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Dynamics of annulus maps II: Periodic points for coverings

by

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Abstract. Let f be a covering map of the open annulus $A = S^1 \times (0,1)$ of degree d, $|d| > 1$. Assume that f preserves an essential (i.e not contained in a disk of A) compact subset K. We show that f has at least the same number of periodic points in each period as the map z^d on S^1.

1. Introduction. Existence of periodic orbits for orientation preserving annulus homeomorphisms has been extensively studied. One of the motivations is a celebrated theorem of dynamical systems, the so-called "last geometric theorem of Poincaré". Roughly this result says that an area-preserving homeomorphism of the closed annulus which rotates one boundary component clockwise and the other counterclockwise possesses at least two fixed points. This result was conjectured and proved in special cases by Poincaré [P], and was finally proved by Birkhoff [Bi]. This problem has been considered by many authors and actually triggered a great deal of research (see the paper [DR] for a historical review). Since Franks' paper [Fr1], who generalized and proved the statement for homeomorphisms of the open annulus, the interest in the problem of existence of periodic orbits for non-compact surface homeomorphisms arose (see, for example, [Fr2], [Fr3], [FH], [LC2]).

More recently, in [PS] the problem of existence and growth rate of periodic orbits for degree two surface endomorphisms was considered. In that paper, Pugh and Shub deal with a particular case of Problem 3 posed in [S]: let S be the 2-sphere, and $f : S \to S$ a continuous map of degree 2; is the

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growth rate inequality

$$\limsup_{n\to\infty} \frac{1}{n} \ln(\#\{\mathrm{Fix}(f^n)\}) \geq \ln(2)$$

true? The answer is no, as the map $(r,\theta) \mapsto (2r, 2\theta)$ has only the poles as periodic points. However, in [PS] it is shown that the growth inequality holds in a particular case: if f is C^1 and preserves the latitude foliation, then for each n, f^n has at least 2^n fixed points.

In this paper, we study the existence of periodic orbits for covering maps of the open annulus $f : A \to A$ of degree d, $|d| > 1$. Note that the growth inequality holds trivially for the closed annulus $\overline{A}$ as each connected component of the boundary of the annulus must be invariant by f or f^2, and we are assuming $|d| > 1$. On the other hand, the covering map $(r,\theta) \mapsto (2r, 2\theta)$ provides a periodic point free example in the open annulus $\mathbb{C} \setminus \{0\}$. Our result relates both to the theory of annulus homeomorphisms, and to work in [PS].

Let us introduce some preliminary definitions. If $f : A \to A$ is a continuous function, then the homomorphism f_* induced by f on the first homology group $H_1(A,\mathbb{Z}) \simeq \mathbb{Z}$ is $n \mapsto dn$, for some integer d. This number d is called the *degree* of f.

We say that an open subset $U \subset A$ is *essential* if $i_*(H_1(U,\mathbb{Z})) = H_1(A,\mathbb{Z}) = \mathbb{Z}$, where $i_* : H_1(U,\mathbb{Z}) \to H_1(A,\mathbb{Z})$ is the map induced in homology by the inclusion $i : U \to A$. We say that a subset $X \subset A$ is *essential* if any neighbourhood of X in A is essential. We say that a subset is *inessential* if it is not essential, or equivalently, if it is contained in a disk of A. If x is a periodic point for f, its *period* is $\min\{n \geq 1 : f^n(x) = x\}$. We write $\mathrm{Per}_n(f)$ for the set of periodic points of period n of a given map f, and $\mathrm{Fix}(f) = \mathrm{Per}_1(f)$.

Let A^* be the compactification of the annulus A with two points so that A^* is homeomorphic to the two-sphere. Each connected component of $A^* \setminus A$ is called an *end* of A. Note that if f is a proper mapping, then f extends continuously to A^*, and either fixes both ends or interchanges them.

We need one last definition: what it means for f to be complete; we postpone this until Section 2 because the definition involves some Nielsen theory. We recall that $x, y \in \mathrm{Fix}(f)$ are *Nielsen equivalent* if there exists an arc γ joining x and y such that γ is homotopic to $f(\gamma)$ with endpoints fixed. If f is complete then for each n, f^n has exactly $|d^n - 1|$ Nielsen classes of fixed points (see Lemma 3 in Section 3).

We prove the following:

THEOREM 1. *Let $f : A \to A$ be a covering map of degree d, $|d| > 1$. Suppose there exists an essential continuum $K \subset A$ such that $f(K) \subset K$. Then f is complete.*

As explained above, the growth rate inequality holds for annulus maps under the standing hypothesis.

Note that this result is strictly a consequence of degree; an irrational rotation of the open annulus has no periodic points, and has every essential circle as a compact invariant subset. Results along the same lines have been obtained by Boronski [Bo1], [Bo2].

The periodic points given by Theorem 1 do not necessarily belong to K (see Subsection 5.1). The problem of whether or not the fixed points of a given map with a compact invariant set K belong to K is known in the literature as Cartwright–Littlewood theory. In a seminal paper, M. Cartwright and J. Littlewood [CL] proved that if K is a nonseparating continuum of the plane, invariant under an orientation preserving homeomorphism h, then h has a fixed point *in* K. Existence of a fixed point under that hypothesis was already known on account of Brouwer's plane translation theorem [Brou], the novelty being that the fixed point must belong to K. An easy proof of Cartwright–Littlewood's theorem can be found in the extraordinary one-page paper of M. Brown [Brow]. H. Bell [Be] proved Cartwright–Littlewood's theorem for orientation reversing plane homeomorphisms, and his results were later generalized by K. Kuperberg [K] to arbitrary plane continua (not necessarily nonseparating). In Subsection 5.1, we construct a degree two covering map f of the annulus with a totally invariant essential continuum K ($f^{-1}(K) = K$) such that $\mathrm{Fix}(f) \cap K = \emptyset$. However, K is not *filled*, that is, its complement has bounded connected components. If $K \subset A$ is any compact set, the set $\mathrm{Fill}(K)$ is defined as the union of K with the bounded connected components of its complement. The definition of Nielsen classes of periodic points is given in Section 2. We prove the following:

THEOREM 2. *Let $f : A \to A$ be a covering map of degree d, $|d| > 1$. Suppose there exists an essential continuum $K \subset A$ such that $f(K) \subset K$. Then there exists a representative $x \in \mathrm{Fill}(K)$ for each Nielsen class of periodic points of f.*

Notation. Throughout, $A = S^1 \times (0,1)$ is the open annulus, $\tilde{A} = \mathbb{R} \times (0,1)$ its universal covering space, and $\pi : \tilde{A} \to A$ the universal covering projection. We will denote by F any lift of $f : A \to A$ to the universal covering, that is, F is a map satisfying $f\pi = \pi F$. Note that $F(x+1,y) = F(x,y) + (d,0)$ if f has degree d. To lighten notation, if $z \in \tilde{A}$, we write $z + k$ for the point $z + (k,0)$, $k \in \mathbb{Z}$. The map $m_d : S^1 \to S^1$ is defined as $m_d(z) = z^d$.

2. Nielsen theory background. In this section, we gather the necessary information on Nielsen theory. In what follows, $f : A \to A$ is any continuous map. If $p, q \in \mathrm{Fix}(f)$, then p and q are said to be *Nielsen equiva-*

lent if there exists a curve γ from p to q such that $f(\gamma)$ and γ are homotopic with endpoints fixed. If p and q are periodic points of f, then p and q are Nielsen equivalent if they are equivalent as fixed points of some f^k, $k \geq 1$. The definition of Nielsen equivalence does not depend on the choice of k:

LEMMA 1. *Let $p, q \in \mathrm{Fix}(f)$ and let γ be a curve from p to q. If $\gamma \sim f^k\gamma$ for some $k > 1$, then $\gamma \sim f\gamma$.*

Proof. Let $\tilde{p}$ be a lift of p and let F be the lift of f that fixes $\tilde{p}$. Let $\tilde{\gamma}$ be the lift of γ starting at $\tilde{p}$, and let $\tilde{q}$ be the endpoint of $\tilde{\gamma}$. Assume that γ is not homotopic to $f(\gamma)$. This implies that $F(\tilde{q}) \neq \tilde{q}$, and so there exists $l \in \mathbb{Z}$, $l \neq 0$ such that $F(\tilde{q}) = \tilde{q} + l$. Then $F^k(\tilde{q}) = \tilde{q} + \sum_{i=0}^{k-1} d^i l \neq \tilde{q}$, and so $f^k\gamma$ is not homotopic to γ. ∎

LEMMA 2. *Let p and q be fixed points of f. The following conditions are equivalent:*

(1) *p and q are Nielsen equivalent.*
(2) *If F is any lift of f, and $\tilde{p}$ is a lift of p, there exists a lift $\tilde{q}$ of q such that $F(\tilde{p}) - \tilde{p} = F(\tilde{q}) - \tilde{q}$.*

Proof. (1)⇒(2): Let F be a lift of f, and $\tilde{p}$ any lift of p. Then, as $p \in \mathrm{Fix}(f)$, there exists $l \in \mathbb{Z}$ such that $F(\tilde{p}) = \tilde{p} + l$. Let $\tilde{q}$ be the endpoint of $\tilde{\gamma}$, the lift of γ starting at $\tilde{p}$, where γ is the arc given by the Nielsen equivalence. As $\gamma \sim f(\gamma)$, the lift of $f(\gamma)$ starting at $\tilde{p} + l$ must end at $\tilde{q} + l$. On the other hand, this lift must coincide with $F(\tilde{\gamma})$, which gives $F(\tilde{q}) = \tilde{q} + l$.

(2)⇒(1): Let $\tilde{p}$ be a lift of p, and let F be the lift of f such that $F(\tilde{p}) = \tilde{p}$. There exists a lift $\tilde{q}$ of q such that $F(\tilde{q}) = \tilde{q}$. Take any arc $\tilde{\gamma}$ joining $\tilde{p}$ and $\tilde{q}$. Then $F(\tilde{\gamma})$ is obviously homotopic to $\tilde{\gamma}$. So, $\gamma = \pi(\tilde{\gamma})$ realizes the Nielsen equivalence between p and q. ∎

Note that the number of Nielsen classes of fixed points for the map m_d coincides with its number of fixed points, which is $|d-1|$. We will give a simple proof of the following fact:

THEOREM 3. *If f is a map of degree d, $|d| > 1$, of the annulus, then the number of equivalence classes of fixed points of f is less than or equal to $|d-1|$.*

Proof. Let f be a degree d map of the annulus, let F be a lift of f and $p \in \mathrm{Fix}(f)$. If $\tilde{p}$ is any lift of p then there exists an integer l such that $F(\tilde{p}) = \tilde{p} + l$. Moreover, as $F(x+1) = F(x) + d$, one can choose $\tilde{p}$ so that l is an integer between 1 and $|d-1|$. To see this, note that if $j \in \mathbb{Z}$, then $F(\tilde{p} + j) = F(\tilde{p}) + dj = \tilde{p} + l + dj = \tilde{p} + j + j(d-1) + l$. So, there exists a unique $j \in \mathbb{Z}$ such that $F(\tilde{p} + j) = \tilde{p} + j + L$ with $1 \leq L \leq |d-1|$.

This number $L = \ell_p$ is uniquely determined by p and the lift F. The previous lemma implies that p and q are equivalent iff $\ell_p = \ell_q$, which clearly implies the assertion. ∎

The previous result also follows from the fact that the number of all classes is a homotopy type invariant and that Theorem 3 holds for the circle (see [J, pp. 618 and 630]).

The *period* of a Nielsen class is defined as the minimum of the periods of periodic points of f in the class. Let $N_k(f)$ be the number of different Nielsen classes of period k of the map f.

DEFINITION 1. A degree d map f of the annulus is said to be *complete* if $N_k(f) = N_k(m_d)$ for each positive integer k.

Note that Theorem 3 implies that a complete map has the maximum possible number of Nielsen classes in each period.

The following two results will be used in the proof of Theorem 1.

LEMMA 3. *A map f is complete iff $N_1(f^k) = N_1(m_d^k)$ for every positive k.*

Proof. It is clear that f complete implies that condition. For the converse, we proceed by induction to prove that $N_k(f) = N_k(m_d)$ for all k; for $k = 1$ it is obvious by hypothesis.

Now assume that $N_{k'}(f) = N_{k'}(m_d)$ for every $k' < k$. Then

$$|d^k - 1| = N_1(f^k) = N_k(f) + \sum_{k'|k} N_{k'}(f),$$

where $k' \mid k$ means k is a multiple of k' and $k' < k$. On the other hand, the same equality holds with m_d in place of f, and using the induction hypothesis, we obtain $N_k(f) = N_k(m_d)$. ∎

LEMMA 4. *Suppose that* $\mathrm{Fix}(F) \neq \emptyset$ *for every lift $F : \tilde{A} \to \tilde{A}$ of f. Then f has $|d-1|$ different Nielsen classes of fixed points.*

Proof. Fix a lift $F_0 = F$ of f, and for $k \in \mathbb{Z}$ define $F_k(x) = F(x) + k$. Note that every lift of f belongs to the family $(F_k)_{k\in\mathbb{Z}}$. For every $k \in \mathbb{Z}$, let $x_k \in \tilde{A}$ be such that $F_k(x_k) = x_k$. We want to show that there are $|d-1|$ different Nielsen classes of fixed points.

Suppose there exists $i \neq 0$ such that $\pi(x_i)$ is Nielsen equivalent to $\pi(x_0)$. As $F(x_0) = x_0$, by Lemma 2 there exists $l \in \mathbb{Z}$ such that $F(x_i + l) = x_i + l$.

As $F_i(x_i) = x_i$ and $F_i(x_i) = F(x_i) + i$, we have $F(x_i) = x_i - i$. So,

$$x_i + l = F(x_i + l) = F(x_i) + ld = x_i - i + ld.$$

Thus $i = l(d-1)$. As $i \neq 0$, it follows that $l \neq 0$ and the points $\pi(x_0), \pi(x_1), \ldots, \pi(x_{|d-1|-1})$ are all in different Nielsen classes. So, there are at least $|d-1|$ different Nielsen classes, and by Theorem 3 there are exactly $|d-1|$ of them. ∎

In most cases we will prove completeness by means of the following corollary:

Corollary 1. *If for every k every lift of f^k has a fixed point, then f is complete.*

Remarks. (1) Note that the fact that *some* lift of f has fixed points does not imply that *every* lift does (see Subsection 5.3).

(2) The definition of completeness can also be applied to circle maps. If f is a degree d map of the circle, then whether it is a covering or not, the number of Nielsen classes of fixed points of f is $|d-1|$. In general, $N_k(f) = N_k(m_d)$ for every k. It follows that every circle map is complete.

(3) Theorem 3 implies that $N_1(f) \leq |d-1|$, and obviously $N_1(f^k) \leq |d^k - 1|$. However, it is not true in general that $N_j(f) \leq N_j(m_d)$ as the example in Figure 1 shows. It is a map of degree -2 that has two fixed points r_1 and r_2, the rays S_1 and S_2 satisfy $f(S_1) = S_2$, $f(S_2) = S_1$ and $\{p, q\}$ is a two-periodic cycle. The open invariant region bounded by S_1 and S_2 contains no fixed point. Note that f has two fixed points (against three of m_{-2}) and has one two-periodic cycle formed by p and q (against zero of m_{-2}).

Here f^2 has exactly four fixed points, but just three Nielsen classes, one of them contains the two-periodic cycle. According to our definition, this map is not complete and consequently it cannot leave invariant an essential compact set.

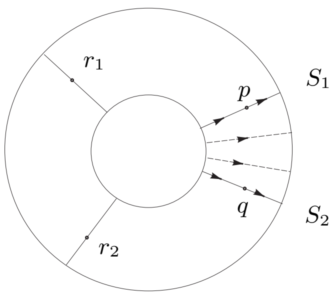


Fig. 1

3. Proof of completeness. In this section we prove Theorem 1. To find periodic orbits we proceed in the standard fashion: we lift to the universal covering space $\mathbb{R} \times (0,1) \sim \mathbb{R}^2$ and try to use some fixed point theorems for self-maps of the plane. If f happens to be an orientation preserving covering map, then the lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving plane homeomorphism, and existence of fixed points is guaranteed by any kind of recurrence:

Theorem 4 ([Brou]). *If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a fixed point free orientation preserving homeomorphism, then every point is wandering.*

For a modern exposition of this theorem in its maximum expression, see [LC1].

Although this technique is quite useful when f is isotopic to the identity, recurrence in the lift for maps of degree d, $|d| > 1$, is not easy to get. Indeed, the lifted map F satisfies $F(x+1, y) = F(x, y)+d$ and so every point wants to escape to infinity exponentially fast. Of course we may impose some strong hypothesis immediately implying recurrence for the lift:

LEMMA 5. *If $f : A \to A$ is an orientation preserving covering map of degree $d \neq 0$ preserving an inessential continuum $K \subset A$, then* $\mathrm{Fix}(f) \neq \emptyset$.

A continuum is *inessential* if it is contained in a disk of A. The proof is immediate from Brouwer's Theorem 4, as the hypothesis implies that there exists a lift of f that preserves a compact subset of the plane (namely, a connected component of the preimage of K by the covering projection).

However, if K is essential, no connected component of its lift to the universal covering space is compact. The proof of Theorem 1 in the orientation preserving case is based on a simple (though key) observation that was already made in [IPRX]. If $f : A \to A$ is a continuous map of degree d, $|d| > 1$, and $K \subset A$ is a compact set such that $f(K) \subset K$, then $f|_K$ is semiconjugate to the restriction of m_d to an invariant subset. Existence and properties of the semiconjugacy $h : K \to S^1$ are stated in Lemma 6. Using Brouwer's Theorem, existence of fixed points is proved in Lemma 8 if $h^{-1}(1) \neq \emptyset$, as this guarantees existence of a compact invariant set for the lift. We prove completeness of f using standard Nielsen theory if K is essential (note that Lemma 5 only gives a fixed point, not completeness); this is done in Lemmas 4 and 9.

If f reverses orientation, we use Kuperberg's theorem [K] to find fixed points for orientation reversing plane homeomorphisms.

THEOREM 5 ([K]). *Let f be an orientation reversing homeomorphism of the plane, and X a continuum of the plane invariant under f. Then f has at least one fixed point in X.*

The following lemma, which is esentially the Shadowing Lemma for expanding maps, is key for the purposes of this paper. See [IPRX, Lemmas 1 and 2].

LEMMA 6. *Let $f : A \to A$ be a continuous map of degree d, $|d| > 1$, and let $F : \tilde{A} \to \tilde{A}$ be a lift of f. Let K be a compact f-invariant ($f(K) \subset K$) subset of the annulus, and $\tilde{K} = \pi^{-1}(K)$. Then there exists a continuous map $H_F : \tilde{K} \to \mathbb{R}$ such that:*

(1) $H_F(x + 1, y) = H_F(x, y) + 1$,
(2) $H_F F = dH_F$,
(3) $|H_F(x, y) - x|$ *is bounded on* $\tilde{K}$,

(4) *$H_F(x) = \lim_{n\to\infty} (F^n(x))_1/d^n$, where $()_1$ denotes projection over the first coordinate.*

The function H_F appears as a fixed point of the contracting operator $H \mapsto \frac{1}{d}HF$ acting on the space of continuous functions $H : \tilde{K} \to \mathbb{R}$ that satisfy (1).

Let h be the quotient function of H_F. It is well defined because of (1).

The previous lemma gives:

Corollary 2. *Let $f : A \to A$ be a continuous map of degree d, $|d| > 1$, and $K \subset A$ be compact and forward invariant. Then the function $h : A \to S^1$, projection of H_F, is a semiconjugacy from the restriction of f to K to the restriction of m_d to an invariant subset.*

Remark 1. Note that we have not yet assumed that f is a covering and thus Lemma 6 and Corollary 2 are valid for all continuous maps of degree d, $|d| > 1$.

There is still one preliminary result needed in the proof of the theorem.

Lemma 7. *Let g be a covering map of the open annulus A, and K a compact subset of A. Then there exists a covering g' of the closed annulus such that $g' = g$ on K.*

Proof. The proof is given in the case that g fixes the ends of A; in the other case the proof is analogous.

Let G be a lift of g and let $V_\epsilon = \{(x, y) \in \mathbb{R} \times (0, 1) : \epsilon < y < 1 - \epsilon\}$ be a neighbourhood of $\tilde{K} = \pi^{-1}(K)$. It is claimed that there exists a homeomorphism G' of $\mathbb{R} \times [0, 1]$ satisfying $G'(x, y) = (dx, y)$ for $y = 0$ and $y = 1$, $G' = G$ on V_ϵ and $G'(x + 1, y) = G'(x, y) + (d, 0)$ for every (x, y), where d is the degree of g.

To see this, let R be the rectangle $0 \le x \le 1$, $0 \le y \le \epsilon$. Note that the above requirements already define G' on the horizontal sides of R. Choose a simple arc s in A joining $G'(0, 0)$ to $G'(0, \epsilon)$ and disjoint from $G(y = \epsilon)$. Next, define G' on the segment $x = 0$, $0 \le y \le \epsilon$ as a homeomorphism onto s. Then define G' on $x = 1$, $0 \le y \le \epsilon$ so as to satisfy the condition $G'(1, y) = G'(0, y) + (d, 0)$. Until now, a map G' was defined on the boundary of R and is a homeomorphism from the boundary of R to a simple closed curve α. By the Jordan–Schoenflies theorem, G' can be extended to a homeomorphism from R to the closure of the bounded component of the complement of α. Once G' is defined in R, extend it to the whole horizontal strip $0 \le y \le \epsilon$ so as to satisfy $G'(x + 1, y) = G'(x, y) + (d, 0)$.

Repeat the construction in the horizontal strip between $y = 1 - \epsilon$ and $y = 1$. The map G' obtained is a homeomorphism from the closure of $\tilde{A}$ onto itself and satisfies the claim. Then project G' to the annulus, giving the required map g'. ∎

For the remainder of this section, we assume that f is a covering map and that $K \subset A$ is a compact subset such that $f(K) \subset K$. If $F : \tilde{A} \to \tilde{A}$ is any lift of f, H_F is the map given by Lemma 6. Note that $H_F \neq H_{F'}$ if F and F' are different lifts of f. If no confusion can arise, we will write H instead of H_F.

The proof of Theorem 1 will be divided into two cases.

3.1. The orientation preserving case

Lemma 8. *If f preserves orientation, and there exists a lift $F : \tilde{A} \to \tilde{A}$ of f such that $H^{-1}(0) \neq \emptyset$, then* $\operatorname{Fix}(F) \neq \emptyset$ *(and so* $\operatorname{Fix}(f) \neq \emptyset$*).*

Proof. Note that as $f : A \to A$ is a covering, $F : \tilde{A} \to \tilde{A}$ is a homeomorphism. Moreover, $\tilde{A}$ is homeomorphic to $\mathbb{R}^2$ and F preserves orientation because f does. So, by Brouwer's Theorem 4 it is enough to prove that $H^{-1}(0)$ is a compact F-invariant set. The invariance follows from the equality $HF = dH$ (Lemma 6(2)). To see the compactness, recall from Lemma 6 that the function $(x, y) \mapsto H(x, y) - x$ defined on $\tilde{K}$ is bounded. So, we may take $C \in \mathbb{R}$ such that $|H(x, y) - x| < C$ on $\tilde{K}$. Then $(x, y) \in H^{-1}(0)$ implies $x \in [-C, C]$, proving that $H^{-1}(0)$ is compact. ∎

Remark 2. The fixed point found in the previous lemma does not necessarily belong to K (see Subsection 5.1).

The following remark resembles rotation theory for surface homeomorphisms.

Remark 3. The previous lemma can be restated as follows: if there exists $x \in K$ and a lift F of f such that $\lim_{n\to\infty} (F^n(\tilde{x}))_1/d^n = 0$ for a lift $\tilde{x}$ of x, then $\operatorname{Fix}(F) \neq \emptyset$ (see Lemmas 6(4) and 8).

Note, however, that the mere existence of a point $\tilde{x} \in \tilde{A}$ such that $\lim_{n\to\infty} (F^n(\tilde{x}))_1/d^n = 0$ for some lift F of f does not imply the existence of a fixed point; the set K is key. An example is given in Subsection 5.2.

The following is [IPRX, Lemma 3].

Lemma 9. *If K is an essential subset of A, then for any lift F of f, the function $H_F : \tilde{K} \to \mathbb{R}$ is surjective.*

Lemma 10. *Let $g : A \to A$ be an orientation preserving covering map of degree d, $|d| > 1$, and let $K \subset A$ be an essential continuum such that $g(K) \subset K$. Then every lift of g has a fixed point.*

Proof. Lemma 9 states that H_G is surjective for any lift G of g. In particular, $H_G^{-1}(0) \neq \emptyset$ for any lift G of g. Then Lemma 8 implies that $\operatorname{Fix}(G) \neq \emptyset$. ∎

3.2. The orientation reversing case

Lemma 11. *Let $g : A \to A$ be an orientation reversing covering map of degree d, $|d| > 1$, and let $K \subset A$ be an essential continuum such that $g(K) \subset K$. Then every lift of g to the universal covering $\tilde{A}$ has a fixed point.*

Proof. There are two options:

Case 1: *g has negative degree and fixes both ends of A.* Let G be a lift of g and note that by Lemma 7, the map G can be modified without changing its restriction to $\pi^{-1}(K)$ so as to obtain a map that extends to the closure of $\mathbb{R} \times (0,1)$. This extension (also denoted G) induces a homeomorphism of the compactification of $\mathbb{R} \times [0,1]$ with two points $\{-\infty, \infty\}$. Note that G carries $-\infty$ to ∞ and vice versa. The map G is a homeomorphism of a closed ball, and it can be extended to the whole plane. Theorem 5 implies that G has a fixed point in $\pi^{-1}(K) \cup \{-\infty, \infty\}$, as this set is connected (even if $\pi^{-1}(K)$ is not; see Figure 2). Note that this fixed point cannot be $-\infty$ or ∞, as $\{-\infty, \infty\}$ is a two-periodic orbit. So, every lift of g has a fixed point. Note moreover that in this case the fixed point belongs to $\pi^{-1}(K)$.

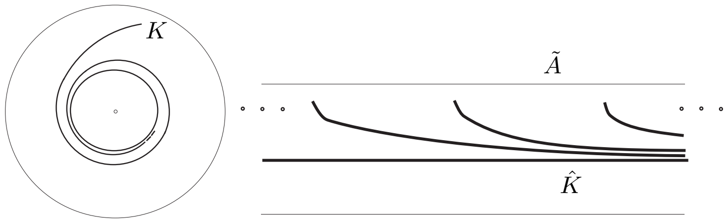
Case 2: *g has positive degree and interchanges the two ends of A.* Then Lemma 7 can be used to modify the map g so that it can be extended to a covering of the closed annulus, and as g reverts the ends, the modification can be performed without changing the fixed point set.

So, by Corollary 2 with $K = \overline{A}$, we may assume that g is semiconjugate to m_d. This in its turn implies that g has an invariant connector, meaning an inessential continuum of the closed annulus connecting the two boundary components (see [IPRX, Corollary 9]). Moreover, g has an invariant connector contained in each of the preimages under the semiconjugacy of a fixed point of m_d. Then any lift G of g must fix one of the lifts of these invariant connectors. Then extend G as a homeomorphism of the whole plane and apply Kuperberg's theorem to conclude that G has a fixed point in that invariant connector. ∎

Proof of Theorem 1. By Corollary 1, it is enough to prove that for all n, every lift of f^n has a fixed point. Note that $f^n(K) \subset K$ for all n. If f is orientation preserving, so is f^n for all n, and applying Lemma 10 we obtain the result. If f is orientation reversing, we obtain the result by applying Lemma 10 to even powers of f, and Lemma 11 to odd powers. ∎

4. Location of periodic orbits. In this section, we prove Theorem 2, that is, that the periodic points given by Theorem 1 can be found in $\mathrm{Fill}(K)$. We assume throughout this section that K is an essential continuum such that $f(K) \subset K$. Recall that f is complete in view of Theorem 1.

We will denote $\pi^{-1}(\mathrm{Fill}\,(K)) = \hat{K}$. Note that $\hat{K}$ is not necessarily connected (see Figure 2). However, if K' denotes the closure of $\hat{K}$ in D, then K' is a connected subset of D, the two-point compactification of $\mathbb{R} \times [0,1]$, as there are no bounded connected components of $\hat{K}$ and because K is essential. Also, the set $K' \subset D$ is filled.



(a) (b)

Fig. 2

To prove Theorem 2 it is enough to prove that for all $n \in \mathbb{N}$, every lift of f^n has a fixed point in $\hat{K}$ (see Corollary 1).

Lemma 12. *Let $g : A \to A$ be a covering map of degree d, $|d| > 1$, and let $K \subset A$ be an essential continuum such that $g(K) \subset K$. Then every lift of g to the universal covering $\tilde{A}$ has a fixed point in $\hat{K}$.*

Proof. The proof will be divided into three cases.

Case 1: *g is orientation preserving.* Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of g and let $H_G : \hat{K} \to \mathbb{R}$ be the map given by Lemma 6. Let $K_0 = H_G^{-1}(0)$ and recall from Lemmas 6 and 9 that $K_0 \neq \emptyset$, $K_0 \subset \hat{K}$, $G(K_0) \subset K_0$ and K_0 is a compact subset of $\mathbb{R}^2$. Suppose for a contradiction that $\mathrm{Fix}(G) \cap \hat{K} = \emptyset$. Let D be the compactification of $\mathbb{R} \times [0,1]$ with two points $-\infty$ and ∞. Note that D is a closed disk and that we may assume that G extends to the boundary of D, by Lemma 7. Moreover, the closure K' of $\hat{K}$ is a connected subset of D. Let P be the set of fixed points of G in the interior of D. Note that P does not accumulate at $-\infty$ or ∞. Define U as the connected component of $D \setminus P$ containing K', and let $(\tilde{U}, p)$ be the universal covering of U. Note that U is G-invariant, and we claim that there exists a lift $\tilde{G} : \tilde{U} \to \tilde{U}$ of $G|_U$ having a compact invariant set in the interior of $\tilde{U}$. To see this, take an open, connected and simply connected neighbourhood $V \subset U$ of K' (whose existence is guaranteed as the set is filled), and note that each connected component of $p^{-1}(V)$ is mapped homeomorphically onto V by p. Moreover, as K' is connected, there is only one connected component of $p^{-1}(K')$ in each connected component of $p^{-1}(V)$. Fix a connected component K'' of $p^{-1}(K')$ and take the lift $\tilde{G}$ of G such that $\tilde{G}(K'') = K''$. Note that $p^{-1}(H_G^{-1}(0)) \cap K''$ is $\tilde{G}$-invariant, compact and contained in the interior of $\tilde{U}$. So, as $\tilde{G}$ is orientation preserving, Brouwer's theorem gives a fixed point of $\tilde{G}$ in the interior of $\tilde{U}$. This is a contradiction because by definition, there are no fixed points in the interior of U.

Case 2: *g is orientation reversing and* $d < -1$. This has already been proved in Case 1 of the proof of Theorem 1.

Case 3: *g is orientation reversing and* $d > 1$. Let U_1 and U_2 be the connected components of $\tilde{A} \setminus \hat{K}$. Note that our hypothesis implies that $G(U_i) \cap U_i = \emptyset$, $i = 1, 2$. So, $\mathrm{Fix}(G) \subset \hat{K}$. It is then enough to prove that $\mathrm{Fix}(G) \neq \emptyset$. This has already been proved in Case 2 of the proof of Theorem 1. ■

5. Examples. In this section we exhibit a series of examples illustrating all the ideas in this article. Examples in Subsections 5.1, 5.4 and 5.6 are particularly interesting, regardless of their connection to the theorems of this paper.

5.1. Location of periodic orbits. Our first example shows that the periodic points given by Theorem 1 do not necessarily belong to K.

We will show that there exists a degree two covering map f of the annulus having an essential continuum K, totally invariant, which contains no fixed points of f.

We construct an isotopy from $f_0 = p_2$, $p_2(z) = z^2$, to $f_1 = f$ in the annulus $A = \mathbb{C} \setminus \{0\}$. For every t, $f_t(z) = f_0(z)$ for every z outside a neighbourhood V of the fixed point 1. Every f_t will be a homeomorphism from V to $f_0(V)$. For points in V, the restriction of f_t to V will have a unique fixed point at 1. Around this point f_t performs a Hopf bifurcation (see Figure 3). That is, for t close to 0, the eigenvalues at the fixed point 1 of

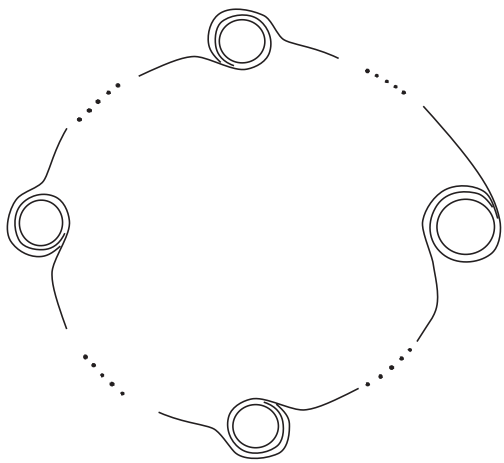


Fig. 3

f_t have nonzero imaginary part; the modulus is decreasing, and for t equal to $1/2$ the Hopf bifurcation takes place: the modulus of the eigenvalues is equal to 1, while the imaginary part is different from 0. Then let f_t for $t > 1/2$ be a generic family through the Hopf bifurcation. The following facts hold for f_1: 1 is an attracting fixed point, there is a repeller simple closed curve C where f is conjugate to a rotation with nonzero rotation number, and every point $z \in V \setminus \{1\}$ has a preorbit in V which converges to C.

Now let K be the boundary of the basin of ∞. Then K is a totally invariant essential continuum. It is clear that K contains no fixed point of $f = f_1$.

5.2. A fixed point free example having a point with zero rotation number. It may happen that $\lim_{n\to\infty} (F^n(x))_1/d^n = 0$ for some lift F of f and $x \in \tilde{A}$, but $\mathrm{Fix}(F) = \mathrm{Fix}(f) = \emptyset$. Just consider a degree 2 map preserving a ray of the annulus in which the dynamics is north-south, and lift it preserving a lift of that ray.

5.3. Changing the lift. This example shows that the map f may have a lift with fixed points and another lift which is fixed point free.

Let $f : [0, 2\pi] \times (0, 1) \to [0, 2\pi] \times (0, 1)$ with $f(\theta, r) = (3\theta, \phi(r, \theta))$, where ϕ fixes the rays $\theta = 0$ and $\theta = \pi$. On the ray $\theta = 0$, the dynamics of ϕ is as in Figure 4(a), and on the ray $\theta = \pi$, ϕ is as in Figure 4(b). So, $(0, 1/2)$ is fixed by f and one can lift f by fixing any of the lifts of $(0, 1/2)$. However, if you take a lift F of f fixing any preimage of $\theta = \pi$, then $\mathrm{Fix}(F) = \emptyset$.

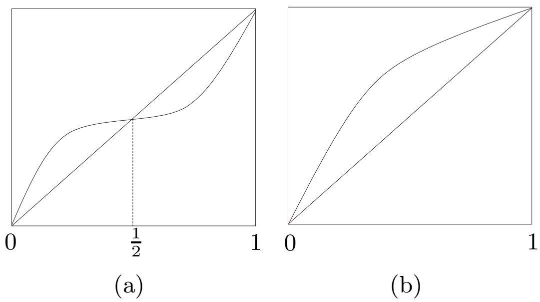


Fig. 4

5.4. Recurrence and periodic orbits. As in the fixed point free degree 2 covering example $(r, \theta) \mapsto (2r, 2\theta)$ every point is wandering, one may ask if the existence of a nonwandering point is enough to ensure the existence of a fixed point. The next example shows that this is not the case.

We will construct a degree 2 covering $f : (0, \infty) \times S^1 \to (0, \infty) \times S^1$ such that there is a compact set K satisfying $f(K) = K$ and $\mathrm{Per}(f) = \emptyset$. Of course, K must be inessential and disconnected (see Theorem 1 and

Lemma 5). In fact, in this example K is a Cantor set. We recall that in [IPRX] we showed that for a degree $d > 1$ covering g of the circle, $\overline{\mathrm{Per}(g)} = \Omega(g)$. This example also shows that this is no longer the case for annulus coverings.

We start with a degree 2 circle covering having a wandering interval. Let $g_1 : S^1 \to S^1$ be a Denjoy homeomorphism with a wandering interval I. Take an open interval $I_0 \subsetneq I$ and an increasing function $h : I \to S^1$ such that $h(I_0) = S^1$ and $h|_{I \setminus I_0} \equiv g_1$ (see Figure 5(a)). Let $g : S^1 \to S^1$ be the map

$$g(x) = \begin{cases} g_1(x) & \text{if } x \notin I, \\ h(x) & \text{if } x \in I. \end{cases}$$

So, g is a degree 2 covering of the circle and $g_1(I)$ is a wandering interval for g. Moreover, if $x_0 \in g_1(I)$ then $K_1 = \omega_g(x_0)$ is a Cantor set and $K_1 \cap \mathrm{Per}(g) = \emptyset$.

Our example $f : (0, \infty) \times S^1 \to (0, \infty) \times S^1$ has the form $f(r, \theta) = (\phi(r, \theta), g(\theta))$, where ϕ is to be constructed. Let $\psi : S^1 \to \mathbb{R}$, $\psi(\theta) = \mathrm{dist}(\theta, K_1)$, and let $\varphi : (0, \infty) \to (0, \infty)$ be as in Figure 5(b). Define $\phi(r, \theta) = \varphi(r) + r\psi(\theta)$.

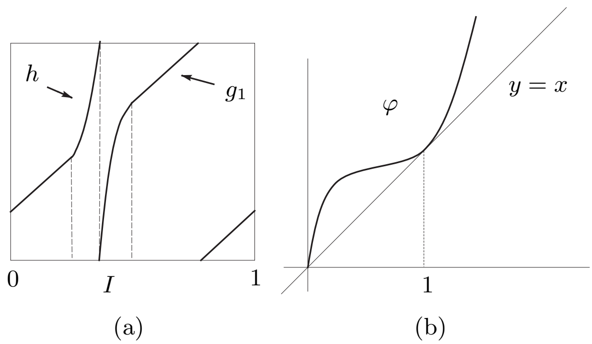


Fig. 5

Note that f has the following properties:

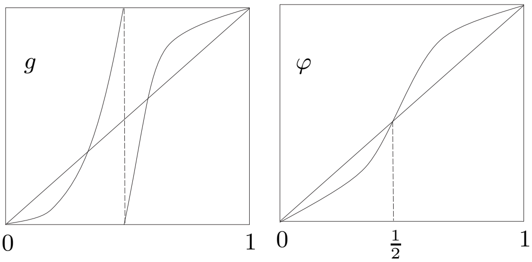
(1) For fixed θ, let $\phi_\theta(r) = \phi(r, \theta)$. Then ϕ_θ has fixed points if and only if $\theta \in K_1$, and for $\theta \in K_1$, ϕ_θ has a unique fixed point at $r = 1$.
(2) $K = \{1\} \times K_1$ is compact and $f(K) = K$.

Furthermore, $\mathrm{Per}(f) = \emptyset$. Indeed, if (r_0, θ_0) is f-periodic, then θ_0 must be g-periodic. So, $\theta_0 \notin K_1$. But this is imposible, as dynamics in the lines $\{(r, \theta) : r > 0,\ \theta \notin K_1\}$ is wandering.

Note also that this example can be made C^1 if we use the square of the distance in the definition of ψ, and other functions sufficiently regular.

5.5. Inessential totally invariant subset. We give an example of a degree 2 covering of the annulus with a Cantor set $K \subset A$ such that $f^{-1}(K) = K$. This implies, by Proposition 1(4) in the next section, that f is complete.

Let $g : S^1 \to S^1$ be as in Figure 6(a). Note that $\Omega(g) = \{0\} \cup K_1$ where 0 is an attracting fixed point and K_1 is an expanding Cantor set with $g^{-1}(K_1) = K_1$. Let $f : (0,1) \times S^1 \to (0,1) \times S^1$, $f(r,\theta) = (\varphi(r), g(\theta))$, where φ is as in Figure 6(b). Then $K = 1/2 \times K_1$ is a Cantor set and $f^{-1}(K) = K$.



(a) (b)

Fig. 6

5.6. Failure of Brouwer's theory. We construct a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ of Brouwer degree 1 such that $\Omega(F) \neq \emptyset$ and $\mathrm{Per}(F) = \emptyset$ (compare with Brouwer's Theorem 4).

Let $g : S^1 \to S^1$ be the degree 2 covering map of Subsection 5.4 (Figure 5(a)) and let K be the Cantor set such that $g(K) = K$ and $K \cap \mathrm{Per}(g) = \emptyset$. For any degree d, $|d| > 1$, covering $g : S^1 \to S^1$ there exists an increasing semiconjugacy h_1 between g and $q(z) = z^2$, that is, $h_1 g = q h_1$ [IPRX, Prop. 1].

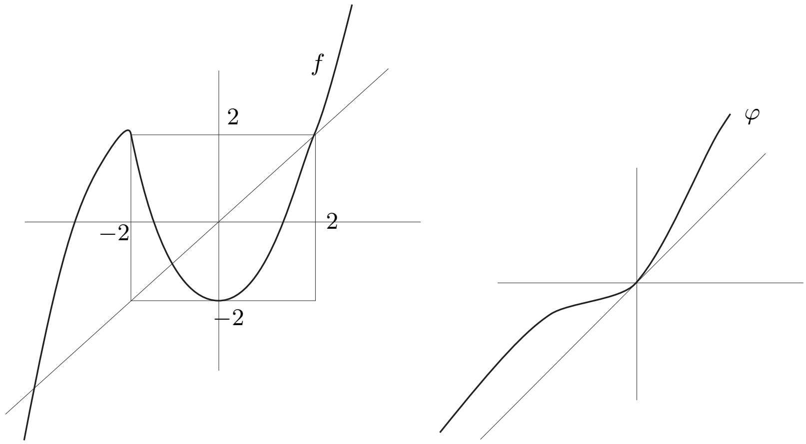
Then $h_1(K)$ is compact, q-invariant and $\mathrm{Per}(q) \cap h_1(K) = \emptyset$. Now, consider the maps $h_2 : S^1 \to [-2,2]$, $h_2(z) = z + 1/z$, and $p : [-2,2] \to [-2,2]$, $p(z) = z^2 - 2$. Note that h_2 is continuous, surjective and $h_2 q = p h_2$. Moreover:

- $K_1 = h_2(h_1(K))$ is compact and $p(K_1) \subset K_1$.
- $\mathrm{Per}(p) \cap K_1 = \emptyset$.

We need one more auxiliary function to make our example $F : \mathbb{R}^2 \to \mathbb{R}^2$ have degree 1. Let $f : \mathbb{R} \to \mathbb{R}$ be as in Figure 7(a), so that $f|_{[-2,2]} = z^2 - 2$. Now we proceed in the same fashion as in Subsection 5.4. Let $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = \mathrm{dist}(x, K_1)$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be as in Figure 7(b).

Define $\phi(x,y) = \varphi(y) + \psi(x)$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = (f(x), \phi(x,y)).$$



(a) (b)

Fig. 7

Note that $K_2 = K_1 \times \{0\}$ is compact and F-invariant, so $\Omega(F) \neq \emptyset$. However, $\mathrm{Per}(f) = \emptyset$ as the lines $\{(x, y) : x \notin K_1\}$ have wandering dynamics, and no line $\{(x, y) : x \in K_1\}$ is periodic.

5.7. Another example without periodic points. There are essentially two examples of covering maps of the annulus without periodic points: The first one, given in the introduction, is conjugate to $p_d(z) = z^d$ acting in the punctured unit disk. The second one was given in Subsection 5.4, in this case the map has nonempty nonwandering set. As in Subsection 5.3, examples with any finite number of periodic points can be constructed; the question is if there exist examples of covering maps which are not complete but satisfy the growth rate inequality for periodic points. Note also that if both ends are attracting or both repelling, then the map is complete. In all the examples of noncomplete maps, one end is attracting and the other repelling. Is this necessary? For example, consider the following concrete question: Let $f : S^1 \times (0, 1) \ni (z, x) \mapsto (z^3, \varphi_z(x))$, where φ_z is an increasing homeomorphism of the interval $(0, 1)$ for each z, such that $\varphi_1^n(x) \to 1$ and $\varphi_{-1}^n(x) \to 0$ for every x. This implies that the ends are neither attracting nor repelling and that the map is not complete (it has no fixed points). It will be shown here that a map like f can be constructed without any periodic point. This example was communicated to us by the referee.

Let $A = S^1 \times \mathbb{R}$ and assume that the map is given by

$$f(z, x) = (z^3, x + t_z)$$

where t_z varies continuously with z and the set of numbers $\{t_p\}$ with p periodic of m_3 is rationally independent. This means that f has no periodic points, because if $\{p_1, \ldots, p_n\}$ is a periodic orbit of z^3, then $f^n(p_1, x) = (p_1, x + \sum_i t_{p_i})$, but by assumption $\sum_i t_{p_i} \neq 0$. To construct the function t_z choose a rationally independent sequence $\{a_i : i \geq 0\}$ of positive numbers and enumerate the periodic points of m_3 as $\{p_n : n \geq 0\}$, with $p_0 = 1$, $p_1 = -1$. Define by induction a sequence of functions $z \mapsto t_z^n$ beginning with any continuous function t_z^0 such that $t_{p_0}^0 = a_0$ and $t_{p_1}^0 = 0$. Given $n > 0$ define t_z^n as follows: $t_z^n = t_z^{n-1}$ outside a neighbourhood of p_n not containing any p_i for $i < n$, $0 \leq t_z^{n-1} - t_z^n < 2^{-n}$ and $t_{p_n}^n \in a_n\mathbb{Q}$. The sequence of functions $z \mapsto t_z^n$ converges uniformly to a function $z \mapsto t_z$ satisfying the required properties.

6. Applications. We devote this section to applications of Theorem 1 and Lemma 8 to dynamics. Throughout this section, $f : A \to A$ is a degree d, $|d| > 1$, covering.

By *attracting set* we mean a proper open subset U such that $\overline{f(U)} \subset U$. A subset $X \subset A$ is *totally invariant* if $f^{-1}(X) = X$.

PROPOSITION 1. *Any of the following hypotheses implies that f is complete.*

(1) *There is an essential attracting set.*
(2) *Each end of A is attracting.*
(3) *f extends to a map of the two-point compactification of A in such a way that it is C^1 at the poles.*
(4) *f preserves orientation and there exists a compact totally invariant* (*not necessarily connected*) *subset.*
(5) *f preserves orientation and there exists an invariant continuum K such that $h(K)$ is not reduced to a point* (*h was defined in Corollary* 2).

Proof. (1) Let $U \subset A$ be an essential open set such that $\overline{f(U)} \subset U$. Then $K = \bigcap_{n \geq 0} f^n(U)$ is an invariant continuum. Moreover, it is essential, because $f^n(U)$ is essential for each n as $|d| > 1$. The result now follows from Theorem 1.

(2) If both ends are attracting, then the complement of both basins of attraction is an essential invariant continuum, and we may apply Theorem 1.

(3) Note that in this case both ends must be attracting, as the derivative in the compactification must be 0 at the poles. Indeed, note that f is a $d:1$ branched covering in a neighbourhood of the pole. So the winding number $I_{f\gamma}(p)$ is d whenever γ is a small circle with centre p. On the other hand, if a map g has a fixed point at p isolated in $g^{-1}(p)$ and nonvanishing differential at p, then $|I_{g\gamma}(p)| \leq 1$ for a small curve γ such that $I_\gamma(p) = 1$.

(4) Let X be a compact set such that $f^{-1}(X) = X$. It is enough to show that $m_d^{-1}(h(X)) = h(X)$. Indeed, $h(X)$ is then dense in S^1. As $h(X)$ is also

compact, $h(X) = S^1$. So, H_F (from Lemma 6) is surjective for any lift F of f. The same argument shows that H_G is surjective for any lift G of f^n, and we are done by Lemma 8 and Corollary 1.

To prove that $m_d^{-1}(h(X)) = h(X)$ we first claim that if $x \neq y$ and $f(x) = f(y)$, then $h(x) \neq h(y)$. Let F be a lift of f and $\tilde{x}$ a lift of x. Define $\tilde{y}_j = F^{-1}(\tilde{x} + j)$, $j = 0, \ldots, |d| - 1$. Then $\pi(\bigcup_{j=0}^{|d|-1} \tilde{y}_j) = f^{-1}(x)$. Moreover, $dH(\tilde{y}_j) = HF(\tilde{y}_j) = H(\tilde{x} + j) = H(\tilde{x}) + j$, and so $H(\tilde{y}_j) = H(\tilde{x})/d + j/d$. This proves the claim, because $\pi H(\tilde{y}_j) = h\pi(\tilde{y}_j)$ and $\{\pi H(\tilde{y}_j) : 0 \leq j \leq |d| - 1\}$ has $|d|$ different elements by the above computation.

It is obvious that $h(X) \subset m_d^{-1}(h(X))$ by the semiconjugacy equation. Conversely, let $z \in h(X)$; we will prove that $m_d^{-1}(z) \subset h(X)$. We have $z = h(x)$ for some $x \in X$. Let $f^{-1}(x) = \{x_1, \ldots, x_d\}$. By hypothesis $x_i \in X$ for all $i = 1, \ldots, d$. Now,

$$m_d h(x_i) = hf(x_i) = h(x) = z.$$

So, $h(x_1), \ldots, h(x_d) \in m_d^{-1}(z)$. As $m_d^{-1}(z)$ has exactly d elements, and $h(x_1), \ldots, h(x_d)$ are distinct by the claim, we have

$$m_d^{-1}(z) = \{h(x_1), \ldots, h(x_d)\}.$$

(5) Note that $h(K)$ is an invariant interval that is not reduced to a point, and so $h(K) = S^1$, which implies that H_F is surjective for any lift F of f, and we conclude as in the previous item. ■

The following application shows how the existence of a periodic orbit can imply existence of infinitely many of them. The proof is immediate from item (3) in the previous proposition.

Corollary 3. *Let $f : S^2 \to S^2$ be a C^1 degree d map, $|d| > 1$, and p, q a two-periodic totally invariant orbit ($f^{-1}(\{p, q\}) = \{p, q\}$, $f(p) = q$, $f(q) = p$). If $f : S^2 \setminus \{p, q\} \to S^2 \setminus \{p, q\}$ is a covering, then f has periodic points of arbitrarily large period.*

We will make some calculations that will be used in the following lemma.

Fix a lift $F = F_0$ of f, and for any $k \in \mathbb{Z}$, define the maps $F_k(x) = F(x) + k$. Then, for any $m \in \mathbb{N}$ and $x \in \tilde{A}$,

$$F_k^m(x) = F^m(x) + \sum_{i=0}^{m-1} kd^i = F^m(x) + \frac{k(1 - d^m)}{1 - d}. \tag{1}$$

This is a straightforward consequence of the fact that $F(x + k) = F(x) + dk$ for any $k \in \mathbb{Z}$ and $x \in \tilde{A}$.

Lemma 13. *Suppose that there exists a compact set $K \subset A$ such that $f(K) \subset K$. If there exists $x \in K$ and a lift F of f such that*

$$\lim_{m \to \infty} (F^m(\tilde{x}))_1/d^m = k/(d^n - 1)$$

for some $k \in \mathbb{Z}$ and $n \geq 1$, for a lift $\tilde{x}$ of x, then there exists $z \in \tilde{A}$ such that $F^n(z) = z + k$. In particular, $\mathrm{Per}(f) \neq \emptyset$.

Proof. We have to show that the map $F^n - k$ has a fixed point. By Remark 3, it is enough to show that $\lim_{m\to\infty} (G^m(\tilde{x}))_1/(d^n)^m = 0$, where $G = F^n - k$ (see Remark 3 and note that G is a lift of f^n). Indeed

$$\lim_{m\to\infty} \frac{(G^m(\tilde{x}))_1}{(d^n)^m} = \lim_{m\to\infty} \frac{((F^n - k)^m(\tilde{x}))_1}{d^{nm}} \overset{(1)}{=} \lim_{m\to\infty} \frac{(F^{nm}(\tilde{x}))_1 - \sum_{i=0}^{m-1} kd^{ni}}{d^{nm}}$$
$$= \frac{k}{d^n - 1} - \lim_{m\to\infty} \frac{k}{d^{nm}} \sum_{i=0}^{m-1} d^{ni}.$$

Now,

$$\lim_{m\to\infty} \frac{k}{d^{nm}} \sum_{i=0}^{m-1} d^{ni} = \lim_{m\to\infty} \frac{k}{d^{nm}} \frac{1 - d^{nm}}{1 - d^n} = \frac{k}{d^n - 1}. \blacksquare$$

Proposition 2. *Any of the following hypotheses implies that f has periodic points.*

1. *There exists $x \in A$ with bounded forward orbit such that $\lim_{m\to\infty} (F^m(\tilde{x}))_1/d^m = k/(d^n - 1)$ for some lift $\tilde{x}$ of x and some $k \in \mathbb{Z}$ and $n \geq 1$.*
2. *There exists an invariant continuum.*

Proof. (1) Let K be the closure of the forward orbit of x. Then K is compact, $f(K) \subset K$, and $\lim_{m\to\infty} (F^m(\tilde{x}))_1/d^m = k/(d^n - 1)$ for a lift $\tilde{x}$ of x and some $k \in \mathbb{Z}$ and $n \geq 1$. Then $\mathrm{Per}(f) \neq \emptyset$ by Lemma 13.

(2) If the continuum happens to be essential, then f is complete by Theorem 1. Otherwise, if f preserves orientation, this follows from Lemma 5. If f reverses orientation, one applies Kuperberg's theorem [K]. $\blacksquare$

We finish this paper with an open question: if the covering assumption is dropped, so f is just a degree d map of the annulus ($|d| > 1$), and if K is an invariant essential continuum, is f necessarily complete?

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