1)$; if $j \in F_{k+1}' = F_{k+1} \setminus D$ we set $z_i = \omega_1$; and if $j \in D$ we define $z_j = a_j \in \Delta$. According to the construction of K_n, if $1 \leq i \leq k$ then $(G_r^i(\gamma))_{r\in\mathbb{N}}$ converges to $\gamma + (i-1)$. We will now also argue that $(G_r^{k+1}(\gamma))_{r\in\mathbb{N}}$ converges to ω_1. Indeed, the fact that then $(G_r^i(\gamma))_{r\in\mathbb{N}}$ is a discrete family of sets in $([0, \gamma + (n-1)), \tau_{\gamma+(n-1)})$ (every point in the space has a neighborhood which meets at most one element of the family) follows directly from the construction. Conditions (2) and (4) imply that this property holds in $([0, \omega_1), \tau_{\omega_1}^*)$. It follows that $(G_r^{k+1}(\gamma))_{r\in\mathbb{N}}$ converges to the point ω_1 of the one-point compactification. It follows that z is an accumulation point of the original sequence $\{(x_1^\alpha, \ldots, x_m^\alpha)\}_{\alpha<\omega_1}$.

If z is not $(F_1, \ldots, F_{k+1})$-diverse, then there is $x \in \{z_j : j \in F_{i_1}\} \cap \{z_j : j \in F_{i_2}\}$ for some $i_1 \neq i_2$. By definition of z, there are $j_1 \in F_{i_1} \cap D$ and $j_2 \in F_{i_2} \cap D$ such that $x = a_{j_1} = a_{j_2} \in \Delta$ or $x = \omega_1$. Since our original sequence is $(F_1, \ldots, F_{k+1})$-diverse, it follows that $\{a_j : j \in F_{i_1} \cap D\}$ $\cap \{a_j : j \in F_{i_2} \cap D\} = \emptyset$. Also if $x = \omega_1$, then one of the i_ls for $l \in \{1, 2\}$ is $k+1$ and ω_1 is the value of z_j for all $j \in F_{k+1}'$ and for the other element

i_{3-l} there is $j \in F_{i_{3-l}}$ with $\omega_1 = a_j$. But we have chosen such an element i_{3-l} to be $k+1$, so $i_1 = i_2$, a contradiction.

Claim 4. *The space K_n has height $\omega + 1$.*

By Claim 1 it is enough to note that the height cannot be finite. If the height were finite, considering the last uncountable Cantor–Bendixson level, we would construct an uncountable 2-diverse sequence of its elements whose members are larger than all the members of ω_1 which are in the following Cantor–Bendixson levels. This would imply that there can only be one accumulation point of this sequence, which would have to have all coordinates equal to ω_1, which contradicts the 2-diversity. ∎

Proposition 3.5 (♣). *There is a nonseparable Hausdorff compact scattered topology τ on $[0, \omega_1]$ which is n-diverse for every n, has height $\omega + 1$ and weight ω_1 and for every $\alpha < \omega_1$ there is $n_\alpha \in \mathbb{N}$ such that the set $[0, \alpha + (n_\alpha - 1)] \cup \{\omega_1\}$ is closed. Moreover, there is a finite-to-one surjective function $\phi : [0, \omega_1) \to [0, \omega_1)$ which is τ-to-order-topology continuous.*

Proof. Do a similar construction to the previous one but allowing the limit points γ to split into n points $\{\gamma + i : i < n\}$ without limiting $n \in \mathbb{N}$. ∎

4. The spaces of A. Dow, H. Junnila and J. Pelant. In this section, the symbol DJP_1 stands for the compact Hausdorff space constructed in [4, Example 2.17], denoted by K_3 there, under the assumption of $\diamondsuit$, and DJP_2 stands for the ZFC compact space from [4, Example 2.16], denoted by K_2 there. It is proved in [4] that $C(DJP_1)$ is weakly pcc and admits a finite-to-one continuous map onto $[0, \omega_1]$ with the order topology. The proof in [4] of $C(DJP_1)$ being weakly pcc relies on the fact that it is pointwise pcc, which means that every point-finite family of open sets in the topology of pointwise convergence is at most countable. In fact, it is proved in [4, Lemma 2.13] that for $C(K)$ with K a scattered compact space, being weakly pcc is the same as being pointwise pcc. On the other hand, Arhangel'skii and Tkačuk implicitly proved in [2] (see [4, Proposition 2.2]) that a compact Hausdorff K is pointwise pcc if and only if $K^n \setminus \Delta_n$ is ω_1-compact for all $n \in \mathbb{N}$, where ω_1-*compact* means that every set of cardinality ω_1 has an accumulation point and $\Delta_n = \{(x_1, \ldots, x_n) \in K^n : x_i = x_j \text{ for some } i \neq j\}$. Our main observation concerning these notions is the following:

Lemma 4.1. *Suppose K is a compact space. If $C(K)$ is pointwise pcc, then so is $C(K^2)$.*

Proof. Suppose K is pointwise pcc. Then by [4, Proposition 2.2] (based on [2, Proposition 2.7]) we find that $K^n \setminus \Delta_n$ is ω_1-compact for each $n \in \mathbb{N}$.

By the same result we need to prove that $(K^2)^n \setminus \Delta_n(K^2)$ is ω_1-compact for every $n \in \mathbb{N}$, where

$$\Delta_n(K^2) = \{(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) : \exists i \neq j \ \ (x_{2i+1}, x_{2i+2}) = (x_{2j+1}, x_{2j+2})\}.$$

So $X = (K^2)^n \setminus \Delta_n(K^2)$ is

$$\{(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) : \forall i \neq j \ \ (x_{2i+1}, x_{2i+2}) \neq (x_{2j+1}, x_{2j+2})\}.$$

Let $\{x^\xi = (x_1^\xi, x_2^\xi, \ldots, x_{2n-1}^\xi, x_{2n}^\xi)) : \xi < \omega_1\}$ be an uncountable subset of X. By passing to an uncountable subset we may assume that $x_j^\xi = x_l^\xi$ if and only if $x_j^\eta = x_l^\eta$ for all $\xi < \eta < \omega_1$ and $1 \leq j, l \leq n$. In this way we obtain a partition of $\{1, \ldots, 2n\}$ into sets $(A_k)_{1 \leq k \leq m}$ for some $m \leq 2n$ such that for every $\xi < \omega_1$ we have $x_j^\xi = x_l^\xi$ if and only if j and l are in the same set of the partition. Since the points x^ξ are in $X = (K^2)^n \setminus \Delta_n(K^2)$, for every $0 \leq i < j < n$ it is not true that at the same time $2i+1, 2j+1$ are in the same part of the partition and $2i+2, 2j+2$ are in the same part of the partition.

Choose a representative j_k of each element A_k of the partition and consider a point $v^\xi = (x_{j_1}^\xi, \ldots, x_{j_m}^\xi)$ for each $\xi < \omega_1$. We have $v^\xi \in K^m \setminus \Delta_m$. So by the pointwise pcc property of $C(K)$, we conclude that $(v^\xi)_{\xi < \omega_1}$ has a cluster point, say $(v_1, \ldots, v_m)$, in $K^m \setminus \Delta_m$, that is, with all coordinates distinct.

Now define a point x of $(K^2)^n$ by putting v_k in all the coordinates from A_k. Since for $0 \leq i < j < n$ it is not true that at the same time $2i+1, 2j+1$ are in the same part of the partition and $2i+2, 2j+2$ are in the same part of the partition, x is in $X = (K^2)^n \setminus \Delta_n(K^2)$ and x must be a cluster point of the x^ξs. ∎

Corollary 4.2. *Let K be a compact scattered space. If $C(K)$ is weakly pcc, then so is $C(K^n)$ for all $n \in \mathbb{N}$. In particular $C(K^n)$ does not contain a complemented copy of $c_0(\omega_1)$ for any $n \in \mathbb{N}$.*

Proof. If K is compact scattered and $C(K)$ is weakly pcc, then by [4, Lemma 2.13], $C(K)$ is pointwise pcc. By Lemma 4.1 we conclude that $C(K^{2^n})$ is pointwise pcc for every $n \in \mathbb{N}$ and so weakly pcc for every $n \in \mathbb{N}$ again by [4, Lemma 2.13]. As K^n is homeomorphic to a closed subset of K^{2^n}, $C(K^n)$ is a quotient Banach space of $C(K^{2^n})$ and so is weakly pcc as well by [4, Lemma 1.8]. As weakly pcc implies half-pcc, by Theorem 1.1 we conclude that $C(K^n)$ does not contain a complemented copy of $c_0(\omega_1)$ for any $n \in \mathbb{N}$. ∎

Corollary 4.3 (♣). *$C(DJP_1^n)$ is weakly pcc for any $n \in \mathbb{N}$ and $C(DJP_1)$ is not Lindelöf in the weak topology. In particular $C(DJP_1^n)$ does not contain a complemented copy of $c_0(\omega_1)$ for any $n \in \mathbb{N}$.*

Proof. It is shown in [4, Example 2.17] that $C(DJP_1)$ is weakly pcc. The examination of the construction leads to the conclusion that the same can be achieved under $\clubsuit$ rather than $\diamondsuit$. As proved in [4, Example 2.17], DJP_1 maps continuously onto $[0,\omega_1]$ and so $C([0,\omega_1])$ is a closed subspace of $C(DJP_1)$. It is well-known however that $C([0,\omega_1])$ is not Lindelöf in the weak topology (consider the open cover by the sets $V_\alpha = \{f \in C([0,\omega_1]) : |f(\alpha)| > 0\}$ for $\alpha < \omega_1$ and by $\{f : |f(\omega_1)| < 1\}$), so $C(DJP_1)$ cannot be Lindelöf in the weak topology. ∎

Proposition 4.4. *The space DJP_2 has countable height and contains a point ∞ such that $DJP_2 \setminus \{\infty\}$ maps injectively and continuously onto a subset of $\mathbb{R}$.*

Proof. The sets A_i of [4, Example 2.15] have heights no greater than $i \in \mathbb{N}$, and the entire space is obtained as the one-point compactification of $\bigcup_{i\in\mathbb{N}} A_i$, so the height of DJP_2 is $\omega + 1$. The topology on $\bigcup_{i\in\mathbb{N}} A_i$ is a refinement of the topology inherited from $\mathbb{R}$ and so the identity is the desired continuous map on $DJP_2 \setminus \{\infty\}$. ∎

The following should be compared with the fact that our space K_1 and the space of [10] are 2-diverse and not weakly pcc by Theorems 3.1 and 1.7.

Proposition 4.5. *Suppose that K is a compact scattered space which contains a point ∞ such that $K \setminus \{\infty\}$ maps injectively and continuously onto a subset of $\mathbb{R}$. If K is 2-diverse, then $C(K)$ is weakly pcc.*

Proof. Let $\phi : K \setminus \{\infty\} \to \mathbb{R}$ denote the continuous injective map. By [4, Lemma 2.13 and Proposition 2.2] (cf. [2, Proposition 2.7]) it is enough to prove that $K^n \setminus \Delta_n$ is ω_1-compact for every $n \in \mathbb{N}$. So let $(x_1^\xi, \ldots, x_n^\xi)$ be points of $K^n \setminus \Delta_n$ for $\xi < \omega_1$. By passing to a smaller power we may assume that they have no coordinate ∞. By passing to an uncountable subset we may assume that there is $\varepsilon > 0$ such that $|\phi(x_i^\xi) - \phi(x_j^\xi)| \geq \varepsilon$ for every $\xi < \omega_1$ and any distinct $i, j \leq n$. Whenever $(x_1, \ldots, x_n)$ is an accumulation point of $\{(x_1^\xi, \ldots, x_n^\xi) : \xi < \omega_1\}$ in K^n and $x_i, x_j \in K \setminus \{\infty\}$, (x_i, x_j) is an accumulation point of $\{(x_i^\xi, x_j^\xi) : \xi < \omega_1\}$ in $(K \setminus \{\infty\})^2$ and so $|\phi(x_i) - \phi(x_j)| \geq \varepsilon$ because otherwise $\{(y_1, y_2) : |\phi(y_1) - \phi(y_2)| < \varepsilon\}$ is an open neighborhood of the point (x_i, x_j) in $(K \setminus \{\infty\})^2$ which separates it from the set.

Now the points $(\infty, x_1^\xi, \ldots, x_n^\xi)$ are $(\{1\}, \{2, \ldots, n+1\})$-diverse, so by the hypothesis they should have a $(\{1\}, \{2, \ldots, n+1\})$-diverse accumulation point (see Definitions 1.4 and 1.5). But such a point would give rise to an accumulation point of $\{(x_1^\xi, \ldots, x_n^\xi) : \xi < \omega_1\}$ in $(K \setminus \{\infty\})^n$, which, as we noted, must have all coordinates distinct, and so is in $K^n \setminus \Delta_n$ as required for the weak pcc. ∎

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