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Compactifications of ω and the Banach space c_0

by

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Abstract. We investigate for which compactifications $\gamma\omega$ of the discrete space of natural numbers ω, the natural copy of the Banach space c_0 is complemented in $C(\gamma\omega)$. We show, in particular, that the separability of the remainder $\gamma\omega \setminus \omega$ is neither sufficient nor necessary for c_0 to be complemented in $C(\gamma\omega)$ (the latter result is proved under the continuum hypothesis). We analyse, in this context, compactifications of ω related to embeddings of the measure algebra into $P(\omega)/\mathrm{fin}$.

We also prove that a Banach space $C(K)$ contains a rich family of complemented copies of c_0 whenever the compact space K admits only measures of countable Maharam type.

1. Introduction. If X is a Banach space and Y is a closed subspace of X then Y is said to be *complemented* in X if there is a closed subspace Z of X such that $X = Y \oplus Z$. This is equivalent to saying that there is a bounded linear operator P from X onto Y which is a projection, i.e. $P \circ P = P$. Recall that typically, unless a Banach space X is isomorphic to a Hilbert space, there are many closed uncomplemented subspaces of X.

The classical Banach space c_0 plays a special role when we speak of complementability: by Sobczyk's theorem [29] every isomorphic copy of c_0 is complemented in any separable superspace. Cabello Sánchez, Castillo and Yost [6] offer an interesting discussion of various proofs and aspects of Sobczyk's theorem; see also the survey paper by Godefroy [20].

Complementability of isomorphic copies of c_0 has been investigated for nonseparable spaces. A Banach space X is said to have the *Sobczyk property* if every subspace of X isomorphic to c_0 is complemented in X. Moltó [25] singled out a certain topological property of the weak* topology in X^* en-

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suring that X has the Sobczyk property. Correa and Tausk [11] proved that the space $C(K)$ has the Sobczyk property whenever K is a compact line, considerably generalizing an earlier result from [26], showing that *isometric* copies of c_0 inside $C(K)$, with K being the double arrow space, are always complemented; see also [2], [7], [17], [19] for related results.

Let $\gamma\omega$ be a compactification of the discrete space ω of natural numbers. Then c_0 can be naturally identified with the subspace Y of $C(\gamma\omega)$, where

$$Y = \{f \in C(\gamma\omega) : f|(\gamma\omega \setminus \omega) \equiv 0\},$$

simply by identifying the unit vector e_n in c_0 with $\chi_{\{n\}} \in C(\gamma\omega)$. We shall call the space Y the *natural copy* of c_0 in $C(\gamma\omega)$. We also use the following terminology.

Definition 1.1. We say that a compactification $\gamma\omega$ is *tame* if the natural copy of c_0 is complemented in $C(\gamma\omega)$.

The main problem considered in the present paper may be stated informally as follows.

Question 1.2. *Which compactifications $\gamma\omega$ are tame?*

Note that every metrizable compactification $\gamma\omega$ is tame because $C(\gamma\omega)$ is then separable and *every* copy of c_0 is complemented in $C(\gamma\omega)$ by Sobczyk's theorem. On the other hand, the maximal compactification $\beta\omega$ is not tame: $C(\beta\omega)$ is isometric to l_∞, and by Phillips' theorem [27], c_0 is not complemented in l_∞. In fact, $C(\beta\omega)$ is a Grothendieck space, so it contains no complemented copies of c_0 (see the next section). Note also that if we have two comparable compactifications $\gamma_1\omega \le \gamma_2\omega$, in the sense that there is a continuous surjection $\gamma_2\omega \to \gamma_1\omega$ that does not move points of ω, then $\gamma_1\omega$ is tame provided $\gamma_2\omega$ is. Thus tame compactifications form a natural subclass of all compactifications of ω, and Question 1.2 calls for a reasonable characterization of tameness.

Question 1.2 has been motivated by Castillo [8] and by conversations with Wiesław Kubiś and Piotr Koszmider. In particular, W. Kubiś observed that if $\gamma\omega$ is tame then the remainder $\gamma\omega \setminus \omega$ must carry a strictly positive measure (see Section 5), and asked if the converse holds.

The paper is organized as follows. In Section 2 we recall the standard facts related to complementability of c_0. In Section 3 we introduce the terminology and notation concerning Boolean algebras and finitely additive measures and then translate facts from Section 2 to the setting of the paper.

In Section 4 we consider compactifications of ω defined by subalgebras $\mathfrak{A}$ of $P(\omega)$ containing all finite sets and such that the quotient map $\mathfrak{A} \to \mathfrak{A}/\mathrm{fin}$ admits a lifting. We prove in particular that every separable zerodimensional compact space is homeomorphic to the remainder of a tame compactification (Theorem 4.3).

Our main results read as follows:

1. If a compactification $\gamma\omega$ is related to an embedding of the measure algebra into $P(\omega)/\mathrm{fin}$ then $\gamma\omega$ is not tame (Theorem 5.2). Since in that case $\gamma\omega \setminus \omega$ is homeomorphic to the Stone space of the measure algebra, $\gamma\omega\setminus\omega$ does carry a strictly positive measure. A related result on such $\gamma\omega$ is Theorem 5.6.
2. Under CH there is a tame compactification with a nonseparable remainder (Theorem 6.1).
3. There is a nontame compactification of ω with a separable remainder (Theorem 7.1).

The conclusion is that, as it seems, the tameness of $\gamma\omega$ is not directly related to simple topological properties of $\gamma\omega \setminus \omega$. In fact, a tame compactification may have the same remainder as another nontame one (Corollary 4.4).

We have not been able to prove (2) above in the usual set theory; it is likely that for our argument we can relax CH to the assumption $\mathfrak{b} = \mathfrak{c}$. However, Borodulin-Nadzieja and Inamdar [5] have recently proved (2) without any set-theoretic assumptions by a completely different approach, at the same time answering a problem posed in [15].

In the final section we prove a general result on $C(K)$ spaces containing complemented copies of c_0. Theorem 8.4 says that if a compact space K has a certain measure-theoretic property then every isomorphic copy of c_0 inside $C(K)$ contains a complemented subcopy of c_0. Our result is related to the work of Moltó [25] and Galego and Plichko [19].

The referee has pointed out that Question 1.2 is partially connected with the question whether for every nonmetrizable compact space K, the Banach space $C(K)$ has a nontrivial twisted sum with c_0 (see Castillo [9] and Correa and Tausk [12]). We shall investigate that aspect elsewhere.

2. Preliminaries.

In the following, K (possibly with some subscript) always denotes a compact Hausdorff space, and $C(K)$ stands for the Banach space of (real-valued) continuous functions on K equipped with the usual supremum norm. The dual space $C(K)^*$ is identified with the space $\mathcal{M}(K)$ of all signed Radon measures of bounded variation defined on the Borel σ-algebra on K. For $\mu \in \mathcal{M}(K)$ and $f \in C(K)$ we write $\mu(g) = \int_K f\,d\mu$ for simplicity. Recall that every $\mu \in \mathcal{M}(K)$ can be written as $\mu = \mu^+ - \mu^-$, where μ^+, μ^- are nonnegative orthogonal measures. Then $|\mu|$, the total variation of μ, is defined as $|\mu| = \mu^+ + \mu^-$, and the norm of μ is $\|\mu\| = |\mu|(K)$. If $x \in K$ then $\delta_x \in \mathcal{M}(K)$ denotes the probability Dirac measure at x. The basic facts on $C(K)$ and $\mathcal{M}(K)$ may be found in Albiac and Kalton [1] or Diestel [13].

The following well-known lemma, establishing a connection between sequences of measures on K and complementability of c_0 in $C(K)$, originates in Veech's proof of Sobczyk's theorem [30]. Here we write $\delta(n,k)$ for the Kronecker symbol.

Lemma 2.1. *Let* $T\colon c_0 \to C(K)$ *be an isomorphic embedding and let* $Te_n = \varphi_n$. *Then the following conditions are equivalent:*

(i) $T[c_0]$ *is complemented in* $C(K)$*;*
(ii) *there exist bounded sequences* $(\mu_n)_n$ *and* $(\nu_n)_n$ *in* $\mathcal{M}(K)$ *such that*
  - $\nu_n(\varphi_k) = 0$ *for all* $n, k \in \omega$,
  - $\mu_n(\varphi_k) = \delta(n,k)$ *for all* $n, k \in \omega$,
  - $\mu_n - \nu_n \to 0$ *in the weak* topology.*

Proof. To prove (ii)⇒(i) define $P\colon C(K) \to C(K)$ by

$$Pf = \sum_{n\in\omega} (\mu_n - \nu_n)(f) \cdot \varphi_n.$$

Then P is easily seen to be a bounded projection from $C(K)$ onto $T[c_0]$.

For the converse, consider the dual operator $T^*\colon \mathcal{M}(K) \to c_0^* = l_1$. Since T is an isomorphic embedding, T^* is a surjection, so for each $e_n^* = e_n \in l_1$ there exists $\mu_n \in \mathcal{M}(K)$ such that $T^*\mu_n = e_n^*$, and the sequence of μ_n is norm bounded. We have $T^*(\mu_n)(e_k) = e_n^*(e_k) = \delta(n,k)$ and $T^*(\mu_n)(e_k) = \mu_n(Te_k) = \mu_n(\varphi_k)$, so $\mu_n(\varphi_k) = \delta(n,k)$. For every n define $\nu_n \in \mathcal{M}(K)$ by setting $\nu_n(f) = \mu_n(f) - \mu_n(Pf)$ for $f \in C(K)$. Then ν_n vanishes on $P[C(K)]$, and for every $f \in C(K)$, taking $x \in c_0$ such that $Tx = Pf$, we get

$$\mu_n(f) - \nu_n(f) = \mu_n(Pf) = \mu_n(Tx) = e_n^*(x) \to 0,$$

as required. ∎

Here is the most obvious illustration of the previous lemma.

Corollary 2.2. *If* K *contains a nontrivial converging sequence then* $C(K)$ *contains a complemented copy of* c_0.

Proof. Let $(x_n)_n$ be a sequence in K converging to $x \in K$ and such that $x_n \neq x$ for every n. Then it is easy to construct a pairwise disjoint family $\{U_n : n \in \omega\}$ of open subsets of K such that $x_n \in U_n$ for each $n \in \omega$.

For every n we can find a continuous function $f_n : K \to [0,1]$ such that $f_n(x_n) = 1$ and f_n vanishes outside U_n. Now if we define $T : c_0 \to C(K)$ by $Te_n = f_n$ then $T[c_0]$ is complemented in $C(K)$. Indeed, we can apply Lemma 2.1 with $\mu_n = \delta_{x_n}$ and $\nu_n = \delta_x$ for every n. ∎

The way we stated Lemma 2.1 is motivated by its application to compactifications of ω.

COROLLARY 2.3. *A compactification $\gamma\omega$ is tame if and only if there exists a bounded sequence $(\nu_n)_n$ in $\mathcal{M}(\gamma\omega)$ such that $|\nu_n|(\omega) = 0$ for every n and $\nu_n - \delta_n \to 0$ in the weak* topology.*

Let us recall that the tameness of $\gamma\omega$ is directly related to the existence of a certain extension operator. If L is a closed subspace of a compact space K then an *extension operator* $E : C(L) \to C(K)$ is a bounded linear operator such that $E(f)|L = f$ for every $f \in C(L)$; see Avilés and Marciszewski [3] for a recent result on extension operators and for related references.

LEMMA 2.4. *A compactification $\gamma\omega$ is tame if and only if there is an extension operator $C(\gamma\omega \setminus \omega) \to C(\gamma\omega)$.*

Proof. Suppose that $P : C(\gamma\omega) \to c_0$ is a bounded projection (where c_0 is identified with its natural copy inside $C(\gamma\omega)$). For $f \in C(\gamma\omega \setminus \omega)$ take any extension $g \in C(\gamma\omega)$ of f and define $E(f) = g - Pg$. Note that $E(f)$ is uniquely defined: if $g' \in C(\gamma\omega)$ is another extension of f then $g' - g$ vanishes on the remainder, so $g' - g \in c_0$ and $P(g' - g) = g' - g$, that is, $g' - Pg' = g - Pg$.

Suppose now that $E : C(\gamma\omega\setminus\omega) \to C(\gamma\omega)$ is an extension operator. Then $Pg = g - E(g|\gamma\omega \setminus \omega)$ defines a projection from $C(\gamma\omega)$ onto c_0. ■

We shall now recall the notion of a Grothendieck space, which for $C(K)$ spaces means being anti-Sobczyk.

DEFINITION 2.5. A Banach space is said to be a *Grothendieck space* if every weak* null sequence $(x_n^*)_n$ in X^* converges weakly (i.e. $x^{**}(x_n^*) \to 0$ for every $x^{**} \in X^{**}$).

For the proof of the following see Cembranos [10].

THEOREM 2.6. *Given a compact space K, the space $C(K)$ is Grothendieck if and only if $C(K)$ does not contain a complemented copy of c_0.*

Recall that typical examples of Grothendieck spaces are l_∞ and, more generally, $C(K)$ spaces where K is an extremally disconnected compact space (see [1] or [13]).

3. Boolean algebras and compactifications.

We shall consider mainly zerodimensional compactifications of ω, and those are naturally related to Boolean subalgebras of $P(\omega)$. If $\mathfrak{A}$ is any Boolean algebra then $\mathrm{ULT}(\mathfrak{A})$ denotes the Stone space of all ultrafilters on $\mathfrak{A}$. If $a \in \mathfrak{A}$ then $\widehat{a}$ is the corresponding clopen set in $\mathrm{ULT}(\mathfrak{A})$, that is,

$$\widehat{a} = \{x \in \mathrm{ULT}(\mathfrak{A}) : a \in x\}.$$

A family $\mathcal{U} \subseteq p$ is a *base* of the ultrafilter p if every $A \in p$ contains some $U \in \mathcal{U}$, in other words, if $\{\widehat{U} : U \in \mathcal{U}\}$ is a local base at $p \in \mathrm{ULT}(\mathfrak{A})$.

Let $\mathfrak{A}$ be any Boolean algebra. Then $\mathrm{ba}(\mathfrak{A})$ will stand for the family of all finitely additive measures μ on $\mathfrak{A}$ that have bounded variation, and $\mathrm{ba}_+(\mathfrak{A})$ stands for the finitely additive nonnegative measures. We call any $\mu \in \mathrm{ba}(\mathfrak{A})$ simply a measure.

We now recall the following standard facts (see e.g. Semadeni [28, 18.7]). Every $\mu \in \mathrm{ba}_+(\mathfrak{A})$ can be transferred onto the algebra of clopen subsets of $\mathrm{ULT}(\mathfrak{A})$ by the formula $\widehat{\mu}(\widehat{a}) = \mu(a)$, and then uniquely extended to a Radon measure on $\mathrm{ULT}(\mathcal{A})$ (that Radon measure is still denoted by $\widehat{\mu}$). Note that the weak* topology on a bounded subset of $\mathcal{M}(\mathrm{ULT}(\mathfrak{A}))$ may be seen as the topology of convergence of clopen subsets of $\mathrm{ULT}(\mathfrak{A})$. Hence for a bounded sequence μ_n in $\mathrm{ba}(\mathfrak{A})$ we have $\widehat{\mu}_n \to 0$ in the weak* topology of $\mathcal{M}(\mathrm{ULT}(\mathfrak{A}))$ if and only if $\mu_n(a) \to 0$ for every $a \in \mathfrak{A}$.

If $\mathfrak{A}$ is a subalgebra of $P(\omega)$ and $\mathfrak{A}$ contains fin, the family of all finite subsets of ω, then $\mathrm{ULT}(\mathfrak{A})$ is a compactification of ω—we simply identify points in ω with principal ultrafilters on $\mathfrak{A}$. Consider the canonical quotient map

$$\mathfrak{A} \to \mathfrak{A}/\mathrm{fin}, \qquad \mathfrak{A} \ni A \mapsto A^\bullet \in \mathfrak{A}/\mathrm{fin}\,.$$

There is a natural correspondence $x \to x' = \{A^\bullet : A \in x\}$ between nonprincipal ultrafilters $x \in \mathrm{ULT}(\mathfrak{A})$ and ultrafilters x' in $\mathrm{ULT}(\mathfrak{A}/\mathrm{fin})$. We usually identify x with x', so that $\mathrm{ULT}(\mathfrak{A}/\mathrm{fin})$ becomes $\mathrm{ULT}(\mathfrak{A}) \setminus \omega$ and is seen as the remainder of the compactification $\mathrm{ULT}(\mathfrak{A})$ of ω.

Using the terminology and notation introduced above we can rewrite Corollary 2.3 as follows.

Lemma 3.1. *Let $\mathfrak{A}$ be an algebra such that* $\mathrm{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$. *Then the compactification* $\mathrm{ULT}(\mathfrak{A})$ *of ω is tame if and only if there exists a bounded sequence $(\nu_n)_n$ in* $\mathrm{ba}(\mathfrak{A})$ *such that*

(i) $\nu_n|\mathrm{fin} \equiv 0$ *for every n, and*
(ii) $\nu_n - \delta_n \to 0$ *on $\mathfrak{A}$, that is, $(\nu_n - \delta_n)(A) \to 0$ for every $A \in \mathfrak{A}$.*

Note that in case $A \in \mathfrak{A}$ is infinite and co-infinite, condition (ii) above is equivalent to

$$\lim_{n\in A} \nu_n(A) = 1, \qquad \lim_{n\notin A} \nu_n(A) = 0. \tag{3.1.1}$$

We shall often enlarge a given algebra $\mathfrak{A} \subseteq P(\omega)$ by adding a new set $X \subseteq \omega$; let $\mathfrak{A}[X]$ be the algebra generated by $\mathfrak{A}$ and X. Then

$$\mathfrak{A}[X] = \{(A \cap X) \cup (A' \setminus X) : A, A' \in \mathfrak{A}\}.$$

If $\mu \in \mathrm{ba}_+(\mathfrak{A})$ then μ_* and μ^* denote the corresponding inner and outer measures defined as

$$\mu_*(X) = \sup\{\mu(A) : A \in \mathfrak{A}, A \subseteq X\}, \quad \mu^*(X) = \inf\{\mu(A) : A \in \mathfrak{A}, A \supseteq X\}.$$

The following fact on extensions of finitely additive measures is due to Łoś and Marczewski [24].

Theorem 3.2. *Given an algebra $\mathfrak{A}$, $\mu \in \mathrm{ba}_+(\mathfrak{A})$ and any X, the following formulas define extensions $\mu_1, \mu_2 \in \mathrm{ba}_+(\mathfrak{A}[X])$ of μ:*

$$\mu_1((A \cap X) \cup (A' \setminus X)) = \mu_*(A \cap X) + \mu^*(A' \setminus X),$$
$$\mu_2((A \cap X) \cup (A' \setminus X)) = \mu^*(A \cap X) + \mu_*(A' \setminus X).$$

Consequently, for every t satisfying $\mu_*(X) \le t \le \mu^*(X)$, there is an extension $\mu_t \in \mathrm{ba}_+(\mathfrak{A}[X])$ of μ such that $\mu_t(X) = t$.

The following lemma on convergence of extended probability measures will be needed in Section 6.

Lemma 3.3. *Let $\mathfrak{A}$ be an algebra such that* $\mathrm{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ *and let $(\nu_n)_n$ be a sequence of probability measures from* $\mathrm{ba}_+(\mathfrak{A})$ *with $\nu_n - \delta_n \to 0$. Further, for every n, let $\widetilde{\nu}_n \in \mathrm{ba}_+(\mathfrak{A}[X])$ be any extension of ν_n. If $\widetilde{\nu}_n(X) - \delta_n(X) \to 0$ then $\widetilde{\nu}_n - \delta_n \to 0$ on $\mathfrak{A}[X]$.*

Proof. We use (3.1.1). For any $A \in \mathfrak{A}$, if n runs through $A \cap X$ then $\widetilde{\nu}_n(X) \to 1$ and $\widetilde{\nu}_n(A) = \nu_n(A) \to 1$, so $\widetilde{\nu}_n(A \cap X) \to 1$ (use $\widetilde{\nu}_n(\omega) = \nu_n(\omega) = 1$).

Take any $\varepsilon > 0$. Then $\nu_n(A) < \varepsilon$ if $n \notin A$ and $n \ge n_0$, and $\widetilde{\nu}_n(X) < \varepsilon$ whenever $n \ge n_1$ and $n \notin X$. Hence for $n \ge \max(n_0, n_1)$, if $n \notin A \cap X$ then either $n \notin A$ and $\widetilde{\nu}_n(A \cap X) \le \nu_n(A) < \varepsilon$, or $n \notin X$ and $\widetilde{\nu}_n(A \cap X) \le \widetilde{\nu}_n(X) < \varepsilon$.

The convergence of $\widetilde{\nu}_n(A \setminus X)$ may be checked in a similar way. ■

A Boolean algebra $\mathfrak{A}$ is *nonatomic* if every nonzero $A \in \mathfrak{A}$ is a disjoint union of two nonzero elements of $\mathfrak{A}$, which is equivalent to saying that $\mathrm{ULT}(\mathfrak{A})$ has no isolated points.

Recall that a nonnegative measure $\mu \in \mathrm{ba}(\mathfrak{A})$ is said to be *nonatomic* if for every $\varepsilon > 0$ there is a finite partition of $1_{\mathfrak{A}}$ into pieces of measure $< \varepsilon$. A signed measure μ is nonatomic if its variation $|\mu|$ is nonatomic. We shall use the following two simple observations.

Lemma 3.4. *Given a Boolean algebra $\mathfrak{A}$ and a signed measure μ on $\mathfrak{A}$, μ is nonatomic if and only if* $\inf\{|\mu|(A) : A \in p\} = 0$ *for every* $p \in \mathrm{ULT}(\mathfrak{A})$, *or equivalently $\widehat{\mu}$ is a Radon measure on* $\mathrm{ULT}(\mathfrak{A})$ *vanishing on all points of* $\mathrm{ULT}(\mathfrak{A})$.

If μ is nonatomic on $\mathfrak{A}$, $a \in \mathfrak{A}$ and $t < \mu(a)$ then for every $\varepsilon > 0$ there is $b \in \mathfrak{A}$ such that $b \le a$ and $|\mu(b) - t| < \varepsilon$.

4. Liftings.

Let $\mathfrak{A}$ be an algebra such that $\mathrm{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$. By a *lifting* of the quotient map we mean a Boolean homomorphism $\rho\colon \mathfrak{A}/\mathrm{fin} \to \mathfrak{A}$ such that $\rho(a)^\bullet = a$ for every $a \in \mathfrak{A}/\mathrm{fin}$. By the well-known duality between

Boolean algebras and their Stone spaces, a lifting $\rho\colon \mathfrak{A}/\mathrm{fin} \to \mathfrak{A}$ induces a continuous surjection

$$r : \mathrm{ULT}(\mathfrak{A}) \to \mathrm{ULT}(\mathfrak{A}/\mathrm{fin}), \qquad r(x) = \{a \in \mathfrak{A}/\mathrm{fin} : \rho(a) \in x\},$$

which is a retraction of the compactification $\mathrm{ULT}(\mathfrak{A})$ of ω onto its remainder.

Given some compactification $\gamma\omega$ admitting a retraction r onto $\gamma\omega \setminus \omega$, we have an obvious extension operator $E : C(\gamma\omega \setminus \omega) \to C(\gamma\omega)$ defined as $E(g) = g \circ r$. This leads to the following observation.

THEOREM 4.1. *If* $\mathrm{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ *is an algebra such that the quotient map* $\mathfrak{A} \to \mathfrak{A}/\mathrm{fin}$ *admits a lifting then the compactification* $\mathrm{ULT}(\mathfrak{A})$ *of* ω *is tame.*

The following simple lemma enables us to define subalgebras of $P(\omega)$ admitting a lifting.

LEMMA 4.2. *For an algebra* $\mathfrak{A}$ *such that* $\mathrm{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$*, the quotient map* $\mathfrak{A} \to \mathfrak{A}/\mathrm{fin}$ *admits a lifting if and only if there exists a Boolean algebra* $\mathfrak{B} \subseteq P(\omega)$ *such that every* $B \in \mathfrak{B} \setminus \{\emptyset\}$ *is infinite and* $\mathfrak{A}$ *is equal to* $\mathrm{alg}(\mathfrak{B} \cup \mathrm{fin})$*, the algebra generated by* $\mathfrak{B}$ *and* fin.

Proof. If ρ is a lifting then set $\mathfrak{B} = \rho[\mathfrak{A}/\mathrm{fin}]$. For every nonzero $a \in \mathfrak{A}/\mathrm{fin}$ the set $\rho(a)$ is infinite because $\rho(a)^\bullet = a \neq 0$.

If $\mathfrak{A} = \mathrm{alg}(\mathfrak{B} \cup \mathrm{fin})$ then, by the property of $\mathfrak{B}$, for any $a \in \mathfrak{A}/\mathrm{fin}$ there exists exactly one $B_a \in \mathfrak{B}$ such that $B_a^\bullet = a$. Therefore we can define $\rho(a) = B_a$ and ρ is the required homomorphism. ∎

Let us illustrate how Lemma 3.1 works in this context. Suppose that $\mathfrak{A}$ is generated by $\mathfrak{B}$ and fin, where every $B \in \mathfrak{B} \setminus \{\emptyset\}$ is infinite. For every n consider the ultrafilter $p_n = \{B \in \mathfrak{B} : n \in B\}$ on $\mathfrak{B}$. Then p_n extends to the nonprincipal ultrafilter x_n on $\mathfrak{A}$, where

$$x_n = \{B \triangle I : B \in p_n,\ I \in \mathrm{fin}\}.$$

Hence $\delta_{x_n} - \delta_n \to 0$ on $\mathfrak{A}$, since for every $A \in \mathcal{A}$ we have $\delta_{x_n}(A) - \delta_n(A) = 0$ for all but finitely many n's.

Thanks to Theorem 4.1 one can easily define relatively big tame compactifications of ω. Take for instance an independent family $\{B_\alpha : \alpha < \mathfrak{c}\}$ in $P(\omega)$ such that $\bigcap_{\alpha\in I} B_\alpha^{\varepsilon(\alpha)}$ is infinite for every finite $I \subseteq \mathfrak{c}$ and every $\varepsilon\colon I \to \{0,1\}$. Then the algebra $\mathfrak{A}$ generated by all B_α's and fin is such that $\mathfrak{A}/\mathrm{fin}$ has a lifting by Lemma 4.2; moreover, by Theorem 4.1, $\mathrm{ULT}(\mathfrak{A})$ is a tame compactification of ω with remainder homeomorphic to the Cantor cube $2^{\mathfrak{c}}$. This can be generalized as follows (recall that the space $2^{\mathfrak{c}}$ is separable).

THEOREM 4.3. *If* L *is a separable zerodimensional compact space then there exists a tame compactification* $\gamma\omega$ *such that* $\gamma\omega \setminus \omega$ *is homeomorphic to* L.

Proof. We write $\mathrm{Clopen}(L)$ for the algebra of clopen subsets of L. Let D be a countable dense subset of L. We define an embedding $\varphi\colon \mathrm{Clopen}(L) \to P(\omega)$ as follows. Take a partition $\{B_d : d \in D\}$ of ω into infinite sets, and for $V \in \mathrm{Clopen}(L)$ let

$$\varphi(V) = \bigcup_{d \in V \cap D} B_d.$$

Then for $U, V \in \mathrm{Clopen}(L)$, if $\varphi(U) = \varphi(V)$ then $U \cap D = V \cap D$, so $U = V$. It is easy to check that φ is indeed an isomorphic embedding of $\mathrm{Clopen}(L)$ into $P(\omega)$, and the algebra $\mathfrak{B} = \varphi[\mathrm{Clopen}(L)]$ has the property that every nonempty $B \in \mathfrak{B}$ is infinite.

Letting $\mathfrak{A}$ be the algebra in $P(\omega)$ generated by $\mathfrak{B}$ and fin, we conclude from Theorem 4.1 that $\mathrm{ULT}(\mathfrak{A})$ is a tame compactification. Moreover, its remainder $\mathrm{ULT}(\mathfrak{A}/\mathrm{fin})$ can be identified with $\mathrm{ULT}(\mathfrak{B})$, which is homeomorphic to $\mathrm{ULT}(\mathrm{Clopen}(L))$, so to L itself. ∎

We prove below that there is a compactification $\gamma\omega$ which is not tame but $\gamma\omega \setminus \omega$ is separable (and zerodimensional). The conclusion is that tameness of a compactification $\gamma\omega$ cannot be decided by examining topological properties of $\gamma\omega \setminus \omega$ alone (below we write $\simeq$ for homeomorphism).

COROLLARY 4.4. *There are two compactifications $\gamma_1\omega$ and $\gamma_2\omega$ such that $\gamma_1\omega \setminus \omega \simeq \gamma_2\omega \setminus \omega$, while $\gamma_2\omega$ is tame and $\gamma_1\omega$ is not tame.*

Proof. Take $\gamma_1\omega$ as in Theorem 7.1, that is, a nontame zerodimensional compactification with separable remainder $L = \gamma_1\omega \setminus \omega$. By Theorem 4.3 there is a tame compactification $\gamma_2\omega$ such that $L \simeq \gamma_2\omega \setminus \omega$, and we are done. ∎

We finish this section by the following side remark. If $\mathrm{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ is an algebra and the quotient map $\mathfrak{A} \to \mathfrak{A}/\mathrm{fin}$ admits a lifting then the algebra $\mathfrak{A}/\mathrm{fin}$ is *σ-centred*, i.e. $\mathfrak{A}/\mathrm{fin}$ is a countable union of centred families. Note that the converse does not hold: If we take $\mathfrak{A}$ as in Theorem 7.1 then $\mathrm{ULT}(\mathfrak{A})$ is not tame, so $\mathfrak{A} \to \mathfrak{A}/\mathrm{fin}$ does not have a lifting by Theorem 4.1. On the other hand, $\mathfrak{A}/\mathrm{fin}$ is σ-centred since it is isomorphic to the clopen algebra of a separable space.

5. The measure algebra.

We start with the following observation due to W. Kubiś.

THEOREM 5.1. *If $\gamma\omega$ is a tame compactification then its remainder carries a strictly positive measure.*

Proof. Take a sequence $(\nu_n)_n$ as in Lemma 2.3 and consider $\mu = \sum_n 2^{-n}|\nu_n|$. Then μ is a finite nonnegative measure on $\gamma\omega$, and $\mu(\omega) = 0$. Let $U \subseteq \gamma\omega$ be an open set such that $U \cap (\gamma\omega \setminus \omega) \neq \emptyset$. Take a continuous function $g\colon \gamma\omega \to [0,1]$ that vanishes outside U and $g(x_0) = 1$ for some $x_0 \in U \setminus \omega$. Then the set $V = \{g > 1/2\}$ contains infinitely many $n \in \omega$. Since $\nu_n(g) - g(n) \to 0$ we conclude that $\nu_n(g) > 0$ for some n, and this gives

$$\mu(U) \geq \mu(g) \geq 2^{-n}|\nu_n|(g) > 0,$$

so the measure μ is positive on every nonempty open subset of $\gamma\omega \setminus \omega$. ∎

We show in this section that under CH there are nontame compactifications $\gamma\omega$ such that $\gamma\omega \setminus \omega$ carries a strictly positive nonatomic measure.

Let $\mathfrak{M}$ be the *measure algebra*, that is, the quotient $\mathrm{Bor}[0,1]/\mathcal{N}$, where $\mathcal{N}$ is the ideal of Lebesgue null sets. We denote by λ the measure on $\mathfrak{M}$ defined from the Lebesgue measure. Throughout this section we write $S = \mathrm{ULT}(\mathfrak{M})$ for the Stone space of $\mathfrak{M}$, for simplicity. It is well-known that $C(S)$ is isometric $L_\infty[0,1]$.

By the classical Parovichenko theorem, under CH there is an isomorphic embedding $\varphi\colon \mathfrak{M} \to P(\omega)/\mathrm{fin}$. Define $\mathfrak{A} \subseteq P(\omega)$ as the algebra of all finite modifications of elements of $\varphi[\mathfrak{M}]$ and consider the Stone space $\mathrm{ULT}(\mathfrak{A})$. Then $\mathrm{ULT}(\mathfrak{A})$ is a compactification of ω and its remainder is homeomorphic to S. In Theorem 5.4 we shall investigate the space $\mathrm{ULT}(\mathfrak{A})$. Note first that $C(\mathrm{ULT}(\mathfrak{A}) \setminus \omega) = C(S)$ is a Grothendieck space, so it contains no complemented copy of c_0.

LEMMA 5.2. *Let $\mathfrak{A}$ be the above-defined algebra. For every family $\{A_n : n \in \omega\} \subseteq \mathfrak{A}$ there is $B \in \mathfrak{A}$ which is an* almost upper bound *of that family, in the sense that $A_n \subseteq^* B$ for every n and $B \cap A$ is finite whenever $A \in \mathfrak{A}$ is such that $A \cap A_n = \emptyset$ for every n.*

Proof. For every n take $a_n \in \mathfrak{M}$ such that $\varphi(a_n) = A_n^\bullet$. The algebra $\mathfrak{M}$ is complete, so $\{a_n : n \in \omega\}$ has the least upper bound $b \in \mathfrak{M}$. Now $B \in \mathfrak{A}$ such that $B^\bullet = \varphi(b)$ is as required. ∎

THEOREM 5.3. *Let $\mathrm{fin} \subseteq \mathfrak{A} \subseteq P(\omega)$ be an algebra such that the space $\mathrm{ULT}(\mathfrak{A}/\mathrm{fin})$ is homeomorphic to S. Then $\mathrm{ULT}(\mathfrak{A})$ is a compactification of ω that is not tame.*

Proof. By Lemma 3.1 it is enough to check that whenever $(\nu_n)_n \subset \mathrm{ba}(\mathfrak{A})$ is a bounded sequence, where every ν_n vanishes on finite sets, then $\nu_n - \delta_n$ does not converge to 0. Suppose otherwise: let $\nu_n(A) - \delta_n(A) \to 0$ for every $A \in \mathfrak{A}$.

Note first that there is an infinite $T \in \mathfrak{A}$ such that every ν_n is nonatomic on the algebra $\mathfrak{A}_T = \{A \in \mathfrak{A} : A \subseteq T\}$ of subsets of T. Indeed, every $\widehat{\nu}_n$, as a Radon measure on $\mathrm{ULT}(\mathfrak{A})$, is concentrated on S and the set $\{x \in S : \widehat{\nu}_n(\{x\}) \neq 0\}$ is at most countable (since $|\widehat{\nu}_n|(S)$ is finite; see Lemma 3.4). The space S is not separable, so there is a nonzero $a \in \mathfrak{M}$ such that $\widehat{\nu}_n(\{x\}) = 0$ for all n and $x \in \widehat{a}$. Take $T \in \mathfrak{A}$ such that $T^\bullet = a$; then T is as required.

Fix $\varepsilon = 1/8$. Pick any $A_1 \in \mathfrak{A}$ such that $A_1 \subseteq T$ and $A_1, T \setminus A_1$ are infinite. Since $\delta_n(A_1) = 1$ for every $n \in A_1$, we have $\lim_{n\in A_1} \nu_n(A_1) = 1$. Hence there exists $n_1 \in A_1$ such that $\nu_{n_1}(A_1) \geq 1 - \varepsilon$. Moreover, since the variation of ν_{n_1} is finite, there exists an infinite $D_1 \in \mathfrak{A}$ such that $D_1 \subseteq T \setminus A_1$ and $|\nu_{n_1}|(D_1) < \varepsilon$.

In a similar way, for every $k > 1$ there exist infinite sets $A_k, D_k \in \mathfrak{A}$ and $n_1 < n_2 < \cdots$ such that

(a) $A_k \subseteq D_{k-1}$ and $D_k \subseteq D_{k-1} \setminus A_k$;
(b) $D_{k-1} \setminus A_k$ is infinite;
(c) $n_k \in A_k$;
(d) $\nu_{n_k}(A_k) \geq 1 - \varepsilon$ and $|\nu_{n_k}|(D_k) < \varepsilon$.

Since all the measures ν_n are nonatomic on T, by Lemma 3.4 for every $k \in \omega$ there exists a set $B_k \in \mathfrak{A}$ such that $B_k \subseteq A_k$ and $|\nu_{n_k}(B_k) - 1/2| < \varepsilon$.

As A_k are pairwise disjoint and $n_k \in A_k$, it follows from $\nu_n - \delta_n \to 0$ that there is an infinite $N \subseteq \omega$ such that for every $k \in N$,

$$\Big|\nu_{n_k}\Big(\bigcup_{j<k,\, j\in N} B_j\Big)\Big| < \varepsilon. \tag{5.3.1}$$

Let $B \in \mathfrak{A}$ be an almost upper bound for $\{B_k : k \in N\}$ as in Lemma 5.2. Write, for simplicity, $B_{<k} = \bigcup_{j<k,\, j\in N} B_j$, and let $B_{\leq k}$ be defined accordingly. For any $k \in N$ we have

$$\nu_{n_k}(B) = \nu_{n_k}(B_{<k}) + \nu_{n_k}(B_k) + \nu_{n_k}(B \setminus B_{\leq k}),$$

where $B \setminus B_{\leq k} \subseteq^* D_k$. Using (5.3.1), condition (d) and the fact that ν_{n_k} vanishes on finite sets, we get

$$|\nu_{n_k}(B) - 1/2| < 3\varepsilon, \quad \text{so} \quad 1/8 < \nu_{n_k}(B) < 7/8,$$

for every $k \in N$ (note that $1/2 + 3\varepsilon = 1/2 + 3/8 = 7/8$).

On the other hand, consider the set $J = \{k \in N : n_k \in B\}$. If J is infinite then $\nu_{n_k}(B) \to 1$ for $k \in J$. If $N \setminus J$ is infinite, we should have $\nu_{n_k}(B) \to 0$ for $k \in N \setminus J$, and in both cases this is a contradiction. ∎

We shall now prove that for $\mathfrak{A}$ as in Theorem 5.3 the space $C(\mathrm{ULT}(\mathfrak{A}))$ need not be Grothendieck. Consider the family $\mathfrak{D}$ of all subsets $A \subseteq \omega$ having the asymptotic density

$$d(A) = \lim_n |A \cap n|/n.$$

Observe that $d(A)$ does not depend on finite modifications of $A \subseteq \mathfrak{D}$, so we can also treat d as a function defined on $\mathfrak{D}/\mathrm{fin}$.

In the proof of Theorem 5.6 we shall use the following result of Frankiewicz and Gutek and a simple lemma.

Theorem 5.4 ([18]). *Assuming CH there exists an embedding* $\varphi\colon \mathfrak{M} \to P(\omega)/\mathrm{fin}$ *such that* $\varphi(a) \in \mathfrak{D}/\mathrm{fin}$ *and* $d(\varphi(a)) = \lambda(a)$ *for every* $a \in \mathfrak{M}$.

Lemma 5.5. *There exists a family* $\{I_n\}_{n\in\omega}$ *of pairwise disjoint finite subsets of* ω *such that* $d(A) = \lim_n |A \cap I_n|/|I_n|$ *for any* $A \in \mathfrak{D}$.

Proof. Take any increasing sequence of integers k_n such that $\lim_n k_n/k_{n+1} = 0$ and set $I_n = \{i \in \omega : k_n \le i < k_{n+1}\}$. Then $\{I_n\}$ is as required by standard calculations. ∎

Theorem 5.6. *Under CH there is a compactification* $\gamma\omega$ *such that* $\gamma\omega\setminus\omega \simeq S$ *and* $C(\gamma\omega)$ *contains a complemented copy of* c_0.

Proof. We use an embedding $\varphi\colon \mathfrak{M} \to P(\omega)/\mathrm{fin}$ as in Theorem 5.4 and consider $\mathfrak{A}$ as at the beginning of this section and in Theorem 5.3. Take a family of pairwise disjoint intervals I_n as in Lemma 5.5. For every $n \in \omega$ let $f_n = \chi_{I_n} \in C(\mathrm{ULT}(\mathfrak{A}))$.

Set $Y = \overline{\mathrm{lin}\langle f_n\colon n \in \omega\rangle}$. Then Y is a closed subspace of $C(\mathrm{ULT}(\mathfrak{A}))$ isomorphic to c_0; the isomorphism is defined by setting $e_n \mapsto f_n$.

Now Y is complemented in $C(\mathrm{ULT}(\mathfrak{A}))$; indeed, consider the measures

$$\mu_n = \frac{1}{|I_n|}\sum_{i\in I_n} \delta_i \in \mathrm{ba}_+(\mathfrak{A}).$$

Then $\mu_n(f_k)$ equals 1 if $k = n$ and 0 if $n \ne k$ because I_n are pairwise disjoint. Moreover, by Lemma 5.5, $\mu_n \to d$ on $\mathfrak{A}$, so we can apply Lemma 2.1. ∎

6. Large tame compactification.

This section is devoted to proving

Theorem 6.1. *Under CH there exists a tame compactification* $\gamma\omega$ *of* ω *such that* $\gamma\omega \setminus \omega$ *is not separable.*

The desired compactification will be defined as $\mathrm{ULT}(\mathfrak{A})$, where $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$, and the $\mathfrak{A}_\alpha$ are countable algebras defined inductively. Lemma 6.2 describes the starting point of the construction; we only sketch its proof since it closely follows the proof of Sobczyk's theorem in Diestel [13]. The subsequent Lemma 6.3 contains the essence of the inductive step.

Lemma 6.2. *There exist a countable nonatomic Boolean algebra* $\mathfrak{C} \subseteq P(\omega)$ *and a sequence* $(\nu_n)_n$ *of nonatomic probability measures on* $\mathfrak{B} = \mathrm{alg}(\mathfrak{C}\cup \mathrm{fin})$ *such that* $\nu_n|\mathrm{fin} \equiv 0$ *for every* n *and* $\nu_n - \delta_n \to 0$ *on* $\mathfrak{B}$.

Proof. There is an algebra $\mathfrak{C} \subseteq P(\omega)$ isomorphic to $\mathrm{Clopen}(2^\omega)$. We can copy the standard product measure ν on 2^ω onto $\mathfrak{C}$ (and denote it by the same letter). Set $\mathfrak{B} = \mathrm{alg}(\mathfrak{C}\cup\mathrm{fin})$. Then we have a probability measure μ on $\mathfrak{B}$ defined by $\mu(C \triangle I) = \nu(C)$ for all $C \in \mathfrak{C}$ and $I \in \mathrm{fin}$.

Since $\mathfrak{B}$ is countable, $\mathrm{ULT}(\mathfrak{B})$ is metrizable, so $C(\mathrm{ULT}(\mathfrak{B}))$ is a separable Banach space. Hence the unit ball $\mathcal{M}_1(\mathrm{ULT}(\mathfrak{B}))$ is metrizable in its weak* topology, by some metric ρ. Denote by $\mathcal{P}$ the space of probability measures on $\mathrm{ULT}(\mathfrak{B})$ that vanish on ω.

It is not difficult to check that the set of nonatomic $\nu \in \mathcal{P}$ is weak* dense in $\mathcal{P}$: Consider for instance the convex hull of the family $\mu_C \in \mathrm{ba}_+(\mathfrak{B})$, where $\mu_C(B) = (1/\mu(C)) \cdot \mu(C \cap B)$ and $C \in \mathfrak{C}$. For every n we can choose a nonatomic $\nu_n \in P$ such that

$$\rho(\nu_n, \delta_n) \leq 2\,\mathrm{dist}(\delta_n, P).$$

Then $\nu_n - \delta_n \to 0$ in the weak* topology. ■

LEMMA 6.3. *Let $\mathfrak{B} \subseteq P(\omega)$ be a countable Boolean algebra containing* fin. *Moreover, suppose that*

(i) *we are given a set $\{p_j : j \in \omega\}$ of nonprincipal ultrafilters which is dense in* $\mathrm{ULT}(\mathfrak{B}/\mathrm{fin})$*;*
(ii) *$(\nu_n)_{n\in\omega}$ is a sequence of nonatomic probability measures on $\mathfrak{B}$;*
(iii) *$\nu_n(I) = 0$ for every $I \in \mathrm{fin}$ and every n;*
(iv) *$\nu_n - \delta_n \to 0$ on $\mathfrak{B}$.*

Then there exists an infinite set $X \subseteq \omega$ such that

- *for any extension $\widetilde{p}_j$ of p_j to an ultrafilter on $\mathfrak{B}[X]$ the set $\{\widetilde{p}_j : j \in \omega\}$ is not dense in* $\mathrm{ULT}(\mathfrak{B}[X])$*;*
- *we can extend every ν_n to a probability measure $\widetilde{\nu}_n$ on $\mathfrak{B}[X]$ so that $\widetilde{\nu}_n - \delta_n \to 0$ on $\mathfrak{B}[X]$.*

Proof. Since $\mathfrak{B}$ is countable, we fix an enumeration $\mathfrak{B} = \{B_0, B_1, \ldots\}$.

CLAIM. *There are infinite sets $A_j \in \mathfrak{B}$ and $c(j) \in \omega$ for $j \in \omega$ such that*

(1) *$\nu_0(A_j) < 2^{-(j+2)}$ for all $j \in \omega$;*
(2) *$A_j \in p_j$;*
(3) *for all i and $n \in \omega \setminus A_j$ we have $\nu_n(A_j) < 2^{-(j+2)}$;*
(4) *$c(j) \notin A_k$ for all j, k;*
(5) *for every j either $B_j \subseteq \bigcup_{k\leq j} A_k$ or $c(j) \in B_j$.*

Proof of Claim. We proceed by induction on j. Suppose that we have already constructed $A_0, \ldots, A_j$ and $c(0), \ldots c(j)$, and set $m = \max_{i\leq j} c(i)+1$ (of course we can additionally assume that $0 \notin A_i$).

Since all the measures ν_i are nonatomic and p_{j+1} is a nonprincipal ultrafilter, there is $A \in p_{j+1}$ such that $A \cap m = \emptyset$ and $\nu_i(A) < 2^{-(j+3)}$ for every $i \leq m$. Since $\nu_n(A) - \delta_n(A) \to 0$, the set $F = \{n \notin A : \nu_n(A) \geq 2^{-(j+3)}\}$ is finite. Define $A_{j+1} = A \cup F$. Since $\nu_i(F) = 0$ for every i, (1) and (3) are granted by the choice of A. Condition (4) holds since $F \cap m = \emptyset$.

Now we can set $c(j+1) = 0$ or choose $c(j+1) \in B_{j+1} \setminus \bigcup_{k\leq j+1} A_k$, if possible. This proves the Claim.

We take the sets A_j from the Claim and prove that the set $X = \bigcup_j A_j$ is as desired. We first check the following properties of X:

(a) For any $B \in \mathfrak{B}$, if $B \subseteq X$ then $B \subseteq \bigcup_{j \le N} A_j$ for some $N \in \omega$.
(b) $\omega \setminus X$ is infinite.
(c) $\nu_n^*(X) = 1$ for every n.
(d) $\lim_{n \notin X} (\nu_n)_*(X) = 0$.

To check (a) take any $B_j \in \mathfrak{B}$ such that $B_j \subseteq X$. Suppose that $B_j \nsubseteq \bigcup_{i \le j} A_i$. Then $c(j) \in B_j \setminus X$ by (4) and (5), a contradiction.

Suppose $\omega \setminus X$ is finite. Then $X \in \mathfrak{B}$, since every finite set belongs to $\mathfrak{B}$. By (a), $X \subseteq \bigcup_{i \le N} A_i$ for some N. Since ν_0 is insensitive to finite modifications of sets, by condition (1) we get

$$\nu_0(\omega) = \nu_0(X) \le \nu_0\Big(\bigcup_{i \le N} A_i\Big) \le \sum_{i \le N} 2^{-(i+2)} \le 1/2,$$

which contradicts the fact that ν_0 is a probability measure on ω.

(c) follows from the fact that for any n, if $B \in \mathfrak{B}$ and $\nu_n(B) > 0$ then B is infinite, so $B \in p_j$ for some j and hence $B \cap X \neq \emptyset$. This proves $\nu_n^*(X) = 1$.

To prove (d) note that by (a) every $B \in \mathfrak{B}$ that is contained in X is in fact contained in some finite union of A_j's; hence

$$(\nu_n)_*(X) = \sup_N \nu_n\Big(\bigcup_{i \le N} A_i\Big).$$

Observe that by condition (3), if $n \notin X$ and $k < N$ then

$$\nu_n\Big(\bigcup_{i=k}^{N} A_i\Big) \le \sum_{i=k}^{N} \nu_n(A_i) \le \sum_{i=k}^{N} 2^{-(i+2)} < 2^{-(k+1)}. \tag{6.3.1}$$

Using this estimate we can compute the limit of inner measures:

$$\begin{aligned}
\lim_{n \notin X} \nu_{n*}(X) &= \lim_{n \notin X} \sup_N \nu_n\Big(\bigcup_{i \le N} A_i\Big) \\
&\le \lim_{n \notin X} \sup_N \Big(\nu_n\Big(\bigcup_{i<k} A_i\Big) + \nu_n\Big(\bigcup_{i=k}^{N} A_i\Big)\Big) \\
&\le \lim_{n \notin X} \Big(\nu_n\Big(\bigcup_{i<k} A_i\Big) + 2^{-(k+1)}\Big) \quad \text{(by (6.3.1))} \\
&= \lim_{n \notin X} \nu_n\Big(\bigcup_{i<k} A_i\Big) + 2^{-(k+1)}.
\end{aligned}$$

But $\bigcup_{i<k} A_i \in \mathfrak{B}$ for any $k \in \omega$, so $\lim_{n \notin X} \nu_n(\bigcup_{i<k} A_i) = 0$. Since k is arbitrary, this proves (d).

Once we know that X satisfies (a)–(d), we can check that X is indeed the set we are looking for. For every j, let $\widetilde{p}_j \in \mathrm{ULT}(\mathfrak{B}[X])$ be an arbitrary extension of p_j. Because $X \supseteq A_j$ and $A_j \in p_j$, we have $X \in \widetilde{p}_j$. Thus the infinite set $\omega \setminus X$ omits all the ultrafilters $\widetilde{p}_j$, and so they are not dense in $\text{ULT}(\mathfrak{B}[X]/\text{fin})$.

Now appealing to Theorem 3.2 we define the measures $\widetilde{\nu}_n$ on $\mathfrak{B}[X]$ extending ν_n so that

$$\widetilde{\nu}_n(X) = \begin{cases} \nu_n^*(X) & \text{for } n \in X, \\ (\nu_n)_*(X) & \text{for } n \notin X. \end{cases} \tag{6.3.2}$$

Then for $n \in X$ we have $\widetilde{\nu}_n(X) = \nu_n^*(X) \to 1$, and for $n \notin X$ we have $\widetilde{\nu}_n(X) = (\nu_n)_*(X) \to 0$ by (d). Using Lemma 3.3 we conclude that $\widetilde{\nu}_n - \delta_n \to 0$ on $\mathfrak{B}[X]$, and the proof is complete. ∎

We already have all essential ingredients to carry out a diagonal construction leading to Theorem 6.1.

Proof of Theorem 6.1. Let $\mathfrak{A}_0$ be the algebra from Lemma 6.2 and let $(\nu_n^0)_n$ be a sequence of nonatomic probability measures on $\mathfrak{A}_0$ such that $\nu_n^0 - \delta_n \to 0$.

We construct inductively, for $\xi < \omega_1$, a sequence of countable algebras $\mathfrak{A}_\xi \subseteq P(\omega)$, sets $X_\xi \subseteq \omega$ and sequences $(\nu_n^\xi)_n$ of probability measures on $\mathfrak{A}_\xi$ such that

(i) $\mathfrak{A}_\beta \subseteq \mathfrak{A}_\xi$ for all $\beta < \xi < \omega_1$;
(ii) $\mathfrak{A}_\xi$ is generated by $\bigcup_{\beta<\xi} \mathfrak{A}_\beta$ and X_ξ;
(iii) $\nu_n^\xi \,|\, \mathfrak{A}_\beta = \nu_n^\beta$ for all n and $\beta < \xi$;
(iv) $\nu_n^\xi - \delta_n \to 0$ on $\mathfrak{A}_\xi$ for every ξ.

Then we consider the algebra $\mathfrak{A} = \bigcup_{\xi<\omega_1} \mathfrak{A}_\xi$, and for every n let μ_n be the unique probability measure on $\mathfrak{A}$ such that $\mu_n | \mathfrak{A}_\xi = \nu_n^\xi$ for $\beta < \xi$. It is clear that $\mu_n - \delta_n \to 0$ on $\mathfrak{A}$, so $\text{ULT}(\mathfrak{A})$ is a tame compactification of ω by Lemma 3.1. Therefore it is enough to check that by a suitable choice of the sets X_ξ, we can guarantee that $\text{ULT}(\mathfrak{A}/\text{fin})$ is not separable.

Using CH we fix an enumeration $\{D(\xi) : \alpha < \omega_1\}$ of all countable dense sets in $\text{ULT}(\mathfrak{A}_0/\text{fin})$. At step ξ we let $\mathfrak{B} = \bigcup_{\alpha<\xi} \mathfrak{A}_\alpha$ and consider a sequence of measures ν_n defined on $\mathfrak{B}$, where ν_n is the unique extension of ν_n^α, $\alpha < \xi$. Then we apply Lemma 6.3 to find a set X_ξ such that (iv) holds for $\mathfrak{A}_\xi = \mathfrak{B}[X_\xi]$, and at the same time X_ξ witnesses that no extensions of ultrafilters from $D(\xi)$ are dense in $\text{ULT}(\mathfrak{A}_\xi/\text{fin})$.

It follows that $\text{ULT}(\mathfrak{A}/\text{fin})$ is not separable. Indeed, if we had a countable dense set $D \subseteq \text{ULT}(\mathfrak{A}/\text{fin})$ then $D = \{x|\mathfrak{A}_0 : x \in X\}$ would be dense in $\text{ULT}(\mathfrak{A}_0/\text{fin})$. But $D = D(\xi)$ for some $\xi < \omega_1$, and $D(\xi)$ is not dense in $\text{ULT}(\mathfrak{A}_\xi/\text{fin})$. ∎

7. Small and stormy. In this section we construct a relatively small compactification $\gamma\omega$ which is not tame, in contrast to the compactification from Section 6.

THEOREM 7.1. *There exists a nontame compactification $\gamma\omega$ which is first-countable and whose remainder $\gamma\omega \setminus \omega$ is separable.*

We again construct a certain algebra $\mathfrak{A} \subseteq P(\omega)$; this time the main idea is to keep a fixed countable dense set of ultrafilters and to kill all possible sequences of measures that would witness tameness.

We shall use the notion of minimal extensions of algebras introduced by Koppelberg [22], which we now recall in the context of subalgebras od $P(\omega)$. The basic facts we list below can be found in [22] or [4].

If $\mathfrak{A} \subseteq P(\omega)$ and $X \subseteq \omega$ then $\mathfrak{A}[X]$ is said to be a *minimal extension* of $\mathfrak{A}$ if for any algebra $\mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{A}[X]$ then either $\mathfrak{B} = \mathfrak{A}$ or $\mathfrak{B} = \mathfrak{A}[X]$. This is equivalent to saying that for every $A \in \mathfrak{A}$, either $X \cap A \in \mathfrak{A}$ or $X \setminus A \in \mathfrak{A}$.

If $\mathfrak{A}[X] \neq \mathfrak{A}$ is a minimal extension then there is exactly one $q(X) \in \mathrm{ULT}(\mathfrak{A})$ that gets split in $\mathfrak{A}[X]$, i.e. has two different extensions to elements of $\mathrm{ULT}(\mathfrak{A}[X])$; this is $q(X) = \{A \in \mathfrak{A} : A \cap X \notin \mathfrak{A}\}$. Then every ultrafilter $p \neq q(X)$ has a unique extension to $\widetilde{p} \in \mathrm{ULT}(\mathfrak{A}[X])$.

Given a sequence $A_n \in \mathfrak{A}$ and $p \in \mathrm{ULT}(\mathfrak{A})$, it will be convenient to say that A_n *converges to* p if every $B \in p$ contains almost all A_n.

LEMMA 7.2. *If $\mathfrak{A}[X]$ is a minimal extension of $\mathfrak{A}$ and $\mu \in \mathrm{ba}_+(\mathfrak{A})$ does not have an atom at $q(X)$ then μ has a unique extension to $\widetilde{\mu} \in \mathrm{ba}_+(\mathfrak{A}[X])$.*

Proof. For every $\varepsilon > 0$ there is $A \in q(X)$ with $\mu(A) < \varepsilon$. Then $B = X \setminus A \in \mathfrak{A}$, so $B \subseteq X \subseteq B_1 = B \cup A$, where $B, B_1 \in \mathfrak{A}$ and $\mu(B_1 \setminus B) < \varepsilon$. This shows that $\mu_*(X) = \mu^*(X)$ must be equal to $\widetilde{\mu}(X)$ whenever $\widetilde{\mu}$ is a nonnegative extension of μ. We can repeat this argument with $A \cap X$ and $A \setminus X$ for $A \in \mathfrak{A}$ to conclude that $\widetilde{\mu} = \mu^* = \mu_*$ on $\mathfrak{A}[X]$. ∎

LEMMA 7.3. *Let $\mathfrak{B} \subseteq P(\omega)$ be an algebra containing* fin *with $\mathfrak{B}/\mathrm{fin}$ nonatomic. Let $C = \{p_j : j \in \omega\}$ be a dense subset of $\mathrm{ULT}(\mathfrak{B}/\mathrm{fin})$. Further let*

(i) *$q \in \mathrm{ULT}(\mathfrak{B}/\mathrm{fin}) \setminus C$ be a point of countable character;*

(ii) *$(n_k)_k$ be a sequence on ω such that $n_k \to q$ (in $\mathrm{ULT}(\mathfrak{B})$);*

(iii) *$(B_k)_k$ and $(D_k)_k$ be sequences in $\mathfrak{B}$ of infinite sets that converge to q with $B_k \cap D_j = \emptyset$ for all j, k.*

If we let

$$X = \{n_k : k \in \omega\} \cup \bigcup_k B_k,$$

then $\mathfrak{B}[X]$ is a minimal extension of $\mathfrak{B}$ and only $q = q(X)$ may be split in $\mathfrak{B}[X]$. Consequently, every p_j has a unique extension to an ultrafilter $\widetilde{p}_j$ on $\mathfrak{B}[X]$, and $\widetilde{C} = \{\widetilde{p}_j : j \in \omega\}$ is dense in $\mathrm{ULT}(\mathfrak{B}[X]/\mathrm{fin})$.

Proof. For every $B \in q$, we have $X \setminus B \in \mathfrak{B}$ since $B_k \cup \{n_k\}$ converges to q. Hence $\mathfrak{B}[X]$ is a minimal extension.

To complete the proof we have to check the density of $\widetilde{C}$, i.e. that for every $B \in \mathfrak{B}$, if $B \cap X$ is infinite then $B \cap X \in \widetilde{p}_j$ for some j, and if $B \setminus X$ is infinite then $B \setminus X \in \widetilde{p}_j$ for some j.

Take $B \in \mathfrak{B}$ such that $B \cap X$ is infinite. If $B \in q$ then $B_k \subseteq B \cap X$ for some k, and taking j with $B_k \in p_j$, we get $B \cap X \in \widetilde{p}_j$. If $B \notin q$ then $B \cap X \in \mathfrak{B}$ (by (ii) and (iii)), so $B \cap X \in p_j$ for some j.

Suppose now that $B \setminus X$ is infinite. If $B \notin q$ then $B \setminus X \in \mathfrak{B}$, as above. Finally, if $B \in q$ then there is k such that $D_k \subseteq B$. Then $D_k \subseteq B \setminus X$ by (iii), and taking j with $D_k \in p_j$, we get $B \setminus X \in \widetilde{p}_j$, so the proof is complete. ∎

The lemma below constitutes the essence of the inductive step.

Lemma 7.4. *Let $\mathfrak{B} \subseteq P(\omega)$ be an algebra such that $\mathfrak{B}/\mathrm{fin}$ is nonatomic and $\mathrm{ULT}(\mathfrak{B}/\mathrm{fin})$ is first-countable. Let $(p_j)_j$ be a sequence of nonprincipal ultrafilters on $\mathfrak{B}$ such that the set $C = \{p_j : j < \omega\}$ is dense in $\mathrm{ULT}(\mathfrak{B}/\mathrm{fin})$. Suppose also that $(\nu_n)_n$ is a sequence of measures on $\mathfrak{B}$ such that $\nu_n|\mathrm{fin} \equiv 0$ for every n and $\nu_n - \delta_n \to 0$ on $\mathfrak{B}$. Then there exists a set $X \subseteq \omega$ such that*

1. *every p_j has a unique extension to $\widetilde{p}_j \in \mathrm{ULT}(\mathfrak{B}[X])$ and the set $\{\widetilde{p}_j : j < \omega\}$ is dense in $\mathrm{ULT}(\mathfrak{B}[X]/\mathrm{fin})$;*
2. *if $\bar{\nu}_n$ is an extension of ν_n to a measure on $\mathfrak{B}[X]$ and $\|\bar{\nu}_n\| = \|\nu_n\|$ for every n then $\bar{\nu}_n(X) - \delta_n(X) \not\to 0$.*

Proof. We shall choose an ultrafilter q on $\mathfrak{B}$ and construct $B_n, D_n \in \mathfrak{B}$ and numbers n_k as in Lemma 7.1 so that the set $X = \{n_1, n_2, \ldots\} \cup \bigcup_n B_n$ satisfies condition (2). Then (1) is granted by Lemma 7.3.

Take a point q in $\mathrm{ULT}(\mathfrak{B}/\mathrm{fin})$ such that $q \notin C$, which is an atom of no measure ν_n. Let $\{U_k : k \in \omega\} \subseteq \mathfrak{B}$ be a base at $q \in \mathrm{ULT}(\mathfrak{B})$. Choose also a sequence $(m_i)_i$ in ω such that $m_i \to q$ in $\mathrm{ULT}(\mathfrak{B})$ and set $N = \{m_i : i \in \omega\}$.

We construct inductively $C_k, D_k, V_k \in \mathfrak{B}$ and $n_k \in N$ such that for every k,

(a) $V_k \setminus (B_k \cup D_k) \in q$ and $B_k \cup D_k \subseteq V_k \subseteq U_k$;
(b) $n_k \in V_{k-1} \supseteq V_k$;
(c) $B_i \cap D_j = \emptyset$ for all i, j;
(d) $\nu_{n_k}(\bigcup_{j<k} B_j) < 1/k$;
(e) $|\nu_{n_k}|(V_k) < 1/k$.

The inductive construction is straightforward: if we have carried it out below k then we set

$$B = \bigcup_{j \le k-1} B_j, \qquad D = \bigcup_{j \le k-1} D_j, \qquad V = V_{k-1} \cap U_k \setminus (B \cup D).$$

Then $V \in q$, so V contains infinitely many m_i (as $m_i \to q$), and among them we choose n_k so that (d) holds (using $\nu_n - \delta_n \to 0$). Then we choose $V_k \subseteq V$ satisfying (e) (as q is not an atom of ν_{n_k}). Finally, we can choose $B_k, D_k \subseteq V_k \setminus (B \cup D)$ so that (a)–(c) hold.

Recall that $X = \{n_1, n_2, \ldots\} \cup \bigcup_n B_n$. Consider now any extensions $\bar{\nu}_n \in \mathrm{ba}(\mathfrak{B}[X])$ of measures ν_n with $\|\bar{\nu}_n\| = \|\nu_n\|$. Note that in that case $|\bar{\nu}_n|$ is an extension of $|\nu_n|$. Since for any k we have (using (a))

$$\bigcup_{n\in\omega} B_n = \bigcup_{j<k} B_j \cup \bigcup_{j\geq k} B_j \subseteq \bigcup_{j<k} B_j \cup V_k,$$

and the set $N \setminus V_k$ is finite, we get

$$\bar{\nu}_{n_k}(X) \leq \nu_{n_k}\Big(\bigcup_{j<k} B_j\Big) + |\bar{\nu}_{n_k}|(V_k) < 1/k + |\bar{\nu}_{n_k}|(V_k) = 1/k + |\nu_{n_k}|(V_k) < 2/k.$$

But $n_k \in X$, so $\nu_n(X) - \delta_n(X) \not\to 0$, and we are done. ■

Proof of Theorem 7.1. We first describe a certain operation that will be iterated to construct an algebra defining the required compactification.

Consider some algebra $\mathfrak{B}$ in $P(\omega)$ containing fin and such that $\mathfrak{B}/\mathrm{fin}$ is nonatomic. Suppose also that

(1) $\mathrm{ULT}(\mathfrak{B})$ is first-countable;
(2) $C = \{p_j : j \in \omega\}$ is a dense subset of $\mathrm{ULT}(\mathfrak{B}/\mathrm{fin})$;
(3) $\mathrm{ba}(\mathfrak{B})$ is of size $\mathfrak{c}$ and $\nu^\xi = (\nu^\xi_n)_n$, $\xi < \mathfrak{c}$, is an enumeration of all bounded sequences of measures on $\mathfrak{B}_\xi$ such that $\nu^\xi_n - \delta_n \to 0$ on $\mathfrak{B}_\xi$.

We fix a set $Q \subseteq \mathrm{ULT}(\mathfrak{B}/\mathrm{fin})$ with $Q \cap C = \emptyset$, of cardinality $\mathfrak{c}$, and for every $\xi < \mathfrak{c}$ we apply Lemma 7.4 to the sequence ν^ξ; pick $q_\xi \in Q \setminus \{q_\eta : \eta < \xi\}$ and form the set X_ξ as in Lemma 7.4. Then we let $\mathfrak{B}^\#$ be the algebra generated by $\mathfrak{B}$ and $\{X_\xi : \xi < \mathfrak{c}\}$. Note that

(a) $\mathrm{ULT}(\mathfrak{B}^\#)$ is first-countable;
(b) every p_j has a unique extension $p^\#_j \in \mathrm{ULT}(\mathfrak{B}^\#)$;
(c) $C^\# = \{p^\#_j : j \in \omega\}$ is dense in $\mathrm{ULT}(\mathfrak{B}^\#/\mathrm{fin})$;
(d) $|\mathrm{ba}(\mathfrak{B}^\#)| = \mathfrak{c}$.

To see (a) note that if $p \in \mathrm{ULT}(\mathfrak{B})$ is never split then it has a base in $\mathfrak{B}$. Otherwise $p = q_\xi$ for some $\xi < \mathfrak{c}$ and it is split only by X_ξ into two ultrafilters having bases in $\mathfrak{B}[X_\xi]$.

Properties (b) and (c) follow from Lemma 7.3.

To check (d) it suffices to note that any $\mu \in \mathrm{ba}_+(\mathfrak{B})$ has at most $\mathfrak{c}$ extensions to nonnegative measures on $\mathfrak{B}^\#$. Indeed, take $\mu \in \mathrm{ba}_+(\mathfrak{B})$ and let $N \subseteq \mathrm{ULT}(\mathfrak{B}/\mathrm{fin})$ be the set of all atoms of μ. Then N is countable and the algebra $\mathfrak{B}'$ generated by $\mathfrak{B}$ and $\{X_\xi : q_\xi \in N\}$ is countably generated over $\mathfrak{B}$. Therefore we can extend μ to a nonnegative measure μ' on $\mathfrak{B}'$ in at most $\mathfrak{c}$ ways. In turn, every such μ' extends uniquely to a measure in $\mathrm{ba}_+(\mathfrak{B}^\#)$ by Lemma 7.2.

We shall now iterate the operation $\#$. Let $\mathfrak{A}_0$ be a countable algebra in $P(\omega)$ such that $\mathfrak{A}_0/\mathrm{fin}$ is nonatomic. Fix $C = \{p_j : j \in \omega\}$ as above and choose a pairwise disjoint family $\{Q_\alpha : \alpha < \omega_1\}$ such that $|Q_\alpha| = \mathfrak{c}$, $Q_\alpha \subseteq \mathrm{ULT}(\mathfrak{A}_0/\mathrm{fin})$, and $Q_\alpha \cap C = \emptyset$ for every $\alpha < \omega_1$.

We define $\mathfrak{A}_{\alpha+1} = (\mathfrak{A}_\alpha)^\#$, with Q_α playing the role of Q in the construction. We also let $\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$ for $\alpha < \omega_1$ limit, and claim that $\mathfrak{A} = \bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha$ is the required algebra.

The compactification $\mathrm{ULT}(\mathfrak{A})$ is not tame by Lemma 3.1. Indeed, take any $\mu_n \in \mathrm{ba}(\mathfrak{A})$ with $\sup_n \|\mu_n\| < \infty$. Note that every measure on $\mathfrak{A}$ attains its norm on some countable subalgebra. Hence there is $\alpha < \omega_1$ such that, writing $\nu_n = \mu_n|\mathfrak{A}_\alpha$, we have $\|\nu_n\| = \|\mu_n\|$ for every n. Then $(\nu_n)_n = \nu^\xi$ for some $\xi < \mathfrak{c}$, and by our construction $\nu_n(X_\xi) - \delta_n(X_\xi) \nrightarrow 0$.

Finally, $\mathrm{ULT}(\mathfrak{A}/\mathrm{fin})$ is separable because C remains dense throughout the construction. The fact that $\mathrm{ULT}(\mathfrak{A})$ is first-countable follows from the fact that the Q_α's are pairwise disjoint, as in the proof of (a) above. ∎

In the terminology of [22], if $\mathfrak{B} \subseteq \mathfrak{A} \subseteq P(\omega)$ then $\mathfrak{A}$ is *minimally generated over* $\mathfrak{B}$ if, for some ξ, $\mathfrak{A}$ is a continuous increasing union $\bigcup_{\alpha<\xi} \mathfrak{A}_\alpha$, where $\mathfrak{A}_0 = \mathfrak{B}$ and $\mathfrak{A}_{\alpha+1}$ is a minimal extension of $\mathfrak{A}_\alpha$ for every $\alpha < \xi$. An algebra $\mathfrak{A}$ is *minimally generated* if it is minimally generated over the trivial algebra. It is clear that the algebra $\mathfrak{A}$ constructed in the proof of Theorem 7.1 is minimally generated.

In the language of extension operators, Theorem 7.1 says the following.

COROLLARY 7.5. *There exist a separable first-countable compact space L and a compact superspace K with $K \setminus L$ countable such that there is no extension operator $C(L) \to C(K)$.*

Proof. Take $\gamma\omega$ from Theorem 7.1, set $L = \gamma\omega \setminus \omega$, $K = \gamma\omega$ and apply Lemma 2.4. ∎

8. On hereditarily Sobczyk spaces. In this final section we prove a general result on compacta K for which the space $C(K)$ contains a rich family of complemented copies of c_0. In the definition below we use the terminology from Džamonja and Kunen [16].

DEFINITION 8.1. We say that a compact space K is *in the class* (MS) (of measure separable spaces) if every probability Radon measure μ on K has *countable Maharam type*, i.e. $L_1(\mu)$ is a separable Banach space.

Recall also the following standard notion.

DEFINITION 8.2. Let X be any vector space and let $(x_n)_n$ be a sequence in X. A sequence $(y_n)_n$ is a *convex block subsequence* of $(x_n)_n$ if there exist finite sets $I_n \subset \omega$ with $\max I_n < \min I_{n+1}$ and a function $a : \omega \to \mathbb{R}_+$ such that for all $n \in \omega$,

$$y_n = \sum_{j \in I_n} a(j)\, x_j \quad \text{and} \quad \sum_{j \in I_n} a(j) = 1.$$

Our result is a consequence of (a particular case of) a result due to Haydon, Levy and Odell [21] (see also Krupski and Plebanek [23] for a direct approach).

Theorem 8.3 (Haydon, Levy, Odell). *If K is a compact space in the class* (MS) *then every bounded sequence $(\mu_n)_n$ in $\mathcal{M}(K)$ has a convex block subsequence $(\nu_n)_n$ converging to some $\nu \in \mathcal{M}(K)$.*

Theorem 8.4. *Let K be a compact space in* (MS). *Then for any isomorphic embedding $T\colon c_0 \to C(K)$ the space $T[c_0]$ contains a subspace Y which is isomorphic to c_0 and complemented in $C(K)$.*

Proof. Let (e_n) be the sequence of unit vectors in c_0; we write $e_n^* \in c_0^* = l_1$.

Given an isomorphic embedding $T\colon c_0 \to C(K)$, set $g_n = Te_n$ for every n. Since T is an embedding, the dual operator

$$T^*\colon C(K)^* = \mathcal{M}(K) \to c_0^* = l_1$$

is onto, and therefore there is a bounded sequence of measures $\mu_n \in \mathcal{M}(K)$ such that $T^*\mu_n = e_n^*$. Then $\mu_n(g_k) = \mu_n(Te_k) = T^*\mu_n(e_k) = e_n^*(e_k)$.

Consider the sequence $(\mu_n)_n$. By Theorem 8.3 it has a convex block subsequence $(\nu_n)_n$ converging to some $\nu \in \mathcal{M}(K)$. Say that $\nu_n = \sum_{k\in I_n} a(k)\mu_k$, where I_n and a are as in Definition 8.2.

For every n set $\bar{e}_n = \sum_{k\in I_n} e_k$. Then the $\bar{e}_n$ are norm-one vectors spanning a subspace X of c_0 that is clearly isometric to c_0. Hence the functions $h_n = T\bar{e}_n \in C(K)$ span the subspace $Y = T[X]$ of $T[c_0]$ that is isomorphic to c_0, and it is enough to check that Y is complemented in $C(K)$.

Since $T^*\mu_n = e_n^*$, we have $T^*\nu_n = \sum_{i\in I_n} t(i)e_i^*$, and

$$\nu_n(h_k) = \nu_n(T\bar{e}_k) = T^*\nu_n(\bar{e}_k) = \sum_{i\in I_n} t(i)e_i^*\Big(\sum_{j\in I_k} e_j\Big) = \sum_{i\in I_n,\, j\in I_k} t(i)e_i^*(e_j),$$

which is equal to 0 if $n \neq k$ (since then $I_n \cap I_k = \emptyset$), and is equal to $\sum_{i\in I_n} t(i) = 1$ when $n = k$.

Now, as in Lemma 2.1, we conclude that $P : C(K) \to C(K)$ defined by

$$Pf = \sum_{n\in\omega} (\nu_n - \nu)(f) \cdot h_n$$

is a bounded projection onto Y. Indeed,

$$\nu(h_n) = \lim_j \nu_j(h_n) = \lim_j T^*\nu_j(\bar{e}_n) = \lim_j \Big(\sum_{i\in I_j} t(i)e_i^*\Big)(\bar{e}_n) = 0$$

for every n. This shows that $Ph_n = h_n$; moreover, $Pf \in Y$ for any $f \in C(K)$ since $\nu_n(f) - \nu(f) \to 0$ for every f. ∎

A Banach space X such that every isomorphic copy of c_0 in X has a subspace isomorphic to c_0 and complemented in X is called *hereditarily separably Sobczyk* in [19]. Theorem 8.4 states that $C(K)$ is such a space whenever K is in the class (MS). Moltó [25] gave an example of a Rosenthal compact space K such that $C(K)$ does not have the Sobczyk property. Rosenthal compacta are in (MS), due to a result of Bourgain and Todorčević (see [15] for a more general result and for relevant references). Consequently, $C(K)$ is hereditarily separably Sobczyk whenever K is Rosenthal compact.

The final result is related to our Theorem 7.1.

Corollary 8.5. *If $\mathfrak{A}$ is a minimally generated Boolean algebra then $C(\mathrm{ULT}(\mathfrak{A}))$ is a hereditarily separably Sobczyk space.*

Proof. This follows from Theorem 8.4 and the remark after it, and a result due to Borodulin-Nadzieja [4] stating that $\mathrm{ULT}(\mathfrak{A})$ is in the class (MS) whenever $\mathfrak{A}$ is minimally generated. ∎

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References

[1] F. Albiac and N. J. Kalton, *Topics in Banach Space Theory*, Springer, 2005.

[2] S. A. Argyros, J. M. F. Castillo, A. S. Granero, M. Jiménez and J. P. Moreno, *Complementation and embeddings of $c_0(I)$ in Banach spaces*, Proc. London Math. Soc. 85 (2002), 742–768.

[3] A. Avilés and W. Marciszewski, *Extension operators on balls and on spaces of finite sets*, Studia Math. 227 (2015), 165–182.

[4] P. Borodulin-Nadzieja, *Measures on minimally generated Boolean algebras*, Topology Appl. 154 (2007), 3107–3124.

[5] P. Borodulin-Nadzieja and T. Inamdar, *Measures and slaloms*, Fund. Math., to appear; arXiv:1604.03137 (2016).

[6] F. Cabello Sánchez, J. M. F. Castillo and D. Yost, *Sobczyk's Theorems from A to B*, Extracta Math. 15 (2000), 391–420.

[7] L. Candido and P. Koszmider, *On complemented copies of $c_0(\omega_1)$ in $C(K^n)$ spaces*, Studia Math. 233 (2016), 209–226.

[8] J. M. F. Castillo, *Wheeling around Sobczyk's theorem*, in: General Topology in Banach Spaces, T. Banakh and A. Plichko (eds.), Nova Sci., Huntington, NY, 2001, 103–110.

[9] J. M. F. Castillo, *Nonseparable $C(K)$-spaces can be twisted when K is a finite height compact*, Topology Appl. 198 (2016), 107–116.
[10] P. Cembranos, *$C(K, E)$ contains a complemented copy of c_0*, Proc. Amer. Math. Soc. 91 (1984), 556–558.
[11] C. Correa and D. V. Tausk, *Compact lines and the Sobczyk property*, J. Funct. Anal. 266 (2014), 5765–5778.
[12] C. Correa and D. V. Tausk, *Nontrivial twisted sums of c_0 and $C(K)$*, J. Funct. Anal. 270 (2016), 842–853.
[13] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, 1984.
[14] A. Dow and K. P. Hart, *The measure algebra does not always embed*, Fund. Math. 163 (2000), 163–176.
[15] P. Drygier and G. Plebanek, *Nonseparable growth of the integers supporting a measure*, Topology Appl. 191 (2015), 58–64.
[16] M. Džamonja and K. Kunen, *Properties of the class of measure separable compact spaces*, Fund. Math. 147 (1995), 261–271.
[17] J. Ferrer, J. Kąkol, M. López Pellicer and M. Wójtowicz, *On a three-space property for Lindelöf Σ-spaces, (WCG)-spaces and the Sobczyk property*, Funct. Approx. Comment. Math. 44 (2011), 289–306.
[18] R. Frankiewicz and A. Gutek, *Some remarks on embeddings of Boolean algebras and the topological spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 29 (1981), 471–476.
[19] E. M. Galego and A. Plichko, *On Banach spaces containing complemented and uncomplemented subspaces isomorphic to c_0*, Extracta Math. 18 (2003), 315–319.
[20] G. Godefroy, *The Banach space c_0*, Extracta Math. 16 (2001), 1–25.
[21] R. Haydon, M. Levy and E. Odell, *On sequences without weak* convergent convex block subsequences*, Proc. Amer. Math. Soc. 100 (1987), 94–98.
[22] S. Koppelberg, *Minimally generated Boolean algebras*, Order 5 (1989), 393–406.
[23] M. Krupski and G. Plebanek, *A dichotomy for the convex spaces of probability measures*, Topology Appl. 158 (2011), 2184–2190.
[24] J. Łoś and E. Marczewski, *Extensions of measure*, Fund. Math. 36 (1949), 267–276.
[25] A. Moltó, *On a theorem of Sobczyk*, Bull. Austral. Math. Soc. 43 (1991), 121–130.
[26] W. M. Patterson, *Complemented c_0-subspaces of a nonseparable $C(K)$-space*, Canad. Math. Bull. 36 (1993), 351–357.
[27] R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. 48 (1940), 516–541.
[28] Z. Semadeni, *Banach Spaces of Continuous Functions, Vol. I*, PWN–Polish Sci. Publ., Warszawa, 1971.
[29] A. Sobczyk, *Projection of the space (m) on its subspace (c_0)*, Bull. Amer. Math. Soc. 47 (1941), 938–947.
[30] W. Veech, *Short proof of Sobczyk's theorem*, Proc. Amer. Math. Soc. 28 (1971), 627–628.

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