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Higher Mahler measure of an n-variable family

by

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1. Introduction. For k a positive integer, the *k-higher Mahler measure* of a non-zero, n-variable, rational function $P(x_1,\dots,x_n)\in\mathbb{C}(x_1,\dots,x_n)$ is given by

$$\mathrm{m}_k(P(x_1,\dots,x_n))=\int_0^1\cdots\int_0^1\log^k|P(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n})|\,d\theta_1\cdots d\theta_n.$$

We observe that the case $k=1$ recovers the formula for the "classical" Mahler measure. This function, originally defined as a height on polynomials, has attracted considerable interest in the last decades due to its connection to special values of the Riemann zeta function, and of L-functions associated to objects of arithmetic significance such as elliptic curves as well as special values of polylogarithms and other special functions. Part of such phenomena have been explained in terms of Beilinson's conjectures via relationships with regulators by Deninger [Den97] (see also the crucial articles by Boyd [Boy97] and Rodriguez-Villegas [R-V97]).

Higher (and multiple) Mahler measures were originally defined in [KLO08] and subsequently studied by several authors [Sas10, BS11, LS11, BBSW12, BS12, Sas12, Bis14, BM14]. A related object, the Zeta Mahler measure, was first studied by Akatsuka [Aka09]. As remarked by Deninger, higher Mahler measures are expected to yield different regulators from the ones that appear in the case of the usual Mahler measure, and they may reveal a more complicated structure at the level of periods (see [Lal10] for more details).

In order to continue this program of understanding periods that arise from higher Mahler measure, an essential component is to generate examples

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of formulas for higher Mahler measure involving special functions that can be easily expressed as periods, such as polylogarithms. In the present work we consider the family of rational functions in $\mathbb{C}(x_1, \dots, x_m, z)$ given by

$$R_m(x_1, \dots, x_m, z) := z + \left(\frac{1-x_1}{1+x_1}\right) \cdots \left(\frac{1-x_m}{1+x_m}\right).$$

Let $\zeta(s)$ be the Riemann zeta function and $L(\chi_{-4}, s)$ be the Dirichlet L-function in the character of conductor 4, defined, for $\mathrm{Re}(s) > 1$, as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

$$L(\chi_{-4}, s) := \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^s}, \qquad \chi_{-4}(n) = \begin{cases} \left(\frac{-1}{n}\right) & \text{if } n \text{ odd,} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

We also consider the functions

$$\mathcal{L}_{n_1,\dots,n_m}(w_1, \dots, w_m) = \sum_{(r_1,\dots,r_m)\in\{0,1\}^m} (-1)^{r_m} \mathrm{Li}_{n_1,\dots,n_m}\big((-1)^{r_1} w_1, \dots, (-1)^{r_m} w_m\big), \tag{1.1}$$

given by combinations of multiple polylogarithms (of length m), defined for positive integers n_i by

$$\mathrm{Li}_{n_1,\dots,n_m}(w_1, \dots, w_m) = \sum_{0<j_1<\cdots<j_m} \frac{w_1^{j_1} \cdots w_m^{j_m}}{j_1^{n_1} \cdots j_m^{n_m}}.$$

The series above is absolutely convergent for $|w_i| \leq 1$ and $n_m > 1$.

Finally, for $a_1, \dots, a_m \in \mathbb{C}$, consider

$$s_\ell(a_1, \dots, a_m) = \begin{cases} 1 & \text{if } \ell = 0, \\ \displaystyle\sum_{i_1<\cdots<i_\ell} a_{i_1} \cdots a_{i_\ell} & \text{if } 0 < \ell \leq m, \\ 0 & \text{if } m < \ell. \end{cases}$$

We prove the following result.

Theorem 1.1. *For $n \geq 1$, we have*

$$\mathrm{m}_k(R_{2n}) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2, 4^2, \dots, (2n-2)^2)}{(2n-1)!} \left(\frac{2}{\pi}\right)^{2h} \mathcal{A}_k(h),$$

where

$$\mathcal{A}_k(h) := (2h+k-1)!\left(1-\frac{1}{2^{2h+k}}\right)\zeta(2h+k)+(-1)^k k! \sum_{k/2-1\le n\le k-2}\frac{1}{2^{k-n-1}}$$
$$\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-n-2}} (2h-1)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h}(1,\dots,1)$$
$$+\sum_{j=1}^{k-2}\frac{(-1)^{k+j}k!}{j!}\sum_{(k-j)/2-1\le n\le k-j-2}\frac{1}{2^{k-n-j}}$$
$$\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} (2h+j-1)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+j}(1,\dots,1).$$

For $n\ge 0$*, we have*

$$\mathrm{m}_k(R_{2n+1}) = \sum_{h=0}^{n}\frac{s_{n-h}(1^2,3^2,\dots,(2n-1)^2)}{(2n)!}\left(\frac{2}{\pi}\right)^{2h+1}\mathcal{B}_k(h),$$

where

$$\mathcal{B}_k(h) := (2h+k)!L(\chi_{-4},2h+k+1)$$
$$+(-1)^{k+1}k!\sum_{k/2-1\le n\le k-2}\frac{1}{2^{k-n-1}}$$
$$\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-n-2}} i(2h)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+1}(1,\dots,1,i,i)$$
$$+\sum_{j=1}^{k-2}\frac{(-1)^{k+j+1}k!}{j!}\sum_{(k-j)/2-1\le n\le k-j-2}\frac{1}{2^{k-n-j}}$$
$$\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} i(2h+j)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+j+1}(1,\dots,1,i,i).$$

For the sake of clarity, we record here the case of $k=2$:

$$\mathcal{A}_2(h) := (2h+1)!\left(1-\frac{1}{2^{2h+2}}\right)\zeta(2h+2)+(2h-1)!\mathcal{L}_{2,2h}(1,1),$$

$$\mathcal{B}_2(h) := (2h+2)!L(\chi_{-4},2h+3)-i(2h)!\mathcal{L}_{2,2h+1}(i,i).$$

The case of $k=1$ clearly yields formulas that only depend on $\zeta(s)$, $L(\chi_{-4},s)$ and powers of π. This is equivalent to saying that all the terms can be expressed in terms of polylogarithms of length one. There is another case in which we can prove a similar formula.

Corollary 1.2. *The previous result includes the following particular case:*

$$\mathrm{m}_2(R_2) = -\frac{31\pi^2}{360} + \frac{28}{\pi^2}(\log 2)\zeta(3) + \frac{32}{\pi^2}\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{4}{3\pi^2}(\log^2 2)(\log^2 2 - \pi^2), \tag{1.2}$$

where all the terms are products of polylogarithms of length one.

Our method of proof of Theorem 1.1 relies on the ideas of [Lal06a] combined with key properties of the Zeta Mahler function constructed by Akatsuka [Aka09].

The same method yields a formula for a simpler polynomial. Let

$$Q_m(x_1, \dots, x_m) := \left(\frac{x_1 - 1}{x_1 + 1}\right) \cdots \left(\frac{x_m - 1}{x_m + 1}\right).$$

We can express the higher Mahler measure of this family in terms of rational combinations of powers of π. More precisely, we obtain the following result.

Proposition 1.3. *For $n \geq 1$, we have*

$$\mathrm{m}_{2k}(Q_{2n}) = \pi^{2k} \sum_{h=1}^{n} \frac{s_{n-h}(2^2, 4^2, \dots, (2n-2)^2)}{(2n-1)!} \frac{(-1)^{k+h+1}}{2(k+h)} 2^{2h}(2^{2k+2h} - 1)B_{2(k+h)}.$$

For $n \geq 0$, we have

$$\mathrm{m}_{2k}(Q_{2n+1}) = \left(\frac{\pi}{2}\right)^{2k} \sum_{h=0}^{n} \frac{s_{n-h}(1^2, 3^2, \dots, (2n-1)^2)}{(2n)!}(-1)^{k+h}E_{2(k+h)}.$$

In addition, for $k \geq 0$ and $m \geq 1$,

$$\mathrm{m}_{2k+1}(Q_m) = 0.$$

In the above expressions, B_m and E_m are the Bernoulli and Euler numbers respectively. Formulas for $\mathrm{m}_k(Q_1)$ were found in [BBSW12].

It is not necessary to use the method of [Lal06a] to find formulas for $\mathrm{m}_k(Q_m)$. By using simple properties of the higher Mahler measure of a product of polynomials with different variables, we get alternative expressions for the same formulas. By comparing with the results of Proposition 1.3, we obtain the following identities between Bernoulli and Euler numbers which generalize some results of [Lal06a, Appendix].

Corollary 1.4.

$$\sum_{h=1}^{n} \frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!} \frac{(-1)^{h+1}}{2(k+h)} 2^{2k+2h}(2^{2k+2h}-1)B_{2(k+h)} = \sum_{j_1+\cdots+j_{2n}=k} \binom{2k}{2j_1,\dots,2j_{2n}} E_{2j_1}\cdots E_{2j_{2n}}$$

and

$$\sum_{h=0}^{n} \frac{s_{n-h}(1^2,3^2,\dots,(2n-1)^2)}{(2n)!}(-1)^h E_{2(k+h)} = \sum_{j_1+\cdots+j_{2n+1}=k} \binom{2k}{2j_1,\dots,2j_{2n+1}} E_{2j_1}\cdots E_{2j_{2n+1}}.$$

The present article is organized as follows. In Section 2 we recall previous results on the classical Mahler measure of R_n, as well as similar results for other rational functions, obtained in [Lal06a]. We outline the general method of proof in Section 3. In Section 4 we discuss some properties of the Zeta Mahler measure. Sections 5 and 6 treat certain technical simplifications of the integrals involved. We discuss properties of polylogarithms in Section 7, and we present the final details of the proofs in Section 8. Some technical results were already part of [Lal06a] but we include them for the sake of completeness.

2. Description of similar results for Mahler measure. In this section we present the previous results that were obtained with this method for the classical Mahler measure.

Theorem 2.1 ([Lal06a]). *For $n \geq 1$,*

$$\mathrm{m}(R_{2n}) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!} \left(\frac{2}{\pi}\right)^{2h} \mathcal{A}_1(h),$$

where

$$\mathcal{A}_1(h) := (2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1).$$

For $n \geq 0$,

$$\mathrm{m}(R_{2n+1}) = \sum_{h=0}^{n} \frac{s_{n-h}(1^2,3^2,\dots,(2n-1)^2)}{(2n)!} \left(\frac{2}{\pi}\right)^{2h+1} \mathcal{B}_1(h),$$

where

$$\mathcal{B}_1(h) := (2h+1)!L(\chi_{-4},2h+2).$$

In addition, similar results were proved in [Lal06a] for other rational functions by the same method. Let

$$S_m(x_1,\dots,x_m,x,y,z) := (1+x)z + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)(1+y),$$

$$T_m(x_1,\dots,x_m,x,y)$$
$$:= 1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)x + \left(1-\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)\right)y.$$

Theorem 2.2 ([Lal06a]). *For $n\geq 1$,*

$$\mathrm{m}(S_{2n}) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!}\left(\frac{2}{\pi}\right)^{2h+2}\mathcal{C}_1(h),$$

where

$$\mathcal{C}_1(h) := \sum_{\ell=1}^{h}\binom{2h}{2\ell}\frac{(-1)^{h-\ell}}{4h}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+2)!\left(1-\frac{1}{2^{2\ell+3}}\right)\zeta(2\ell+3).$$

For $n\geq 0$,

$$\mathrm{m}(S_{2n+1}) = \sum_{h=0}^{n} \frac{s_{n-h}(1^2,3^2\dots,(2n-1)^2)}{(2n)!}\left(\frac{2}{\pi}\right)^{2h+3}\mathcal{D}_1(h),$$

where

$$\mathcal{D}_1(h) := \sum_{\ell=0}^{h}\binom{2h+1}{2\ell+1}\frac{(-1)^{h-\ell}}{2(2h+1)}B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell+3)!L(\chi_{-4},2\ell+4).$$

For $n\geq 1$,

$$\mathrm{m}(T_{2n}) = \frac{\log 2}{2} + \sum_{h=1}^{n} \frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!}\left(\frac{2}{\pi}\right)^{2h}\mathcal{E}_1(h),$$

where

$$\mathcal{E}_1(h) := \frac{(2h)!}{2}\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1) + \sum_{\ell=1}^{h}(2^{2(h-\ell)-1}-1)\binom{2h}{2\ell}\frac{(-1)^{h-\ell+1}}{2h}$$
$$\times B_{2(h-\ell)}\pi^{2h-2\ell}(2\ell)!\left(1-\frac{1}{2^{2\ell+1}}\right)\zeta(2\ell+1).$$

For $n\geq 0$,

$$\mathrm{m}(T_{2n+1}) = \frac{\log 2}{2} + \sum_{h=1}^{n} \frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n+1)!}\left(\frac{2}{\pi}\right)^{2h+2}\mathcal{F}_1(h),$$

where

$$\begin{aligned}\mathcal{F}_1(h) := {} & \frac{(2h+2)!}{2}\left(1-\frac{1}{2^{2h+3}}\right)\zeta(2h+3) \\ & + \frac{\pi^2 n^2}{2}(2h)!\left(1-\frac{1}{2^{2h+1}}\right)\zeta(2h+1) \\ & + n(2n+1)\sum_{\ell=1}^{h}(2^{2(h-\ell)-1}-1)\binom{2h}{2\ell}\frac{(-1)^{h-\ell+1}}{4h} \\ & \qquad \times B_{2(h-\ell)}\pi^{2h+2-2\ell}(2\ell)!\left(1-\frac{1}{2^{2\ell+1}}\right)\zeta(2\ell+1).\end{aligned}$$

We remark that some of the above formulas differ in presentation from [Lal06a] as they have been simplified by using recent results of [LL].

3. The general method. Here we describe the general structure of the proof of Theorem 1.1, based on the ideas of [Lal06a].

Let $P_a \in \mathbb{C}(z)$ be such that its coefficients are rational functions in a parameter $a \in \mathbb{C}$. By making the change of variables $a = \left(\frac{x_1-1}{x_1+1}\right)\cdots\left(\frac{x_n-1}{x_n+1}\right)$ we can view the rational function P_a as a new rational function in $n+1$ variables, $\tilde{P} \in \mathbb{C}(x_1,\dots,x_n,z)$. Thus, the k-higher Mahler measure of $\tilde{P}$ is

$$\begin{aligned}\mathrm{m}_k(\tilde{P}) &= \frac{1}{(2\pi i)^{n+1}}\int_{\mathbb{T}^{n+1}}\log^k|\tilde{P}|\,\frac{dx}{x}\,\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\ &= \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\left(\frac{1}{(2\pi i)}\int_{\mathbb{T}}\log^k|\tilde{P}|\,\frac{dx}{x}\right)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\ &= \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\mathrm{m}_k\big(P_{(\frac{x_1-1}{x_1+1})\cdots(\frac{x_n-1}{x_n+1})}\big)\,\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n},\end{aligned}$$

where $\mathrm{m}_k\big(P_{(\frac{x_1-1}{x_1+1})\cdots(\frac{x_n-1}{x_n+1})}\big)$ is a function of $x_1,\dots,x_n$. By making the change of variables $x_j = e^{i\theta_j}$, followed by $y_j = \tan(\theta_j/2)$, the integral above equals

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathrm{m}_k(P_{i^n\tan(\theta_1/2)\cdots\tan(\theta_n/2)})\,d\theta_1\cdots d\theta_n \\ &\qquad = \frac{1}{\pi^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{m}_k(P_{i^n y_1\cdots y_n})\,\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}.\end{aligned}$$

Assume that the function of a given by $\mathrm{m}_k(P_a)$ is even, that is, $\mathrm{m}_k(P_a) = \mathrm{m}_k(P_{-a})$ (this happens if, for instance, $\mathrm{m}_k(P_a)$ only depends on $|a|$). Then the same can be said of the integrand above, and the previous expression equals

$$\frac{2^n}{\pi^n}\int_0^\infty\cdots\int_0^\infty \mathrm{m}_k(P_{i^n y_1\cdots y_n})\,\frac{dy_1}{y_1^2+1}\cdots\frac{dy_n}{y_n^2+1}$$
$$=\frac{2^n}{\pi^n}\int_0^\infty\cdots\int_0^\infty \mathrm{m}_k(P_{i^n\hat{x}_n})\,\frac{\hat{x}_1\,d\hat{x}_1}{\hat{x}_1^2+1}\,\frac{\hat{x}_2\,d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1}\,d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\,\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2},$$

where we have set $\hat{x}_j = \prod_{i=1}^j y_i$. This change of variables is motivated by the goal of recovering a term of the form $\mathrm{m}_k(P_x)$ inside the integral. Indeed, if we further assume that $\mathrm{m}_k(P_a)$ depends only on $|a|$, we find that $\mathrm{m}_k(P_{i^n\hat{x}_n}) = \mathrm{m}_k(P_{\hat{x}_n})$.

Now choose $P_a = z + a$. Then $\tilde{P} = R_n$. As we will see in Section 4, $\mathrm{m}_k(P_a)$ is a function of $|a|$. This implies

$$\mathrm{m}_k(R_n) = \frac{2^n}{\pi^n}\int_0^\infty\cdots\int_0^\infty \mathrm{m}_k(z+\hat{x}_n)\,\frac{\hat{x}_1\,d\hat{x}_1}{\hat{x}_1^2+1}\,\frac{\hat{x}_2\,d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\cdots\frac{\hat{x}_{n-1}\,d\hat{x}_{n-1}}{\hat{x}_{n-1}^2+\hat{x}_{n-2}^2}\,\frac{d\hat{x}_n}{\hat{x}_n^2+\hat{x}_{n-1}^2}. \tag{3.1}$$

Thus, if we have a good expression for $\mathrm{m}_k(z + x)$, then under favorable circumstances we may obtain a good expression for $\mathrm{m}_k(R_n)$.

4. Zeta Mahler measure. In this section we discuss the Zeta Mahler measure, an object that is closely related to higher Mahler measure and that will allow us to compute $\mathrm{m}_k(z + a)$ for any $a \in \mathbb{C}$.

Definition 4.1. Let $P \in \mathbb{C}(x_1,\dots,x_n)$ be a non-zero rational function. Its *Zeta Mahler measure* is defined by

$$Z(s,P) := \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n} |P(x_1,\dots,x_n)|^s\,\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

Remark 4.2. It is not hard to see that

$$\left.\frac{d^k Z(s,P)}{ds^k}\right|_{s=0} = \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n}\log^k|P(x_1,\dots,x_n)|\,\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} = \mathrm{m}_k(P).$$

The simplest possible case of Zeta Mahler measure was computed by Akatsuka.

Theorem 4.3 (Akatsuka [Aka09]). *Let $a \in \mathbb{C}$ with $|a| \neq 1$. Then*

$$Z(s, z+a) = \begin{cases} {}_2F_1(-s/2,-s/2;1;|a|^2) & \text{if } |a|<1,\\ |a|^s\cdot {}_2F_1(-s/2,-s/2;1;|a|^{-2}) & \text{if } |a|>1,\end{cases}$$

where, for $|t| < 1$,

$${}_2F_1(\alpha,\beta;\gamma;t) = \sum_{n=0}^\infty \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{t^n}{n!}$$

denotes the hypergeometric series, and

$$(\alpha)_n := \begin{cases} 1, & n = 0, \\ \alpha(\alpha+1)\cdots(\alpha+n-1), & n \geq 1, \end{cases}$$

is the Pochhammer symbol.

Formulas for $\mathrm{m}_k(z+a)$ can be derived from $Z(s, z+a)$ by means of Remark 4.2. We proceed to compute some derivatives of $Z(s, z+a)$.

LEMMA 4.4. *Let $t \in \mathbb{C}$ with $|t| < 1$ and set*

$$G_t(s) = {}_2F_1(-s/2, -s/2; 1; t) = \sum_{n=0}^{\infty} \left(\frac{-s}{2}\right)_n^2 \frac{t^n}{(n!)^2}.$$

Then

$$G_t(0) = 1, \qquad G_t'(0) = 0.$$

Proof. Indeed, setting $s = 0$ we obtain $(-s/2)_n = 0$ unless $n = 0$ and in that case $(-s/2)_0 = 1$. This implies that $G_t(0) = 1$.

Differentiating $G_t(s)$, we obtain

$$G_t'(s) = \sum_{n=1}^{\infty} 2\left(\frac{-s}{2}\right)_n \left(\frac{-s}{2}\right)_n' \frac{t^n}{(n!)^2}.$$

If we now set $s = 0$, we see that each term in the sum equals 0 and therefore $G_t'(0) = 0$. ∎

Akatsuka [Aka09] found a formula for $\mathrm{m}_k(z+a)$ for $|a| < 1$. This formula can be easily adapted to the case of general $a \in \mathbb{C}$.

THEOREM 4.5. *Let*

$$L_{(n_1,\dots,n_m)}(w) := \sum_{0<j_1<\cdots<j_m} \frac{w^{j_m}}{j_1^{n_1}\cdots j_m^{n_m}}.$$

Then for $|a| \leq 1$ and $k \geq 2$,

$$\mathrm{m}_k(z+a) = (-1)^k k! \sum_{k/2-1\leq n\leq k-2} \frac{1}{2^{2(k-n-1)}} \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n \\ \#\{i:\,\epsilon_i=2\}=k-n-2}} L_{(\epsilon_1,\dots,\epsilon_n,2)}(|a|^2).$$

For $|a| \geq 1$ and $k \geq 2$,

$$\mathrm{m}_k(z+a) = \log^k |a| + \sum_{j=0}^{k-2} \binom{k}{j} (-1)^{k-j}(k-j)! \sum_{(k-j)/2-1\leq n\leq k-j-2} \frac{1}{2^{2(k-j-n-1)}} \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n \\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} (\log^j |a|) L_{(\epsilon_1,\dots,\epsilon_n,2)}(|a|^{-2}).$$

Proof. The case of $|a| < 1$ is [Aka09, Theorem 7]. It is proved by observing that

$$Z(s, z+a) = G_{|a|^2}(s), \qquad \mathrm{m}_k(z+a) = \frac{d^k Z(s,z+a)}{ds^k}\bigg|_{s=0} = G_{|a|^2}^{(k)}(0).$$

Akatsuka applies the eighth formula of [OZ01, p. 485] for $\alpha = \beta$ and $x = 0$ in order to deduce the result.

For $|a| > 1$ we have, by Theorem 4.3,

$$Z(s, z+a) = |a|^s G_{|a|^{-2}}(s).$$

Thus, by Lemma 4.4,

$$\begin{aligned}\mathrm{m}_k(z+a) = \frac{d^k Z(s,z+a)}{ds^k}\bigg|_{s=0} &= \sum_{j=0}^{k} \binom{k}{j} (\log^j |a|) G_{|a|^{-2}}^{(k-j)}(0)\\ &= \log^k |a| + \sum_{j=0}^{k-2} \binom{k}{j} (\log^j |a|)(-1)^{k-j}(k-j)!\\ &\quad\times \sum_{(k-j)/2-1 \le n \le k-j-2} \frac{1}{2^{2(k-j-n-1)}} \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} L_{(\epsilon_1,\dots,\epsilon_n,2)}(|a|^{-2}),\end{aligned}$$

and the result follows.

Finally, the case $|a| = 1$ is considered in the third formula of [KLO08, p. 273]. It is not hard to see that both formulas above remain true for $|a| = 1$. ∎

5. Integral simplification. In this section we discuss how to simplify the integral in (3.1). To do so, we define certain polynomials and prove a recurrence relation for them. We then use these polynomials to compute a certain family of integrals.

DEFINITION 5.1. Let $A_m(x) \in \mathbb{Q}[x]$ be defined by

$$R(T;x) = \frac{e^{xT}-1}{\sin T} = \sum_{m\ge 0} A_m(x)\frac{T^m}{m!}.$$

Thus, $A_0(x) = x$, $A_1(x) = x^2/2$, $A_2(x) = x^3/3 + x/3$, etc.

LEMMA 5.2. *The polynomials $A_m(x)$ satisfy the recurrence*

$$A_m(x) = \frac{x^{m+1}}{m+1} + \frac{1}{m+1} \sum_{j>1\,\mathrm{odd}}^{m+1} (-1)^{(j+1)/2} \binom{m+1}{j} A_{m+1-j}(x). \tag{5.1}$$

Proof. By writing $\sin T = \frac{e^{iT}-e^{-iT}}{2i}$, we obtain

$$e^{xT} - 1 = \left(\frac{e^{iT} - e^{-iT}}{2i}\right) R(T;x).$$

In other words,

$$\sum_{m\geq 1}^{\infty} \frac{x^m T^m}{m!} = \sum_{j>0\,\mathrm{odd}} \frac{(-1)^{(j-1)/2} T^j}{j!} \sum_{\ell\geq 0} A_\ell(x) \frac{T^\ell}{\ell!}.$$

The result is obtained by comparing the coefficient of T^{m+1} on both sides. ∎

REMARK 5.3. More properties of $A_m(x)$ can be found in [Lal06a, Appendix]. For instance,

$$A_m(x) = -\frac{2}{m+1} \sum_{h=0}^{m} B_h \binom{m+1}{h} (2^{h-1} - 1) i^h x^{m+1-h},$$

where the B_n are the Bernoulli numbers.

We will eventually compute a certain integral. For this, we need the following auxiliary result.

LEMMA 5.4. *For $0 < \beta < 1$,*

$$\int_0^\infty \frac{x^\beta dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi(a^{\beta-1} - b^{\beta-1})}{2(b^2 - a^2) \cos \frac{\pi\beta}{2}}. \tag{5.2}$$

Proof. We decompose the integrand into partial fractions:

$$\int_0^\infty \frac{x^\beta\, dx}{(x^2 + a^2)(x^2 + b^2)} = \int_0^\infty \left(\frac{1}{x^2 + a^2} - \frac{1}{x^2 + b^2}\right) \frac{x^\beta\, dx}{(b^2 - a^2)}. \tag{5.3}$$

By integrating over a well-chosen contour (see [Ahl79, p. 159, Section 5.3]), we obtain

$$\int_0^\infty \frac{x^\beta\, dx}{x^2 + a^2} = \frac{1}{1 - e^{2\pi i\beta}} 2\pi i \sum_{x\neq 0} \mathrm{Res}\left\{\frac{x^\beta}{x^2 + a^2}\right\} = \frac{\pi a^{\beta-1}}{2\cos\frac{\pi\beta}{2}}.$$

By inserting this in (5.3), the result follows. ∎

We will use the polynomials $A_m(x)$ to compute certain integrals.

PROPOSITION 5.5. *For $m \geq 0$,*

$$\int_0^\infty \frac{x \log^m x\, dx}{(x^2 + a^2)(x^2 + b^2)} = \left(\frac{\pi}{2}\right)^{m+1} \frac{A_m\left(\frac{2\log a}{\pi}\right) - A_m\left(\frac{2\log b}{\pi}\right)}{a^2 - b^2}.$$

Proof. Let

$$f(\beta) := \int_0^\infty \frac{x^\beta\, dx}{(x^2 + a^2)(x^2 + b^2)}.$$

As the integral converges for $0 < \beta < 3$, the function is well-defined and continuous in this interval. We differentiate m times and obtain

$$f^{(m)}(1) = \int_0^\infty \frac{x \log^m x\, dx}{(x^2+a^2)(x^2+b^2)}.$$

Lemma 5.4 implies, for $0 < \beta < 1$,

$$f(\beta)\cos\frac{\pi\beta}{2} = \frac{\pi(a^{\beta-1}-b^{\beta-1})}{2(b^2-a^2)}.$$

By differentiating m times, we obtain

$$\sum_{j=0}^{m}\binom{m}{j} f^{(m-j)}(\beta)\left(\cos\frac{\pi\beta}{2}\right)^{(j)} = \frac{\pi}{2(b^2-a^2)}(a^{\beta-1}\log^m a - b^{\beta-1}\log^m b).$$

We now take the limit as $\beta \to 1$ to obtain

$$\sum_{j=1\,\mathrm{odd}}^{m}(-1)^{(j+1)/2}\binom{m}{j} f^{(m-j)}(1)\left(\frac{\pi}{2}\right)^{j} = \frac{\pi(\log^m a - \log^m b)}{2(b^2-a^2)}.$$

Changing m to $m+1$, isolating the term $f^{(m)}(1)$, and dividing by $\frac{\pi}{2}(m+1)$ yields

$$f^{(m)}(1) = \frac{1}{m+1}\sum_{j>1\,\mathrm{odd}}^{m+1}(-1)^{(j+1)/2}\binom{m+1}{j} f^{(m+1-j)}(1)\left(\frac{\pi}{2}\right)^{j-1} + \frac{\log^{m+1} a - \log^{m+1} b}{(m+1)(a^2-b^2)}.$$

For $m = 0$ the above equation becomes

$$f^{(0)}(1) = f(1) = \frac{\log^m a - \log^m b}{a^2-b^2} = \frac{\pi}{2}\,\frac{A_0\left(\frac{2\log a}{\pi}\right) - A_0\left(\frac{2\log b}{\pi}\right)}{a^2-b^2}.$$

The rest of the proof proceeds by induction, using the recurrence for $A_m(x)$ that was proved in Lemma 5.2. ■

By Proposition 5.5, we can write the integral in (3.1) as a sum of simpler integrals. For instance, for $n = 2$, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \mathrm{m}_k(z+\hat{x}_2)\frac{\hat{x}_1\, d\hat{x}_1}{\hat{x}_1^2+1}\,\frac{d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\\
&\quad= \int_0^\infty \mathrm{m}_k(z+\hat{x}_2)\left(\int_0^\infty \frac{\hat{x}_1\, d\hat{x}_1}{(\hat{x}_1^2+\hat{x}_2^2)(\hat{x}_1^2+1)}\right)d\hat{x}_2\\
&\quad= \int_0^\infty \mathrm{m}_k(z+\hat{x}_2)\left(\frac{\pi}{2}\,\frac{A_0\left(\frac{2\log\hat{x}_2}{\pi}\right) - A_0\left(\frac{2\log 1}{\pi}\right)}{\hat{x}_2^2-1^2}\right)d\hat{x}_2\\
&\quad= \int_0^\infty \mathrm{m}_k(z+\hat{x}_2)\left(\frac{\pi}{2}\,\frac{\frac{2\log\hat{x}_2}{\pi}}{\hat{x}_2^2-1}\right)d\hat{x}_2 = \int_0^\infty \mathrm{m}_k(z+x)\log x\,\frac{dx}{x^2-1}.
\end{aligned}$$

Analogously, for $n = 3$, we obtain

$$\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{m}_k(z+\hat{x}_3)\,\frac{\hat{x}_1\,d\hat{x}_1}{\hat{x}_1^2+1}\,\frac{\hat{x}_2\,d\hat{x}_2}{\hat{x}_2^2+\hat{x}_1^2}\,\frac{d\hat{x}_3}{\hat{x}_3^2+\hat{x}_2^2}$$
$$= \frac{\pi^2}{8}\int_0^\infty \mathrm{m}_k(z+x)\,\frac{dx}{x^2+1} + \frac{1}{2}\int_0^\infty \mathrm{m}_k(z+x)\log^2 x\,\frac{dx}{x^2+1}.$$

More generally, we can always reduce the computation to a sum of integrals of the form

$$\begin{cases} \displaystyle\int_0^\infty \mathrm{m}_k(z+x)\log^{2h+1} x\,\frac{dx}{x^2-1} & \text{for } n \text{ even},\\ \displaystyle\int_0^\infty \mathrm{m}_k(z+x)\log^{2h} x\,\frac{dx}{x^2+1} & \text{for } n \text{ odd}. \end{cases} \tag{5.4}$$

6. Coefficient formulas. In this section, we find the coefficients that allow us to express the integral from (3.1) as a linear combination of integrals from (5.4). Let $\phi(a)$ be a function that depends on $|a|$. Eventually we will have $\phi(a) = \mathrm{m}_k(P_a)$, where P_a is a rational function such that $\mathrm{m}_k(P_a) = \mathrm{m}_k(P_{|a|})$ (for instance, we could take $P_a = z + a$). In what follows, it is assumed that $\phi(a)$ is such that all the integrals converge.

Definition 6.1. For $n \geq 1$ and $0 \leq h \leq n-1$, let $a_{n,h} \in \mathbb{Q}$ be defined by

$$\int_0^\infty \cdots \int_0^\infty \phi(x_1)\,\frac{x_{2n}\,dx_{2n}}{x_{2n}^2+1}\,\frac{x_{2n-1}\,dx_{2n-1}}{x_{2n-1}^2+x_{2n}^2}\cdots\frac{dx_1}{x_1^2+x_2^2}$$
$$= \sum_{h=1}^{n} a_{n,h-1}\left(\frac{\pi}{2}\right)^{2n-2h}\int_0^\infty \phi(x)\log^{2h-1} x\,\frac{dx}{x^2-1}. \tag{6.1}$$

For $n \geq 0$ and $0 \leq h \leq n$, let $b_{n,h} \in \mathbb{Q}$ be defined by

$$\int_0^\infty \cdots \int_0^\infty \phi(x_1)\,\frac{x_{2n+1}\,dx_{2n+1}}{x_{2n+1}^2+1}\,\frac{x_{2n}\,dx_{2n}}{x_{2n}^2+x_{2n+1}^2}\cdots\frac{dx_1}{x_1^2+x_2^2}$$
$$= \sum_{h=0}^{n} b_{n,h}\left(\frac{\pi}{2}\right)^{2n-2h}\int_0^\infty \phi(x)\log^{2h} x\,\frac{dx}{x^2+1}. \tag{6.2}$$

Lemma 6.2. *We have the following identities:*

$$\sum_{h=0}^{n} b_{n,h}x^{2h} = \sum_{h=1}^{n} a_{n,h-1}(A_{2h-1}(x) - A_{2h-1}(i)), \tag{6.3}$$

$$\sum_{h=1}^{n+1} a_{n+1,h-1} x^{2h-1} = \sum_{h=0}^{n} b_{n,h} A_{2h}(x), \tag{6.4}$$

where the $A_m(x)$ *are the polynomials given in Definition* 5.1.

Proof. By making the change of variables $x_i = y_i x_{2n+1}$ for $i = 1, \dots, 2n$, we have

$$\begin{aligned}&\int_0^\infty \cdots \int_0^\infty \phi(x_1) \frac{x_{2n+1}\, dx_{2n+1}}{x_{2n+1}^2 + 1} \frac{x_{2n}\, dx_{2n}}{x_{2n}^2 + x_{2n+1}^2} \cdots \frac{dx_1}{x_1^2 + x_2^2} \\ &\quad = \int_0^\infty \cdots \int_0^\infty \phi(y_1 x_{2n+1}) \frac{dx_{2n+1}}{x_{2n+1}^2 + 1} \frac{y_{2n}\, dy_{2n}}{y_{2n}^2 + 1} \frac{y_{2n-1}\, dy_{2n-1}}{y_{2n-1}^2 + y_{2n}^2} \cdots \frac{dy_1}{y_1^2 + y_2^2}.\end{aligned}$$

We now ignore the variable x_{2n+1} and apply (6.1) where $\int_0^\infty \phi(y_1 x_{2n+1}) \frac{dx_{2n+1}}{x_{2n+1}^2+1}$ is a function depending on y_1. The above equals

$$\sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty \phi(y_1 x_{2n+1}) \frac{dx_{2n+1}}{x_{2n+1}^2 + 1} \log^{2h-1} y_1 \frac{dy_1}{y_1^2 - 1}.$$

We set $x = y_1 x_{2n+1}$ and rename $y = y_1$. We obtain

$$\sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty \phi(x) \frac{y\, dx}{x^2 + y^2} \log^{2h-1} y \frac{dy}{y^2 - 1}.$$

By applying (6.2), we conclude that

$$\begin{aligned}&\sum_{h=0}^{n} b_{n,h} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \phi(x) \log^{2h} x \frac{dx}{x^2 + 1} \\ &\quad = \sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n-2h} \int_0^\infty \int_0^\infty \phi(x) y \log^{2h-1} y \frac{dy}{y^2 - 1} \frac{dx}{x^2 + y^2}.\end{aligned} \tag{6.5}$$

By Proposition 5.5 with $a = x$ and $b = i$, we obtain

$$\int_0^\infty \frac{y \log^{2h-1} y\, dy}{(y^2 + x^2)(y^2 - 1)} = \left(\frac{\pi}{2}\right)^{2h} \frac{A_{2h-1}\left(\frac{2\log x}{\pi}\right) - A_{2h-1}(i)}{x^2 + 1}.$$

Thus, (6.5) equals

$$\sum_{h=1}^{n} a_{n,h-1} \left(\frac{\pi}{2}\right)^{2n} \int_0^\infty \phi(x) \left(A_{2h-1}\left(\frac{2\log x}{\pi}\right) - A_{2h-1}(i) \right) \frac{dx}{x^2 + 1}.$$

This equality is true for any choice of $\phi(x)$, and therefore (6.3) must hold.

Analogously, we can prove that

$$\sum_{h=1}^{n+1} a_{n+1,h-1}\left(\frac{\pi}{2}\right)^{2n+2-2h}\int_0^\infty \phi(x)\log^{2h-1}x\,\frac{dx}{x^2-1} \tag{6.6}$$
$$= \sum_{h=0}^{n} b_{n,h}\left(\frac{\pi}{2}\right)^{2n-2h}\int_0^\infty\int_0^\infty \phi(x)y\log^{2h}y\,\frac{dy}{y^2+1}\,\frac{dx}{x^2+y^2}.$$

By Proposition 5.5 with $a = x$ and $b = 1$,

$$\int_0^\infty \frac{y\log^{2h}y\,dy}{(y^2+x^2)(y^2+1)} = \left(\frac{\pi}{2}\right)^{2h+1}\frac{A_{2h}\left(\frac{2\log x}{\pi}\right)-A_{2h}(0)}{x^2-1}.$$

Thus, the right-hand side of (6.6) becomes

$$\sum_{h=0}^{n} b_{n,h}\left(\frac{\pi}{2}\right)^{2n+1}\int_0^\infty \phi(x)A_{2h}\left(\frac{2\log x}{\pi}\right)\frac{dx}{x^2-1}.$$

By inserting this in (6.6), we obtain (6.4) by the same argument that was used to obtain (6.3). ∎

We now prove a result that will be key in finding formulas for $a_{n,h}$ and $b_{n,h}$.

Lemma 6.3. *We have the following identities:*

$$2n(-1)^\ell s_{n-\ell}(2^2,4^2,\ldots,(2n-2)^2) = \sum_{h=\ell}^{n}(-1)^h\binom{2h}{2\ell-1}s_{n-h}(1^2,3^2,\ldots,(2n-1)^2), \tag{6.7}$$

$$(2n+1)(-1)^\ell s_{n-\ell}(1^2,3^2,\ldots,(2n-1)^2) = \sum_{h=\ell}^{n}(-1)^h\binom{2h+1}{2\ell}s_{n-h}(2^2,4^2,\ldots,(2n)^2). \tag{6.8}$$

Proof. We multiply by $x^{2\ell}$ on both sides of (6.7) and we sum over $\ell = 1,\ldots,n$ to obtain

$$2n\sum_{\ell=1}^{n} s_{n-\ell}(2^2,4^2,\ldots,(2n-2)^2)(-1)^\ell x^{2\ell}$$
$$= \sum_{\ell=1}^{n}\sum_{h=\ell}^{n}(-1)^h\binom{2h}{2\ell-1}s_{n-h}(1^2,3^2,\ldots,(2n-1)^2)x^{2\ell}.$$

By identifying the left-hand side with the corresponding polynomial and reversing the sums on the right-hand side, we conclude that it suffices to prove that

$$2n\prod_{j=0}^{n-1}((2j)^2-x^2)=\sum_{h=1}^{n}(-1)^h s_{n-h}(1^2,3^2,\dots,(2n-1)^2)\sum_{\ell=1}^{h}\binom{2h}{2\ell-1}x^{2\ell}.$$

The right-hand side equals

$$\sum_{h=1}^{n}(-1)^h s_{n-h}(1^2,3^2,\dots,(2n-1)^2)\frac{x}{2}((x+1)^{2h}-(x-1)^{2h})$$
$$=\frac{x}{2}\Big(\prod_{j=1}^{n}((2j-1)^2-(x+1)^2)-\prod_{j=1}^{n}((2j-1)^2-(x-1)^2)\Big)$$
$$=\frac{x}{2}\Big(\prod_{j=1}^{n}(2j+x)(2j-2-x)-\prod_{j=1}^{n}(2j+x-2)(2j-x)\Big).$$

By inspecting the common zeros in both products, we infer that the above equals

$$((-x)(2n+x)-x(2n-x))\frac{x}{2}\prod_{j=1}^{n-1}((2j)^2-x^2)=2n\prod_{j=0}^{n-1}((2j)^2-x^2).$$

This concludes the proof of (6.7).

For (6.8) we proceed analogously by multiplying each side by $x^{2\ell+1}$ and summing over $\ell=1,\dots,n$:

$$(2n+1)\sum_{\ell=1}^{n}s_{n-\ell}(1^2,3^2,\dots,(2n-1)^2)(-1)^\ell x^{2\ell+1}$$
$$=\sum_{\ell=1}^{n}\sum_{h=\ell}^{n}(-1)^h\binom{2h+1}{2\ell}x^{2\ell+1}.$$

Thus, it suffices to prove that

$$(2n+1)x\prod_{j=1}^{n}((2j-1)-x^2)$$
$$=\sum_{h=1}^{n}(-1)^h s_{n-h}(2^2,4^2,\dots,(2n)^2)\sum_{l=1}^{h}\binom{2h+1}{2l}x^{2l+1}.$$

The right-hand side equals

$$\sum_{h=0}^{n}(-1)^h s_{n-h}(2^2,4^2,\dots,(2n)^2)\frac{x}{2}((x+1)^{2h+1}-(x-1)^{2h+1})$$
$$=\frac{x}{2}\Big((x+1)\prod_{j=1}^{n}((2j)^2-(x+1)^2)-(x-1)\prod_{j=1}^{n}((2j)^2-(x-1)^2)\Big)$$

$$= \frac{x}{2}\Big((x+1)\prod_{j=1}^{n}(2j+1+x)(2j-1-x) - (x-1)\prod_{j=1}^{n}(2j-1+x)(2j+1-x)\Big)$$

$$= ((2n+1+x)+(2n+1-x))\frac{x}{2}\prod_{j=1}^{n}((2j-1)^2-x^2)$$

$$= (2n+1)x\prod_{j=1}^{n}((2j-1)^2-x^2),$$

and this concludes the proof of (6.8). ∎

REMARK 6.4. Mathew Rogers has remarked that the sequences under consideration are related to Stirling numbers of the first kind via

$$s_{n-h}(2^2,4^2,\ldots,(2n-2)^2) = 2^{2n-2h}\sum_{m=0}^{2h}(-1)^{h-m}S_n^{(m)}S_n^{(2h-m)}.$$

We are now ready to compute the coefficients $a_{n,h}$ and $b_{n,h}$ from Definition 6.1.

THEOREM 6.5. *For* $n \geq 1$,

$$\sum_{h=0}^{n-1} a_{n,h}x^{2h} = \frac{(x^2+2^2)\cdots(x^2+(2n-2)^2)}{(2n-1)!},$$

and for $n \geq 0$,

$$\sum_{h=0}^{n} b_{n,h}x^{2h} = \frac{(x^2+1^2)(x^2+3^2)\cdots(x^2+(2n-1)^2)}{(2n)!}.$$

In other words,

$$a_{n,h} = \frac{s_{n-h-1}(2^2,4^2,\ldots,(2n-2)^2)}{(2n-1)!}, \tag{6.9}$$

$$b_{n,h} = \frac{s_{n-h}(1^2,3^2,\ldots,(2n-1)^2)}{(2n)!}. \tag{6.10}$$

Proof. We proceed by induction. By definition, when $2n+1=1$, we have $n=0$ and

$$\int_0^\infty \phi(x)\,\frac{dx}{x^2+1} = b_{0,0}\int_0^\infty \phi(x)\,\frac{dx}{x^2+1}.$$

Therefore, $b_{0,0}=1$.

Analogously, when $2n=2$ we have $n=1$. We have seen that

$$\int_0^\infty\int_0^\infty \phi(x)\,\frac{y\,dy}{y^2+1}\,\frac{dx}{x^2+y^2} = \int_0^\infty \phi(x)\frac{\log x\,dx}{x^2-1} = a_{1,0}\int_0^\infty \phi(x)\frac{\log x\,dx}{x^2-1}.$$

Thus $a_{1,0}=1$ and the result holds for the first two cases.

Now assume that for a fixed $n \geq 1$ and all $0 \leq h \leq n-1$ we have

$$a_{n,h} = \frac{s_{n-h-1}(2^2, 4^2, \ldots, (2n-2)^2)}{(2n-1)!}.$$

We will prove that for all $0 \leq h \leq n$,

$$b_{n,h} = \frac{s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2)}{(2n)!}.$$

By Lemma 6.2, it suffices to show that

$$\sum_{h=0}^{n} s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2) x^{2h} \\ = 2n \sum_{h=1}^{n} s_{n-h}(2^2, \ldots, (2n-2)^2)\big(A_{2h-1}(x) - A_{2h-1}(i)\big).$$

Taking $m = 2h-1$ in (5.1), we obtain

$$x^{2h} = \sum_{j=0}^{h-1} (-1)^j \binom{2h}{2j+1} A_{2h-2j-1}(x).$$

Multiplying by $s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2)$ and summing over h, we get

$$\sum_{h=0}^{n} s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2) x^{2h} \\ = \sum_{h=1}^{n} s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2) \sum_{j=0}^{h-1} (-1)^j \binom{2h}{2j+1} A_{2h-2j-1}(x) \\ + s_n(1^2, 3^2, \ldots, (2n-1)^2).$$

Evaluating this equation at $x = i$, we obtain

$$\sum_{h=0}^{n} s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2)(-1)^h \\ = \sum_{h=1}^{n} s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2) \sum_{j=0}^{h-1} (-1)^j \binom{2h}{2j+1} A_{2h-2j-1}(i) \\ + s_n(1^2, 3^2, \ldots, (2n-1)^2).$$

Since

$$\sum_{h=0}^{n} s_{n-h}(1^2, 3^2, \ldots, (2n-1)^2)(-1)^h = (x+1^2)\cdots(x+(2n-1)^2)|_{x=-1} = 0,$$

we deduce that

$$\sum_{h=0}^{n} s_{n-h}(1^2,3^2,\ldots,(2n-1)^2)x^{2h} = \sum_{h=1}^{n} s_{n-h}(1^2,3^2,\ldots,(2n-1)^2) \times \sum_{j=0}^{h-1}(-1)^j\binom{2h}{2j+1}(A_{2h-2j-1}(x)-A_{2h-2j-1}(i)).$$

By setting $\ell = h-j$, the right-hand side becomes

$$\sum_{h=1}^{n} s_{n-h}(1^2,3^2,\ldots,(2n-1)^2)\sum_{\ell=1}^{h}(-1)^{h-\ell}\binom{2h}{2\ell-1}(A_{2\ell-1}(x)-A_{2\ell-1}(i))$$
$$= \sum_{\ell=1}^{n}\left(\sum_{h=\ell}^{n}(-1)^h\binom{2h}{2\ell-1}s_{n-h}(1^2,3^2,\ldots,(2n-1)^2)\right) \times (-1)^{\ell}(A_{2\ell-1}(x)-A_{2\ell-1}(i)).$$

Lemma 6.3 then implies (6.10).

Now suppose that for fixed $n \geq 1$ and all $0 \leq h \leq n$, we have

$$b_{n,h} = \frac{s_{n-h}(1^2,3^2,\ldots,(2n-1)^2)}{(2n)!}.$$

We wish to show that all $0 \leq h \leq n$,

$$a_{n+1,h} = \frac{s_{n-h}(2^2,4^2,\ldots,(2n)^2)}{(2n+1)!}.$$

By Lemma 6.2, it suffices to show that

$$\sum_{h=0}^{n} s_{n-h}(2^2,4^2,\ldots,(2n)^2)x^{2h+1} = (2n+1)\sum_{h=0}^{n} s_{n-h}(1^2,3^2,\ldots,(2n-1)^2)A_{2h}(x).$$

By setting $m=2h$ in (5.1), we obtain

$$x^{2h+1} = \sum_{j=0}^{h}(-1)^j\binom{2h+1}{2j+1}A_{2h-2j}(x).$$

Therefore,

$$\sum_{h=0}^{n} s_{n-h}(2^2,4^2,\ldots,(2n)^2)x^{2h+1} = \sum_{h=0}^{n} s_{n-h}(2^2,4^2,\ldots,(2n)^2)\sum_{j=0}^{h}(-1)^j\binom{2h+1}{2j+1}A_{2h-2j}(x).$$

Setting $\ell = h - j$, we see that this equals

$$\sum_{h=0}^{n} s_{n-h}(2^2, 4^2, \ldots, (2n)^2) \sum_{\ell=0}^{h} (-1)^{h-\ell} \binom{2h+1}{2\ell} A_{2\ell}(x)$$
$$= \sum_{\ell=0}^{h} \Big(\sum_{h=\ell}^{n} (-1)^h \binom{2h+1}{2\ell} s_{n-h}(2^2, 4^2, \ldots, (2n)^2) \Big) (-1)^\ell A_{2\ell}(x).$$

Thus, (6.9) follows from Lemma 6.3. ∎

7. Polylogarithms and hyperlogarithms. To complete the proof of Theorem 1.1, we need to compute integrals of the form

$$\int_0^\infty \mathrm{m}_k(z+x) \log^j x \, \frac{dx}{x^2 \pm 1}.$$

These integrals are related to polylogarithms.

Definition 7.1. Let $w_1, \ldots, w_m$ be complex variables and $n_1, \ldots, n_m$ be positive integers. Define the *multiple polylogarithm* by the power series

$$\mathrm{Li}_{n_1,\ldots,n_m}(w_1, \ldots, w_m) := \sum_{0<j_1<\cdots<j_m} \frac{w_1^{j_1} \cdots w_m^{j_m}}{j_1^{n_1} \cdots j_m^{n_m}}.$$

We say that the above series has *length* m and *weight* $\omega = n_1 + \cdots + n_m$. It is absolutely convergent for $|w_i| \le 1$ and $n_m > 1$.

Remark 7.2. We remark that Akatsuka's polylogarithm from Theorem 4.5 is a particular case of multiple polylogarithms:

$$L_{(n_1,\ldots,n_m)}(w) := \sum_{0<j_1<\cdots<j_m} \frac{w^{j_m}}{j_1^{n_1} \cdots j_m^{n_m}} = \mathrm{Li}_{n_1,\ldots,n_m}(1, \ldots, 1, w).$$

Multiple polylogarithms have meromorphic continuations to the complex plane.

Definition 7.3. *Hyperlogarithms* are defined by the following iterated integral:

$$\mathrm{I}_{n_1,\ldots,n_m}(a_1 : \cdots : a_{m+1})$$
$$:= \int_0^{a_{m+1}} \underbrace{\frac{dt}{t-a_1} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_1} \circ \cdots \circ \underbrace{\frac{dt}{t-a_m} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n_m}$$

where

$$\int_0^{b_{h+1}} \frac{dt}{t-b_1} \circ \cdots \circ \frac{dt}{t-b_h} = \int_{0 \le t_1 \le \cdots \le t_h \le b_{h+1}} \frac{dt_1}{t_1-b_1} \cdots \frac{dt_h}{t_h-b_h}.$$

Remark 7.4. The path of integration should be interpreted as any path connecting 0 and b_{h+1} in $\mathbb{C} \setminus \{b_1, \dots, b_h\}$. The integral depends on the homotopy class of this path. For our purposes, we will always integrate on the real line.

Multiple polylogarithms and hyperlogarithms are related by the following identities (see [Gon95]).

Lemma 7.5.

$$\mathrm{I}_{n_1,\dots,n_m}(a_1 : \dots : a_{m+1}) = (-1)^m \mathrm{Li}_{n_1,\dots,n_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \dots, \frac{a_{m+1}}{a_m}\right),$$
$$\mathrm{Li}_{n_1,\dots,n_m}(w_1, \dots, w_m) = (-1)^m \mathrm{I}_{n_1,\dots,n_m}((w_1 \cdots w_m)^{-1} : \dots : w_m^{-1} : 1).$$

The following example will be useful later.

Example 7.6. The second equality in Lemma 7.5 implies, for $(\epsilon_1, \dots, \epsilon_n) \in \{1, 2\}^n$ and $\epsilon_1 + \dots + \epsilon_n = k - 2$,

$$L_{(\epsilon_1,\dots,\epsilon_n,2)}(w^2) = (-1)^{n+1} \int_0^1 \frac{dt}{t - 1/w^2} \circ \dots \circ \frac{dt}{t - 1/w^2} \circ \frac{dt}{t},$$

where there is a term of the form $\frac{dt}{t}$ after each $\frac{dt}{t-1/w^2}$ term corresponding to $\epsilon_i = 2$ and there is no $\frac{dt}{t}$ after any $\frac{dt}{t-1/w^2}$ corresponding to $\epsilon_i = 1$, and the last two terms $\frac{dt}{t-1/w^2} \circ \frac{dt}{t}$ correspond to the subindex 2.

By setting $s^2 = w^2 t$, the above equals

$$(-1)^{n+1} \int_0^w \frac{2s\,ds}{s^2 - 1} \circ \dots \circ \frac{2s\,ds}{s^2 - 1} \circ \frac{2\,ds}{s}$$
$$= (-1)^{n+1} 2^{k-n-1} \int_0^w \left(\frac{1}{s-1} + \frac{1}{s+1}\right) \circ \dots \circ \left(\frac{1}{s-1} + \frac{1}{s+1}\right) \circ \frac{ds}{s}.$$

In the previous formula, each $\frac{ds}{s}$ has contributed a factor of 2 to the leading coefficient.

To express our results more clearly, we recall the notation from (1.1).

Definition 7.7. We will work with combinations of polylogarithms given by

$$\mathcal{L}_{n_1,\dots,n_m}(w_1, \dots, w_m) = \sum_{(r_1,\dots,r_m)\in\{0,1\}^m} (-1)^{r_m} \mathrm{Li}_{n_1,\dots,n_m}((-1)^{r_1} w_1, \dots, (-1)^{r_m} w_m).$$

The following result expresses the integrals that we need to evaluate in terms of polylogarithms.

Lemma 7.8. *We have*

$$\int_0^1 \log^j x \, \frac{dx}{x^2-1} = (-1)^{j+1} j! \left(1 - \frac{1}{2^{j+1}}\right)\zeta(j+1),$$

$$\int_0^1 \log^j x \, \frac{dx}{x^2+1} = (-1)^j j! L(\chi_{-4}, j+1).$$

For $(\epsilon_1, \dots, \epsilon_n) \in \{1,2\}^n$ *and* $\epsilon_1 + \dots + \epsilon_n = k-2$, *we have*

$$\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2) \log^j x \, \frac{dx}{x^2-1} = (-1)^{j+1} 2^{k-n-2} j! \mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,j+1}(1,\dots,1), \tag{7.1}$$

$$\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2) \log^j x \, \frac{dx}{x^2+1} = i(-1)^{j+1} 2^{k-n-2} j! \mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,j+1}(1,\dots,1,i,i). \tag{7.2}$$

Proof. The first two identities are proved in [Lal06a, Lemma 9]. By applying Example 7.6, we get

$$\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2) \log^j x \, \frac{dx}{x^2-1}$$
$$= \int_0^1 \left((-1)^{n+1} 2^{k-n-1} \int_0^x \left(\frac{1}{s-1} + \frac{1}{s+1}\right) \circ \dots \circ \left(\frac{1}{s-1} + \frac{1}{s+1}\right) \circ \frac{ds}{s}\right) \times \log^j x \, \frac{dx}{x^2-1}.$$

We now use the fact that $\int_x^1 \frac{dt}{t} = -\log x$ to deduce that the above equals

$$(-1)^{n+1+j} 2^{k-n-2} j! \int_0^1 \left(\frac{1}{s-1} + \frac{1}{s+1}\right) \circ \dots \circ \left(\frac{1}{s-1} + \frac{1}{s+1}\right) \circ \frac{ds}{s} \circ \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{j}$$

$$= (-1)^{n+1+j} 2^{k-n-2} j! \sum_{(r_1,\dots,r_{n+1},r) \in \{0,1\}^{n+2}} (-1)^r \times \mathrm{I}_{\epsilon_1,\dots,\epsilon_n,2,j+1}\big((-1)^{r_1} : \dots : (-1)^{r_n} : (-1)^{r_{n+1}} : (-1)^r : 1\big)$$

$$= (-1)^{j+1} 2^{k-n-2} j! \sum_{(r_1,\dots,r_{n+1},r)\in\{0,1\}^{n+2}} (-1)^r$$
$$\times \mathrm{Li}_{\epsilon_1,\dots,\epsilon_n,2,j+1}\big((-1)^{r_1+r_2},\dots,(-1)^{r_n+r_{n+1}},(-1)^{r_{n+1}+r},(-1)^r\big)$$
$$= (-1)^{j+1} 2^{k-n-2} j! \mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,j+1}(1,\dots,1).$$

This yields (7.1).

We proceed similarly to prove (7.2):

$$\int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^j x\,\frac{dx}{x^2+1}$$
$$= \int_0^1 \bigg((-1)^{n+1}2^{k-n-1}\int_0^x \Big(\frac{1}{s-1}+\frac{1}{s+1}\Big)\circ\cdots\circ\Big(\frac{1}{s-1}+\frac{1}{s+1}\Big)\circ\frac{ds}{s}\bigg)$$
$$\times \log^j x\,\frac{dx}{x^2+1}$$
$$= i(-1)^{n+j}2^{k-n-2}j!\int_0^1 \Big(\frac{1}{s-1}+\frac{1}{s+1}\Big)\circ\cdots\circ\Big(\frac{1}{s-1}+\frac{1}{s+1}\Big)\circ\frac{ds}{s}$$
$$\circ\Big(\frac{1}{x-i}-\frac{1}{x+i}\Big)dx\circ\underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{j}$$
$$= i(-1)^{n+j}2^{k-n-2}j! \sum_{(r_1,\dots,r_{n+1},r)\in\{0,1\}^{n+2}} (-1)^r$$
$$\times \mathrm{I}_{\epsilon_1,\dots,\epsilon_n,2,j+1}\big((-1)^{r_1}:\cdots:(-1)^{r_n}:(-1)^{r_{n+1}}:(-1)^r i:1\big)$$
$$= i(-1)^{j}2^{k-n-2}j! \sum_{(r_1,\dots,r_{n+1},r)\in\{0,1\}^{n+2}} (-1)^r$$
$$\times \mathrm{Li}_{\epsilon_1,\dots,\epsilon_n,2,j+1}\big((-1)^{r_1+r_2},\dots,(-1)^{r_n+r_{n+1}},(-1)^{r_{n+1}+r}i,(-1)^{r+1}i\big)$$
$$= i(-1)^{j+1}2^{k-n-2}j!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,j+1}(1,\dots,1,i,i). \quad \blacksquare$$

8. The final steps in the proof of Theorem 1.1. By combining (3.1), Definition 6.1, and Theorem 6.5, we obtain

$$\pi^{2n}\mathrm{m}_k(R_{2n})$$
$$= \sum_{h=1}^{n} \frac{s_{n-h}(2^2,4^2,\dots,(2n-2)^2)}{(2n-1)!}2^{2h}\pi^{2n-2h}\int_0^\infty \mathrm{m}_k(z+x)\log^{2h-1}x\,\frac{dx}{x^2-1},$$
$$\pi^{2n+1}\mathrm{m}_k(R_{2n+1})$$
$$= \sum_{h=0}^{n} \frac{s_{n-h}(1^2,\dots,(2n-1)^2)}{(2n)!}2^{2h+1}\pi^{2n-2h}\int_0^\infty \mathrm{m}_k(z+x)\log^{2h}x\,\frac{dx}{x^2+1}.$$

We are now ready to express the above two integrals in terms of polylogarithms.

For the case of $2h-1$, we have, by Theorem 4.5,

$$\int_0^\infty \mathrm{m}_k(z+x)\log^{2h-1} x\,\frac{dx}{x^2-1}$$

$$= (-1)^k k! \sum_{k/2-1\le n\le k-2} \frac{1}{2^{2(k-n-1)}} \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h-1} x\,\frac{dx}{x^2-1}$$

$$+ \int_1^\infty \log^{2h+k-1} x\,\frac{dx}{x^2-1}$$

$$+ \sum_{j=0}^{k-2}\binom{k}{j}(-1)^{k-j}(k-j)! \sum_{(k-j)/2-1\le n\le k-j-2}\frac{1}{2^{2(k-j-n-1)}}$$

$$\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} \int_1^\infty \log^j x\, L_{(\epsilon_1,\dots,\epsilon_n,2)}\left(\frac{1}{x^2}\right)\log^{2h-1} x\,\frac{dx}{x^2-1}.$$

By making the change of variables $y = 1/x$ in the integrals over $x \ge 1$, and then replacing y by x again, the above expression becomes

$$(-1)^k k! \sum_{k/2-1\le n\le k-2} \frac{2}{2^{2(k-n-1)}} \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h-1} x\,\frac{dx}{x^2-1}$$

$$+ \int_0^1 (-1)^k \log^{2h+k-1} x\,\frac{dx}{x^2-1}$$

$$+ \sum_{j=1}^{k-2}\binom{k}{j}(-1)^{k}(k-j)! \sum_{(k-j)/2-1\le n\le k-j-2}\frac{1}{2^{2(k-j-n-1)}}$$

$$\times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h+j-1} x\,\frac{dx}{x^2-1}.$$

Finally, by Lemma 7.8,

$$\begin{aligned}\int_0^\infty & \mathrm{m}_k(z+x)\log^{2h-1} x \,\frac{dx}{x^2-1}\\ &= (2h+k-1)!\left(1-\frac{1}{2^{2h+k}}\right)\zeta(2h+k)\\ &\quad + (-1)^k k! \sum_{k/2-1\le n\le k-2} \frac{1}{2^{k-n-1}}\\ &\qquad\qquad \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-n-2}} (2h-1)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h}(1,\dots,1)\\ &\quad + \sum_{j=1}^{k-2}\binom{k}{j}(-1)^k(k-j)! \sum_{(k-j)/2-1\le n\le k-j-2} \frac{1}{2^{k-n-j}}\\ &\qquad \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} (-1)^j(2h+j-1)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+j}(1,\dots,1).\end{aligned}$$

The case of $2h$ is handled similarly. First, by Theorem 4.5,

$$\begin{aligned}\int_0^\infty & \mathrm{m}_k(z+x)\log^{2h} x \,\frac{dx}{x^2+1}\\ &= (-1)^k k! \sum_{k/2-1\le n\le k-2} \frac{1}{2^{2(k-n-1)}}\\ &\qquad \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h} x\,\frac{dx}{x^2+1}\\ &\quad + \int_1^\infty \log^{2h+k} x\,\frac{dx}{x^2+1}\\ &\quad + \sum_{j=0}^{k-2}\binom{k}{j}(-1)^{k-j}(k-j)! \sum_{(k-j)/2-1\le n\le k-j-2} \frac{1}{2^{2(k-j-n-1)}}\\ &\qquad \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} \int_1^\infty \log^j x\, L_{(\epsilon_1,\dots,\epsilon_n,2)}\left(\frac{1}{x^2}\right)\log^{2h} x\,\frac{dx}{x^2+1}.\end{aligned}$$

Then, by making the change of variables $y=1/x$ in the integrals over $x\ge 1$, and then replacing y by x again, we see that the above equals

$$\begin{aligned}
&(-1)^k k! \sum_{k/2-1\leq n\leq k-2} \frac{2}{2^{2(k-n-1)}} \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h} x \frac{dx}{x^2+1}\\
&\quad + \int_0^1 (-1)^k \log^{2h+k} x \frac{dx}{x^2+1}\\
&\quad + \sum_{j=1}^{k-2}\binom{k}{j}(-1)^k(k-j)! \sum_{(k-j)/2-1\leq n\leq k-j-2} \frac{1}{2^{2(k-j-n-1)}}\\
&\qquad\qquad \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} \int_0^1 L_{(\epsilon_1,\dots,\epsilon_n,2)}(x^2)\log^{2h+j} x \frac{dx}{x^2+1}.
\end{aligned}$$

Finally, by Lemma 7.8,

$$\begin{aligned}
&\int_0^\infty \mathrm{m}_k(z+x)\log^{2h} x \frac{dx}{x^2+1}\\
&\quad = (2h+k)!L(\chi_{-4}, 2h+k+1)\\
&\qquad + (-1)^{k+1}k! \sum_{k/2-1\leq n\leq k-2} \frac{1}{2^{k-n-1}}\\
&\qquad\qquad \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-n-2}} i(2h)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+1}(1,\dots,1,i,i)\\
&\qquad + \sum_{j=1}^{k-2}\binom{k}{j}(-1)^k(k-j)! \sum_{(k-j)/2-1\leq n\leq k-j-2} \frac{1}{2^{k-n-j}}\\
&\qquad \times \sum_{\substack{(\epsilon_1,\dots,\epsilon_n)\in\{1,2\}^n\\ \#\{i:\,\epsilon_i=2\}=k-j-n-2}} i(-1)^{j+1}(2h+j)!\mathcal{L}_{\epsilon_1,\dots,\epsilon_n,2,2h+j+1}(1,\dots,1,i,i).
\end{aligned}$$

This concludes the proof of Theorem 1.1.

8.1. The reduction to Corollary 1.2. We now proceed to deduce Corollary 1.2 from Theorem 1.1.

Recall that Theorem 1.1 implies in particular that

$$\mathrm{m}_2(R_2) = \frac{\pi^2}{4} + \frac{4}{\pi^2}\mathcal{L}_{2,2}(1,1).$$

By using the formulas (see [BJOP02])

$$\begin{aligned}
\mathrm{Li}_{2,2}(1,1) &= \frac{3}{10}\zeta(2)^2,\\
\mathrm{Li}_{2,2}(1,-1) &= \frac{1}{8}\zeta(2)^2 - 2\mathrm{Li}_{1,3}(1,-1),\\
\mathrm{Li}_{2,2}(-1,1) &= \frac{-11}{40}\zeta(2)^2 + 2\mathrm{Li}_{1,3}(1,-1),\\
\mathrm{Li}_{2,2}(-1,-1) &= \frac{-3}{40}\zeta(2)^2,
\end{aligned}$$

one can replace $\mathcal{L}_{2,2}(1,1)$ in order to obtain

$$\mathrm{m}_2(R_2) = \frac{89\pi^2}{360} + \frac{16}{\pi^2}\mathrm{Li}_{1,3}(1,-1).$$

By combining the identity

$$\mathrm{Li}_{1,3}(-1,-1) = -\frac{11}{20}\zeta(2)^2 + \frac{7}{4}(\log 2)\zeta(3) - \mathrm{Li}_{1,3}(1,-1)$$

from [BJOP02] with

$$\mathrm{Li}_{1,3}(-1,-1) = \frac{\zeta(4)}{2} - 2\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{1}{24}(\log^2 2)(\log^2 2 - \pi^2)\right)$$

(see [BBG95]) one finally obtains (1.2).

8.2. The simpler case of Proposition 1.3. To find formulas for $\mathrm{m}_k(Q_m)$, we start by taking $P_a = az$. Indeed, replacing a by $\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)$ yields zQ_m which has the same higher Mahler measures as Q_m. This computation is particularly easy to do because $\mathrm{m}_k(az) = \log^k|a|$. We obtain

$$\pi^{2n}\mathrm{m}_k(Q_{2n}) = \sum_{h=1}^{n} \frac{s_{n-h}(2^2, 4^2, \ldots, (2n-2)^2)}{(2n-1)!} 2^{2h}\pi^{2n-2h}\int_0^\infty \log^{k+2h-1} x \,\frac{dx}{x^2-1}$$

and

$$\pi^{2n+1}\mathrm{m}_k(Q_{2n+1}) = \sum_{h=0}^{n} \frac{s_{n-h}(1^2, \ldots, (2n-1)^2)}{(2n)!} 2^{2h+1}\pi^{2n-2h}\int_0^\infty \log^{k+2h} x \,\frac{dx}{x^2+1}.$$

Proposition 1.3 follows with the same arguments that we used in the final steps of the proof of Theorem 1.1 and the combination of the well-known formulas

$$\zeta(2j) = \frac{(-1)^{j+1}B_{2j}(2\pi)^{2j}}{2(2j)!}$$

and

$$L(\chi_{-4}, 2j+1) = \frac{(-1)^j E_{2j}\pi^{2j+1}}{2^{2j+2}(2j)!}.$$

Finally, Corollary 1.4 follows from the simple observation that

$$\mathrm{m}_k(Q_m) = \sum_{j_1+\cdots+j_m=k} \binom{k}{j_1,\ldots,j_m} \mathrm{m}_{j_1}\left(\frac{1-x_1}{1+x_1}\right)\cdots \mathrm{m}_{j_m}\left(\frac{1-x_m}{1+x_m}\right),$$

and the fact that

$$\mathrm{m}_j\left(\frac{1-x}{1+x}\right) = \mathrm{m}_j(Q_1) = \begin{cases} (-1)^{j/2}(\pi/2)^j E_j, & j \text{ even},\\ 0, & j \text{ odd}.\end{cases}$$

9. Concluding remarks. We have proved exact formulas for m_k of a particular family of rational functions with an arbitrary number of variables. Much as in the case of the classical Mahler measure for this family, we have obtained formulas involving multiple polylogarithms evaluated at roots of unity. It is expected that many, if not all, of the formulas from Theorem 1.1 could be reduced to expressions solely involving terms of length 1. For example, $\mathcal{L}_{2,2h+1}(1,1)$ can be reduced to combinations of products of $\zeta(n)$ by means of a result in [BBB97] that generalizes a result of Euler on the reduction of multizeta values of length 2. Additional efforts in this direction may be found in [Lal06b, LL], but they are currently insufficient to reduce all the terms involved in such expressions. Another direction for future exploration is the search for formulas for $\mathrm{m}_k(S_m)$ and $\mathrm{m}_k(T_m)$.

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