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On Lusternik–Schnirelmann category of $\mathbf{SO}(10)$

by

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Abstract. Let G be a compact connected Lie group and $p : E \to \Sigma A$ be a principal G-bundle with a characteristic map $\alpha : A \to G$, where $A = \Sigma A_0$ for some A_0. Let $\{K_i \to F_{i-1} \hookrightarrow F_i \mid 1 \leq i \leq m\}$ with $F_0 = \{*\}$, $F_1 = \Sigma K_1$ and $F_m \simeq G$ be a cone-decomposition of G of length m and $F'_1 = \Sigma K'_1 \subset F_1$ with $K'_1 \subset K_1$ which satisfy $F_i F'_1 \subset F_{i+1}$ up to homotopy for all i. Then $\mathrm{cat}(E) \leq m+1$, under suitable conditions, which is used to determine $\mathrm{cat}(\mathbf{SO}(10))$. A similar result was obtained by Kono and the first author (2007) to determine $\mathrm{cat}(\mathbf{Spin}(9))$, but that result could not yield $\mathrm{cat}(E) \leq m+1$.

1. Introduction. Throughout the paper, we work in the homotopy category of based CW-complexes, and often identify a map with its homotopy class.

The *Lusternik–Schnirelmann category* of a connected space X, denoted by $\mathrm{cat}(X)$, is the least integer n such that there is an open covering $\{U_i \mid 0 \leq i \leq n\}$ of X with each U_i contractible in X. If no such integer exists, we write $\mathrm{cat}(X) = \infty$. Let R be a commutative ring with unit. The *cup-length* of X with respect to R, denoted by $\mathrm{cup}(X; R)$, is the supremum of all non-negative integers k such that there is a non-zero k-fold cup product in the ordinary reduced cohomology $\tilde{H}^*(X; R)$.

In 1967, Ganea [3] introduced a strong category $\mathrm{Cat}(X)$ by modifying Fox's strong category (see Fox [2]), which is characterized as follows: for a connected space X, $\mathrm{Cat}(X)$ is 0 if X is contractible and, otherwise, is equal to the smallest integer n such that there is a series of cofibre sequences $\{K_i \to F_{i-1} \hookrightarrow F_i \mid 1 \leq i \leq m\}$ with $F_0 = \{*\}$ and $F_m \simeq X$ (a *cone-*

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decomposition of length m). $\mathrm{Cat}(X)$ is often called the *cone-length* of X. The following theorem is well-known.

Theorem 1.1 (Ganea [3]). $\mathrm{cup}(X;R) \le \mathrm{cat}(X) \le \mathrm{Cat}(X)$.

In 1968, Berstein and Hilton [1] gave a criterion for $\mathrm{cat}(C_f) = 2$ in terms of their Hopf invariant $H_1(f) \in [\Sigma X, \Omega\Sigma Y * \Omega\Sigma Y]$ for a map $f : \Sigma X \to \Sigma Y$, where $A * B$ denotes the join of the spaces A and B. In addition, its higher version H_m was used to disprove the Ganea conjecture (see Iwase [6, 8]).

We summarize here known L-S categories of special orthogonal groups: since $\mathbf{SO}(2) = S^1$, $\mathbf{SO}(3) = \mathbb{R}P^3$ and $\mathbf{SO}(4) = \mathbb{R}P^3 \times S^3$, we know that

$$\mathrm{cat}(\mathbf{SO}(2)) = 1, \quad \mathrm{cat}(\mathbf{SO}(3)) = 3, \quad \mathrm{cat}(\mathbf{SO}(4)) = 4.$$

In 1999, James and Singhof [12] gave the first non-trivial result:

$$\mathrm{cat}(\mathbf{SO}(5)) = 8.$$

In 2005, Mimura, Nishimoto and the first author [11] gave an alternative proof of $\mathrm{cat}(\mathbf{SO}(5)) = 8$ and determined $\mathrm{cat}(\mathbf{SO}(n))$ up to $n = 9$:

$$\mathrm{cat}(\mathbf{SO}(6)) = 9,\ \mathrm{cat}(\mathbf{SO}(7)) = 11,\ \mathrm{cat}(\mathbf{SO}(8)) = 12,\ \mathrm{cat}(\mathbf{SO}(9)) = 20.$$

Let $G \hookrightarrow E \to \Sigma A$ be a principal bundle with a characteristic map $\alpha : A \to G$, where A is a suspension space and G is a connected compact Lie group with a cone-decomposition of length m, i.e., there is a series of cofibre sequences $\{K_i \to F_{i-1} \hookrightarrow F_i \mid 1 \le i \le m\}$ with $F_0 = \{*\}$, $F_1 \simeq \Sigma K_1$ and $F_m \simeq G$. Then the multiplication of G is, up to homotopy, a map $\mu : F_m \times F_m \to F_m$, since $G \simeq F_m$. The main result of this paper is as follows.

Theorem 1.2. *Let $F_1' = \Sigma K_1'$, where K_1' is a connected subspace of K_1 such that F_1' is simply-connected, and let $\mu|_{F_i \times F_1'} : F_i \times F_1' \to F_m$ be compressible into $F_{i+1} \subset F_m$ as $\mu_{i,1} : F_i \times F_1' \to F_{i+1}$, $1 \le i < m$, so that $\mu_{i,1}|_{F_{i-1} \times F_1'} \sim \mu_{i-1,1}$ in F_{i+1}. Then the following three conditions together imply* $\mathrm{cat}(E) \le m + 1$*:*

(1) *α is compressible into F_1',*
(2) *$H_1(\alpha) = 0$ in $[A, \Omega F_1' * \Omega F_1']$,*
(3) *$K_m = S^{\ell-1}$ with $m, \ell \ge 3$.*

Remark. Under the conditions in Theorem 1.2, [9, Theorem 0.8] does *not* imply $\mathrm{cat}(E) \le m+1$, but only $\mathrm{cat}(E) \le m+2$, since its key lemma [9, Lemma 1.1] cannot properly manage the case when $\mathrm{im}\,\alpha \subset F_1$.

Theorem 1.2 yields the following result on the L-S category of $\mathbf{SO}(10)$.

Theorem 5.1. $\mathrm{cat}(\mathbf{SO}(10)) = \mathrm{cup}(\mathbf{SO}(10); \mathbb{F}_2) = 21$.

All the results on $\mathrm{cat}(\mathbf{SO}(n))$ with $n \le 10$ support the following "folklore conjecture".

Conjecture 1. $\mathrm{cat}(\mathbf{SO}(n)) = \mathrm{cup}(\mathbf{SO}(n); \mathbb{F}_2)$.

Let us explain the method we employ in this paper. To study L-S category, we must understand Ganea's criterion of L-S category as a basic idea, given in terms of a fibre-cofibre construction in [3]: Let X be a connected space. Then there is a fibre sequence $F_nX \hookrightarrow G_nX \to X$, natural with respect to X, such that $\mathrm{cat}(X) \le n$ if and only if the fibration $G_nX \to X$ has a cross-section.

However, four years before [3], a more understandable description of the fibre sequence $F_n(X) \hookrightarrow G_n(X) \to X$ was already published by Stasheff [15]: following [6, 7, 8], we may replace the inclusion $F_nX \hookrightarrow G_nX$ with the fibration $p_n^{\Omega X} : E^{n+1}\Omega X \to P^n\Omega X$ associated with the A_∞-structure of ΩX, the based loop space of X in the sense of Stasheff, where $E^{n+1}\Omega X$ has the homotopy type of $(\Omega X)^{*(n+1)}$, the $n+1$-fold join of ΩX, and $P^n\Omega X$ satisfies $P^0\Omega X = *$, $P^1\Omega X = \Sigma\Omega X$ and $P^\infty\Omega X \simeq X$. Let $\iota_{m,n}^{\Omega X} : P^m\Omega X \hookrightarrow P^n\Omega X$ be the canonical inclusion, for $m \le n$, and $e_\infty^X : P^\infty\Omega X \simeq X$ be the natural equivalence. Then the fibration $G_nX \to X$ may be replaced with the map $e_n^X = e_\infty^X \circ \iota_{n,\infty}^{\Omega X} : P^n\Omega X \to X$, where $e_1^X : \Sigma\Omega X \to X$ equals the evaluation.

Thus, we may restate Ganea's criterion as follows: Let X be a connected space. Then $\mathrm{cat}(X) \le n$ if and only if $e_n^X : P^n\Omega X \to X$ has a right homotopy inverse. That is why we use A_∞-structures to determine L-S category.

In this paper, instead of using [9, Lemma 1.1], we show Proposition 2.4 and Lemmas 3.3, 4.4. This is a key process to obtain Theorem 1.2. In Sections 2 and 3, we construct a structure map associated to a given cone-decomposition. In Section 4, we introduce a map $\hat{\lambda}$ from $\hat{F}_{m+1} = P_m^m \times \Sigma\Omega F_1'$ to $P^{m+1}\Omega F_m$, which is the main tool to construct a complex D with $\mathrm{Cat}(D) \le m+1$ dominating E. Finally, in Section 5 we prove Theorem 5.1.

2. Structure map associated with cone-decomposition. In this section, we generalize the following well-known fact to the case of filtered spaces and maps.

FACT 2.1. *Let* $K \xrightarrow{a} A \hookrightarrow C(a)$ *and* $L \xrightarrow{b} B \hookrightarrow C(b)$ *be cofibre sequences with canonical copairings* $\nu : C(a) \to C(a) \vee \Sigma K$ *and* $\hat{\nu} : C(b) \to C(b) \vee \Sigma L$. *If there are maps* $f : A \to B$ *and* $f^0 : K \to L$ *such that* $f \circ a = b \circ f^0$, *then they induce a map* $f' : C(a) \to C(b)$ *satisfying* $(f' \vee \Sigma f^0) \circ \nu = \hat{\nu} \circ f'$.

DEFINITION 2.2. A space X with a series of subspaces $\{X_n; n \ge 0\}$,

$$\{*\} = X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \subset X,$$

is said to be *filtered* by $\{X_n; n \ge 0\}$ and denoted by $(X, \{X_n\})$. We also denote by $i_{m,n}^X : X_m \hookrightarrow X_n$, $m < n$, the canonical inclusion.

Definition 2.3. Let X and Y be spaces filtered by $\{X_n\}$ and $\{Y_n\}$, respectively. A map $f : X \to Y$ is a *filtered map* if $f(X_n) \subset Y_n$ for all n.

Proposition 2.4. *Let X and Y be filtered by $\{X_n\}$ and $\{Y_n\}$, respectively, and $f : X \to Y$ be a filtered map. If $\{X_n\}$ is a cone-decomposition of X, i.e. there is a series of cofibre sequences $\{K_n \xrightarrow{h_n} X_{n-1} \overset{i^X_{n-1,n}}{\hookrightarrow} X_n \mid n \geq 1\}$ with $X_0 = *$, then there exist families $\{\hat{f}_n : X_n \to P^n\Omega Y_n \mid n \geq 0\}$ and $\{\hat{f}^0_n : K_n \to E^n\Omega Y_n \mid n \geq 0\}$ of maps such that:*

(1) *The following diagram is commutative:*

$$\begin{tikzcd}
K_n \arrow[r, "h_n"] \arrow[dd, "\hat{f}^0_n"'] & X_{n-1} \arrow[r, hook, "i^X_{n-1,n}"] \arrow[d, "\hat{f}_{n-1}"] & X_n \arrow[dd, "\hat{f}_n"] \arrow[ddr, "f|_{X_n}"] & \\
 & P^{n-1}\Omega Y_{n-1} \arrow[d, hook, "P^{n-1}\Omega i^Y_{n-1,n}"] & & \\
E^n\Omega Y_n \arrow[r, "p^{\Omega Y_n}_{n-1}"'] & P^{n-1}\Omega Y_n \arrow[r, hook, "\iota^{\Omega Y_n}_{n-1,n}"'] & P^n\Omega Y_n \arrow[r, "e^{Y_n}_n"'] & Y_n
\end{tikzcd}$$

(2) *Denote by $f'_n = (P^{n-1}\Omega i^Y_{n-1,n} \circ \hat{f}_{n-1}) \cup C(\hat{f}^0_n) : X_n \to P^n\Omega Y_n$ the induced map from the commutativity of the left square in* (1). *Then the middle square in* (1) *with $\hat{f}_n$ replaced with f'_n is commutative. The difference of $\hat{f}_n$ and f'_n is given by a map $\delta^f_n : \Sigma K_n \to P^{n-1}\Omega Y_n$ composed with the inclusion $\iota^{\Omega Y_n}_{n-1,n} : P^{n-1}\Omega Y_n \hookrightarrow P^n\Omega Y_n$, $n \geq 1$.*

Proof. First of all, we set $\hat{f}_0 = *$, the trivial map.

Next, we use induction on $n \geq 1$. When $n = 1$, we set $\hat{f}^0_1 = \mathrm{ad}(f|_{X_1})$ and $\hat{f}_1 = \Sigma\,\mathrm{ad}(f|_{X_1}) = f'_1$ to obtain the commutative diagram

$$\begin{tikzcd}
K_1 \arrow[r] \arrow[d, "\hat{f}^0_1"'] & * \arrow[r] \arrow[d, "\hat{f}_0"] & \Sigma K_1 \arrow[d, "\hat{f}_1"] \arrow[dr, "f|_{X_1}"] & \\
\Omega Y_1 \arrow[r] & * \arrow[r] & \Sigma\Omega Y_1 \arrow[r, hook, "e^{Y_1}_1"'] & Y_1
\end{tikzcd}$$

Then (1) is clear, and (2) is trivial in this case.

When $n = k > 1$, suppose we have already obtained $\{\hat{f}_i\}$ and $\{\hat{f}^0_i\}$ for $i < k$, which satisfy conditions (1) and (2).

Firstly, we define $\hat{f}^0_k : K_k \to E^k\Omega Y_k$ as follows: The homotopy class of a map $P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1} \circ h_k : K_k \to P^{k-1}\Omega Y_k$ can be described as

$$h_{k*}(P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1}) \in [K_k, Y_k] \quad \text{with} \quad P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1} \in [X_{k-1}, Y_k]$$

in the following ladder of exact sequences induced from the fibre sequence

$E^k\Omega Y_k \to P^{k-1}\Omega Y_k \to Y_k$:

$$\begin{array}{ccccc}
[X_{k-1}, E^k\Omega Y_k] & \xrightarrow{p^{\Omega Y_k}_{k-1\,*}} & [X_{k-1}, P^{k-1}\Omega Y_k] & \xrightarrow{e^{Y_k}_{k-1\,*}} & [X_{k-1}, Y_k] \\
\Big\downarrow{\scriptstyle h_k^*} & & \Big\downarrow{\scriptstyle h_k^*} & & \Big\downarrow{\scriptstyle h_k^*} \\
[K_k, E^k\Omega Y_k] & \xrightarrow{p^{\Omega Y_k}_{k-1\,*}} & [K_k, P^{k-1}\Omega Y_k] & \xrightarrow{e^{Y_k}_{k-1\,*}} & [K_k, Y_k]
\end{array}$$

Since we know that the naturality of e^Z_{k-1} at Z implies $e^{Y_k}_{k-1} \circ P^{k-1}\Omega i^Y_{k-1,k} = i^Y_{k-1,k} \circ e^{Y_{k-1}}_{k-1}$, that the induction hypothesis implies $e^{Y_{k-1}}_{k-1} \circ \hat{f}_{k-1} = f|_{X_{k-1}}$ and that the naturality of $i^Z_{k-1,k}$ at Z implies $i^Y_{k-1,k} \circ f|_{X_{k-1}} = f|_{X_k} \circ i^X_{k-1,k}$, we obtain $e^{Y_k}_{k-1*}(P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1}) = i^Y_{k-1,k} \circ e^{Y_{k-1}}_{k-1} \circ \hat{f}_{k-1} = f|_{X_k} \circ i^X_{k-1,k} \in [X_{k-1}, Y_k]$. On the other hand, since $K_k \to X_{k-1} \hookrightarrow X_k$ is a cofibre sequence, we get

$$e^{Y_k}_{k-1*}(h_k^*(P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1})) = [f|_{X_k} \circ i^X_{k-1,k} \circ h_k] = 0.$$

Thus we have $e^{Y_k}_{k-1*}(P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1} \circ h_k) = 0$, and there exists a map $\hat{f}^0_k : K_k \to E^k\Omega Y_k$ such that $p^{\Omega Y_k}_{k-1*}(\hat{f}^0_k) = P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1} \circ h_k$, which implies the commutativity of the left square in (1).

Secondly, let $f'_k : X_k \to P^k\Omega Y_k$ be the map induced from the commutativity of the left square in (1). By the induction hypothesis, we have

$$\begin{aligned}
(i^X_{k-1,k})^*(e^{Y_k}_k \circ f'_k) &= [e^{Y_k}_k \circ f'_k \circ i^X_{k-1,k}] = [e^{Y_k}_k \circ \iota^{\Omega Y_k}_{k-1,k} \circ P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1}] \\
&= [i^Y_{k-1,k} \circ e^{Y_{k-1}}_{k-1} \circ \hat{f}_{k-1}] = [i^Y_{k-1,k} \circ f|_{X_{k-1}}] = [f|_{X_k} \circ i^X_{k-1,k}] = (i^X_{k-1,k})^*(f|_{X_k}).
\end{aligned}$$

By a standard argument of homotopy theory applied to the cofibre sequence $K_k \to X_{k-1} \hookrightarrow X_k$ (see Hilton [5] or Oda [13]), there is a map $\delta^{f,0}_k : \Sigma K_k \to Y_k$ such that

$$f|_{X_k} = \nabla_{Y_k} \circ (e^{Y_k}_k \circ f'_k \vee \delta^{f,0}_k) \circ \nu_k,$$

where $\nabla_Y : Y \vee Y \to Y$ denotes the folding map for a space Y and $\nu_k : X_k \to X_k \vee \Sigma K_k$ denotes the canonical copairing.

Let $\delta^f_k = \iota^{\Omega Y_k}_{1,k-1} \circ \Sigma\,\mathrm{ad}(\delta^{f,0}_k) : \Sigma K_k \to \Sigma\Omega Y_k \hookrightarrow P^{k-1}\Omega Y_k$. Since $e^{Y_k}_1 = e^{Y_k}_{k-1} \circ \iota^{\Omega Y_k}_{1,k-1}$, we have $\delta^{f,0}_k = e^{Y_k}_1 \circ \Sigma\mathrm{ad}(\delta^{f,0}_k) = e^{Y_k}_{k-1} \circ \delta^f_k$. Hence, the map $\hat{f}_k = \nabla_{P^k\Omega Y_k} \circ (f'_k \vee \iota^{\Omega Y_k}_{k-1,k} \circ \delta^f_k) \circ \nu_k$ satisfies condition (2).

Thirdly, by using the above homotopy relations, we obtain

$$\begin{aligned}
f|_{X_k} &= \nabla_{Y_k} \circ (e^{Y_k}_k \circ f'_k \vee e^{Y_k}_{k-1} \circ \delta^f_k) \circ \nu_k \\
&= e^{Y_k}_k \circ \nabla_{P^k\Omega Y_k} \circ (f'_k \vee \iota^{\Omega Y_k}_{k-1,k} \circ \delta^f_k) \circ \nu_k = e^{Y_k}_k \circ \hat{f}_k.
\end{aligned}$$

This implies the commutativity of the right triangle in (1).

Finally, since ν_k is a copairing, we have
$pr_1 \circ \nu_k \circ i^X_{k-1,k} = 1_{X_k} \circ i^X_{k-1,k} = i^X_{k-1,k}$ and $pr_2 \circ \nu_k \circ i^X_{k-1,k} = q \circ i^X_{k-1,k} = *$, where $pr_1 : X_k \vee \Sigma K_k \to X_k$ and $pr_2 : X_k \vee \Sigma K_k \to \Sigma K_k$ are the first and second projections, respectively. Then

$$\begin{aligned}\hat{f}_k \circ i^X_{k-1,k} &= \nabla_{P^k\Omega Y_k} \circ (f'_k \vee \iota^{\Omega Y_k}_{k-1,k} \circ \delta^f_k) \circ \nu_k \circ i^X_{k-1,k}\\ &= f'_k \circ i^X_{k-1,k} = \iota^{\Omega Y_k}_{k-1,k} \circ P^{k-1}\Omega i^Y_{k-1,k} \circ \hat{f}_{k-1},\end{aligned}$$

which implies the commutativity of the middle square in (1). This completes the induction step for $n = k$, and we obtain the proposition for all n. ∎

Corollary 2.4.1. *Let $\hat{\nu}_n : P^n\Omega Y_n \to P^n\Omega Y_n \vee \Sigma E^n\Omega Y_n$ be the canonical copairing. If K_n is a co-H-space, then the following diagram is commutative:*

$$\begin{tikzcd} X_n \arrow[r, "\nu_n"] \arrow[d, "\hat{f}_n"'] & X_n \vee \Sigma K_n \arrow[d, "\hat{f}_n \vee \Sigma \hat{f}^0_n"] \\ P^n\Omega Y_n \arrow[r, "\hat{\nu}_n"] & P^n\Omega Y_n \vee \Sigma E^n\Omega Y_n \end{tikzcd}$$

Proof. Let P and E denote $P^n\Omega Y_n$ and $E^n\Omega Y_n$, respectively. By Proposition 2.4(2), the difference of $\hat{f}_n$ and f'_n is given by $\iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n$, and hence

$$\begin{aligned}(\hat{f}_n \vee \Sigma\hat{f}^0_n) \circ \nu_n &= \{(\nabla_P \circ (f'_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n) \circ \nu_n) \vee \Sigma\hat{f}^0_n\} \circ \nu_n\\ &= (\nabla_P \vee 1_{\Sigma E}) \circ (f'_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n \vee \Sigma\hat{f}^0_n) \circ (\nu_n \vee 1_{\Sigma K_n}) \circ \nu_n.\end{aligned}$$

Since K_n is a co-H-space, we have the following homotopy relations:

$$\upsilon_n = T \circ \upsilon_n \quad \text{and} \quad (\nu_n \vee 1_{\Sigma K_n}) \circ \nu_n = (1_{X_n} \vee \upsilon_n) \circ \nu_n,$$

where $\upsilon_n : \Sigma K_n \to \Sigma K_n \vee \Sigma K_n$ is the comultiplication and where $T : \Sigma K_n \vee \Sigma K_n \to \Sigma K_n \vee \Sigma K_n$ is the switching map. Hence

$$\begin{aligned}(\hat{f}_n \vee \Sigma\hat{f}^0_n) \circ \nu_n &= (\nabla_P \vee 1_{\Sigma E}) \circ (f'_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n \vee \Sigma\hat{f}^0_n) \circ (1_{X_n} \vee \upsilon_n) \circ \nu_n\\ &= (\nabla_P \vee 1_{\Sigma E}) \circ (f'_n \vee (\iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n \vee \Sigma\hat{f}^0_n)) \circ (1_{X_n} \vee T \circ \upsilon_n) \circ \nu_n\\ &= (\nabla_P \vee 1_{\Sigma E}) \circ \{f'_n \vee T' \circ (\Sigma\hat{f}^0_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n)\} \circ (\nu_n \vee 1_{\Sigma K_n}) \circ \nu_n\\ &= (\nabla_P \vee 1_{\Sigma E}) \circ (1_P \vee T') \circ \{(f'_n \vee \Sigma\hat{f}^0_n) \circ \nu_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n\} \circ \nu_n,\end{aligned}$$

where $T' : \Sigma E \vee P \to P \vee \Sigma E$ is the switching map. Then we can easily see that $(\nabla_P \vee 1_{\Sigma E}) \circ (1_P \vee T') = \nabla_{P \vee \Sigma E} \circ \mathrm{in}_{\Sigma E}$, where, for any space Y, we denote by $\mathrm{in}_{\Sigma E} : Y \hookrightarrow Y \vee \Sigma E$ the first inclusion. Hence

$$\begin{aligned}(\hat{f}_n \vee \Sigma\hat{f}^0_n) \circ \nu_n &= \nabla_{P\vee\Sigma E} \circ \mathrm{in}_{\Sigma E} \circ \{(f'_n \vee \Sigma\hat{f}^0_n) \circ \nu_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n\} \circ \nu_n\\ &= \nabla_{P\vee\Sigma E} \circ \{(f'_n \vee \Sigma\hat{f}^0_n) \circ \nu_n \vee \mathrm{in}_{\Sigma E} \circ \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n\} \circ \nu_n.\end{aligned}$$

Here, since the copairing $\hat{\nu}_n$ is associated to the cofibre sequence

$$P^{n-1}\Omega Y_n \overset{\iota^{\Omega Y_n}_{n-1,n}}{\hookrightarrow} P^n\Omega Y_n \to \Sigma E^n\Omega Y_n,$$

we have the following equality up to homotopy:

$$\hat{\nu}_n \circ \iota^{\Omega Y_n}_{n-1,n} = \mathrm{in}_{\Sigma E} \circ \iota^{\Omega Y_n}_{n-1,n} : P^{n-1}\Omega Y_n \hookrightarrow P^n\Omega Y_n \hookrightarrow P^n\Omega Y_n \vee \Sigma E^n\Omega Y_n.$$

Then, by Theorem 2.1,

$$\begin{aligned}(\hat{f}_n \vee \Sigma\hat{f}^0_n) \circ \nu_n &= \nabla_{P\vee\Sigma E} \circ \{(f'_n \vee \Sigma\hat{f}^0_n) \circ \nu_n \vee \hat{\nu}_n \circ \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n\} \circ \nu_n \\ &= \nabla_{P\vee\Sigma E} \circ (\hat{\nu}_n \circ f'_n \vee \hat{\nu}_n \circ \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n) \circ \nu_n \\ &= \hat{\nu}_n \circ \nabla_P \circ (f'_n \vee \iota^{\Omega Y_n}_{n-1,n} \circ \delta^f_n) \circ \nu_n = \hat{\nu}_n \circ f_n. \ \blacksquare\end{aligned}$$

3. Cone-decomposition associated with projective spaces. Let G be a compact Lie group of dimension ℓ with a cone-decomposition of length m, that is, there is a series of cofibre sequences

$$\{K_i \xrightarrow{h_i} F_{i-1} \hookrightarrow F_i \mid 1 \le i \le m\} \tag{3.1}$$

with $F_0 = \{*\}$ and $F_m \simeq G$. We also denote by $i^F_{i-1,i} : F_{i-1} \hookrightarrow F_i$ the canonical inclusion and by $q^F_{i-1,i} : F_i \to F_i/F_{i-1} = \Sigma K_i$ its successive quotient.

LEMMA 3.1. *If $K_m = S^{\ell-1}$ with $m, \ell \ge 3$, then:*

(1) *$(E^m\Omega F_m, E^m\Omega F_{m-1})$ is an ℓ-connected pair.*
(2) *There exists an ℓ-connected map $\hat{\phi}_S : P^m_m = P^m\Omega F_{m-1} \cup CS^{\ell-1} \to P^m\Omega F_m$ extending the inclusion $P^m\Omega F_{m-1} \hookrightarrow P^m\Omega F_m$.*

Proof. Let $q_E : \mathfrak{F}_E \to E^m\Omega F_{m-1}$, $q_P : \mathfrak{F}_P \to P^{m-1}\Omega F_{m-1}$ and $q_F : \mathfrak{F}_F \to F_{m-1}$ be homotopy fibres of inclusion maps $E^m\Omega i^F_{m-1,m}$, $P^{m-1}\Omega i^F_{m-1,m}$ and $i^F_{m-1,m}$, respectively, which fit in with the following commutative diagram of fibre sequences. Thus we obtain a fibre sequence $\mathfrak{F}_E \to \mathfrak{F}_P \to \mathfrak{F}_F$:

$$\begin{tikzcd}
\mathfrak{F}_E \arrow[r] \arrow[d, "q_E"] & \mathfrak{F}_P \arrow[r] \arrow[d, "q_P"] & \mathfrak{F}_F \arrow[d, "q_F"'] \\
E^m\Omega F_{m-1} \arrow[r, "p^{\Omega F_{m-1}}_{m-1}"] \arrow[d, hook, "E^m\Omega i^F_{m-1,m}"] & P^{m-1}\Omega F_{m-1} \arrow[r, "e^{F_{m-1}}_{m-1}"] \arrow[d, hook, "P^{m-1}\Omega i^F_{m-1,m}"] & F_{m-1} \arrow[d, hook, "i^F_{m-1,m}"] \\
E^m\Omega F_m \arrow[r, "p^{\Omega F_m}_{m-1}"] & P^{m-1}\Omega F_m \arrow[r, "e^{F_m}_{m-1}"] & F_m
\end{tikzcd}$$

Firstly, since the pair (F_m, F_{m-1}) is $(\ell-1)$-connected, $(\Omega F_m, \Omega F_{m-1})$ is $(\ell-2)$-connected and $(E^m\Omega F_m, E^m\Omega F_{m-1})$ is $(\ell+m-3)$-connected. Therefore, $\mathfrak{F}_F$ is $(\ell-2)$-connected and $\mathfrak{F}_E$ is $(\ell+m-4)$-connected. We remark that $\mathfrak{F}_E$ is at least $(\ell-1)$-connected, since $m \ge 3$, Then, by the homotopy exact sequence for the fibre sequence $\mathfrak{F}_E \to \mathfrak{F}_P \to \mathfrak{F}_F$,

$$\pi_k(\mathfrak{F}_P) \cong \pi_k(\mathfrak{F}_F), \quad k \le \ell - 1,$$

and hence $\mathfrak{F}_P$ is $(\ell-2)$-connected. Thus $\mathfrak{F}_P$ is 1-connected, since $\ell \geq 3$. By a general version of the Blakers–Massey Theorem (see [4, Corollary 16.27], for example) and the hypothesis that $K_m = S^{\ell-1}$, it follows that

$$\pi_{\ell-1}(\mathfrak{F}_P) \cong \pi_{\ell-1}(\mathfrak{F}_F) \cong \pi_\ell(F_m, F_{m-1}) \cong \pi_\ell(\varSigma K_m) \cong \pi_\ell(S^\ell) \cong \mathbb{Z}.$$

Thus, $\mathfrak{F}_P$ has the following homology decomposition, up to homotopy:

$$\mathfrak{F}_P = (S^{\ell-1} \vee S^\ell \vee \cdots \vee S^\ell) \cup (\text{cells in dimension} \geq \ell+1).$$

Secondly, $P^{m-1}\varOmega F_{m-1} \cup_{q_P} C\mathfrak{F}_P$ is described as the homotopy pushout of $q_P : \mathfrak{F}_P \to P^{m-1}\varOmega F_{m-1}$ and the trivial map $* : \mathfrak{F}_P \to \{*\}$. Then we obtain

$$\tag{3.2} \begin{array}{ccc} P^{m-1}\varOmega F_{m-1} \cup_{q_P} C\mathfrak{F}_P & \longrightarrow & \begin{array}{c} P^{m-1}\varOmega F_{m-1} \times P^{m-1}\varOmega F_m \\ \cup\, P^{m-1}\varOmega F_m \times \{*\} \end{array} \\ {\scriptstyle \phi_P}\big\downarrow & \text{HPB} & \big\downarrow \\ P^{m-1}\varOmega F_m & \xrightarrow{\quad \varDelta \quad} & P^{m-1}\varOmega F_m \times P^{m-1}\varOmega F_m \end{array}$$

(see [6, Lemma 2.1], for example, with $(X, A) = (P^{m-1}\varOmega F_m, P^{m-1}\varOmega F_{m-1})$, $(Y, B) = (P^{m-1}\varOmega F_m, \{*\})$ and $Z = P^{m-1}\varOmega F_m$), where we denote by $\varDelta$ the diagonal map. Thus, there is a map

$$\phi_P : P^{m-1}\varOmega F_{m-1} \cup_{q_P} C(\mathfrak{F}) \to P^{m-1}\varOmega F_m,$$

the left down arrow in diagram (3.2). On the other hand, by the proof of [6, Lemma 2.1], the subspace $P^{m-1}\varOmega F_{m-1} \subset P^{m-1}\varOmega F_{m-1} \cup_{q_P} C\mathfrak{F}_P$ can be described as the pullback of $\varDelta$ above and the inclusion map

$$P^{m-1}\varOmega i^F_{m-1,m} \times 1 : P^{m-1}\varOmega F_{m-1} \times P^{m-1}\varOmega F_m \hookrightarrow P^{m-1}\varOmega F_{m-1} \times P^{m-1}\varOmega F_m,$$

and hence we obtain

$$\phi_P|_{P^{m-1}\varOmega F_{m-1}} = P^{m-1}\varOmega i^F_{m-1,m} : P^{m-1}\varOmega F_{m-1} \hookrightarrow P^{m-1}\varOmega F_m.$$

Thirdly, the homotopy fibre $\mathfrak{F}^0_P$ of ϕ_P is the homotopy pullback of the inclusion

$$P^{m-1}\varOmega F_{m-1} \times P^{m-1}\varOmega F_m \cup P^{m-1}\varOmega F_m \times \{*\} \hookrightarrow P^{m-1}\varOmega F_m \times P^{m-1}\varOmega F_m$$

and the trivial map $\{*\} \to P^{m-1}\varOmega F_m \times P^{m-1}\varOmega F_m$. Then we obtain

$$\begin{array}{ccc} \mathfrak{F}_P \times \varOmega P^{m-1}\varOmega F_m & \xrightarrow{\text{proj}_2} & P^{m-1}\varOmega F_{m-1} \\ {\scriptstyle \text{proj}_1}\big\downarrow & \text{HPO} & \big\downarrow \\ \mathfrak{F}_P & \longrightarrow & \mathfrak{F}^0_P \end{array}$$

(see [6, Lemma 2.1], for example, with $(X,A)=(P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1})$, $(Y,B)=(P^{m-1}\Omega F_m,\{*\})$ and $Z=\{*\}$). Hence $\mathfrak{F}^0_P$ has the homotopy type of the join $\mathfrak{F}_P * \Omega P^{m-1}\Omega F_m$ which is $(\ell-1)$-connected. Thus ϕ_P is ℓ-connected.

Finally, let $q_S = q_P|_{S^{\ell-1}} : S^{\ell-1} \to P^{m-1}\Omega F_{m-1}$. Then the inclusion $j : P^{m-1}\Omega F_{m-1} \cup_{q_S} CS^{\ell-1} \hookrightarrow P^{m-1}\Omega F_{m-1} \cup_{q_P} C\mathfrak{F}_P$ is ℓ-connected, since

$$\begin{aligned}&P^{m-1}\Omega F_{m-1} \cup_{q_P} C\mathfrak{F}_P\\ &\qquad = P^{m-1}\Omega F_{m-1} \cup_{q_S} CS^{\ell-1} \cup (\text{cells in dimension} \geq \ell+1).\end{aligned}$$

Then the composition $\phi_S = \phi_P \circ j : (P^{m-1}\Omega F_{m-1} \cup_{q_S} CS^{\ell-1}, P^{m-1}\Omega F_{m-1}) \hookrightarrow (P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1})$ of ℓ-connected maps is again ℓ-connected.

Since $m \geq 3$, the pair $(E^m\Omega F_m, E^m\Omega F_{m-1})$ is ℓ-connected, which implies (1). Thus, the inclusion

$$P^{m-1}\Omega F_m \cup C(E^m\Omega F_{m-1}) \hookrightarrow P^{m-1}\Omega F_m \cup C(E^m\Omega F_m)$$

is ℓ-connected, and we obtain an ℓ-connected map

$$\begin{aligned}\hat{\phi}_S : P^m\Omega F_{m-1} \cup CS^{\ell-1} &= P^{m-1}\Omega F_{m-1} \cup_{q_S} CS^{\ell-1} \underset{p_{m-1}}{\cup_{\Omega F_{m-1}}} C(E^m\Omega F_{m-1})\\ &\to P^{m-1}\Omega F_m \cup C(E^m\Omega F_{m-1}) \hookrightarrow P^{m-1}\Omega F_m \cup C(E^m\Omega F_m) = P^m\Omega F_m,\end{aligned}$$

which implies (2). This completes the proof of Lemma 3.1. ∎

From now on, we assume $K_m = S^{\ell-1}$ with $m,\ell \geq 3$. Thus, by Lemma 3.1, we may assume that $P^m_m = P^m\Omega F_{m-1} \cup CS^{\ell-1} \subset P^m\Omega F_m$ is such that $(P^m\Omega F_m, P^m_m)$ is ℓ-connected. In this section, we define cone-decompositions of $F_m \times F'_1$, P^m_m and $P^m_m \times \Sigma\Omega F'_1$.

Firstly, we give a cone-decomposition of $F_m \times F'_1$ of length $m+1$:

$$\text{(3.3)}\quad \{K^{m,1}_i \xrightarrow{w^{m,1}_i} F^{m,1}_{i-1} \hookrightarrow F^{m,1}_i \mid 1 \leq i \leq m+1\} \quad \text{with} \quad F^{m,1}_{m+1} = F_m \times F'_1,$$

here $K^{m,1}_i$, $F^{m,1}_{i-1}$ and $w^{m,1}_i$ $(1 \leq i \leq m+1)$ are defined by

$$K^{m,1}_1 = K_1 \vee K'_1, \quad F^{m,1}_0 = \{*\}, \quad w^{m,1}_1 = * : K^{m,1}_1 \to F^{m,1}_0,$$

$$\left\{\begin{array}{l} K^{m,1}_i = K_i \vee (K_{i-1} * K'_1), \quad F^{m,1}_{i-1} = F_{i-1} \times \{*\} \cup F_{i-2} \times F'_1,\\ w^{m,1}_i|_{K_i} = \mathrm{incl} \circ (h_i \times *) : K_i \to F_{i-1} = F_{i-1} \times \{*\} \subset F^{m,1}_{i-1},\\ w^{m,1}_i|_{K_{i-1}*K'_1} = [\chi_{i-1}, \Sigma 1_{K'_1}]^r :\\ \qquad K_{i-1} * K'_1 \to F_{i-1} \times \{*\} \cup F_{i-2} \times \Sigma K'_1 = F^{m,1}_{i-1}; \end{array}\right. \qquad i \geq 2,$$

here $K_{m+1} = \{*\}$, incl is the canonical inclusion and $[\chi_i, \Sigma 1_{K'_1}]^r$ is the relative Whitehead product of the characteristic map $\chi_i : (CK_i, K_i) \to (F_i, F_{i-1})$ and the suspension of the identity map $\Sigma 1_{K'_1} : \Sigma K'_1 \to \Sigma K'_1$.

Secondly, a cone-decomposition of P^m_m of length m is

$$\begin{cases} \Omega F_{m-1} \to \{*\} \hookrightarrow \Sigma\Omega F_{m-1}, \\ \qquad\qquad\qquad \vdots \\ E^i\Omega F_{m-1} \to P^{i-1}\Omega F_{m-1} \hookrightarrow P^i\Omega F_{m-1}, \qquad 1 \le i < m, \\ \qquad\qquad\qquad \vdots \\ E^m\Omega F_{m-1} \vee K_m \to P^{m-1}\Omega F_{m-1} \hookrightarrow P^m_m. \end{cases}$$

Finally, a cone-decomposition of $P^m_m \times \Sigma\Omega F'_1$ of length $m+1$ is

$$\{\hat{E}_i \xrightarrow{\hat{w}_i} \hat{F}_{i-1} \hookrightarrow \hat{F}_i \mid 1 \le i \le m+1\} \quad \text{with} \quad \hat{F}_{m+1} = P^m_m \times \Sigma\Omega F'_1, \tag{3.4}$$

where $\hat{E}_{i+1}$, $\hat{F}_i$ and $\hat{w}_{i+1}$, $0 \le i \le m$, are defined by

$$\hat{E}_1 = \Omega F_{m-1} \vee \Omega F'_1, \qquad \hat{F}_0 = \{*\}, \qquad \hat{w}_1 = * : \hat{E}_1 \to \hat{F}_0,$$

$$\left.\begin{cases} \hat{E}_{i+1} = E^{i+1}\Omega F_{m-1} \vee \{E^i\Omega F_{m-1} * \Omega F'_1\}, \\ \hat{F}_i = P^i\Omega F_{m-1} \times \{*\} \cup P^{i-1}\Omega F_{m-1} \times \Sigma\Omega F'_1, \\ \hat{w}_{i+1}|_{E^{i+1}\Omega F_{m-1}} : E^{i+1}\Omega F_{m-1} \xrightarrow{p_i^{\Omega F_{m-1}}} P^i\Omega F_{m-1} \times \{*\} \subset \hat{F}_i, \\ \hat{w}_{i+1}|_{E^i\Omega F_{m-1} * \Omega F'_1} = [\chi'_i, 1_{\Sigma\Omega F'_1}]^r : E^i\Omega F_{m-1} * \Omega F'_1 \to \hat{F}_i, \end{cases}\right\} 1 \le i < m-1,$$

$$\begin{cases} \hat{E}_m = \{E^m\Omega F_{m-1} \vee K_m\} \vee \{E^{m-1}\Omega F_{m-1} * \Omega F'_1\}, \\ \hat{F}_{m-1} = P^{m-1}\Omega F_{m-1} \times \{*\} \cup P^{m-2}\Omega F_{m-1} \times \Sigma\Omega F'_1, \\ \hat{w}_m|_{E^m\Omega F_{m-1} \vee K_m} : E^m\Omega F_{m-1} \vee K_m \xrightarrow{p'_S} P^{m-1}\Omega F_{m-1} \times \{*\} \subset \hat{F}_{m-1}, \\ \hat{w}_m|_{E^{m-1}\Omega F_{m-1} * \Omega F'_1} = [\chi'_{m-1}, 1_{\Sigma\Omega F'_1}]^r : E^{m-1}\Omega F_{m-1} * \Omega F'_1 \to \hat{F}_{m-1}, \end{cases}$$

$$\begin{cases} \hat{E}_{m+1} = \{E^m\Omega F_{m-1} \vee K_m\} * \Omega F'_1, \\ \hat{F}_m = P^m_m \times \{*\} \cup P^{m-1}\Omega F_{m-1} \times \Sigma\Omega F'_1, \\ \hat{w}_{m+1} = [\chi'_m, 1_{\Sigma\Omega F'_1}]^r : \hat{E}_{m+1} \to \hat{F}_m; \end{cases}$$

here $p'_S : E^m\Omega F_{m-1} \vee K_m \to P^{m-1}\Omega F_{m-1}$ is given by $p'_S|_{E^m\Omega F_{m-1}} = p^{\Omega F_{m-1}}_{m-1}$ and $p'_S|_{K_m} = q_S$, and χ'_i is a relative homeomorphism given by

$$\begin{cases} \chi'_i : (CE^i\Omega F_{m-1}, E^i\Omega F_{m-1}) \to (P^i\Omega F_{m-1}, P^{i-1}\Omega F_{m-1}), \quad 1 \le i < m, \\ \chi'_m : (CE', E') \to (P^m_m, P^{m-1}\Omega F_{m-1}),\ E' = E^m\Omega F_{m-1} \vee K_m. \end{cases}$$

From now on, we denote by $\iota^{m,1}_i : F^{m,1}_i \hookrightarrow F^{m,1}_{i+1}$ and $\hat{\iota}_i : \hat{F}_i \hookrightarrow \hat{F}_{i+1}$ the canonical inclusions. Let us denote $1_m = 1_{F_m} : F_m \to F_m$.

Definition 3.2. The identity 1_m is filtered with respect to the filtration $* = F_0 \subset F_1 \subset \cdots \subset F_m$. Then by Proposition 2.4 for $f = 1_m$, we obtain $\sigma_i = \widehat{(1_m)}_i : F_i \to P^i\Omega F_i$ for $1 \le i \le m$, and $\widehat{(1_m)}^0_j : K_j \to E^j\Omega F_j$ for

$1 \le j \le m$. Let $g_j = \widehat{(1_m)}^0_j : K_j \to E^j \Omega F_j$ for $1 \le j \le m$. We also obtain $g' = \mathrm{ad}(1_{K'_1}) : K'_1 \to \Omega\Sigma K'_1 = \Omega F'_1$ and $\sigma' = \Sigma g' : F'_1 \to \Sigma\Omega F'_1$.

Since K_m and F_m are of dimension $\ell - 1$ and ℓ, respectively, we may assume that the images of g_m and σ_m are in $E^m \Omega F_{m-1}$ and P^m_m, respectively.

LEMMA 3.3. *Let $\nu^{m,1}_k : F^{m,1}_k \to F^{m,1}_k \vee \Sigma K^{m,1}_k$ and $\hat\nu_k : \hat F_k \to \hat F_k \vee \Sigma \hat K_k$ be the canonical copairings for $1 \le k \le m+1$, and $\sigma^{m,1}_m = \sigma_m \times \{*\} \cup \sigma_{m-1} \times \sigma' : F^{m,1}_m \to \hat F_m$. Then the following diagram is commutative:*

$$\begin{tikzcd}
K^{m,1}_{m+1} \arrow[r, "w^{m,1}_{m+1}"] \arrow[d, "g_m * g'"] & F^{m,1}_m \arrow[r, "\iota^{m,1}_m"] \arrow[d, "\sigma^{m,1}_m"] & F^{m,1}_{m+1} \arrow[r, "\nu^{m,1}_{m+1}"] \arrow[d, "\sigma_m \times \sigma'"] & F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1} \arrow[d, "\sigma_m \times \sigma' \vee \Sigma g_m * g'"] \\
\hat E_{m+1} \arrow[r, "\hat w_{m+1}"] & \hat F_m \arrow[r, "\hat\iota_m"] & \hat F_{m+1} \arrow[r, "\hat\nu_{m+1}"] & \hat F_{m+1} \vee \Sigma \hat E_{m+1}
\end{tikzcd}$$

To prove Lemma 3.3, we need the following propositions.

PROPOSITION 3.4. *Let $K \xrightarrow{a} A \hookrightarrow C(a)$ and $L \xrightarrow{b} B \hookrightarrow C(b)$ be cofibre sequences, and let $\nu_a : C(a) \to C(a) \vee \Sigma K$, $\nu_b : C(b) \to C(b) \vee \Sigma L$ and $\nu = \nu(a,b) : C(a) \times C(b) \to C(a) \times C(b) \vee \Sigma K * L$ be the canonical copairings.*

(1) *ν is given by the following composition, natural with respect to g, h:*

$$C(a) \times C(b)$$
$$\xrightarrow{\nu_a \times \nu_b} C(a) \times C(b) \underset{C(a)}{\cup} C(a) \times \Sigma L \underset{C(b)}{\cup} \Sigma K \times C(b) \underset{\Sigma K \vee \Sigma L}{\cup} \Sigma K \times \Sigma L$$
$$\xrightarrow{\Phi} C(a) \times C(b) \vee \Sigma K \times \Sigma L / (\Sigma K \vee \Sigma L) \xrightarrow{\approx} C(a) \times C(b) \vee \Sigma(K * L),$$

where Φ is given by $\Phi|_{C(a) \times \Sigma L} = \mathrm{proj}_1$, $\Phi|_{\Sigma K \times C(b)} = \mathrm{proj}_2$ and $\Phi|_{\Sigma K \times \Sigma L} =$ (collapsing) $: \Sigma K \times \Sigma L \twoheadrightarrow \Sigma K \times \Sigma L / (\Sigma K \vee \Sigma L)$.

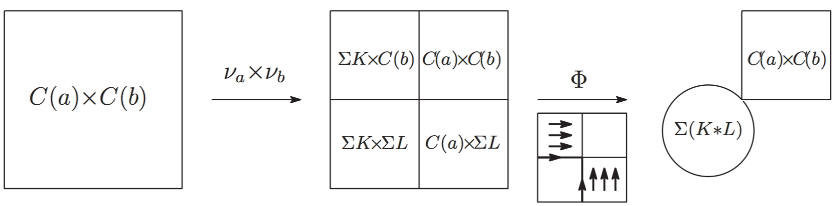

Fig. 1

(2) *Let $K' \xrightarrow{a'} A' \hookrightarrow C(a')$ and $L' \xrightarrow{b'} B' \hookrightarrow C(b')$ be cofibre sequences and $\hat\nu = \nu(a', b') : C(a') \times C(b') \to C(a') \times C(b') \vee \Sigma(K' * L')$. If $f^0 : K \to K'$, $f : A \to A'$, $g^0 : L \to L'$ and $g : B \to B'$ satisfy $f \circ a = a' \circ f^0$ and $g \circ b = b' \circ g^0$, then (f, f^0) and (g, g^0) induce $f' : C(a) \to C(a')$ and $g' : C(b) \to C(b')$ as in Theorem 2.1, which satisfy $\hat\nu \circ (f' \times g') = (f' \times g' \vee \Sigma(f^0 * g^0)) \circ \nu : C(a) \times C(b) \to C(a') \times C(b') \vee \Sigma(K' * L')$.*

Proof. Let us recall the definition of $C(h)$ for $h : X \to Z$ and related spaces:

$$CX = ([0,1] \times X) \amalg \{*\}/\!\sim,\ (0,x) \sim *;\ C(h) = Z \amalg CX/\!\sim,\ 1 \wedge x \sim h(x),$$
$$C_{\leq 1/2}X = \{t \wedge x \in CX \mid t \leq 1/2\} \approx CX,$$
$$C_{\geq 1/2}(h) = \{t \wedge x \in C(h) \mid t \geq 1/2\}, \quad (t,x) \in [0,1] \times X.$$

Firstly, we define a homeomorphism

$$\hat{\alpha} : (C(K * L), K * L) \approx (CK \times CL, CK \times L \cup K \times CL)$$

by $\hat{\alpha}(t \wedge (s \wedge x, y)) = ((ts) \wedge x, t \wedge y)$ and $\hat{\alpha}(t \wedge (x, s \wedge y)) = (t \wedge x, (ts) \wedge y)$ for $(x,y) \in K \times L$ and $s,t \in [0,1]$ (see Figure 2).

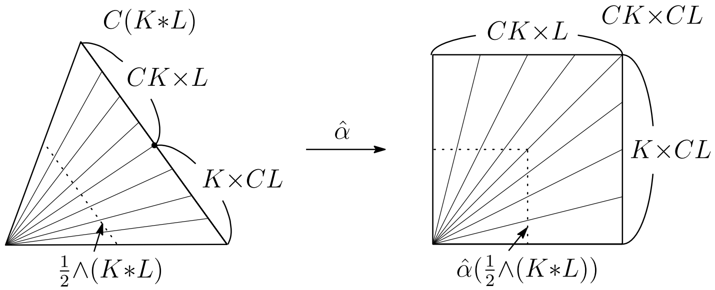

Fig. 2

Since $C([\chi_a, \chi_b]) = C(a) \times B \cup A \times C(b) \cup_{[\chi_a,\chi_b]} C(K*L)$ and $C(a) \times C(b) = (C(a) \times B \cup A \times C(b)) \cup_{[\chi_a,\chi_b]} CK \times CL$, $\hat{\alpha}$ induces a homeomorphism $\alpha : C([\chi_a, \chi_b]) \approx C(a) \times C(b)$. Thus the canonical copairing ν is given by

$$\nu : C(a) \times C(b) \to \frac{C(a) \times C(b)}{\alpha(\{C_{\leq 1/2}(K * L)\})} \vee \frac{\alpha(C_{\leq 1/2}(K * L))}{\alpha(\{1/2\} \times (K * L))}.$$

Since we can easily see that $\alpha(C_{\leq 1/2}(K * L))/\alpha(\{1/2\} \times (K * L)) \approx \Sigma(K * L)$ and $C(a) \times C(b)/\alpha(\{C_{\leq 1/2}(K * L)\}) = C(a) \times C(b)/C_{\leq 1/2}K \times C_{\leq 1/2}L$, ν is given as

$$\nu : C(a) \times C(b) \to \frac{C(a) \times C(b)}{C_{\leq 1/2}K \times C_{\leq 1/2}L} \vee \Sigma(K * L).$$

Since $C_{\leq 1/2}X$ is contractible, the inclusion $(C(a), \{*\}) \times (C(b), \{*\}) \hookrightarrow (C(a), C_{\leq 1/2}K) \times (C(b), C_{\leq 1/2}L)$ is homotopy equivalence, and so is the inclusion $C(a) \times \{*\} \cup \{*\} \times C(b) \hookrightarrow C(a) \times C_{\leq 1/2}L \cup C_{\leq 1/2}K \times C(b)$.

Hence, the following collapsing map is a homotopy equivalence:

$$\frac{C(a) \times C_{\leq 1/2}L \cup C_{\leq 1/2}K \times C(b)}{C_{\leq 1/2}K \times C_{\leq 1/2}L} \to \frac{C_{\geq 1/2}(a)}{\{1/2\} \times K} \vee \frac{C_{\geq 1/2}(b)}{\{1/2\} \times L}$$
$$\approx C(a) \vee C(b).$$

Finally, since $C_{\le 1/2}K\times C_{\le 1/2}L=\alpha(\{C_{\le 1/2}(K*L)\})$, by taking pushout of this collapsing with the inclusion

$$C(a)\times C_{\le 1/2}L\cup\frac{C_{\le 1/2}K\times C(b)}{C_{\le 1/2}K\times C_{\le 1/2}L}\hookrightarrow\frac{C(a)\times C(b)}{\alpha(\{C_{\le 1/2}(K*L)\})},$$

we obtain a homotopy equivalence

$$\frac{C(a)\times C(b)}{\alpha(\{C_{\le 1/2}(K*L)\})}\to\frac{C_{\ge 1/2}(a)}{\{1/2\}\times K}\times\frac{C_{\ge 1/2}(b)}{\{1/2\}\times L}\approx C(a)\times C(b).$$

Therefore, ν is homotopic to the map $\hat\nu$ given by

$$\hat\nu(s\wedge x,t\wedge y)=\begin{cases}(s\wedge x,t\wedge y)\in\frac{C_{\ge 1/2}(a)}{\{1/2\}\times K}\times\frac{C_{\ge 1/2}(b)}{\{1/2\}\times L}, & s,t\ge 1/2,\\ (*,t\wedge y)\in\{*\}\times\frac{C_{\ge 1/2}(b)}{\{1/2\}\times L}, & s\le 1/2, t\ge 1/2,\\ (s\wedge x,*)\in\frac{C_{\ge 1/2}(a)}{\{1/2\}\times K}\times\{*\}, & s\ge 1/2, t\le 1/2,\\ ((s\wedge x)\wedge(t\wedge y))\in\frac{C_{\le 1/2}K}{\{1/2\}\times K}\wedge\frac{C_{\le 1/2}L}{\{1/2\}\times L}, & s,t\le 1/2,\end{cases}$$

which coincides with $\Phi\circ(\nu_a\times\nu_b)$ which implies (1). As (2) is clear by concrete definitions of these maps, we obtain the proposition. ∎

PROPOSITION 3.5. *Let $\nu_m:F_m\to F_m\vee\Sigma K_m$ be the canonical copairing and $T_1:F^{m,1}_{m+1}\cup_{F'_1}(\Sigma K_m\times F'_1)\vee\Sigma K^{m,1}_{m+1}\to(F^{m,1}_{m+1}\vee\Sigma K^{m,1}_{m+1})\cup_{F'_1}(\Sigma K_m\times F'_1)$ be an appropriate homeomorphism. Then*

$$T_1\circ((\nu_m\times 1_{F'_1})\vee 1_{\Sigma K^{m,1}_{m+1}})\circ\nu^{m,1}_{m+1}=(\nu^{m,1}_{m+1}\cup 1_{\Sigma K_m\times F'_1})\circ(\nu_m\times 1_{F'_1}).$$

Proof. First, Proposition 3.4 implies the commutative diagram

$$\begin{tikzcd} F_m\times F'_1 \arrow[r, "\nu^{m,1}_{m+1}"] \arrow[d, "\nu_m\times 1_{F'_1}"'] & F_m\times F_1\vee\Sigma(K_m*K'_1) \\ F_m\times F'_1\cup_{F'_1}\Sigma K_m\times F'_1 \arrow[r, "1_m\times\nu_1"] & \begin{array}{c}F_m\times F'_1\cup_{F'_1}\Sigma K_m\times F'_1\cup_{F_m}F_m\times\Sigma K'_1\\ \cup\,\Sigma K_m\times\Sigma K'_1\end{array} \arrow[u, "\Phi"'] \end{tikzcd}$$

Now Φ goes through $(F_m\times F'_1\cup_{F'_1}\Sigma K_m\times F'_1)\cup\Sigma K_m\times\Sigma K'_1/\{*\}\times\Sigma K'_1$ as

$$\begin{aligned}\Phi:(F_m\times F'_1\cup_{F'_1}\Sigma K_m\times F'_1\cup_{F_m}F_m\times\Sigma K'_1)\cup\Sigma K_m\times\Sigma K'_1&\\ \xrightarrow{\Phi'}(F_m\times F'_1\cup_{F'_1}\Sigma K_m\times F'_1)\cup\frac{\Sigma K_m\times\Sigma K'_1}{\{*\}\times\Sigma K'_1}&\\ \xrightarrow{\mathrm{pr}'}F_m\times F'_1\vee\Sigma(K_m*K'_1),&\end{aligned}$$

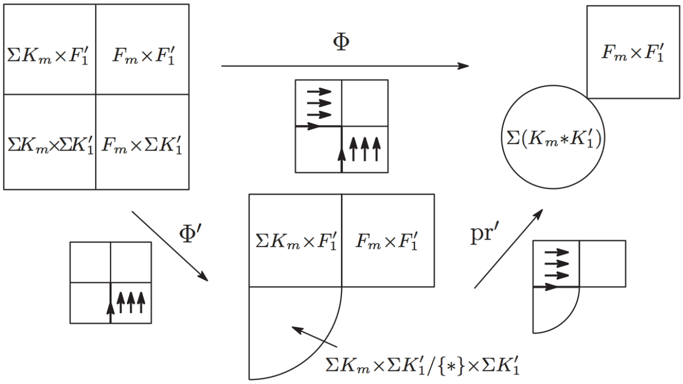

Fig. 3

where Φ' and pr' are given by

$$\Phi'|_{F_m \times F_1'} = 1_{F_m \times F_1'}, \quad \Phi'|_{\Sigma K_m \times F_1'} = 1_{\Sigma K_m \times F_1'}, \quad \Phi'|_{F_m \times \Sigma K_1'} = \mathrm{proj}_1,$$
$$\Phi'|_{\Sigma K_m \times \Sigma K_1'} = (\text{collapsing}) : \Sigma K_m \times \Sigma K_1' \twoheadrightarrow \frac{\Sigma K_m \times \Sigma K_1'}{\{*\} \times \Sigma K_1'},$$
$$\mathrm{pr}'|_{F_m \times F_1'} = 1_{F_m \times F_1'}, \quad \mathrm{pr}'|_{\Sigma K_m \times F_1'} = \mathrm{proj}_2,$$
$$\mathrm{pr}'|_{\Sigma K_m \times \Sigma K_1'/\{*\} \times \Sigma K_1'} = (\text{collapsing}) : \frac{\Sigma K_m \times \Sigma K_1'}{\{*\} \times \Sigma K_1'} \twoheadrightarrow \Sigma(K_m * K_1').$$

Since there is a natural homotopy equivalence $h : \Sigma K_m \times \Sigma K_1'/\{*\} \times \Sigma K_1' \simeq \Sigma K_m \vee \Sigma(K_m * K_1')$ such that $h|_{\Sigma K_m \times \{*\}} = 1_{\Sigma K_m}$, pr' can be decomposed as

$$\mathrm{pr}' = \mathrm{pr}_1' \circ \mathrm{pr}_0',$$

where pr_0' and pr_1' are given by

$$\mathrm{pr}_0'|_{F_m \times F_1'} = 1_{F_m \times F_1'}, \quad \mathrm{pr}_0'|_{\Sigma K_m \times F_1'} = 1_{\Sigma K_m \times F_1'}, \quad \mathrm{pr}_0'|_{\Sigma K_m \times \Sigma K_1'/\{*\} \times \Sigma K_1'} = h,$$
$$\mathrm{pr}_1'|_{F_m \times F_1'} = 1_{F_m \times F_1'}, \quad \mathrm{pr}_1'|_{\Sigma K_m \times F_1'} = \mathrm{proj}_2, \quad \mathrm{pr}_1'|_{\Sigma(K_m * K_1')} = 1_{\Sigma(K_m * K_1')}.$$

Hence $\Phi = \mathrm{pr}' \circ \Phi' = \mathrm{pr}_1' \circ \mathrm{pr}_0' \circ \Phi'$, and $\mathrm{pr}_0' \circ \Phi'$ is given by

$$\mathrm{pr}_0' \circ \Phi'|_{F_m \times F_1'} = 1_{F_m \times F_1'}, \quad \mathrm{pr}_0' \circ \Phi'|_{\Sigma K_m \times F_1'} = 1_{\Sigma K_m \times F_1'},$$
$$\mathrm{pr}_0' \circ \Phi'|_{F_m \times \Sigma K_1'} = \mathrm{proj}_1,$$
$$\mathrm{pr}_0' \circ \Phi'|_{\Sigma K_m \times \Sigma K_1'} = (\text{retraction}) : \Sigma K_m \times \Sigma K_1' \to \Sigma K_m \vee \Sigma(K_m * K_1'),$$

and so $\mathrm{pr}_0' \circ \Phi' \circ (1_m \times \nu_1)$ is given by

$$\mathrm{pr}_0' \circ \Phi' \circ (1_m \times \nu_1)|_{F_m \times F_1'} = 1_{F_m \times F_1'},$$
$$\mathrm{pr}_0' \circ \Phi' \circ (1_m \times \nu_1)|_{\Sigma K_m \times F_1'} = \nu' : \Sigma K_m \times F_1' \to \Sigma K_m \times F_1' \vee \Sigma(K_m * K_1'),$$

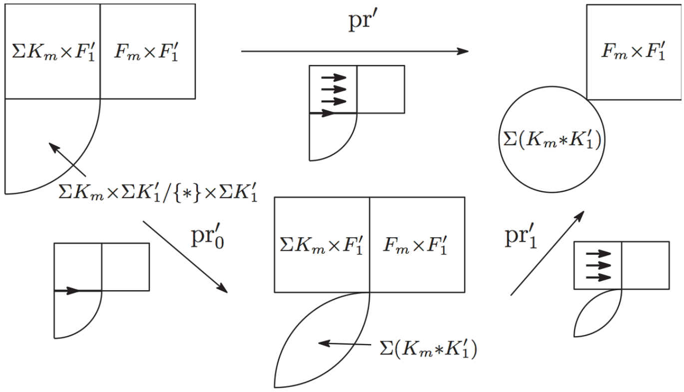

Fig. 4

where ν' is the canonical copairing. Thus we obtain a commutative diagram

$$\text{(3.5)}\qquad \begin{array}{ccc} F^{m,1}_{m+1} = F_m \times F'_1 & \xrightarrow{\nu_m \times 1_{F'_1}} & F_m \times F'_1 \cup_{F'_1} (\Sigma K_m \times F'_1) \\ \downarrow{\scriptstyle \nu^{m,1}_{m+1}} & & \downarrow{\scriptstyle 1_{F_m \times F'_1} \cup \nu'} \\ F_m \times F'_1 \vee \Sigma K_m * K'_1 & \xleftarrow{p_1} & F_m \times F'_1 \cup_{F'_1} (\Sigma K_m \times F'_1) \vee \Sigma K_m * K'_1 \end{array}$$

Therefore

$$\begin{aligned} &T_1 \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} \\ &\quad = T_1 \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ p_1 \circ (1_{F^{m,1}_{m+1}} \cup \nu') \circ (\nu_m \times 1_{F'_1}). \end{aligned}$$

Let us denote by $p_2 : F^{m,1}_{m+1} \cup_{F'_1} (\Sigma K_m \times F'_1) \cup_{F'_1} (\Sigma K_m \times F'_1) \vee \Sigma K^{m,1}_{m+1} \to F^{m,1}_{m+1} \cup_{F'_1} (\Sigma K_m \times F'_1) \vee \Sigma K^{m,1}_{m+1}$ the map pinching the second $\Sigma K_m \times F'_1$ to F'_1, by $p_3 : F^{m,1}_{m+1} \cup_{F'_1} ((\Sigma K_m \times F'_1) \vee \Sigma K^{m,1}_{m+1}) \cup_{F'_1} (\Sigma K_m \times F'_1) \to (F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1}) \cup_{F'_1} \Sigma K^{m,1}_{m+1}$ the map pinching the first $\Sigma K_m \times F'_1$ to one point, by $\nu_0 : \Sigma K_m \to \Sigma K_m \vee \Sigma K_m$ the canonical comultiplication and by $T_0 : \Sigma K_m \vee \Sigma K_m \to \Sigma K_m \vee \Sigma K_m$ the switching map. It is then easy to check that

$$\begin{aligned} &T_1 \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} \\ &\quad = T_1 \circ p_2 \circ \big((\nu_m \times 1_{F'_1}) \cup 1_{\Sigma K_m \times F'_1} \vee 1_{\Sigma K_m * K'_1}\big) \circ (1_{F^{m,1}_{m+1}} \cup \nu') \circ (\nu_m \times 1_{F'_1}) \\ &\quad = p_3 \circ (1_{F^{m,1}_{m+1}} \cup \nu' \cup 1_{\Sigma K_m \times F'_1}) \circ (1_{F^{m,1}_{m+1}} \cup (T_0 \times 1_{F'_1})) \\ &\qquad \circ ((\nu_m \times 1_{F'_1}) \cup 1_{\Sigma K_m \times F'_1}) \circ (\nu_m \times 1_{F'_1}). \end{aligned}$$

Using $(1_{F_m} \vee \nu_0) \circ \nu_m = (\nu_m \vee 1_{\Sigma K_m}) \circ \nu_m$ and $T_0 \circ \nu_0 = \nu_0$ from the assumption that K_m is a co-H-space together with $F^{m,1}_{m+1} = F_m \times F'_1$, we have

$$\begin{aligned}
&T_1 \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1}\\
&\quad = p_3 \circ (1_{F^{m,1}_{m+1}} \cup \nu' \cup 1_{\Sigma K_m \times F'_1}) \circ (1_{F^{m,1}_{m+1}} \cup (T_0 \times 1_{F'_1}))\\
&\qquad \circ (1_{F^{m,1}_{m+1}} \cup (\nu_0 \times 1_{F'_1})) \circ (\nu_m \times 1_{F'_1})\\
&\quad = p_3 \circ (1_{F^{m,1}_{m+1}} \cup \nu' \cup 1_{\Sigma K_m \times F'_1}) \circ ((1_{F_m} \vee \nu_0) \times 1_{F'_1})) \circ (\nu_m \times 1_{F'_1})\\
&\quad = p_3 \circ (1_{F^{m,1}_{m+1}} \cup \nu' \cup 1_{\Sigma K_m \times F'_1}) \circ ((\nu_m \vee 1_{\Sigma K_m}) \times 1_{F'_1}) \circ (\nu_m \times 1_{F'_1}).
\end{aligned}$$

Using diagram (3.5) yields

$$T_1 \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1} = (\nu^{m,1}_{m+1} \cup 1_{\Sigma K_m \times F'_1}) \circ (\nu_m \times 1_{F'_1}).$$

This completes the proof of Proposition 3.5. ∎

Proof of Lemma 3.3. The commutativity of the left square follows from [14, Proposition 2.9], and the middle square is clearly commutative.

So we are left to show $(\sigma_m \times \sigma' \vee \Sigma g_m * g') \circ \nu^{m,1}_{m+1} = \hat{\nu}_{m+1} \circ (\sigma_m \times \sigma')$. Recall that $\sigma_m = \widehat{1_m}$ by Proposition 2.4(1) for $f = 1_m$. On the other hand, by Proposition 2.4(2), we have $\sigma_m = \nabla_{P^m \Omega F_m} \circ ((1_m)'_m \vee \iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m) \circ \nu_m$, and hence

$$\begin{aligned}
&(\sigma_m \times \sigma' \vee \Sigma g_m * g') \circ \nu^{m,1}_{m+1}\\
&\quad = \{(\nabla_{P^m \Omega F_m} \circ ((1_m)'_m \vee (\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m)) \circ \nu_m) \times \sigma' \vee \Sigma g_m * g'\} \circ \nu^{m,1}_{m+1}\\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma \Omega F'_1} \vee 1_{\Sigma \hat{E}_{m+1}})\\
&\qquad \circ \{((1_m)'_m \times \sigma') \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta_m) \times \sigma') \vee \Sigma g_m * g'\}\\
&\qquad \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1}\\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma \Omega F'_1} \vee 1_{\Sigma \hat{E}_{m+1}})\\
&\qquad \circ T_2 \circ \{((1_m)'_m \times \sigma' \vee \Sigma g_m * g') \cup ((\iota^{\Omega F_m}_{m-1,m} \circ \delta^{1_m}_m) \times \sigma')\} \circ T_1\\
&\qquad \circ ((\nu_m \times 1_{F'_1}) \vee 1_{\Sigma K^{m,1}_{m+1}}) \circ \nu^{m,1}_{m+1},
\end{aligned}$$

where $T_1 : F^{m,1}_{m+1} \cup_{F'_1} (\Sigma K_m \times F'_1) \vee \Sigma K^{m,1}_{m+1} \to (F^{m,1}_{m+1} \vee \Sigma K^{m,1}_{m+1}) \cup_{F'_1} (\Sigma K_m \times F'_1)$ and $T_2 : (\hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}) \cup_{\Sigma \Omega F'_1} \hat{F}_{m+1} \to (\hat{F}_{m+1} \cup_{\Sigma \Omega F'_1} \hat{F}_{m+1}) \vee \Sigma \hat{E}_{m+1}$ are appropriate homeomorphisms. Then by Proposition 3.5, Proposition 3.4(2) and the definitions of $(1_m)'_m$ and σ', we proceed as follows:

$$\begin{aligned}
&(\sigma_m \times \sigma' \vee \Sigma g_m * g') \circ \nu_{m+1}^{m,1} \\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma\Omega F_1'} \vee 1_{\Sigma \hat{E}_{m+1}}) \\
&\qquad \circ T_2 \circ \{((1_m)'_m \times \sigma' \vee \Sigma g_m * g') \cup ((\iota_{m-1,m}^{\Omega F_m} \circ \delta_m^{1_m}) \times \sigma')\} \\
&\qquad \circ (\nu_{m+1}^{m,1} \cup 1_{\Sigma K_m \times F_1'}) \circ (\nu_m \times 1_{F_1'}) \\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma\Omega F_1'} \vee 1_{\Sigma \hat{E}_{m+1}}) \circ T_2 \\
&\qquad \circ \{(((1_m)'_m \times \sigma' \vee \Sigma g_m * g') \circ \nu_{m+1}^{m,1}) \cup ((\iota_{m-1,m}^{\Omega F_m} \circ \delta_m^{1_m}) \times \sigma')\} \circ (\nu_m \times 1_{F_1'}). \\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma\Omega F_1'} \vee 1_{\Sigma \hat{E}_{m+1}}) \circ T_2 \\
&\qquad \circ \{(\hat{\nu}_{m+1} \circ ((1_m)'_m \times \sigma')) \cup ((\iota_{m-1,m}^{\Omega F_m} \circ \delta_m^{1_m}) \times \sigma')\} \circ (\nu_m \times 1_{F_1'}) \\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma\Omega F_1'} \vee \nabla_{\Sigma \hat{E}_{m+1}}) \circ T_3 \\
&\qquad \circ \{\hat{\nu}_{m+1} \circ ((1_m)'_m \times \sigma') \cup i_1 \circ ((\iota_{m-1,m}^{\Omega F_m} \circ \delta_m^{1_m}) \times \sigma')\} \circ (\nu_m \times 1_{F_1'}).
\end{aligned}$$

Here $i_1 : \hat{F}_{m+1} \to \hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}$ is the first inclusion and $T_3 : (\hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}) \cup_{\Sigma\Omega F_1'} (\hat{F}_{m+1} \vee \Sigma \hat{E}_{m+1}) \to (\hat{F}_{m+1} \cup_{\Sigma\Omega F_1'} \hat{F}_{m+1}) \vee \Sigma \hat{E}_{m+1} \vee \Sigma \hat{E}_{m+1}$ is an appropriate homeomorphism. Thus

$$\begin{aligned}
&(\sigma_m \times \sigma' \vee \Sigma g_m * g') \circ \nu_{m+1}^{m,1} \\
&\quad = (\nabla_{P^m \Omega F_m} \times 1_{\Sigma\Omega F_1'} \vee \nabla_{\Sigma \hat{E}_{m+1}}) \circ T_3 \\
&\qquad \circ (\hat{\nu}_{m+1} \cup \hat{\nu}_{m+1}) \circ \{((1_m)'_m \times \sigma') \cup ((\iota_{m-1,m}^{\Omega F_m} \circ \delta_m) \times \sigma')\} \circ (\nu_m \times 1_{F_1'}) \\
&\quad = \hat{\nu}_{m+1} \circ \{\nabla_{P^m \Omega F_m} \circ ((1_m)'_m \vee (\iota_{m-1,m}^{\Omega F_m} \circ \delta_m^{1_m})) \circ \nu_m \times \sigma'\} \\
&\quad = \hat{\nu}_{m+1} \circ (\sigma_m^{1_m} \times \sigma').
\end{aligned}$$

This completes the proof of Lemma 3.3. ■

4. Proof of Theorem 1.2. In the fibre sequence $G \hookrightarrow E \to \Sigma A$, by the James–Whitehead decomposition (see Whitehead [17, VII. Theorem (1.15)]), the total space E has the homotopy type of the space $G \cup_\psi G \times CA$, where

$$\psi : G \times A \xrightarrow{1_G \times \alpha} G \times G \xrightarrow{\mu} G.$$

Since $G \simeq F_m$ and, by condition (1) of Theorem 1.2, α is compressible into F_1', we see that

$$\psi : G \times A \simeq F_m \times A \xrightarrow{1_{F_m} \times \alpha} F_m \times F_1' \subset F_m \times F_1 \subset F_m \times F_m \simeq G \times G \xrightarrow{\mu} G \simeq F_m$$

and E is the homotopy pushout of the sequence

$$F_m \xleftarrow{pr_1} F_m \times A \xrightarrow{1_{F_m} \times \alpha} F_m \times F_1' \xrightarrow{\mu_{m,1}} F_m.$$

We construct spaces and maps such that the homotopy pushout of these maps dominates E. Let $e' = e_1^{F_1'} : \Omega\Sigma F_1' \to F_1'$ and $\sigma_A = \Sigma\,\mathrm{ad}(1_A) : A \to \Sigma\Omega A$, since A is a suspended space. By condition (2) of Theorem 1.2, we have $H_1(\alpha) = 0$ in $[A, \Omega F_1' * \Omega F_1']$, which immediately implies

$$\sigma' \circ \alpha = \Sigma\,\mathrm{ad}(\alpha) = e' \circ \sigma_A : A \to \Sigma\Omega F_1'. \tag{4.1}$$

By condition (3) of Theorem 1.2, we have $K_m = S^{\ell-1}$ with $m, \ell \geq 3$, and so $(P^m\Omega F_m, P^m_m)$ is ℓ-connected by Lemma 3.1.

Proposition 4.1. *The following diagram is commutative:*

$$\begin{tikzcd}
F_m & F_m \times A \arrow[l, "pr_1"'] \arrow[r, "1_{F_m}\times\alpha"] \arrow[d, "\sigma_m\times\sigma_A"'] & F_m \times F_1' \arrow[r, "\mu_{m,1}"] \arrow[d, "\sigma_m\times\sigma'"'] & F_m \arrow[d, "\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m"] \\
P^{m+1}\Omega F_m \arrow[d, "e^{F_m}_{m+1}"'] & P^m_m \times \Sigma\Omega A \arrow[l, "\phi"'] \arrow[r, "\chi"] \arrow[d, "e^{F_m}_m\times e^A_1"'] & \hat{F}_{m+1} \arrow[d, "e^{F_m}_m\times e'"'] & P^{m+1}\Omega F_m \arrow[d, "e^{F_m}_{m+1}"] \\
F_m & F_m \times A \arrow[l, "pr_1"'] \arrow[r, "1_{F_m}\times\alpha"] & F_m \times F_1' \arrow[r, "\mu_{m,1}"] & F_m
\arrow[from=1-1, to=2-1, "\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m"']
\end{tikzcd}$$

where $\phi = \iota^{\Omega F_m}_{m,m+1} \circ pr_1$ *and* $\chi = 1_{P^m_m} \times \Sigma\Omega\alpha$.

Proof. The upper left square is clearly commutative. The equality $e^{F_m}_m = e^{F_m}_{m+1} \circ \iota^{\Omega F_m}_{m,m+1}$ implies that the lower left square is commutative. The equality $\alpha \circ e^A_1 = e' \circ \Sigma\Omega\alpha$ implies the commutativity of the lower middle square. The commutativity of the upper middle square is obtained by (4.1). Proposition 2.4(2) for $f = 1_m$ and the fact that $e' \circ \sigma' = 1_{F_1'}$ imply that the right rectangle is commutative. ∎

Definition 4.2. $\lambda = \mu_{m,1} \circ \{e^{F_m}_m \times e'\} : \hat{F}_{m+1} \to F_m \times F_1' \to F_m$.

Then λ is a well-defined filtered map with respect to the filtration (3.4) of $\hat{F}_{m+1}$ and the trivial filtration ($(F_m)_i = F_m$ for all i) of F_m, where $\{e^{F_m}_m \times e'\}(\hat{F}_k) = \{e^{F_{m-1}}_k \times * \cup e^{F_{m-1}}_{k-1} \times e'\}(\hat{F}_k) \subset F_{m-1} \times F_1'$ for $0 \leq k < m$, and $\{e^{F_m}_m \times e'\}(\hat{F}_m) = \{e^{F_m}_m \times * \cup e^{F_{m-1}}_{m-1} \times e'\}(\hat{F}_m) \subset F_m \times \{*\} \cup F_{m-1} \times F_1'$ for $k = m$.

Definition 4.3. By Proposition 2.4 for $f = \lambda$, we obtain a series of maps $\hat{\lambda}_k : \hat{F}_k \to P^k\Omega F_m$, $0 \leq k \leq m+1$.

By the hypothesis of Theorem 1.2, we have $\mu_{k,1} : F_k \times F_1' \to F_{k+1}$ for $k < m$, and $\mu_{m,1} : F_m \times F_1' \to F_m$, both of which are restrictions of μ.

Lemma 4.4. *There is a map $\hat{\lambda} : \hat{F}_{m+1} \to P^{m+1}\Omega F_m$ which fits in with the following commutative diagram obtained by dividing the right square of* *the diagram in Proposition* 4.1 *by* $\hat{\lambda}$ *into upper and lower squares.*

$$\begin{tikzcd}
F_m \arrow[d, "\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m"'] & F_m\times A \arrow[l, "pr_1"'] \arrow[r, "1\times\alpha"] \arrow[d, "\sigma_m\times\sigma_A"'] & F_m\times F'_1 \arrow[r, "\mu_{m,1}"] \arrow[d, "\sigma_m\times\sigma'"'] & F_m \arrow[d, "\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m"] \\
P^{m+1}\Omega F_m \arrow[d, "e^{F_m}_{m+1}"'] & P^m_m\times\Sigma\Omega A \arrow[l, "\phi"'] \arrow[r, "\chi"] \arrow[d, "e^{F_m}_m\times e^A_1"'] & \hat{F}_{m+1} \arrow[r, "\hat{\lambda}"] \arrow[d, "e^{F_m}_m\times e'"'] & P^{m+1}\Omega F_m \arrow[d, "e^{F_m}_{m+1}"] \\
F_m & F_m\times A \arrow[l, "pr_1"'] \arrow[r, "1\times\alpha"] & F_m\times F'_1 \arrow[r, "\mu_{m,1}"] & F_m
\end{tikzcd}$$

Proof. Let $\mu^{m,1}_k = 1_{F_k}\cup\mu_{k-1,1} : F^{m,1}_k = F_k\times\{*\}\cup F_{k-1}\times F'_1 \to F_k$, $\sigma^{m,1}_k = \sigma_k\times * \cup\sigma_{k-1}\times\sigma' : F^{m,1'}_k \to P^k\Omega F_k\times\{*\}\cup P^{k-1}\Omega F_{k-1}\times\Sigma\Omega F'_1$ and $j_k = P^k\Omega i^F_{k,m-1}\times * \cup P^{k-1}\Omega i^F_{k-1,m-1}\times 1_{\Sigma\Omega F'_1}$, $0\le k<m$.

Firstly, we show the following by induction on $k<m$:

$$\iota^{\Omega F_m}_{k,k+1}\circ P^k\Omega i^F_{k,m}\circ\sigma_k\circ\mu^{m,1}_k = \iota^{\Omega F_m}_{k,k+1}\circ\hat{\lambda}_k\circ j_k\circ\sigma^{m,1}_k : F^{m,1}_k \to P^{k+1}\Omega F_m. \tag{4.2}$$

The case $k=0$ is clear, since both maps are constant maps. Assume that (4.2) holds for some k. By Proposition 2.4(1) for $f=1_m$, the diagram

$$\begin{tikzcd}
F_k \arrow[r, "\sigma_k"] \arrow[d, "i^F_{k,k+1}"] & P^k\Omega F_k \arrow[r, hook, "P^k\Omega i^F_{k,k+1}"] & P^k\Omega F_{k+1} \arrow[r, hook, "P^k\Omega i^F_{k+1,m-1}"] \arrow[d, hook, "\iota^{\Omega F_{k+1}}_{k,k+1}"] & P^k\Omega F_{m-1} \arrow[d, hook, "\iota^{\Omega F_{m-1}}_{k,k+1}"] \\
F_{k+1} \arrow[rr, "\sigma_{k+1}"] & & P^{k+1}\Omega F_{k+1} \arrow[r, hook, "P^{k+1}\Omega i^F_{k+1,m-1}"] & P^{k+1}\Omega F_{m-1}
\end{tikzcd}$$

is commutative for $k+1<m$, and hence

$$\begin{aligned}
&j_{k+1}\circ\sigma^{m,1}_{k+1}\circ\iota^{m,1}_k\\
&\quad = (P^{k+1}\Omega i^F_{k+1,m-1}\circ\sigma_{k+1}\circ i^F_{k,k+1})\times *\cup(P^k\Omega i^F_{k,m-1}\circ\sigma_k\circ i^F_{k-1,k})\times\sigma'\\
&\quad = (\iota^{\Omega F_{m-1}}_{k,k+1}\circ P^k\Omega i^F_{k,m-1}\circ\sigma_k)\times *\cup(\iota^{\Omega F_{m-1}}_{k-1,k}\circ P^k\Omega i^F_{k-1,m-1}\circ\sigma_{k-1})\times\sigma'\\
&\quad = \hat{\iota}_k\circ j_k\circ\sigma^{m,1}_k.
\end{aligned}$$

By Proposition 2.4(1) for $f=\lambda$, we have $\hat{\lambda}_{k+1}\circ\hat{\iota}_k = \iota^{\Omega F_m}_{k,k+1}\circ\hat{\lambda}_k$, and hence

$$\hat{\lambda}_{k+1}\circ j_{k+1}\circ\sigma^{m,1}_{k+1}\circ\iota^{m,1}_k = \hat{\lambda}_{k+1}\circ\hat{\iota}_k\circ j_k\circ\sigma^{m,1}_k = \iota^{\Omega F_m}_{k,k+1}\circ\hat{\lambda}_k\circ j_k\circ\sigma^{m,1}_k.$$

Then, by Proposition 2.4(1) for $f=1_m$ and the induction hypothesis,

$$\begin{aligned}
&(\iota^{m,1}_k)^*(\hat{\lambda}_{k+1}\circ j_{k+1}\circ\sigma^{m,1}_{k+1})\\
&= [\iota^{\Omega F_m}_{k,k+1}\circ P^k\Omega i^F_{k,m}\circ\sigma_k\circ\mu^{m,1}_k] = [P^{k+1}\Omega i^F_{k+1,m}\circ\sigma_{k+1}\circ i^F_{k,k+1}\circ\mu^{m,1}_k]\\
&= [P^{k+1}\Omega i^F_{k+1,m}\circ\sigma_{k+1}\circ\mu^{m,1}_{k+1}\circ\iota^{m,1}_k] = (\iota^{m,1}_k)^*(P^{k+1}\Omega i^F_{k+1,m}\circ\sigma_{k+1}\circ\mu^{m,1}_{k+1}).
\end{aligned}$$

By a standard argument of homotopy theory applied to the cofibre sequence $K^{m,1}_{k+1} \to F^{m,1}_{k} \hookrightarrow F^{m,1}_{k+1}$, we obtain the difference map $\delta_{k+1} : \Sigma K^{m,1}_{k+1} \to P^{k+1}\Omega F_m$ of $\hat{\lambda}_{k+1} \circ j_{k+1} \circ \sigma^{m,1}_{k+1}$ and $P^{k+1}\Omega i^F_{k+1,m} \circ \sigma_{k+1} \circ \mu^{m,1}_{k+1}$, $k+1 < m$:

$$\begin{aligned}(4.3)\qquad & P^{k+1}\Omega i^F_{k+1,m} \circ \sigma_{k+1} \circ \mu^{m,1}_{k+1} \\ &\qquad = \nabla_{P^{k+1}\Omega F_m} \circ (\hat{\lambda}_{k+1} \circ j_{k+1} \circ \sigma^{m,1}_{k+1} \vee \delta_{k+1}) \circ \nu^{m,1}_{k+1}.\end{aligned}$$

Then, by Proposition 2.4(1) for $f = \lambda$, we have

$$e^{F_m}_{k+1} \circ \hat{\lambda}_{k+1} = \mu_{m-1,1} \circ \{e^{F_{m-1}}_{k+1} \times * \cup e^{F_{m-1}}_{k} \times e'\},$$

and hence, by the commutative diagram

$$\begin{tikzcd} F_i \arrow[r, "\sigma_i"] \arrow[dr, "1_{F_i}"'] & P^i\Omega F_i \arrow[r, hook, "P^i\Omega i^F_{i,m-1}"] \arrow[d, "e^{F_i}_i"] & P^i\Omega F_{m-1} \arrow[r, "e^{F_{m-1}}_i"] & F_{m-1} \\ & F_i \arrow[urr, hook, "i^F_{i,m-1}"'] & & \end{tikzcd}$$

for $i = k,\ k+1 \le m-1$, we obtain

$$\{e^{F_{m-1}}_{k+1} \times * \cup e^{F_{m-1}}_{k} \times e'\} \circ j_{k+1} \circ \sigma^{m,1}_{k+1} = \iota^{m,1}_{k+1,m},$$

where $\iota^{m,1}_{k+1,m} : F^{m,1}_{k+1} \hookrightarrow F^{m,1}_{m}$ is the canonical inclusion. Thus

$$\begin{aligned} & e^{F_m}_{k+1} \circ \hat{\lambda}_{k+1} \circ j_{k+1} \circ \sigma^{m,1}_{k+1} = \mu_{m-1,1} \circ \iota^{m,1}_{k+1,m} = i^F_{k+1,m} \circ \mu^{m,1}_{k+1} \\ &\quad = i^F_{k+1,m} \circ e^{F_{k+1}}_{k+1} \circ \sigma_{k+1} \circ \mu^{m,1}_{k+1} = e^{F_m}_{k+1} \circ P^{k+1}\Omega i^F_{k+1,m} \circ \sigma_{k+1} \circ \mu^{m,1}_{k+1},\end{aligned}$$

and hence, by (4.3),

$$\begin{aligned} i^F_{k+1,m} \circ \mu^{m,1}_{k+1} &= \nabla_{F_m} \circ (e^{F_m}_{k+1} \circ \hat{\lambda}_{k+1} \circ j_{k+1} \circ \sigma^{m,1}_{k+1} \vee e^{F_m}_{k+1} \circ \delta_{k+1}) \circ \nu^{m,1}_{k+1} \\ &= \nabla_{F_m} \circ (i^F_{k+1,m} \circ \mu^{m,1}_{k+1} \vee e^{F_m}_{k+1} \circ \delta_{k+1}) \circ \nu^{m,1}_{k+1}.\end{aligned}$$

Using [13, Theorem 2.7(1)] and the multiplication μ on $G \simeq F_m$, we see that $e^{F_m}_{k+1} \circ \delta_{k+1} : \Sigma K^{m,1}_{k+1} \to F_m$ is null-homotopic. Hence by a standard argument of homotopy theory applied to the fibre sequence $E^{k+2}\Omega F_m \to P^{k+1}\Omega F_m \to F_m$, we get a lift $\delta'_{k+1} : \Sigma K^{m,1}_{k+1} \to E^{m+1}\Omega F_m$ of δ_{k+1} as $p^{\Omega F_m}_{k+1} \circ \delta'_{k+1} = \delta_{k+1}$, $k+1 < m$. Since $\iota^{\Omega F_m}_{k+1,k+2} \circ p^{\Omega F_m}_{k+1} = *$, we obtain $\iota^{\Omega F_m}_{k+1,k+2} \circ \delta_{k+1} = \iota^{\Omega F_m}_{k+1,k+2} \circ p^{\Omega F_m}_{k+1} \circ \delta'_{k+1} = *$ and

$$\begin{aligned} & \iota^{\Omega F_m}_{k+1,k+2} \circ \nabla_{P^{k+1}\Omega F_m} \circ (\hat{\lambda}_{k+1} \circ j_{k+1} \circ \sigma^{m,1}_{k+1} \vee \delta_{k+1}) \circ \nu^{m,1}_{k+1} \\ &\qquad = \nabla_{P^{k+2}\Omega F_m} \circ (\iota^{\Omega F_m}_{k+1,k+2} \circ \hat{\lambda}_{k+1} \circ j_{k+1} \circ \sigma^{m,1}_{k+1} \vee *) \circ \nu^{m,1}_{k+1} \\ &\qquad = \iota^{\Omega F_m}_{k+1,k+2} \circ \hat{\lambda}_{k+1} \circ j_{k+1} \circ \sigma^{m,1}_{k+1},\end{aligned}$$

and hence, by (4.3),

$$\iota^{\Omega F_m}_{k+1,k+2}\circ P^{k+1}\Omega i^F_{k+1,m}\circ\sigma_{k+1}\circ\mu^{m,1}_{k+1}=\iota^{\Omega F_m}_{k+1,k+2}\circ\hat\lambda_{k+1}\circ j_{k+1}\circ\sigma^{m,1}_{k+1}.$$

This completes the proof of the induction step and we obtain (4.2) for $k<m$.

Secondly, we show that

$$\iota^{\Omega_{F_m}}_{m,m+1}\circ\sigma_m\circ\mu^{m,1}_m=\iota^{\Omega_{F_m}}_{m,m+1}\circ\hat\lambda_m\circ\sigma^{m,1}_m. \tag{4.4}$$

By Proposition 2.4(1) for $f=1_m$, we obtain

$$\sigma_t\circ i^F_{t-1,t}=i^{\Omega F_t}_{t-1,t}\circ P^{t-1}\Omega i^F_{t-1,t}\circ\sigma_{t-1}\quad\text{for}\quad t=m-1,m.$$

Hence

$$\begin{aligned}\sigma^{m,1}_m\circ\iota^{m,1}_{m-1}&=\big((\sigma_m\circ i^F_{m-1,m})\times *\cup(\sigma_{m-1}\circ i^F_{m-2,m-1})\times\sigma'\big)\\&=(\iota^{\Omega F_m}_{m-1,m}\circ P^{m-1}\Omega i^F_{m-1,m}\circ\sigma_{m-1})\times *\\&\quad\cup(\iota^{\Omega F_{m-1}}_{m-2,m-1}\circ P^{m-2}\Omega i^F_{m-2,m-1}\circ\sigma_{m-1})\times\sigma'\\&=\hat{i}_{m-1}\circ j_{m-1}\circ\sigma^{m,1}_{m-1}.\end{aligned}$$

By Proposition 2.4(1) for $f=\lambda$, we obtain $\hat\lambda_m\circ\hat\iota_{m-1}=\iota^{\Omega F_m}_{m-1,m}\circ\hat\lambda_{m-1}$ and

$$\begin{aligned}(\iota^{m,1}_{m-1})^*(\hat\lambda_m\circ\sigma^{m,1}_m)&=[\hat\lambda_m\circ\sigma^{m,1}_m\circ\iota^{m,1}_{m-1}]=[\hat\lambda_m\circ\hat{i}_{m-1}\circ j_{m-1}\circ\sigma^{m,1}_{m-1}]\\&=[\iota^{\Omega F_m}_{m-1,m}\circ\hat\lambda_{m-1}\circ j_{m-1}\circ\sigma^{m,1}_{m-1}]=[\iota^{\Omega F_m}_{m-1,m}\circ P^m\Omega i^F_{m-1,m}\circ\sigma_{m-1}\circ\mu^{m,1}_{m-1}]\\&=[\sigma_m\circ i^F_{m-1,m}\circ\mu^{m,1}_{m-1}]=(\iota^{m,1}_{m-1})^*(\sigma_m\circ\mu^{m,1}_m)\end{aligned}$$

using (4.2) for $k=m-1$. Thus by a standard argument of homotopy theory applied to the cofibre sequence $K^{m,1}_m\to F_m\hookrightarrow F_{m+1}$, we obtain a difference map $\delta_m:\Sigma K^{m,1}_m\to P^m\Omega F_m$ of $\hat\lambda_m\circ\sigma^{m,1}_m$ and $\sigma_m\circ\mu^{m,1}_m$:

$$\sigma_m\circ\mu^{m,1}_m=\nabla_{P^m\Omega F_m}\circ(\hat\lambda_m\circ\sigma^{m,1}_m\vee\delta_m)\circ\nu^{m,1}_m. \tag{4.5}$$

By Proposition 2.4(1) for $f=\lambda$,

$$e^{F_m}_m\circ\hat\lambda_m\circ\sigma^{m,1}_m=\mu^{m,1}_m\circ\{e^{F_m}_m\times *\cup e^{F_{m-1}}_{m-1}\times e'\}\circ(\sigma_m\times *\cup\sigma_{m-1}\times\sigma')=\mu^{m,1}_m,$$

and hence, by (4.5),

$$\mu^{m,1}_m=\nabla_{F_m}\circ(e^{F_m}_m\circ\hat\lambda_m\circ\sigma^{m,1}_m\vee e^{F_m}_m\circ\delta_m)\circ\nu^{m,1}_m=\nabla_{F_m}\circ(\mu^{m,1}_m\vee e^{F_m}_m\circ\delta_m)\circ\nu^{m,1}_m.$$

Thus $e^{F_m}_m\circ\delta_m=*$. Then, by a standard argument of homotopy theory applied to the fibre sequence $E^{m+1}\Omega F_m\to P^m\Omega F_m\to F_m$, we obtain a lift $\delta'_m:\Sigma K^{m,1}_m\to E^{m+1}\Omega F_m$ which satisfies $\delta_m=p^{\Omega F_m}_m\circ\delta'_m$. Since $\iota^{\Omega F_m}_{m,m+1}\circ p^{\Omega F_m}_m=*$, we have $\iota^{\Omega F_m}_{m,m+1}\circ\delta_m=\iota^{\Omega F_m}_{m,m+1}\circ p^{\Omega F_m}_m\circ\delta'_m=*$. Then by (4.5), we obtain (4.4) as follows:

$$\begin{aligned}\iota^{\Omega F_m}_{m,m+1}\circ\sigma_m\circ\mu^{m,1}_m&=\iota^{\Omega F_m}_{m,m+1}\circ\nabla_{P^m\Omega F_m}\circ(\hat\lambda_m\circ\sigma^{m,1}_m\vee\delta_m)\circ\nu^{m,1}_m\\&=\nabla_{P^{m+1}\Omega F_m}\circ(\iota^{\Omega F_m}_{m,m+1}\circ\hat\lambda_m\circ\sigma^{m,1}_m\vee *)\circ\nu^{m,1}_m=\iota^{\Omega F_m}_{m,m+1}\circ\hat\lambda_m\circ\sigma^{m,1}_m.\end{aligned}$$

Finally, we construct a map $\hat{\lambda} : \hat{F}_{m+1} \to P^{m+1}\Omega F_m$. By Proposition 2.4(1) for $f = 1_m$, we have $\sigma_m \circ i^F_{m-1,m} = i^{\Omega F_m}_{m-1,m} \circ P^{m-1}\Omega i^F_{m-1,m} \circ \sigma_{m-1}$, and hence

$$\begin{aligned}(\sigma_m \times \sigma') \circ \iota^{m,1}_m &= (\sigma_m \times \sigma') \circ (1_{F_m} \times * \cup\ i^F_{m-1,m} \times 1_{F'_1}) \\ &= \hat{\iota}_m \circ (\sigma_m \times * \cup\ \sigma_{m-1} \times \sigma') = \hat{\iota}_m \circ \sigma^{m,1}_m.\end{aligned}$$

Also by Proposition 2.4(1) for $f = \lambda$, we get $\hat{\lambda}_{m+1} \circ \hat{\iota}_m = \iota^{\Omega F_m}_{m,m+1} \circ \hat{\lambda}_m$ and

$$\hat{\lambda}_{m+1} \circ (\sigma_m \times \sigma') \circ \iota^{m,1}_m = \hat{\lambda}_{m+1} \circ \hat{\iota}_m \circ \sigma^{m,1}_m = \iota^{\Omega F_m}_{m,m+1} \circ \hat{\lambda}_m \circ \sigma^{m,1}_m,$$

and hence, by (4.4),

$$(\iota^{m,1}_m)^*(\hat{\lambda}_{m+1} \circ (\sigma_m \times \sigma')) = \iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu^{m,1}_m = (\iota^{m,1}_m)^*(\iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu_{m,1}).$$

By a standard argument of homotopy theory applied to the cofibre sequence $K^{m,1}_{m+1} \to F^{m,1}_m \hookrightarrow F^{m,1}_{m+1}$, we obtain $\delta_{m+1} : \Sigma K^{m,1}_{m+1} \to P^{m+1}\Omega F_m$ such that

$$\iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu_{m,1} = \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \circ (\sigma_m \times \sigma') \vee \delta_{m+1}) \circ \nu^{m,1}_{m+1}. \tag{4.6}$$

To proceed further, let us consider the dashed map $\bar{e} : \Sigma\hat{E}_{m+1} \to \Sigma K^{m+1}_m$, induced from the commutativity of the lower left square in the diagram

$$\begin{tikzcd}
F^{m,1}_m \arrow[r, hook, "\iota^{m,1}_m"] \arrow[d, "\sigma^{m,1}_m"] & F^{m,1}_{m+1} \arrow[r, "q_P"] \arrow[d, "\sigma_m \times \sigma'"] & \Sigma K^{m,1}_m \arrow[d, "\Sigma g_m * g'"] \\
\hat{F}_m \arrow[r, hook, "\hat{\iota}_m"] \arrow[d, "\hat{e}_m"] & \hat{F}_{m+1} \arrow[r, "\bar{q}_F"] \arrow[d, "e^{F_m}_m \times e'"] & \Sigma\hat{E}_{m+1} \arrow[d, dashed, "\bar{e}"] \\
F^{m,1}_m \arrow[r, hook, "\iota^{m,1}_m"] & F^{m,1}_{m+1} \arrow[r, "q_P"] & \Sigma K^{m,1}_m
\end{tikzcd}$$

where the map $\hat{e}_m : \hat{F}_m \to F^{m,1}_m$ is $e^{F_m}_m \times * \cup\ e^{F_{m-1}}_{m-1} \times e'$. Since $\hat{e}_m \circ \sigma^{m,1}_m$ and $(e^{F_m}_m \times e') \circ (\sigma_m \times \sigma')$ are homotopy equivalences, so is $\bar{e} \circ \Sigma g_m * g_1$ (see [4, Lemma 16.24]). We denote by $h : \Sigma K^{m+1}_m \to \Sigma K^{m+1}_m$ its homotopy inverse. Then, by (4.6),

$$\begin{aligned}\iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu_{m,1} &= \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \circ (\sigma_m \times \sigma') \vee \delta_{m+1}) \circ \nu^{m,1}_{m+1} \\ &= \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \circ (\sigma_m \times \sigma') \vee \delta_{m+1} \circ h \circ \bar{e} \circ \Sigma g_m * g') \circ \nu^{m,1}_{m+1} \\ &= \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \vee \delta_{m+1} \circ h \circ \bar{e}) \circ ((\sigma_m \times \sigma') \vee \Sigma g_m * g') \circ \nu^{m,1}_{m+1},\end{aligned}$$

which, by Lemma 3.3, can be continued as

$$= \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \vee \delta_{m+1} \circ h \circ \bar{e}) \circ \hat{\nu}_{m+1} \circ (\sigma_m \times \sigma').$$

This suggests defining $\hat{\lambda}$ by $\nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \vee \delta_{m+1} \circ h \circ \bar{e}) \circ \hat{\nu}_{m+1}$ to obtain

$$\iota^{\Omega F_m}_{m,m+1} \circ \sigma_m \circ \mu_{m,1} = \hat{\lambda} \circ (\sigma_m \times \sigma') : F_m \times F'_1 \to P^{m+1}\Omega F_m,$$

which gives the commutativity of the upper right square in Lemma 4.4. So it remains to show the commutativity of the lower right square in Lemma 4.4. By Proposition 2.4(1) for $f = \lambda$, we have

$$e^{F_m}_{m+1} \circ \hat{\lambda}_{m+1} \circ (\sigma_m \times \sigma') = \mu_{m,1} \circ \{e^{F_m}_m \times e'\} \circ (\sigma_m \times \sigma') = \mu_{m,1},$$

and hence, by the equalities $e^{F_m}_{m+1} \circ \iota^{\Omega F_m}_{m,m+1} \circ \sigma_m = 1_{F_m}$ and (4.6),

$$\begin{aligned}\mu_{m,1} &= e^{F_m}_{m+1} \circ \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \circ (\sigma_m \times \sigma') \vee \delta_{m+1}) \circ \nu^{m,1}_{m+1} \\ &= \nabla_{F_m} \circ (\mu_{m,1} \vee e^{F_m}_{m+1} \circ \delta_{m+1}) \circ \nu^{m,1}_{m+1}.\end{aligned}$$

Thus, $e^{F_m}_{m+1} \circ \delta_{m+1} = *$. Therefore,

$$\begin{aligned}e^{F_m}_{m+1} \circ \hat{\lambda} &= e^{F_m}_{m+1} \circ \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \vee \delta_{m+1} \circ h \circ \bar{e}) \circ \hat{\nu}_{m+1} \\ &= \nabla_{F_m} \circ (e^{F_m}_{m+1} \circ \hat{\lambda}_{m+1} \vee *) \circ \hat{\nu}_{m+1} = e^{F_m}_{m+1} \circ \hat{\lambda}_{m+1},\end{aligned}$$

and hence, by Proposition 2.4(1) for $f = \lambda$, we finally get

$$e^{F_m}_{m+1} \circ \hat{\lambda} = \mu_{m,1} \circ \{e^{F_m}_m \times e'\} : \hat{F}_{m+1} \to F_m. \ \blacksquare$$

Now we are ready to define a cone-decomposition $\{\hat{E}'_k \xrightarrow{\hat{w}'_k} \hat{F}'_{k-1} \xhookrightarrow{\hat{i}'_{k-1}} \hat{F}'_k \mid 1 \le k \le m+1\}$ of $P^m_m \times \Sigma\Omega A$ of length $m+1$ by replacing F'_1 with A in the cone-decomposition of $P^m_m \times \Sigma\Omega F'_1$. The series of cofibre sequences

$$\{E^k\Omega F_m \xrightarrow{p^{\Omega F_m}_{k-1}} P^{k-1}\Omega F_m \xhookrightarrow{\iota^{\Omega F_m}_{k-1}} P^k\Omega F_m \mid 1 \le k \le m+1\}$$

gives a cone-decomposition of $P^{m+1}\Omega F_m$ of length $m+1$. Let D be the homotopy pushout of $\phi = \iota^{\Omega F_m}_{m,m+1} \circ pr_1$ and $\hat{\lambda} \circ \chi = \hat{\lambda} \circ (1_{P^m_m} \times \Sigma\Omega\alpha)$:

$$\begin{tikzcd} P^m_m \times \Sigma\Omega A \arrow[r, "\hat{\lambda}\circ\chi"] \arrow[d, "\phi"] & P^{m+1}\Omega F_m \arrow[d] \\ P^{m+1}\Omega F_m \arrow[r] & D \end{tikzcd}$$

We give a cone-decomposition of D as follows: $\hat{\lambda} \circ \hat{\iota}_m = \nabla_{P^{m+1}\Omega F_m} \circ (\hat{\lambda}_{m+1} \vee \delta_{m+1} \circ h \circ \bar{e}) \circ \hat{\nu}_{m+1} \circ \hat{\iota}_m = \hat{\lambda}_{m+1} \circ \hat{\iota}_m$, we may identify the restriction of $\hat{\lambda}$ on $\hat{F}_k$ with $\hat{\lambda}_k$, and hence $\hat{\lambda} \circ \chi$ is a filtered map up to homotopy, i.e., $(\hat{\lambda} \circ \chi)|_{\hat{F}'_k} = \hat{\lambda}_k \circ \chi|_{\hat{F}'_k}$ for $1 \le k \le m$. Since $\chi|_{\hat{F}'_{k-1}} = \chi|_{\hat{F}'_k} \circ \hat{i}'_{k-1}$ and $\hat{i}'_{k-1} \circ \hat{w}'_k = *$, we have

$$\begin{aligned}e^{F_m}_{k-1} \circ ((\hat{\lambda} \circ \chi)|_{\hat{F}'_{k-1}} \circ \hat{w}'_k) &= e^{F_m}_k \circ \hat{\lambda}_k \circ \chi|_{\hat{F}'_k} \circ \hat{i}'_{k-1} \circ \hat{w}'_k \\ &= e^{F_m}_k \circ \hat{\lambda}_k \circ \chi|_{\hat{F}'_k} \circ * = *.\end{aligned}$$

By a standard argument of homotopy theory applied to the fibre sequence $E^k\Omega F_m \to P^{k-1}\Omega F_m \to F_m$, we have a lift $\kappa_k : \hat{E}'_k \to E^k\Omega F_m$ which fits in with the following commutative diagrams:

$$
\begin{tikzcd}
\hat{E}'_k \arrow[r, "\hat{w}'_k"] \arrow[d, "\kappa_k"] & \hat{F}'_{k-1} \arrow[r, hook, "\hat{i}'_{k-1}"] \arrow[d, "\hat{\lambda}_{k-1}\circ\chi|_{\hat{F}'_{k-1}}"] & \hat{F}'_k \arrow[d, "\hat{\lambda}_k\circ\chi|_{\hat{F}'_k}"] \\
E^k\Omega F_m \arrow[r, "p^{\Omega F_m}_{k-1}"] & P^{k-1}\Omega F_m \arrow[r, hook, "\iota^{\Omega F_m}_{k-1,k}"] & P^k\Omega F_m
\end{tikzcd}
\quad (1 \le k \le m), \tag{4.7}
$$

$$
\begin{tikzcd}
\hat{E}'_{m+1} \arrow[r, "\hat{w}'_{m+1}"] \arrow[d, "\kappa_{m+1}"] & \hat{F}'_m \arrow[r, hook, "\hat{i}'_m"] \arrow[d, "\hat{\lambda}_m\circ\chi|_{\hat{F}'_m}"] & \hat{F}'_{m+1} \arrow[d, "\hat{\lambda}\circ\chi"] \\
E^{m+1}\Omega F_m \arrow[r, "p^{\Omega F_m}_m"] & P^m\Omega F_m \arrow[r, hook, "\iota^{\Omega F_m}_{m,m+1}"] & P^{m+1}\Omega F_m
\end{tikzcd}
\quad (k = m+1). \tag{4.8}
$$

By definition of ϕ, it is clear that there exists a map $\psi_k : \hat{E}'_k \to E^k\Omega F_m$ which fits in with the commutative diagram

$$
\begin{tikzcd}
\hat{E}'_k \arrow[r, "\hat{w}'_k"] \arrow[d, "\psi_k"] & \hat{F}'_{k-1} \arrow[r, hook, "\hat{i}'_{k-1}"] \arrow[d, "\phi|_{\hat{F}'_{k-1}}"] & \hat{F}'_k \arrow[d, "\phi|_{\hat{F}'_k}"] \\
E^k\Omega F_m \arrow[r, "p^{\Omega F_m}_{k-1}"] & P^{k-1}\Omega F_m \arrow[r, hook, "\iota^{\Omega F_m}_{k-1,k}"] & P^k\Omega F_m
\end{tikzcd}
\tag{4.9}
$$

Let E^D_k be a homotopy pushout of κ_k and ψ_k, and F^D_k be a homotopy pushout of $(\hat{\lambda}\circ\chi)|_{\hat{F}'_k}$ and $\phi|_{\hat{F}'_k}$. Then using diagrams (4.7)–(4.9) and the universal property of homotopy pushouts, we obtain the following commutative diagram whose front column $E^D_k \to F^D_{k-1} \to F^D_k$ is a cofibre sequence:

$$
\begin{tikzcd}
 & \hat{E}'_k \arrow[dl, "\psi_k"'] \arrow[drr, "\kappa_k"] \arrow[d, "\hat{w}'_k"] & & \\
E^k\Omega F_m \arrow[d, "p^{\Omega F_m}_{k-1}"] \arrow[drr] & \hat{F}'_{k-1} \arrow[dl, "\phi|_{\hat{F}'_{k-1}}"'] \arrow[drr, "(\hat{\lambda}\circ\chi)|_{\hat{F}'_{k-1}}"] \arrow[d, hook, "\hat{i}'_{k-1}"] & & E^k\Omega F_m \arrow[dl] \arrow[d, "p^{\Omega F_m}_{k-1}"] \\
P^{k-1}\Omega F_m \arrow[d, hook, "\iota^{\Omega F_m}_{k-1,k}"] \arrow[drr] & \hat{F}'_k \arrow[dl, "\phi|_{\hat{F}'_k}"'] \arrow[drr, "(\hat{\lambda}\circ\chi)|_{\hat{F}'_k}"] & E^D_k \arrow[d] & P^{k-1}\Omega F_m \arrow[dl] \arrow[d, hook, "\iota^{\Omega F_m}_{k-1,k}"] \\
P^k\Omega F_m \arrow[drr] & & F^D_{k-1} \arrow[d, hook] & P^k\Omega F_m \arrow[dl] \\
 & & F^D_k &
\end{tikzcd}
$$

Thus we obtain a cone-decomposition $\{E^D_k \to F^D_{k-1} \hookrightarrow F^D_k \mid 1 \le k \le m+1\}$

of D of length $m+1$, which immediately implies

$$\mathrm{cat}(D) \leq \mathrm{Cat}(D) \leq m+1.$$

The homotopy pushout of the top and bottom rows in (4.4) are $G \cup_\psi G \times CA$. Also, since the dimensions of F_m, F_1 and A are less than or equal to ℓ, all compositions of columns in (4.4) are homotopy equivalences. Thus, the composite map $D \to G \cup_\psi G \times CA \simeq E \to D$ is a homotopy equivalence (see [4, Lemma 16.24], for example). Hence D dominates E, and we obtain

$$\mathrm{cat}(E) \leq \mathrm{cat}(D) \leq \mathrm{Cat}(D) \leq m+1.$$

5. L-S category of $\mathbf{SO}(10)$. In this section, we determine $\mathrm{cat}(\mathbf{SO}(10))$ and prove Theorem 5.1.

To give a lower bound of $\mathrm{cat}(\mathbf{SO}(10))$, let us recall the algebra structure of the well-known cohomology algebra $H^*(\mathbf{SO}(10);\mathbb{F}_2)$:

$$H^*(\mathbf{SO}(10);\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_3,x_5,x_7,x_9]/(x_1^{16},x_3^4,x_5^2,x_7^2,x_9^2),$$

where x_k is a generator in dimension k. Then by Theorem 1.1,

$$21 = \mathrm{cup}(\mathbf{SO}(10);\mathbb{F}_2) \leq \mathrm{cat}(\mathbf{SO}(10)). \tag{5.1}$$

On the other hand, to give the upper bound using Theorem 1.2, we first recall the cone-decomposition of $\mathbf{Spin}(7)$ in [10]:

$$* \subset F'_1 = \Sigma\mathbb{C}\mathrm{P}^3 \subset F'_2 \subset F'_3 \subset F'_4 \subset F'_5 \simeq \mathbf{Spin}(7).$$

In [11], the cone-decomposition of $\mathbf{SO}(9)$ is given by using the above filtration F'_i of $\mathbf{Spin}(7)$ together with the principal bundle $\mathbf{Spin}(7) \hookrightarrow \mathbf{SO}(9) \to \mathbb{R}\mathrm{P}^{15}$. Let e^k be a k-cell in $\mathbf{SO}(9)$ corresponding to the k-cell in $\mathbb{R}\mathrm{P}^{15}$. The cone-decomposition $\{F_i\}$ of $\mathbf{SO}(9)$ introduced in [11] is

$$\begin{aligned}
F_0 &= \{*\}\\
&\ \vdots \qquad \ddots\\
F_j &= F'_j \cup (e^1 \times F'_{j-1}) \cup \cdots \cup (e^{j-1} \times F'_1) \cup e^j\\
&\ \vdots \qquad\qquad\qquad \ddots\\
F_5 &= F'_5 \cup (e^1 \times F'_4) \cup (e^2 \times F'_3) \cup (e^3 \times F'_2) \cup (e^4 \times F'_1) \cup e^5\\
&\ \vdots \qquad\qquad\qquad\qquad\qquad \ddots\\
F_{i+5} &= F'_5 \cup (e^1 \times F'_5) \cup \cdots \cup (e^i \times F'_5) \cup (e^{i+1} \times F'_4) \cup \cdots \cup (e^{i+4} \times F'_1) \cup e^{i+5}\\
&\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
F_{15} &= F'_5 \cup (e^1 \times F'_5) \cup \cdots \cup (e^{10} \times F'_5) \cup (e^{11} \times F'_4) \cup \cdots \cup (e^{14} \times F'_1) \cup e^{15}\\
&\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
F_{15+j} &= F'_5 \cup (e^1 \times F'_5) \cup \cdots \cup (e^{10+j} \times F'_5) \cup (e^{11+j} \times F'_4) \cup \cdots \cup (e^{15} \times F'_{5-j})\\
&\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
F_{20} &= F'_5 \cup (e^1 \times F'_5) \cup \cdots \cup (e^{15} \times F'_5) \simeq \mathbf{SO}(9)
\end{aligned}$$

where $0 \leq i \leq 10$ and $0 \leq j \leq 5$, which is given with a series of cofibre sequences $\{K_i \to F_{i-1} \to F_i \mid 1 \leq i \leq 20\}$.

Secondly, a cofibre sequence $S^{20} \to F'_4 \hookrightarrow F'_4 \cup e^{21}$ $(= F'_5 \simeq \mathbf{Spin}(9))$ in [10] induces a cofibre sequence $K_{20} = S^{14} * S^{20} = S^{35} \to F_{19} \hookrightarrow F_{20}$.

Thirdly, since $\mu'|_{F'_i \times F'_1}$ is compressible into F'_{i+1} for $1 \leq i < 5$ by [11, proof of Theorem 2.9], $\mu|_{F_i \times F'_1}$ is compressible into F_{i+1} for $1 \leq i < 20$, where μ and μ' are the multiplications of $\mathbf{SO}(9)$ and $\mathbf{Spin}(7)$, respectively.

Fourthly, let us consider two principal bundles $p : \mathbf{SO}(10) \to S^9$ and $p' : \mathbf{SU}(5) \to S^9$, together with the commutative diagram

$$\begin{tikzcd}
\Sigma\mathbb{C}\mathrm{P}^3 \arrow[r, hook] & \mathbf{SU}(4) \arrow[r, hook] \arrow[d, hook] & \mathbf{SO}(9) \arrow[d, hook] \\
 & \mathbf{SU}(5) \arrow[r, hook] \arrow[dr, "p'"] & \mathbf{SO}(10) \arrow[d, "p"] \\
 & S^8 \arrow[r, hook] \arrow[uul, dashed, "\Sigma\gamma_3"] \arrow[uu, dashed, bend left, "\alpha'"] \arrow[uur, "\alpha"'] & S^9
\end{tikzcd}$$

The map $\alpha : S^8 \to \mathbf{SO}(9)$ in the above diagram is the characteristic map of $p : \mathbf{SO}(10) \to S^9$. By Steenrod [16], α is homotopic in $\mathbf{SO}(9)$ to a map $\alpha' : S^8 \to \mathbf{SU}(4)$, the characteristic map of $p' : \mathbf{SU}(5) \to S^9$. Further, by Yokota [18], the suspension $\Sigma\gamma_3 : S^8 \to \Sigma\mathbb{C}\mathrm{P}^3$ of the canonical projection $\gamma_3 : S^7 \to \mathbb{C}\mathrm{P}^3$ is the attaching map of the top cell of $\Sigma\mathbb{C}\mathrm{P}^4 \subset \mathbf{SU}(5)$, which is homotopic to α'. Therefore, the characteristic map α is compressible into $\Sigma\mathbb{C}\mathrm{P}^3 \subset F_1$. Since α is homotopic to a suspension map to $\Sigma\mathbb{C}\mathrm{P}^3$ in $\mathbf{SO}(9)$, we have $H_1(\alpha) = 0 \in \pi_8(\Omega\Sigma\mathbb{C}\mathrm{P}^3 * \Omega\Sigma\mathbb{C}\mathrm{P}^3)$ when α is regarded as a map to $\Sigma\mathbb{C}\mathrm{P}^3$.

Thus, finally by Theorem 1.2 with $F'_1 = \Sigma\mathbb{C}\mathrm{P}^3$, we obtain

$$\operatorname{cat}(\mathbf{SO}(10)) \leq 20 + 1 = 21. \tag{5.2}$$

Combining (5.2) with (5.1), we obtain our desired result.

THEOREM 5.1. $\operatorname{cat}(\mathbf{SO}(10)) = 21 = \operatorname{cup}(\mathbf{SO}(10); \mathbb{F}_2)$.

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