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ON THE CONTINUATION OF THE LIMIT DISTRIBUTION OF CENTRAL ORDER STATISTICS UNDER POWER NORMALIZATION

Abstract. An important stability property of central order statistics under power normalization is discussed. It is proved that the restricted convergence of power normalized central order statistics on an arbitrary nondegenerate interval implies their weak convergence.

1. Introduction. For a long time, the limit distribution functions (df's) of order statistics with fixed rank sequence (i.e., extremes) or with variable rank sequence (i.e., intermediate and central order statistics) were obtained by using linear normalization $G_n(x) = a_n x + b_n$, where $a_n > 0$. The advantage of using this traditional transformation is that it provides us with a sufficiently simple approximation for the exact df's of order statistics. However, Pantcheva (1985) showed that any nonlinear strictly monotone continuous transformation may serve to construct a simplified approximation, provided one can prove a suitable limit theorem. In the last two decades Pantcheva and her collaborators have been investigating various limit theorems for extremes and extremal processes using a wider class of normalizing mappings than the linear ones to get a wider class of limit laws. This wider class of extreme limit laws can be used in solving approximation problems. Another reason for using nonlinear normalization is to refine the accuracy of approximation in the limit theorems. Actually, by using relatively nondifficult monotone mappings in certain cases we may achieve a better rate of convergence: see e.g. Weinstein (1973) and Barakat et al. (2010). Although no one can claim that the employment of nonlinear normalization is prefer-

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able in general, Pantcheva (1994) (and many other authors) showed that in some cases of practical interest it is not only better to use a nonlinear transformation, but we have to use it.

Pantcheva (1985) considered the power normalization $G_n(x) = b_n\,|x|^{a_n} \times \operatorname{sign}(x)$, $a_n, b_n > 0$, to derive all the possible limit distributions of maximum order statistics. These limit distributions are usually called p-max stable df's. We say that two df's Ψ_1 and Ψ_2 are *of the same power type* (p-type) if for some $A, B > 0$, $\Psi_1(x) = \Psi_2(A|x|^B \operatorname{sign}(x))$ for all x. Mohan and Ravi (1992) showed that the p-max stable df's (six p-types of df's) attract more df's than linear max stable df's. Therefore, using power normalization, we get a wider class of limit df's which can be used in solving approximation problems.

Barakat and Omar (2011) extended the work of Pantcheva (1985) to order statistics with variable ranks. They showed that, unlike the case of extreme order statistics, the class of possible limit df's of central order statistics under linear normalization (this class contains four types) coincides with the class of possible limit df's of central order statistics under power normalization.

Specifically, let $X_1, \dots, X_n$ be i.i.d. random variables (rv's) with common df $F(x) = P(X_n \le x)$. Furthermore, let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the order statistics of $X_1, \dots, X_n$. Thus, $X_{r:n}$ is the rth order statistic with rank r. If $r = r_n \to \infty$ as $n \to \infty$ and $r_n/n \to \lambda$, $0 < \lambda < 1$, then r is called a *central rank*.

Assume $\sqrt{n}(r/n - \lambda) \to 0$ as $n \to \infty$, $0 < \lambda < 1$. Then, following Smirnov (1952), a df F is said to belong to the *domain of normal λ-attraction* of a nondegenerate df Φ, written $F \in D_\lambda(\Phi)$, if there exist normalizing constants $a_n > 0$ and b_n such that

$$\Phi_{\lambda:n}(a_n x + b_n) = P(X_{r:n} \le a_n x + b_n) \xrightarrow[n]{w} \Phi(x),$$

where $\xrightarrow[n]{w}$ stands for weak convergence as $n \to \infty$ (everywhere in what follows, $\xrightarrow[n]{}$ means convergence as $n \to \infty$). Smirnov (1952) showed that the class of limit laws of linearly normalized central order statistics consists of the following types:

(i) $\Phi_1(x; c, \alpha) = \mathcal{N}(cx^\alpha) I_{[0,\infty)}(x)$, $c, \alpha > 0$.
(ii) $\Phi_2(x; c, \alpha) = \mathcal{N}(-c(-x)^\alpha) I_{(-\infty,0)}(x) + I_{[0,\infty)}(x)$, $c, \alpha > 0$.
(iii) $\Phi_3(x; c_1, c_2, \alpha) = \mathcal{N}(-c_1(-x)^\alpha) I_{(-\infty,0)}(x) + \mathcal{N}(c_2 x^\alpha) I_{[0,\infty)}(x)$, $c_1, c_2, \alpha > 0$.
(iv) $\Phi_4(x) = \frac{1}{2} I_{[-1,1)}(x) + I_{[1,\infty)}(x)$,

where $\mathcal{N}(\cdot)$ denotes the standard normal distribution. Moreover $F \in D_\lambda(\Phi_i)$,

$i \in \{1,2,3,4\}$, if and only if

$$\sqrt{n}\,\frac{F(a_n x + b_n) - \lambda}{C_\lambda} \underset{n}{\rightarrow} \mathcal{N}^{-1}(\Phi_i(x)),$$

where $C_\lambda = \sqrt{\lambda(1-\lambda)}$.

Barakat and Omar (2011) showed that the possible limit types for central order statistics under power normalization are

$$\Psi_1(x) = \Phi_1(x;1,1),\ \Psi_2(x) = \Phi_2(x;1,1),\ \Psi_3(x) = \Phi_3(x;c_1,c_2,1),\ \Psi_4(x),$$

where $\Psi_4(x)$ is a family of df's which consists of the following six power types:

$$\begin{aligned}
\Phi_4^{[1]}(x) &= \frac{1}{2} I_{[-A,A)}(x) + I_{[A,\infty)}(x),\\
\Phi_4^{[2]}(x) &= \frac{1}{2} I_{[-A,B)}(x) + I_{[B,\infty)}(x),\\
\Phi_4^{[3]}(x) &= \frac{1}{2} I_{[-A,0)}(x) + I_{[0,\infty)}(x),\\
\Phi_4^{[4]}(x) &= \frac{1}{2} I_{[0,A)}(x) + I_{[A,\infty)}(x),\\
\Phi_4^{[5]}(x) &= \frac{1}{2} I_{[A,B)}(x) + I_{[B,\infty)}(x),\\
\Phi_4^{[6]}(x) &= \frac{1}{2} I_{[-A,-B)}(x) + I_{[-B,\infty)}(x),
\end{aligned}$$

where $A, B > 0$.

REMARK 1.1. Note that, under power normalization, the function $c|x|^\alpha$ has the same type as $|x|$, while $\Phi_3(x;c_1,c_2,1)$ represents a family of two power types, which correspond to the cases $c_1 \neq c_2$ and $c_1 = c_2$, respectively.

The main aim of this paper is to prove the weak convergence continuation property of the limit df of central order statistics under power normalization.

In the last two decades the subject of continuation of convergence either in sums of independent rv's or in order statistics gained considerable importance in probability theory and its applications. Perhaps one of the most important reasons for this is that when we get a sample for studying any random quantity, we are often faced with a major difficulty that the range of values of that sample is limited. Therefore, the data actually enables us to identify the limit df of the given random quantity only on a finite interval. This difficulty becomes serious in some situations, such as medical research and for agencies which regulate food or drug safety standards. However, the proof of continuation of convergence of these possible limits allows us to overcome this difficulty and to deal with the identified limit distribution on the whole real line regardless of the length of the interval to which the data belongs.

The theory of continuation of convergence started with the work of Rossberg and Siegel (1975), in which an elegant hypothesis due to V. M. Zolotarev is proved. This hypothesis states that if the distribution of the normalized sum of i.i.d. rv's converges weakly to the normal distribution, then this convergence holds on the whole real line. More recently this result has been generalized in various directions (e.g., Riedel, 1977 and Rossberg, 1995). Moreover, some pertaining results concerning the asymptotic theory of order statistics have been obtained (e.g., Gnedenko, 1983, Gnedenko and Senocy, 1982, 1983, Barakat, 1997, 2000 and Barakat and Ramchandran, 2001). Barakat et al. (2002) proved that the restricted convergence of power normalized extremes on an arbitrary nondegenerate interval implies weak convergence. More recently, Barakat et al. (2003) proved the continuation property of power normalized extremes with random sample indices.

We end this introductory section with a definition and a lemma, which help us establish our results.

Definition 1.1. Let $\{F_n\}_n$ be a sequence of df's. Then the *restricted convergence* $F_n(x) \xrightarrow[n]{S} F(x)$, where S is a set of real numbers and F is a nondecreasing function, means that the convergence of $\{F_n\}_n$ to the limit F is restricted to S, for all continuity points of F. Moreover, a function $F(x)$ is said to be *nondegenerate* on S if it has at least two growth points on S.

Lemma 1.1. *Let $\{u_n, n \geq 1\}$ be a sequence of constants and $0 \leq \tau \leq \infty$. Then*

$$\varPhi_{\lambda:n}(u_n) \underset{n}{\to} \mathcal{N}(\tau) \quad \text{if and only if} \quad \sqrt{n}\frac{[F(u_n)-\lambda]}{C_\lambda} \underset{n}{\to} \tau.$$

Using the power parametrization $u_n = a_n|x|^{b_n}\operatorname{sign}(x)$ and $\tau = V_i(x)$, $i \in \{1,2,3,4\}$, where $V_i(x)$ is defined by $\mathcal{N}^{-1}(\varPsi_i(x))$, Lemma 1.1 gives a necessary and sufficient condition for $\varPhi_{\lambda:n}(a_n|x|^{b_n}\operatorname{sign}(x)) \xrightarrow[n]{w} \varPsi_i(x) = \mathcal{N}(V_i(x))$.

2. Main results

Theorem 2.1. *Let $F(x)$ be a df for which there exist real constants $a_n > 0$ and $b_n > 0$ such that*

$$\varPhi_{\lambda:n}(a_n|x|^{b_n}\operatorname{sign}(x)) \xrightarrow[n]{[c,d]} \varPhi^*(x), \tag{2.1}$$

where $\sqrt{n}(r/n-\lambda) \underset{n}{\to} 0$ and $\varPhi^*(x)$ is any nondecreasing (right continuous) function, which has at least two nonzero growth points in (c,d). Moreover, assume that $\varPhi^*(x)$ takes at least two different values in $[c,d]$, one of which is less than $1/2$, while the other is greater than $1/2$. Then $\varPhi_{\lambda:n}(a_n|x|^{b_n}\operatorname{sign}(x)) \xrightarrow[n]{w} \varPhi(x)$, where $\varPhi(x) = \varPhi^*(x)$ for all $x \in [c,d]$. Moreover, $\varPhi(x) = \mathcal{N}(V_i(x))$,

$i \in \{1,2,3,4\}$, i.e., $\Phi(x)$ has one and only one of the types $\Phi_1(x;1,1)$, $\Phi_2(x;1,1)$, $\Phi_3(x;c_1,c_2,1)$, $\Phi_4^{[1]}(x)$, $\Phi_4^{[2]}(x)$, $\Phi_4^{[5]}(x)$ or $\Phi_4^{[6]}(x)$.

REMARK 2.1. The assumption that the two growth points are nonzero implies that the types $\Phi_4^{[3]}$ and $\Phi_4^{[4]}$ will be excluded in the proof of Theorem 2.1.

Proof of Theorem 2.1. Since the proof is somewhat lengthy, we split it up into several steps, some of which are of independent interest.

STEP 1. *Under the conditions of Theorem* 2.1, *the sequence* $\{\Phi_{\lambda:n}(a_n|x|^{b_n} \times \operatorname{sign}(x))\}_n$ *is stochastically bounded.*

Proof. It is sufficient to show that, for any subsequence $\{n_k\}$ for which

$$\Phi_{\lambda:n_k}(a_{n_k}|x|^{b_{n_k}}\operatorname{sign}(x)) \xrightarrow[k]{\mathbb{R}} \tilde{\Phi}(x), \tag{2.2}$$

where $\tilde{\Phi}(x)$ is a nondecreasing right continuous function, we must have $\tilde{\Phi}(-\infty)=0$ and $\tilde{\Phi}(\infty)=1$. It is easy to see that, in view of Lemma 1.1, the last two equalities are equivalent to $\tilde{V}(-\infty)=-\infty$ and $\tilde{V}(\infty)=\infty$, respectively, where $\tilde{V}(x)=\mathcal{N}^{-1}(\tilde{\Phi}(x))$ for $x\in\mathbb{R}$, $\tilde{V}(x)=V^*(x)$ for $x\in[c,d]$ and $V^*(x)=\mathcal{N}^{-1}(\Phi^*(x))$ for $x\in[c,d]$. By using Lemma 1.1, we get the limit relations

$$V_n(a_n|x|^{b_n}\operatorname{sign}(x)) = \frac{\sqrt{n}(F(a_n|x|^{b_n}\operatorname{sign}(x))-\lambda)}{C_\lambda} \xrightarrow[n]{[c,d]} V^*(x)$$

and

$$V_{n_k}(a_{n_k}|x|^{b_{n_k}}\operatorname{sign}(x)) = \frac{\sqrt{n_k}(F(a_{n_k}|x|^{b_{n_k}}\operatorname{sign}(x))-\lambda)}{C_\lambda} \xrightarrow[k]{\mathbb{R}} \tilde{V}(x).$$

Now, for any positive real number t, (2.1) implies that

$$\Phi_{\lambda:[nt]}(a_{[nt]}|x|^{b_{[nt]}}\operatorname{sign}(x)) \xrightarrow[n]{[c,d]} \Phi^*(x),$$

which again, by Lemma 1.1, is equivalent to

$$V_{\lambda:[nt]}(a_{[nt]}|x|^{b_{[nt]}}\operatorname{sign}(x)) \xrightarrow[n]{[c,d]} V^*(x).$$

Clearly, if there exists $x\in[c,d]$ for which $\Phi^*(x)=0$ or $\Phi^*(x)=1$, then $\tilde{\Phi}(-\infty)=0$ or $\tilde{\Phi}(\infty)=1$, respectively. Therefore (without any loss of generality), we assume that $0<\Phi^*(x)<1$ for all $x\in[c,d]$, or equivalently $-\infty<V^*(x)<\infty$ for all $x\in[c,d]$. This implies

$$\frac{F(a_{[nt]}|x|^{b_{[nt]}}\operatorname{sign}(x))-\lambda}{C_\lambda} \xrightarrow[n]{[c,d]} 0.$$

Therefore,

$$V_n(a_{[nt]}|x|^{b_{[nt]}}\operatorname{sign}(x)) \xrightarrow[n]{[c,d]} \frac{V^*(x)}{\sqrt{t}}, \tag{2.3}$$

which by Lemma 1.1 yields

$$\varPhi_{\lambda,n}(a_{[nt]}|x|^{b_{[nt]}}\operatorname{sign}(x)) \xrightarrow[n]{[c,d]} \mathcal{N}\Big(\frac{V^*(x)}{\sqrt{t}}\Big). \tag{2.4}$$

From (2.1), (2.2) and (2.4) and by applying Lemma 4 of Barakat et al. (2002), we deduce that there exist real functions $\alpha(t), \beta(t) > 0$ such that

$$\mathcal{N}\Big(\frac{V^*(x)}{\sqrt{t}}\Big) = \tilde{\varPhi}(\alpha(t)|x|^{\beta(t)}\operatorname{sign}(x)) = \mathcal{N}\big(\tilde{V}(\alpha(t)|x|^{\beta(t)}\operatorname{sign}(x))\big).$$

Therefore,

$$V^*(x) = \sqrt{t}\,\tilde{V}(\alpha(t)|x|^{\beta(t)}\operatorname{sign}(x)), \qquad \forall x \in [c,d]. \tag{2.5}$$

Now, if the assumption that $\tilde{V}(-\infty) = -\infty$ is violated, (2.5) implies that $\sqrt{t}\,\tilde{V}(-\infty) \le V^*(x)$ for all $x \in [c,d]$, for arbitrarily small values of t. Hence, letting $t \to 0$ we get $V^*(x) \ge 0$, i.e. $\varPhi^*(x) \ge 1/2$ for all $x \in [c,d]$, which contradicts our assumptions. Furthermore, if $\tilde{V}(\infty) < \infty$, (2.5) leads to $\sqrt{t}\,\tilde{V}(\infty) \ge V^*(x)$ for all $x \in [c,d]$, for arbitrarily small values of t. Therefore, $V^*(x) \le 0$ for all $x \in [c,d]$, i.e., $\varPhi^*(x) \le 1/2$ for all $x \in [c,d]$, which again contradicts our assumptions.

STEP 2. *If there exist $t' < t''$ such that $0 < t' < 1 < t'' < \infty$, $c \le \alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c) < d$ and $c < \alpha(t')|d|^{\beta(t')}\operatorname{sign}(d) \le d$, then $\varPhi^*(c) = 0$ and $\varPhi^*(d) = 1$, which immediately proves Theorem* 2.1.

Proof. If such t' and t'' exist then, by (2.5), we have

$$V^*(c) \le V^*(\alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c)) = \tilde{V}\big(\alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c)\big) = \frac{V^*(c)}{\sqrt{t''}},\ t''>1,$$

$$V^*(d) \ge V^*(\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d)) = \tilde{V}\big(\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d)\big) = \frac{V^*(d)}{\sqrt{t'}},\ t'<1.$$

These relations hold only if $V^*(c) = \infty$ (i.e., $\varPhi^*(c) = 1$), or $V^*(c) = 0$ (i.e., $\varPhi^*(c) = 1/2$), or $V^*(c) = -\infty$ (i.e., $\varPhi^*(c) = 0$), and $V^*(d) = -\infty$ (i.e., $\varPhi^*(d) = 0$), or $V^*(d) = 0$ (i.e., $\varPhi^*(d) = 1/2$), or $V^*(d) = \infty$ (i.e., $\varPhi^*(d) = 1$). Clearly, the first two values in each of the above cases contradict our assumptions: e.g., when $V^*(c) = \infty$ or equivalently $\varPhi^*(c) = 1$ we get $\varPhi^*(x) > 1/2$ for all $x \in [c,d]$, and when $V^*(d) = -\infty$ or equivalently $\varPhi^*(d) = 0$ we get $\varPhi^*(x) < 1/2$ for all $x \in [c,d]$. Hence $\varPhi^*(c) = 0$ and $\varPhi^*(d) = 1$ as required.

STEP 3. *Under the conditions of Theorem* 2.1, *there are no* $0<t'<1<t''<\infty$ *such that* $d \le \alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c)$ *or* $\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d) \le c$.

Proof. If such t', t'' exist then, by (2.5), we obtain

$$\tilde{V}\big(\alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c)\big) \ge \tilde{V}(c) = V^*(c) = \sqrt{t''}\big(\tilde{V}(\alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c))\big), t''>1,$$

$$\tilde{V}\big(\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d)\big) \le \tilde{V}(d) = V^*(d) = \sqrt{t'}\big(\tilde{V}(\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d))\big), t'<1,$$

which hold only if $\tilde{V}(\alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c))=-\infty$, 0, or ∞ and $\tilde{V}(\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d)) = -\infty$, 0, or ∞. The first value in the above two cases gives $\Phi^*(d) = 0$. Indeed, in the first case, $0 = \tilde{\Phi}(\alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c)) \geq \tilde{\Phi}(d) = \Phi^*(d)$, while in the second case $V^*(d) = \sqrt{t'}\,\tilde{V}(\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d)) = -\infty$. When the second value 0 is assumed, we get $1/2 = \tilde{\Phi}(\alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c)) \geq \tilde{\Phi}(d) = \Phi^*(d)$ in the first case, while in the second case we have $1/2 = \tilde{\Phi}(\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d)) \leq \tilde{\Phi}(c) = \Phi^*(c)$. Finally, the third value (∞) implies that $\Phi^*(c) = 1$. Indeed, $V^*(c) = \sqrt{t''}\,\tilde{V}(\alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c)) = \infty$ in the first case, while in the second case $1 = \tilde{\Phi}(\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d)) \leq \tilde{\Phi}(c) = \Phi^*(c)$. Therefore, all the preceding cases contradict the assumptions of Theorem 2.1, and hence Step 3 is proved.

Combining Steps 2 and 3 immediately yields

STEP 4.

(i) *If there exists* $1 < t'' < \infty$ *such that* $c \leq \alpha(t'')|c|^{\beta(t'')}\operatorname{sign}(c)$, *then* $\Phi^*(c) = 0$, *which implies continuation of convergence in* (2.1) *to the left.*

(ii) *If there exists* $0 < t' < 1$ *such that* $\alpha(t')|d|^{\beta(t')}\operatorname{sign}(d) \leq d$, *then* $\Phi^*(d) = 1$, *which implies continuation of convergence in* (2.1) *to the right.*

(iii) *If there exist both* t' *and* t'' *as above, convergence in* (2.1) *will continue weakly, for all* x, *to a nondegenerate df which coincides with* Φ^* *on* $[c,d]$, *and Theorem* 2.1 *follows from the result of Barakat and Omar* (2011).

STEP 5. *Under the conditions of Theorem* 2.1, *there exist at least two growth points* $x_1, x_2 \in (c,d)$. *Assume that* $x_1 < x_2$ *and* $d < \alpha(t)|d|^{\beta(t)}\operatorname{sign}(d)$ *for all* $t < 1$. *Then there exists* $t' < 1$ (*in fact, there exist infinitely many* $t' < 1$) *such that*

$$\alpha(t')|c|^{\beta(t')}\operatorname{sign}(c) < x_1 < d < \alpha(t')|d|^{\beta(t')}\operatorname{sign}(d). \tag{2.6}$$

Proof. Since the limit $\tilde{\Phi}(x)$ in (2.2) is a nondegenerate df, by the power central types theorem it must be of the form $\mathcal{N}(\tilde{V}(x))$ and $\tilde{V}(x)$ must be one and only one of the types $\mathcal{N}^{-1}(\Phi_1(x;1,1))$, $\mathcal{N}^{-1}(\Phi_2(x;1,1))$, $\mathcal{N}^{-1}(\Phi_3(x;c_1,c_2,1))$ or $\mathcal{N}^{-1}(\Phi_4^{[i]}(x))$, $i = 1,2,5,6$. Furthermore, by using the modified Khintchine convergence theorem (Lemma 3 in Barakat et al., 2002) it is easy to prove that $\tilde{V}(x) = \sqrt{t}\,\tilde{V}(\alpha(t)|c|^{\beta(t)}\operatorname{sign}(c))$ for all $x \in \mathbb{R}$ and $t > 0$. On the other hand, it is easy to show that $\alpha(t)$ and $\beta(t)$ are continuous and monotonic functions of t. Indeed, a quick check shows that $\alpha(t) = 1/\sqrt{t}$ for the first three types of central limit laws under power normalization, and $\alpha(t) = 1$ for $\Phi_4^{[i]}(x)$, $i = 1,2,5,6$ (see Remark 1.1). Moreover, $\beta(t) = 1$ for all types.

Let us now define a continuous function $f_c(t) = \alpha(t)|c|^{\beta(t)}\operatorname{sign}(c)$. Clearly $f_c(1) = c$. Hence, there exists $\delta > 0$ such that, whenever $t' < 1$ and $0 < 1 - t' < \delta$, we have $|f_c(t') - f_c(1)| = |\alpha(t')|c|^{\beta(t')}\operatorname{sign}(c) - c| < x_1 - c$, which implies that for all $1-\delta < t' < 1$ (there are infinitely many such t'), we have $\alpha(t')|c|^{\beta(t')}\operatorname{sign}(c) < x_1$, which completes the proof.

Step 6. *Assume that, for all* $t < 1$, *we have* $d < \alpha(t)|d|^{\beta(t)}\operatorname{sign}(d)$. *Then the convergence in* (2.1) *will continue weakly, for all* x, *to the right* (*i.e., for all* $x > d$).

Proof. Let T be the set of all $t < 1$ which satisfy the condition (2.6). Henceforth, we consider only those values $t \in T$. Furthermore, let us consider the following cases:

1. There exists $t \in T$ such that $\beta(t) < 1$.
2. There exists $t \in T$ such that $\beta(t) = 1$.
3. For all $t \in T$, we have $\beta(t) > 1$.

Case 1. Clearly, we have $d < (\alpha(t))^{1/(1-\beta(t))}\operatorname{sign}(d)$. If we show that the convergence of the sequence $\{\Phi_{\lambda:n}(a_n|x|^{b_n}\operatorname{sign}(x))\}_n$ continues to the point $\mathcal{P} = (\alpha(t))^{1/(1-\beta(t))}\operatorname{sign}(d)$ (note that $\operatorname{sign}(\mathcal{P}) = \operatorname{sign}(d)$) then, by Step 4(ii), the convergence will continue, for all x, to the right (since $\mathcal{P} = \alpha(t)|\mathcal{P}|^{\beta(t)}\operatorname{sign}(\mathcal{P})$). Indeed, by (2.4) and (2.5), we have

$$\Phi_{\lambda:n}(a_{[nt]}|x|^{b_{[nt]}}\operatorname{sign}(x)) \xrightarrow[n]{[c,d]} \tilde{\Phi}(\alpha(t)|x|^{\beta(t)}\operatorname{sign}(x)). \tag{2.7}$$

Setting $y = \alpha(t)|x|^{\beta(t)}\operatorname{sign}(x)$, we get

$$\Phi_{\lambda:n}(a_n(t)|y|^{b_n(t)}\operatorname{sign}(y)) \xrightarrow[n]{[c_1,d_1]} \tilde{\Phi}(y), \tag{2.8}$$

where $a_n(t) = a_{[nt]}\{\alpha(t)\}^{-b_{[nt]}/\beta(t)}$, $b_n(t) = b_{[nt]}/\beta(t)$, $c_1 = \alpha(t)|c|^{\beta(t)}\operatorname{sign}(c)$, and $d_1 = \alpha(t)|d|^{\beta(t)}\operatorname{sign}(d)$. Since $\tilde{\Phi}(x) = \Phi^*(x)$ for all $x \in [c,d] \cap [c_1,d_1]$ and $\Phi^*(x)$ has more than two different values in the interval $[c,d] \cap [c_1,d_1]$, by application of Lemma 3 of Barakat et al. (2002) to (2.1) and (2.8) we get

$$\left(\frac{a_n(t)}{a_n}\right)^{1/b_n} \underset{n}{\to} 1 \quad \text{and} \quad \frac{b_n(t)}{b_n} \underset{n}{\to} 1.$$

By a further application of Lemma 3 of Barakat et al. (2002), we see that the sequences $\{a_n(t)\}_n$ and $\{b_n(t)\}_n$ in (2.8) may be replaced, respectively, by a_n and b_n. Thus, we get

$$\Phi_{\lambda:n}(a_n|y|^{b_n}\operatorname{sign}(y)) \xrightarrow[n]{[c_1,d_1]} \tilde{\Phi}(y),$$

which by (2.1) leads to

$$\Phi_{\lambda:n}(a_n|y|^{b_n}\operatorname{sign}(y)) \xrightarrow[n]{[c,d_1]} \tilde{\Phi}(y).$$

Repeating this argument N times yields

$$\Phi_{\lambda:n}(a_n|y|^{b_n}\operatorname{sign}(y)) \xrightarrow[n]{[c,d_N]} \tilde{\Phi}(y),$$

where

$$\begin{aligned} d_N &= (\alpha(t))^{1+\beta(t)+\cdots+\beta^{N-1}(t)}|d|^{\beta^N(t)}\operatorname{sign}(d) \\ &= (\alpha(t))^{\frac{1-\beta^N(t)}{1-\beta(t)}}|d|^{\beta^N(t)}\operatorname{sign}(d) \underset{N}{\longrightarrow} \mathcal{P}. \end{aligned}$$

Therefore, due to the continuity of $\tilde{V}(y)$ for all y, the proof of Step 6 follows in this case.

Case 2. In this case $\alpha(t) \neq 1$ (in fact $\alpha(t) < 1$ if $d < 0$, and $\alpha(t) > 1$ if $d > 0$). If we set $y = \alpha(t)|x|\operatorname{sign}(x)$ with $\beta(t) = 1$ in (2.7), we obtain

$$\Phi_{\lambda:n}(a'_n(t)|y|^{b'_n(t)}\operatorname{sign}(y)) \xrightarrow[n]{[c'_1,d'_1]} \tilde{\Phi}(y), \tag{2.9}$$

where $a'_n(t) = a_{[nt]}(\alpha(t))^{-b_{[nt]}}$, $b'_n(t) = b_{[nt]}$, $c'_1 = \alpha(t)c$ and $d'_1 = \alpha(t)d$. An application of Lemma 3 of Barakat et al. (2002) to (2.1) and (2.9) thus yields

$$\left(\frac{a'_n(t)}{a_n}\right)^{1/b_n} \underset{n}{\longrightarrow} 1 \quad \text{and} \quad \frac{b'_n(t)}{b_n} \underset{n}{\longrightarrow} 1.$$

Furthermore, $\tilde{\Phi}(x) = \Phi^*(x)$ for all $x \in [c,d] \cap [c'_1, d'_1]$ and $\Phi^*(x)$ has more than two different values in $[c,d] \cap [c'_1, d'_1]$. By a further application of Lemma 3 of Barakat et al. (2002), the sequences $a'_n(t)$ and $b'_n(t)$ in (2.9) may be replaced, respectively, by a_n and b_n. Thus, we get

$$\Phi_{\lambda:n}(a_n|y|^{b_n}\operatorname{sign}(y)) \xrightarrow[n]{[c'_1,d'_1]} \tilde{\Phi}(y),$$

which, in view of (2.1), leads to

$$\Phi_{\lambda:n}(a_n|y|^{b_n}\operatorname{sign}(y)) \xrightarrow[n]{[c,d'_1]} \tilde{\Phi}(y).$$

By repeating this procedure N times, we obtain

$$\Phi_{\lambda:n}(a_n|y|^{b_n}\operatorname{sign}(y)) \xrightarrow[n]{[c,d'_N]} \tilde{\Phi}(y),$$

where $d'_N = \alpha^N(t)d \underset{n}{\longrightarrow} \mathcal{P}^*$, with $\mathcal{P}^* = \infty$ if $d > 0$, and $\mathcal{P}^* = 0$ if $d \leq 0$. In the case $\mathcal{P}^* = \infty$, the proof follows immediately, while in the case $\mathcal{P}^* = 0$, we use the result of Step 3 (since $\alpha(t)|\mathcal{P}^*|^{\beta(t)}\operatorname{sign}(\mathcal{P}^*) \leq \mathcal{P}^*$ is trivially satisfied).

Case 3. In this case, in view of the assumption $d < \alpha(t)|d|^{\beta(t)}\operatorname{sign}(d)$, we have $\alpha(t)|d|^{\beta(t)-1} > 1$ if $d > 0$, and $\alpha(t)|d|^{\beta(t)-1} < 1$ if $d < 0$. Therefore, by using the same argument as in Case 1, we can prove that

$$\Phi_{\lambda:n}(a_n|y|^{b_n}\operatorname{sign}(y)) \xrightarrow[n]{[c,d''_N]} \tilde{\Phi}(y),$$

where

$$\begin{aligned} d''_N &= (\alpha(t))^{\frac{\beta^N(t)-1}{\beta(t)-1}}|d|^{\beta^N(t)}\operatorname{sign}(d) \\ &= (\alpha(t))^{-\frac{1}{\beta(t)-1}}(\alpha(t)|d|^{\beta(t)-1})^{\frac{\beta^N(t)}{\beta(t)-1}}\operatorname{sign}(d) \xrightarrow[N]{} \begin{cases} \infty, & d>0, \\ 0, & d\le 0. \end{cases} \end{aligned}$$

This implies that the convergence in (2.1) will continue to $\tilde{\Phi}(x)$ for all $x > d$ (to the right), which completes the proof.

STEP 7. *Assume* $\alpha(t)|c|^{\beta(t)}\operatorname{sign}(c) < c$ *for all* $t > 1$. *Then there exists* $t' > 1$ (*in fact there are infinitely many* $t' > 1$) *such that*

$$\alpha(t')|c|^{\beta(t')}\operatorname{sign}(c) < c < x_2 < \alpha(t')|d|^{\beta(t')}\operatorname{sign}(d).$$

Proof. The proof follows closely the proof of Step 5, with obvious changes.

STEP 8. *Assume* $\alpha(t)|c|^{\beta(t)}\operatorname{sign}(c) < c$ *for all* $t > 1$. *Then the convergence in* (2.1) *will continue weakly, for all* x, *to the left* (*i.e. to* $-\infty$).

Proof. The proof follows closely the proof of Step 6, with obvious changes. The proof of Theorem 2.1 is completed by this step. ∎

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