COLLOQUIUM MATHEMATICUM

VOL. 145 2016 NO. 1

ON SOME UNIVERSAL SUMS OF GENERALIZED POLYGONAL NUMBERS

BY

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Abstract. For $m = 3, 4, \ldots$ those $p_m(x) = (m-2)x(x-1)/2+x$ with $x \in \mathbb{Z}$ are called generalized m-gonal numbers. Sun (2015) studied for what values of positive integers a, b, c the sum $ap_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$ (i.e., any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ has the form $ap_5(x) + bp_5(y) + cp_5(z)$ with $x, y, z \in \mathbb{Z}$). We prove that $p_5 + bp_5 + 3p_5$ $(b = 1, 2, 3, 4, 9)$ and $p_5 + 2p_5 + 6p_5$ are universal over $\mathbb{Z}$, as conjectured by Sun. Sun also conjectured that any $n \in \mathbb{N}$ can be written as $p_3(x) + p_5(y) + p_{11}(z)$ and $3p_3(x) + p_5(y) + p_7(z)$ with $x, y, z \in \mathbb{N}$; in contrast, we show that $p_3 + p_5 + p_{11}$ and $3p_3 + p_5 + p_7$ are universal over $\mathbb{Z}$. Our proofs are essentially elementary and hence suitable for general readers.

1. Introduction. For $m = 3, 4, \ldots$ we set

$$p_m(x) = (m-2)\frac{x(x-1)}{2} + x. \tag{1.1}$$

Those $p_m(n)$ with $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ are the well-known *m-gonal numbers* (or polygonal numbers of order m). We call those $p_m(x)$ with $x \in \mathbb{Z}$ *generalized m-gonal numbers*. Note that (generalized) 3-gonal numbers are triangular numbers and (generalized) 4-gonal numbers are squares of integers.

In 1638, Fermat asserted that each $n \in \mathbb{N}$ can be written as the sum of m polygonal numbers of order m. This was proved by Lagrange, Gauss and Cauchy in the cases $m = 4$, $m = 3$ and $m \geq 5$ respectively (see Moreno and Wagstaff [10, pp. 54–57] or Nathanson [11, Chapter 1, pp. 3–34]). The generalized pentagonal numbers play a crucial role in Euler's famous recurrence for the partition function.

For $a, b, c \in \mathbb{Z}^+ = \{1, 2, \ldots\}$ and $i, j, k \in \{3, 4, \ldots\}$, Sun [13] called the sum $ap_i + bp_j + cp_k$ *universal over* $\mathbb{N}$ (resp., *over* $\mathbb{Z}$) if for any $n \in \mathbb{N}$ the equation $n = ap_i(x) + bp_j(y) + cp_k(z)$ has solutions over $\mathbb{N}$ (resp., over $\mathbb{Z}$). In

2010 *Mathematics Subject Classification*: Primary 11E25; Secondary 11B75, 11D85, 11E20, 11P32.

Key words and phrases: polygonal numbers, representations of integers, ternary quadratic forms.

Received 26 August 2015.

Published online 6 June 2016.

DOI: 10.4064/cm6742-3-2016

1862 Liouville (cf. [4, p. 23]) determined all those universal $ap_3 + bp_3 + cp_3$. The second author [12] initiated the determination of those universal sums $ap_i + bp_j + cp_k$ with $\{i, j, k\} = \{3, 4\}$, and this project was completed via [12, 5, 9]. For almost universal sums $ap_i + bp_j + cp_k$ with $\{i, j, k\} \subseteq \{3, 4\}$, see [8, 1, 2].

It is known that generalized hexagonal numbers are identical with triangular numbers (cf. [6] or [13, (1.3)]).

The second author recently established the following result.

Theorem 1.1 (Sun [13, Theorem 1.1]). *Suppose that $ap_k + bp_k + cp_k$ is universal over $\mathbb{Z}$, where $k \in \{4, 5, 7, 8, 9, \dots\}$, $a, b, c \in \mathbb{Z}^+$ and $a \le b \le c$. Then $k = 5$, $a = 1$, and (b, c) is among the following 20 ordered pairs:*

$$\begin{gathered}(1, c)\ (c \in \{1, 2, 3, 4, 5, 6, 8, 9, 10\}),\\ (2, 2),\ (2, 3),\ (2, 4),\ (2, 6),\ (2, 8),\\ (3, 3),\ (3, 4),\ (3, 6),\ (3, 7),\ (3, 8),\ (3, 9).\end{gathered}$$

Guy [6] realized that $p_5+p_5+p_5$ is universal over $\mathbb{Z}$, and Sun [13] proved that the sums

$$\begin{gathered}p_5 + p_5 + 2p_5, \quad p_5 + p_5 + 4p_5, \ p_5 + 2p_5 + 2p_5,\\ p_5 + 2p_5 + 4p_5, \ p_5 + p_5 + 5p_5, \ p_5 + 3p_5 + 6p_5\end{gathered}$$

are universal over $\mathbb{Z}$. So the converse of Theorem 1.1 reduces to the following conjecture of Sun.

Conjecture 1.2 (Sun [13, Remark 1.2]). *The sum $p_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$ if the ordered pair (b, c) is among*

$$\begin{gathered}(1, 3),\ (1, 6),\ (1, 8),\ (1, 9),\ (1, 10),\ (2, 3),\\ (2, 6),\ (2, 8),\ (3, 3),\ (3, 4),\ (3, 7),\ (3, 8),\ (3, 9).\end{gathered}$$

Our following result confirms this conjecture for six ordered pairs (b, c) for the first time.

Theorem 1.3. *For*

$$(b, c) = (1, 3),\, (2, 3),\, (2, 6),\, (3, 3),\, (3, 4),\, (3, 9),$$

the sum $p_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$.

Remark. This result appeared in the initial preprint version of this paper posted to `arXiv` in 2009.

Sun [13] investigated those universal sums $ap_i + bp_j + cp_k$ over $\mathbb{N}$. By [13, Conjectures 1.10 and 1.13], $p_3 + p_5 + p_{11}$ and $3p_3 + p_5 + p_7$ should be universal over $\mathbb{N}$. Though we cannot prove this, we are able to show the following result.

THEOREM 1.4. *The sums $p_3 + p_5 + p_{11}$ and $3p_3 + p_5 + p_7$ are universal over $\mathbb{Z}$.*

Theorems 1.3 and 1.4 will be shown in Sections 2 and 3 respectively. Our proofs are essentially elementary and hence suitable for general readers.

2. Proof of Theorem 1.3

LEMMA 2.1 (Sun [13, Lemma 3.2]). *Let $w = x^2 + 3y^2 \equiv 4 \pmod 8$ with $x, y \in \mathbb{Z}$. Then there are odd integers u and v such that $w = u^2 + 3v^2$.*

LEMMA 2.2. *Let $w = x^2 + 3y^2$ with x, y odd and $3 \nmid x$. Then there are integers u and v relatively prime to 6 such that $w = u^2 + 3v^2$.*

Proof. It suffices to consider the case $3 \mid y$. Without loss of generality, we may assume that $x \not\equiv y \pmod 4$ (otherwise we may use $-y$ instead of y). Thus $(x - y)/2$ and $(x + 3y)/2 = (x - y)/2 + 2y$ are odd. Observe that

$$x^2 + 3y^2 = \left(\frac{x + 3y}{2}\right)^2 + 3\left(\frac{x - y}{2}\right)^2. \tag{2.1}$$

As $3 \nmid x$ and $3 \mid y$, neither $(x-y)/2$ nor $(x+3y)/2$ is divisible by 3. Therefore $u = (x + 3y)/2$ and $v = (x - y)/2$ are relatively prime to 6. ■

LEMMA 2.3 (Jacobi's identity). *We have*

$$3(x^2 + y^2 + z^2) = (x + y + z)^2 + 2\left(\frac{x + y - 2z}{2}\right)^2 + 6\left(\frac{x - y}{2}\right)^2. \tag{2.2}$$

We need to introduce some more notation. For $a, b, c \in \mathbb{Z}^+$, we set

$$E(ax^2 + by^2 + cz^2) = \{n \in \mathbb{N} : n \neq ax^2 + by^2 + cz^2 \text{ for any } x, y, z \in \mathbb{Z}\}.$$

Proof of Theorem 1.3. Let $b, c \in \mathbb{Z}^+$. For $n \in \mathbb{N}$ we have

$$\begin{aligned} n = p_5(x) + bp_5(y) + cp_5(z) &= \frac{3x^2 - x}{2} + b\frac{3y^2 - y}{2} + c\frac{3z^2 - z}{2} \\ \Leftrightarrow\ 24n + b + c + 1 &= (6x - 1)^2 + b(6y - 1)^2 + c(6z - 1)^2. \end{aligned}$$

If $w \in \mathbb{Z}$ is relatively prime to 6, then w or $-w$ is congruent to -1 modulo 6. Thus, $p_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$ if and only if for any $n \in \mathbb{N}$ the equation $24n + b + c + 1 = x^2 + by^2 + cz^2$ has integral solutions with x, y, z relatively prime to 6.

Below we fix a nonnegative integer n.

(i) By Dickson [3, Theorem III],

$$E(x^2 + y^2 + 3z^2) = \{9^k(9l + 6) : k, l \in \mathbb{N}\}. \tag{2.3}$$

So $24n + 5 = u^2 + v^2 + 3w^2$ for some $u, v, w \in \mathbb{Z}$. As $3w^2 \not\equiv 5 \pmod 4$, u or v is odd. Without loss of generality we assume that $2 \nmid u$. Since $v^2 + 3w^2 \equiv 5 - u^2 \equiv 4 \pmod 8$, by Lemma 2.1 we can rewrite $v^2 + 3w^2$ as $s^2 + 3t^2$

with s,t odd. Now we have $24n+5=u^2+s^2+3t^2$ with u,s,t odd. By $u^2+s^2\equiv 5\equiv 2 \pmod 3$, both u and s are relatively prime to 3. Applying Lemma 2.2 we can express s^2+3t^2 as y^2+3z^2 with y,z relatively prime to 6. Thus $24n+5=u^2+y^2+3z^2$ with u,y,z relatively prime to 6. This proves the universality of $p_5+p_5+3p_5$ over $\mathbb{Z}$.

(ii) By Dickson [3, Theorem X],

$$E(x^2+2y^2+3z^2)=\{4^k(16l+10): k,l\in\mathbb{N}\}. \tag{2.4}$$

So $24n+6=2u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. Clearly v and w have the same parity. Thus $4\mid v^2+3w^2$ and hence $2u^2\equiv 6 \pmod 4$. So u is odd and $v^2+3w^2\equiv 6-2u^2\equiv 4 \pmod 8$. By Lemma 2.1 we can rewrite v^2+3w^2 as s^2+3t^2 with s,t odd. Now we have $24n+6=2u^2+s^2+3t^2$ with u,s,t odd. Note that $s^2+2u^2>0$ and $s^2+2u^2\equiv 0 \pmod 3$. By [7, p. 173] or [13, Lemma 2.1], we can rewrite s^2+2u^2 as x^2+2y^2 with x and y relatively prime to 3. As $x^2+2y^2=s^2+2u^2\equiv 3 \pmod 8$, both x and y are odd. By Lemma 2.2, $x^2+3t^2=r^2+3z^2$ for some integers $r,z\in\mathbb{Z}$ relatively prime to 6. Thus $24n+6=r^2+2y^2+3z^2$ with r,y,z relatively prime to 6. It follows that $p_5+2p_5+3p_5$ is universal over $\mathbb{Z}$.

(iii) By Dickson [3, Theorem IV],

$$E(x^2+3y^2+3z^2)=\{9^k(3l+2): k,l\in\mathbb{N}\}. \tag{2.5}$$

So $24n+7=u^2+3v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. Since $u^2\not\equiv 7 \pmod 4$, without loss of generality we assume that $2\nmid w$. As $u^2+3v^2\equiv 7-3w^2\equiv 4 \pmod 8$, by Lemma 2.1 there are odd integers s and t such that $u^2+3v^2=s^2+3t^2$. Thus $24n+7=s^2+3t^2+3w^2$ with s,t,w odd. Clearly, s is relatively prime to 6. By Lemma 2.2, $s^2+3t^2=x_0^2+3y^2$ for some integers x_0 and y relatively prime to 6, and $x_0^2+3w^2=x^2+3z^2$ for some integers x and z relatively prime to 6. Therefore $24n+7=x^2+3y^2+3z^2$ with x,y,z relatively prime to 6. This proves the universality of $p_5+3p_5+3p_5$ over $\mathbb{Z}$.

(iv) By [13, Theorem 1.7(iii)], $24n+8=u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$ with $2\nmid w$. Clearly $u\not\equiv v \pmod 2$. Without loss of generality, we assume that $u=2r$ with $r\in\mathbb{Z}$. Since $(2r)^2+v^2\equiv 8\equiv 2 \pmod 3$, both r and v are relatively prime to 3. As v and w are odd, $v^2+3w^2\equiv 4 \pmod 8$ and hence r is odd. By Lemma 2.2, we can rewrite v^2+3w^2 as x^2+3y^2 with x and y relatively prime to 6. Note that $24n+8=4r^2+v^2+3w^2=x^2+3y^2+4r^2$ with x,y,r relatively prime to 6. It follows that $p_5+3p_5+4p_5$ is universal over $\mathbb{Z}$.

(v) By (2.3), $24n+13=u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. Since $3w^2\not\equiv 13\equiv 1 \pmod 4$, without loss of generality we may assume that u is odd. As $v^2+3w^2\equiv 13-u^2\equiv 4 \pmod 8$, by Lemma 2.1 we can rewrite v^2+3w^2 as s^2+3t^2 with s and t odd. Thus $24n+13=u^2+s^2+3t^2$ with

u, s, t odd. Since $u^2+s^2 \equiv 13 \equiv 1 \pmod 3$, without loss of generality we may assume that $3 \nmid u$ and $s = 3r$ with $r \in \mathbb{Z}$. By Lemma 2.2, $u^2+3t^2 = x^2+3y_0^2$ for some integers x and y_0 relatively prime to 6, also $y_0^2+3r^2 = y^2+3z^2$ for some integers y and z relatively prime to 6. Thus $24n+13 = x^2+3y_0^2+9r^2 = x^2+3y^2+9z^2$ with x, y, z relatively prime to 6. This proves the universality of $p_5+3p_5+9p_5$ over $\mathbb{Z}$.

(vi) By the Gauss–Legendre theorem (cf. [11, pp. 17–23]), $8n+3 = x^2+y^2+z^2$ for some odd integers x, y, z. Without loss of generality we may assume that $x \not\equiv y \pmod 4$. By Jacobi's identity (2.2), we have $3(8n+3) = u^2+2v^2+6w^2$, where $u = x+y+z$, $v = (x+y)/2-z$ and $w = (x-y)/2$ are odd integers. As u^2+2v^2 is a positive integer divisible by 3, by [7, p. 173] or [13, Lemma 2.1] we can write $u^2+2v^2 = a^2+2b^2$ with a and b relatively prime to 3. Since $a^2+2b^2 = u^2+2v^2 \equiv 3 \pmod 8$, both a and b are odd. By Lemma 2.2, we have $b^2+3w^2 = c^2+3d^2$ for some integers c and d relatively prime to 6. Thus $24n+9 = a^2+2b^2+6w^2 = a^2+2c^2+6d^2$ with a, c, d relatively prime to 6. It follows that $p_5+2p_5+6p_5$ is universal over $\mathbb{Z}$.

In view of the above, we have completed the proof of Theorem 1.3. ∎

3. Proof of Theorem 1.4. (i) Let $n \in \mathbb{N}$. By part (v) in the proof of Theorem 1.3, there are integers $u, v, w \in \mathbb{Z}$ relatively prime to 6 such that

$$72n+61 = 24(3n+2)+13 = 9u^2+3v^2+w^2.$$

Clearly $w^2 \equiv 61-3v^2 \equiv 7^2 \pmod 9$ and hence $w \equiv \pm 7 \pmod 9$. So there are $x, y, z \in \mathbb{Z}$ such that

$$72n+61 = 9(2x+1)^2+3(6y-1)^2+(18z-7)^2$$

and hence $n = p_3(x)+p_5(y)+p_{11}(z)$. (Note that $p_{11}(x) = 9(x^2-x)/2+x = (9x^2-7x)/2$.)

(ii) Let $n \in \mathbb{N}$. It is easy to see that

$$\begin{aligned} &n = 3p_3(x)+p_5(y)+p_7(z) \\ &\qquad \Leftrightarrow\ 120n+77 = 5(3(2x+1))^2+5(6y-1)^2+3(10z-3)^2. \end{aligned}$$

Suppose $120n+77 = 5x^2+5y^2+3z^2$ for some $x, y, z \in \mathbb{Z}$ with z odd. Then $x^2+y^2 \equiv 77-3z^2 \equiv 2 \pmod 4$ and hence x and y are odd. Note that $3z^2 \equiv 77 \equiv 12 \pmod 5$ and hence $z \equiv \pm 3 \pmod{10}$. As $5x^2+5y^2 \equiv 77 \equiv 5 \pmod 3$, exactly one of x and y is divisible by 3. Thus there are $u, v, w \in \mathbb{Z}$ such that

$$120n+77 = 5(3(2u+1))^2+5(6v-1)^2+3(10w-3)^2.$$

By the above, to prove the universality of $3p_3+p_5+p_7$ over $\mathbb{Z}$, we only need to show that $120n+77 = 5x^2+5y^2+3z^2$ for some $x, y, z \in \mathbb{Z}$ with z odd.

By (2.3), there are $u, v, w \in \mathbb{Z}$ such that $120n + 77 = u^2 + v^2 + 3w^2$. As $3w^2 \not\equiv 77 \equiv 1 \pmod 4$, u or v is odd, say, $2 \nmid u$. As $v^2 + 3w^2 \equiv 77 - u^2 \equiv 4 \pmod 8$, by Lemma 2.1 we may assume that v and w are odd without loss of generality.

We claim that $120n + 77 = a^2 + b^2 + 3c^2$ for some odd integers a, b, c with $c \equiv \pm 2 \pmod 5$. This holds if $w \equiv \pm 2 \pmod 5$. Suppose that $w \not\equiv \pm 2 \pmod 5$. If $w \equiv \pm 1 \pmod 5$, then $u^2 + v^2 \equiv 77 - 3w^2 \equiv -1 \pmod 5$ and hence u or v is divisible by 5. If $w \equiv 0 \pmod 5$, then $u^2 + v^2 \equiv 77 \equiv 2 \pmod 5$ and hence $u^2 \equiv v^2 \equiv 1 \pmod 5$. Without loss of generality, we assume that one of v and w is divisible by 5 and the other one is congruent to 1 or -1 modulo 5; we may also suppose that $v \not\equiv w \pmod 4$ (otherwise we may use $-w$ instead of w). By the identity (2.1),

$$v^2 + 3w^2 = \left(\frac{v + 3w}{2}\right)^2 + 3\left(\frac{v - w}{2}\right)^2.$$

Note that both $(v - w)/2$ and $(v + 3w)/2 = (v - w)/2 + 2w$ are odd. Also, $(v - w)/2$ is congruent to 2 or -2 modulo 5. This confirms the claim.

By the above, there are odd integers $a, b, c \in \mathbb{Z}$ with $c \equiv \pm 2 \pmod 5$ such that $120n + 77 = a^2 + b^2 + 3c^2$. Since $3c^2 \equiv 77 \pmod 5$, we have $5 \mid a^2 + b^2$ and hence $a^2 \equiv (2b)^2 \pmod 5$. Without loss of generality we assume that $a \equiv 2b \pmod 5$. Then $x = (2a + b)/5$ and $y = (a - 2b)/5$ are odd integers, and

$$a^2 + b^2 = (2x + y)^2 + (x - 2y)^2 = 5(x^2 + y^2).$$

Now we have $120n + 77 = 5(x^2 + y)^2 + 3c^2$ with x, y, c odd.

This concludes our proof of Theorem 1.4. ■

Acknowledgements. The authors would like to thank Dr. Hao Pan for his helpful comments. The second author was supported by the National Natural Science Foundation of China (grant 11571162).

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