ANNALES
POLONICI MATHEMATICI
117.2 (2016)

On the Banach envelopes of Hardy–Orlicz spaces on an annulus

MICHAŁ RZECZKOWSKI (Poznań)

Abstract. We describe the Banach envelopes of Hardy–Orlicz spaces of analytic functions on an annulus in the complex plane generated by Orlicz functions well-estimated by power-type functions.

1. Introduction. Let $X = (X, \tau)$ be a topological vector space whose dual X^* separates the points of X. The *Mackey topology* $\mu = \mu(X, X^*)$ on X is the strongest locally convex topology on X producing the same topological dual X^* as τ. It turns out that if X is metrizable then μ is the strongest locally convex topology on X which is weaker than the original τ. If $(X, \|\cdot\|)$ is a quasi-Banach space whose dual X^* separates the points of X, then the Mackey topology is generated by a norm $\|\cdot\|_c$, which is the Minkowski functional of the convex hull of the unit ball of X. The completion of $(X, \|\cdot\|_c)$ is called the *Banach envelope* of X and is denoted by $\widehat{X}$.

It is easy to see that the Banach envelope of ℓ_p $(0 < p < 1)$ is ℓ_1. Duren, Romberg and Shields [3] identified the Banach envelope of the Hardy space H^p on the unit disc U of the complex plane for $0 < p < 1$. They proved that $\widehat{H}^p$ is canonically isomorphic to the weighted Bergman space $B_1^{1/p-2}$, which consists of all analytic functions on the unit disc U such that $\int_0^1 M_1(f, r)(1 - r)^{1/p-2}\,dr$ is finite, where $M_1(f, r)$ denotes the integral mean. There are many variants and generalizations of this theorem. Pavlović [7] described the Banach envelopes for Hardy–Orlicz spaces; Michalak and Nawrocki [6] studied the Banach envelopes of vector-valued Hardy spaces. In [1], Boyd studied the Hardy spaces of analytic functions on the annulus of the complex plane and described their Banach envelopes.

2010 *Mathematics Subject Classification*: Primary 46E15.
Key words and phrases: Hardy–Orlicz spaces, Banach envelope, Hardy spaces on planar domains.
Received 29 September 2015; revised 17 February 2016.
Published online 20 June 2016.

In this paper we study Hardy–Orlicz spaces on an annulus. In particular, in Theorem 4.6, using the techniques from Shapiro's paper [10], we describe the Banach envelopes of these spaces in the case when the generating Orlicz function is well-estimated by power-type functions.

2. Preliminaries. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be an *Orlicz function*, i.e., a continuous and nondecreasing function such that $\lim_{t\to\infty} \Phi(t) = \infty$, and $\Phi(t) = 0$ if and only if $t = 0$. Let (Ω, Σ, μ) be a σ-finite measure space and $L^0(\Omega) := L^0(\Omega, \Sigma, \mu)$ the space of all equivalence classes of measurable functions. The *Orlicz space* $L^\Phi(\Omega) := L^\Phi(\Omega, \Sigma, \mu)$ is the space of all (equivalence classes of) Σ-measurable functions $f\colon \Omega \to \mathbb{C}$ for which there is a constant $\lambda > 0$ such that

$$\int_\Omega \Phi(\lambda|f|)\, d\mu < \infty.$$

It is easy to check that if there exists $C > 0$ such that $\Phi(t/C) \le \Phi(t)/2$ for all $t > 0$, then $L^\Phi(\Omega)$ is a quasi-Banach lattice equipped with the quasi-norm

$$\|f\|_\Phi := \inf\Big\{\lambda > 0;\ \int_\Omega \Phi(|f|/\lambda)\, d\mu \le 1\Big\}.$$

It is also well-known that $\|\cdot\|_\Phi$ is a norm when Φ is a convex function. The functional $\rho_\Phi\colon L^\Phi(\Omega) \to [0,\infty]$ defined by the formula

$$\rho_\Phi(f) := \int_\Omega \Phi(|f|)\, d\mu$$

is called a *modular*. For $\Phi(t) = t^p$, $p \in (0,\infty]$, we have $L^\Phi(\Omega) = L^p(\Omega)$ and the norms coincide. It is well-known that if $\mu(\Omega) < \infty$ and $\Phi \in \Delta_2$, i.e., $\limsup_{t\to\infty} \Phi(2t)/\Phi(t) < \infty$, then the following sets are equal:

$$\begin{aligned}
L^\Phi &= \Big\{f \in L^0(\Omega);\ \int_\Omega \Phi(\lambda|f|)\, d\mu < \infty \text{ for some } \lambda > 0\Big\},\\
M^\Phi &= \Big\{f \in L^0(\Omega);\ \int_\Omega \Phi(\lambda|f|)\, d\mu < \infty \text{ for all } \lambda > 0\Big\},\\
E^\Phi &= \Big\{f \in L^0(\Omega);\ \int_\Omega \Phi(|f|)\, d\mu < \infty\Big\}.
\end{aligned}$$

We refer the reader to [8] for more information about Orlicz spaces.

Let U be the unit disc of the complex plane. Throughout the paper we identify ∂U with $\mathbb{T} = [0, 2\pi)$. For a given $f \in H(U)$ and $r \in (0,1)$ we denote by $f_r\colon \mathbb{T} \to \mathbb{C}$ the function given by $f_r(e^{it}) = f(re^{it})$ for every $t \in \mathbb{T}$. Following [4, 5], for a given Orlicz function Φ the *Hardy–Orlicz space* $H^\Phi(U)$ is defined as the space of all $f \in H(\mathbb{C})$ such that

$$\|f\|_{H^\Phi(U)} := \sup_{0\le r<1} \|f_r\|_{L^\Phi(\mathbb{T})} < \infty. \tag{2.1}$$

The formula (2.1) defines a quasi-norm in $H^{\Phi} := H^{\Phi}(U)$, and it is a norm when Φ is a convex function. Notice that if $0 < p < \infty$ and $\Phi(t) = t^p$ for all $t \geq 0$, then we recover the classical Hardy space $H^p(U)$ (see [2]).

If an Orlicz function Φ is such that $\Phi(|f|)$ is subharmonic, then $\|f_r\|$ is an increasing function of r. Recall (see [9]) that for any subharmonic function $v\colon U \to \mathbb{R}$ the condition

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} v(re^{it})\,dt < \infty$$

is equivalent to the existence of a harmonic majorant of v (and so the least harmonic majorant). Hence, if $\Phi \in \Delta_2$, then for each $f \in H^{\Phi}(U)$ the function $\Phi(|f|)$ has a harmonic majorant.

3. Hardy–Orlicz spaces on an annulus. Let r_0 be a real number such that $0 < r_0 < 1$, and let $\mathbb{C}_\infty$ be the Riemann sphere. We define subsets of the Riemann sphere

$$E := \{z \in \mathbb{C}_\infty;\ |z| > r_0\},$$
$$A := E \cap U = \{z \in \mathbb{C}_\infty;\ r_0 < |z| < 1\}.$$

It is well-known that each $f \in H(A)$ has a Laurent series expansion $f = f_1 + f_2$, where $f_1(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in H(U)$ and $f_2(z) = \sum_{n=1}^{\infty} \widehat{f}(-n) z^{-n} \in H(E)$.

The function $\eta\colon E \to U$ given by $\eta(z) = r_0/z$ for all $z \in E$ maps E onto U and has an inverse $\eta^{-1}(z) = r_0/z$. For $f \in H(E)$ define $\widetilde{f} = f \circ \eta^{-1}$. The Hardy–Orlicz space $H^{\Phi}(A)$ consists of all $f \in H(A)$ such that $\|f\|_{H^{\Phi}(A)} < \infty$, where

$$\|f\|_{H^{\Phi}(A)} := \sup_{r_0<r<1} \|f_r\|_{L^{\Phi}(\mathbb{T})}.$$

In the same way we define $H^{\Phi}(E)$. Denote by $H_0^{\Phi}(E)$ the subspace of $H^{\Phi}(E)$ consisting of all functions that vanish at infinity. It is obvious that $f \in H^{\Phi}(E)$ if and only if $\widetilde{f} \in H^{\Phi}$, and the map $f \mapsto \widetilde{f}$ is an isometric isomorphism from $H^{\Phi}(E)$ onto H^{Φ}.

We will consider Orlicz functions of a certain class. Let α, β be positive real numbers with $\alpha \leq \beta$, and let Φ be an Orlicz function. We write $\Phi \in \Delta(\alpha, \beta)$ if there exist $t_0 \geq 0$ and $C > 0$ such that for any $t \geq t_0$ and $\lambda \geq 1$,

$$\Phi(\lambda t) \leq C^{-1} \lambda^{\beta} \Phi(t), \qquad \Phi(\lambda t) \geq C \lambda^{\alpha} \Phi(t).$$

Moreover, if $t_0 = 0$, $C = 1$ and $\Phi(t^{1/\alpha})$ is a convex function, then we write $\Phi \in \overline{\Delta(\alpha, \beta)}$. The subclass $\overline{\Delta(\alpha, \beta)} \subset \Delta(\alpha, \beta)$ is introduced for technical reasons. It was proved in [7] that for each $\Phi \in \Delta(\alpha, \beta)$ there exists a function $\Psi \in \overline{\Delta(\alpha, \beta)}$ equivalent to Φ (i.e., $K^{-1}\Psi(t) \leq \Phi(t) \leq K\Psi(t)$ for some positive constants K and large enought t), so that the Orlicz spaces L^{Ψ} and L^{Φ} are isomorphic. Notice that for a holomorphic function $f \in H(A)$, the function $\Phi(|f|)$ is subharmonic whenever $\Phi \in \overline{\Delta(\alpha,\beta)}$. It can also be easily shown that for $\Phi \in \overline{\Delta(\alpha,\beta)}$ we have

$$\Phi(x+y) \le 2^{\beta}(\Phi(x)+\Phi(y)). \tag{3.1}$$

If $\Phi \in \overline{\Delta(\alpha,\beta)}$, then $H^{\Phi}(A) \subset H^{\alpha}(A)$, hence the radial limits $f(e^{it}) = \lim_{r\to 1^-} f(re^{it})$ and $f(r_0e^{it}) = \lim_{r\to r_0^+} f(re^{it})$ exist almost everywhere on $\mathbb{T} = [0,2\pi)$, for each $f \in H^{\Phi}(A)$ (see [1]).

Theorem 3.1. *Let $\Phi \in \overline{\Delta(\alpha,\beta)}$. The following sets coincide:*

$H = H^{\Phi}(A) = \{f \in H(A);\ \|f\|_{H^{\Phi}(A)} < \infty\}$,

$K = \{f \in H(A);\ \Phi(|f(z)|)$ *has a harmonic majorant on* $A\}$,

$L = \{f \in H(A);\ f = f_1 + f_2,\ f_1 \in H^{\Phi}(U),\ f_2 \in H_0^{\Phi}(E)\}$.

Proof. ($L \subset H$). Set $\|f_1\|_{H^{\Phi}(U)} = x$ and $\|f_2\|_{H_0^{\Phi}(E)} = y$ with $x, y > 0$. Then, for each $r \in (r_0, 1)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi\biggl(\frac{|f(re^{it})|}{2^{(1+\beta)/\alpha}(x+y)}\biggr)\,dt \le 2^{-(1+\beta)}\frac{1}{2\pi}\int_0^{2\pi} \Phi\biggl(\frac{|f(re^{it})|}{x+y}\biggr)\,dt$$
$$\le \frac{2^{\beta}}{2^{\beta+1}}\biggl(\frac{1}{2\pi}\int_0^{2\pi} \Phi\biggl(\frac{|f_1(re^{it})|}{x}\biggr)\,dt + \frac{1}{2\pi}\int_0^{2\pi} \Phi\biggl(\frac{|f_2(re^{it})|}{y}\biggr)\,dt\biggr) \le 1.$$

This shows that $\|f\|_{H^{\Phi}(A)} \le 2^{(1+\beta)/\alpha}(\|f_1\|_{H^{\Phi}} + \|f_2\|_{H_0^{\Phi}(E)})$.

($H \subset L$). Since $\Phi(|f_1|)$ is subharmonic on U and $H^{\Phi}(A) \subset H^{\alpha}(A)$, the radial limits $f_1(r_0e^{it})$ and $f_1(e^{it})$ exist a.e. on $\mathbb{T}$. On the other hand, $f_2 \in H(E)$, hence in particular f_2 is continuous on $\mathbb{T}$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi\biggl(\frac{|f_1(re^{it})|}{\varepsilon}\biggr)\,dt \le \frac{1}{2\pi}\int_0^{2\pi} \Phi\biggl(\frac{|f_1(e^{it})|}{\varepsilon}\biggr)\,dt$$
$$\le 2^{\beta}\biggl(\frac{1}{2\pi}\int_0^{2\pi} \Phi\biggl(\frac{|f(e^{it})|}{\varepsilon}\biggr)\,dt + \frac{1}{2\pi}\int_0^{2\pi} \Phi\biggl(\frac{|f_2(e^{it})|}{\varepsilon}\biggr)\,dt\biggr) \le 1$$

for all $r \in (0,1)$ if $\varepsilon \ge 2^{(1+\beta)/\alpha}\max(\|f\|_{H^{\Phi}(A)}, \max_{t\in\mathbb{T}}|f_2(e^{it})|/\Phi^{-1}(1))$, so $f_1 \in H^{\Phi}(U)$. In a similar manner, it can be shown that $f_2 \in H_0^{\Phi}(E)$.

($L \subset K$). The functions $\Phi(|f_1|)$ and $\Phi(|f_2|)$ have harmonic majorants on U and E, respectively. The inclusion follows from inequality (3.1).

($K \subset L$). Let $f = f_1 + f_2 \in K$, where $f_1 \in H(U)$ and $f_2 \in H_0(E)$. If u_f is the harmonic majorant of $\Phi(|f|)$ and s is a fixed real number such that $r_0 < s < 1$, then $\Phi(|f|) \le 2^{\beta}u_f + C$ for some constant $C \ge 0$. But $u_f = u_1 + u_2$, where u_1 is harmonic on U and u_2 is harmonic on E. Since u_2 is bounded for $|z| \ge s$, we have $\Phi(|f_1|) \le 2^{\beta}u_1 + C_1$ for $s \le |z| < 1$, where C_1 is a nonnegative constant. Since $\Phi(|f_1|)$ is subharmonic on U, we conclude that this inequality is true for all $z \in U$. Similar considerations for $\Phi(|f_2|)$ show that $f_2 \in H_0^{\Phi}(E)$. ■

Let us remark that the map $(f_1, f_2) \mapsto f = f_1 + f_2$ is a linear bijection from $H^{\Phi} \oplus H_0^{\Phi}(E)$ onto $H^{\Phi}(A)$. In the proof of the first inclusion we have seen that this operator is continuous. By the Open Mapping Theorem we get the following statement.

COROLLARY 3.2. *Let $\Phi \in \overline{\Delta(\alpha,\beta)}$. Then $H^{\Phi}(A) \cong H^{\Phi} \oplus H_0^{\Phi}(E)$ with equivalent norms.*

4. Weighted Bergman spaces and Banach envelopes of Hardy–Orlicz spaces. Let $\varphi, \omega \colon [0,1) \to (0,\infty)$ be continuous functions such that

$$\lim_{r\to 1^-} \omega(r) = 0, \tag{4.1}$$

$$\int_0^1 \varphi(r)\, dr < \infty. \tag{4.2}$$

The function ω will be called *normal* if there exist $k > \varepsilon > 0$ and $0 < s < 1$ such that

$$\frac{\omega(r)}{(1-r)^{\varepsilon}} \searrow 0, \qquad \frac{\omega(r)}{(1-r)^{k}} \nearrow \infty, \qquad r \ge s, r \to 1^-.$$

The pair $\{\omega, \varphi\}$ of functions defined on $[0,1)$ will be said to define a *normal pair* if ω is normal and, for some k satisfying the above condition, there exists $\alpha > k-1$ such that

$$\omega(r)\varphi(r) = (1-r^2)^{\alpha}, \qquad 0 \le r < 1.$$

Note that if ω is normal then there exists φ such that the pair $\{\omega, \varphi\}$ is normal.

For ω and φ satisfying (4.1) and (4.2), Shields and Williams [11] defined the following linear spaces of analytic functions:

$$B_{\infty}^{\omega}(U) := \Big\{ f \in H(U);\ \|f\|_{B_{\infty}^{\omega}(U)} := \sup_{0\le r<1} M_{\infty}(f,r)\omega(r) < \infty \Big\},$$

$$B_1^{\varphi}(U) := \Big\{ f \in H(U);\ \|f\| := 2\int_0^1 M_1(f,r)\varphi(r) r\, dr < \infty \Big\},$$

where $M_1(f,r) = (2\pi)^{-1}\int_0^{2\pi} |f(re^{it})|\, dt$ and $M_{\infty}(f,r) = \max_{|z|=r} |f(z)|$.

Spaces of these types are called *weighted Bergman spaces.*

It was shown in [11] that $B_{\infty}^{\omega}(U)$ and $B_1^{\varphi}(U)$ are Banach spaces and $B_1^{\varphi}(U)^* \cong B_{\infty}^{\omega}(U)$. More precisely, for a normal pair $\{\omega, \varphi\}$ and $f \in B_{\infty}^{\omega}(U)$, if we define

$$\lambda_f(g) = \int_U g(z) f(\overline{z}) \varphi(|z|)\omega(|z|)\, dz, \qquad g \in B_1^{\varphi}(U),$$

then $\lambda_f \in B_1^{\varphi}(U)^*$, and conversely, given $\lambda \in B_1^{\varphi}(U)^*$ there is a unique $f \in B_\infty^{\omega}(U)$ such that $\lambda = \lambda_f$.

Below we define $B_1^{\varphi}(U)$ in a slightly different manner. We set

$$B_1^{\varphi}(U) := \Big\{ f \in H(U);\ \|f\|_{B_1^{\varphi}(U)} := \int_0^1 M_1(f,r)\varphi(r)\,dr < \infty \Big\}.$$

It can be shown that the norms $\|\cdot\|$ and $\|\cdot\|_{B_1^{\varphi}(U)}$ are equivalent.

Let φ, ω be as above and let $\psi, \upsilon \colon (r_0,\infty) \to (0,\infty)$ be continuous functions such that

$$\lim_{r\to r_0^+} \upsilon(r) = 0, \tag{4.3}$$

$$\int_{r_0}^{\infty} \psi(r)\,dr < \infty. \tag{4.4}$$

We define

$$B_1^{\psi}(E) := \Big\{ f \in H(E);\ \|f\|_{B_1^{\psi}(E)} := \int_{r_0}^{\infty} M_1(f,r)\psi(r)\,dr < \infty \Big\},$$

$$B_\infty^{\upsilon}(E) := \Big\{ f \in H(E);\ \|f\|_{B_\infty^{\upsilon}(E)} := \sup_{r_0<r<\infty} M_\infty(f,r)\upsilon(r) < \infty \Big\}.$$

Denote by $B_{1,0}^{\psi}(E)$ the subspace of $B_1^{\psi}(E)$ consisting of all functions that vanish at ∞. Similarly we define $B_{\infty,0}^{\upsilon}(E) \subset B_\infty^{\upsilon}(E)$. For $f \in B_1^{\psi}(E)$ the map $f \mapsto \widetilde{f}$ is an isometric isomorphism between $B_1^{\psi}(E)$ and $B_1^{\varphi}(U)$, where $\psi(|z|) = \varphi(\eta(|z|))r_0/|z|^2$ for $z \in E$.

Let $\varphi, \psi, \omega, \upsilon$ be as before (in fact we now consider the restrictions of those functions to $(r_0,1)$). We define

$$B_1^{\varphi,\psi}(A) := \Big\{ f \in H(A);\ \|f\|_{B_1^{\varphi,\psi}(A)} := \int_{r_0}^{1} M_1(f,r)\varphi(r)\psi(r)\,dr < \infty \Big\},$$

$$B_\infty^{\omega,\upsilon}(A) := \Big\{ f \in H(A);\ \|f\|_{B_\infty^{\omega,\upsilon}(A)} := \sup_{r_0<r<1} M_\infty(f,r)\omega(r)\upsilon(r) < \infty \Big\}.$$

It is clear that $B_1^{\varphi,\psi}(A)$ and $B_\infty^{\omega,\upsilon}(A)$ are normed vector spaces. Using methods from [1] we can prove the following propositions.

PROPOSITION 4.1. *A function f is in $B_1^{\varphi,\psi}(A)$ if and only if there exist $f_1 \in B_1^{\varphi}(U)$ and $f_2 \in B_{1,0}^{\psi}(E)$ such that $f = f_1 + f_2$.*

PROPOSITION 4.2. *If F is a bounded set in $B_1^{\varphi,\psi}(A)$, then the functions in F are uniformly bounded on compact subsets of A.*

PROPOSITION 4.3. *The space $B_1^{\varphi,\psi}(A)$ is a Banach space and convergence in norm implies uniform convergence on compact sets of A.*

A representation similar to that in Proposition 4.1 is true in the case of the spaces $B_\infty^{\omega,\upsilon}(A)$.

PROPOSITION 4.4. *A function f is in $B_\infty^{\omega,\upsilon}(A)$ if and only if there exist $f_1 \in B_\infty^\omega(U)$ and $f_2 \in B_{\infty,0}^\upsilon(E)$ such that $f = f_1 + f_2$.*

Proof. If $f_1 \in B_\infty^\omega(U)$, then $\sup_{z\in U}|f_1(z)|\omega(|z|) < \infty$, and hence $\sup_{z\in A}|f_1(z)|\upsilon(|z|)\omega(|z|) < \infty$. Analogously, for $f_2 \in B_{\infty,0}^\upsilon(E)$, we have $\sup_{z\in A}|f_2(z)|\upsilon(|z|)\omega(|z|) < \infty$. Thus $\sup_{z\in A}|f(z)|\upsilon(|z|)\omega(|z|)$ is finite. Let $f = f_1 + f_2 \in B_\infty^{\omega,\upsilon}(A)$ and fix $s \in (r_0, 1)$. Then

$$\begin{aligned}\sup_{z\in U}|f_1(z)|\omega(|z|) &= \sup_{|z|<1}|f_1(z)|\omega(|z|)\\ &\le \sup_{|z|\le s}|f_1(z)|\omega(|z|) + \max_{|z|\ge s}\upsilon(|z|)\sup_{s\le|z|<1}|f(z)|\omega(|z|)\\ &\quad + \sup_{s\le z\le 1}|f_2(z)|\omega(|z|) < \infty.\end{aligned}$$

In a similar way we infer that $f_2 \in B_{\infty,0}^\upsilon(E)$. ■

THEOREM 4.5. *Suppose that the functions $\varphi, \psi, \omega, \upsilon$ satisfy conditions* (4.1)–(4.4). *Then we have the following isomorphisms:*

(a) $B_1^{\varphi,\psi}(A) \cong B_1^\varphi(U) \oplus B_{1,0}^\psi(E)$,
(b) $B_\infty^{\omega,\upsilon}(A) \cong B_\infty^\omega(U) \oplus B_{\infty,0}^\upsilon(E)$.

Proof. (a) We know that the map $(f_1, f_2) \mapsto f$ is a linear bijection between $B_\infty^{\omega,\upsilon}(A)$ and $B_\infty^\omega(U) \oplus B_{\infty,0}^\upsilon(E)$. It suffices to show that it is continuous; then by the Open Mapping Theorem we deduce continuity of the inverse. The problem of continuity can be reduced to the following inequalities:

$$\|f_1\|_{B_1^{\varphi,\psi}(A)} \le C\|f_1\|_{B_1^\varphi(U)}, \qquad \|f_2\|_{B_1^{\varphi,\psi}(A)} \le C'\|f_2\|_{B_{1,0}^\psi(E)},$$

where C, C' are constants that depend only on φ, ψ.

For f_1, fix $t, s \in (r_0, 1)$ with $s < t$. Then

$$\int_{r_0}^{s} \varphi(r)\psi(r)\,dr \le \max_{r\in[r_0,s]}\varphi(r)\int_{r_0}^{s}\psi(r)\,dr =: K.$$

Since $M_1(g, r)$ is an increasing function of r for any $g \in H(U)$, we have

$$\int_{r_0}^{s} M_1(f_1, r)\varphi(r)\psi(r)\,dr \le KM_1(f_1, s)$$

and

$$\begin{aligned}\int_{r_0}^{t} M_1(f_1, r)\varphi(r)\psi(r)\,dr &\le KM_1(f_1, s) + \int_{s}^{t} M_1(f_1, r)\varphi(r)\psi(r)\,dr\\ &\le KM_1(f_1, s) + \max_{r\in[s,t]}\psi(r)\int_{s}^{t} M_1(f_1, r)\varphi(r)\,dr\end{aligned}$$

$$\leq \frac{K}{t-s}\int_s^t M_1(f_1,r)\,dr + \max_{r\in[s,t]}\psi(r)\int_s^t M_1(f_1,r)\varphi(r)\,dr$$
$$\leq \frac{K}{t-s}\Big(\inf_{r\in[s,t]}\varphi(r)\Big)^{-1}\int_s^t M_1(f_1,r)\varphi(r)\,dr + \max_{r\in[s,t]}\psi(r)\int_s^t M_1(f_1,r)\varphi(r)\,dr.$$

Thus,

$$\int_{r_0}^t M_1(f_1,r)\varphi(r)\psi(r)\,dr \leq C_1\|f_1\|_{B_1^{\varphi}(U)}, \tag{4.5}$$

where $C_1 = \max\big(\frac{K}{t-s}(\inf_{r\in[s,t]}\varphi(r))^{-1}, \max_{r\in[s,t]}\psi(r)\big)$. On the other hand,

$$\int_t^1 M_1(f_1,r)\varphi(r)\psi(r)\,dr \leq \sup_{r\in[t,1)}\psi(r)\int_t^1 M_1(f_1,r)\varphi(r)\,dr,$$

and this implies

$$\int_t^1 M_1(f_1,r)\varphi(r)\psi(r)\,dr \leq C_2\|f_1\|_{B_1^{\varphi}(U)} \tag{4.6}$$

with $C_2 = \sup_{r\in[t,1)}\psi(r)$. Summing (4.5) and (4.6), we obtain the desired inequality with $C = \max(C_1, C_2)$. The second inequality can be proved in a similar manner.

(b) As above, the problem reduces to the inequalities

$$\|f_1\|_{B_\infty^{\omega,v}(A)} \leq C\|f_1\|_{B_\infty^{\omega}(U)}, \qquad f_1 \in B_\infty^{\omega}(U),$$
$$\|f_2\|_{B_\infty^{\omega,v}(A)} \leq C\|f_2\|_{B_{\infty,0}^{v}(E)}, \qquad f_2 \in B_{\infty,0}^{v}(E),$$

which are trivial in this case. ∎

Let us remark that Theorem 4.5 could be proved in a different manner. We have seen that the topologies of $B_1^{\varphi,\psi}(A)$, $B_1^{\varphi}(U)$ and $B_{1,0}^{\psi}(E)$ are stronger than the respective compact-open topologies. The same is also obvious in the case of $B_\infty^{\omega,v}(A)$, $B_\infty^{\omega}(U)$ and $B_{\infty,0}^{v}(E)$. It can be shown that the graph of the operator $(f_1, f_2) \mapsto f$ is closed in the compact-open topology. Then, by applying the Closed Graph Theorem, we would deduce the continuity of this operator.

Now we formulate the main result of the paper concerning the description of the Banach envelopes of Hardy–Orlicz spaces on an annulus. The special case when Φ is a power function was presented in [1].

Main Theorem 4.6. *Let $\Phi \in \overline{\Delta(\alpha,\beta)}$, $\beta < 1$, and let*

$$\varphi(r) = (1-r)^{-2}/\Phi^{-1}\left(\frac{1}{1-r}\right), \qquad \psi(r) = (\varphi\circ\eta^{-1})(r)\frac{r_0}{r^2}, \qquad \textit{for all } r \in (r_0, 1).$$

Then the Banach envelope of $H^{\Phi}(A)$ is isomorphic to $B_1^{\varphi,\psi}(A)$.

To prove Theorem 4.6 we need to show that the topology induced by the norm $\|\cdot\|_{B_1^{\varphi,\psi}(A)}$ is weaker than the topology of $H^\Phi(A)$, and that the $H^\Phi(A)$-closure of the absolutely convex hull of each $H^\Phi(A)$-neighbourhood of zero contains a $B_1^{\varphi,\psi}(A)$-neighbourhood of zero. For the first condition, from Corollary 3.2 and Theorem 4.5 we have $H^\Phi(A)\cong H^\Phi(U)\oplus H_0^\Phi(E)$ and $B_1^{\varphi,\psi}(A)\cong B_1^\varphi(U)\oplus B_{1,0}^\psi(E)$. Since $H^\Phi(U)\hookrightarrow B_1^\varphi(U)$ (see [7, Lemma 5]), it follows that

$$H^\Phi(A)\cong H^\Phi(U)\oplus H_0^\Phi(E)\hookrightarrow B_1^\varphi(U)\oplus B_{1,0}^\psi(E)\cong B_1^{\varphi,\psi}(A).$$

The proof of the second condition is based on some ideas from [10] (in the case of $H^p(U)$ with $0<p<1$). We will prove that for every $f\in H^\Phi(A)$ such that $\|f\|_{B_1^{\varphi,\psi}(A)}\le C$ there exist a Borel measure μ on A and a family of analytic functions $F^\xi\colon A\to\mathbb{C}$, where $\xi\in A$, such that

$$f(z)=\int_A F^\xi(z)\,d\mu(\xi),\qquad z\in A,\tag{4.7}$$

$$\|F^\xi\|_{H^\Phi(A)}\le M,\tag{4.8}$$

$$|\mu|(A)\le 1,\tag{4.9}$$

for some constant $M>0$. Notice that conditions (4.7)–(4.9) express f as a sort of generalized convex combination of the functions F^ξ.

To do so, let $\gamma>1/\alpha-2$ and $\xi\in A$. Define $K(\xi,z):=K_1(\xi,z)+K_2(\xi,z)$, where

$$K_1(\xi,z):=\sum_{n=0}^{\infty}(1-|\xi|^2)^\gamma\big(B_{r_0}^1(n+1,\gamma+1)\big)^{-1}(\bar{\xi})^n z^n,$$

$$K_2(\xi,z):=\sum_{n=0}^{\infty}\bigg(1-\frac{r_0^2}{|\xi|^2}\bigg)^\gamma\big(B_{r_0}^1(n+1,\gamma+1)\big)^{-1}r_0^{2n+2}|\xi|^{-4}(\bar{\xi})^{-n}z^{-n},$$

with

$$B_{r_0}^1(n+1,\gamma+1):=\int_{r_0^2}^1 u^n(1-u)^\gamma\,du,$$

$$B(n+1,\gamma+1):=B_0^1(n+1,\gamma+1).$$

For a fixed $\xi\in A$, $K(\xi,z)$ is holomorphic on the annulus A (as a function of z). In fact, it is analytic on a bigger annulus, because the radii of convergence of K_1 and K_2 equal $1/|\xi|>1$ and $r_0^2/|\xi|<r_0$, respectively. Moreover, for $f\in B_1^{\varphi,\psi}(A)$, we have

$$f(z)=\int_A K(z,\xi)f(\xi)\,d\lambda(\xi),$$

where λ denotes the two-dimensional Lebesgue measure on A divided by π.

Set $F^\xi = F_1^\xi + F_2^\xi$, where

$$F_1^\xi(z) = (1-|\xi|)^2 \Phi^{-1}\Big(\frac{1}{1-|\xi|}\Big) K_1(\xi, z),$$
$$F_2^\xi(z) = \Big(1-\frac{r_0}{|\xi|}\Big)^2 \Phi^{-1}\Big(\frac{1}{1-r_0/|\xi|}\Big)\frac{|\xi|^2}{r_0} K_2(\xi, z),$$
$$d\mu(\xi) = \big(\varphi(|\xi|)f_1(\xi) + \psi(|\xi|)f_2(\xi)\big)d\lambda(\xi).$$

We now show that F^ξ and μ satisfy (4.7)–(4.9) for each $f \in H^\Phi(A)$ with $\|f\|_{B_1^{\varphi,\upsilon}(A)} \le C$. Indeed,

$$\begin{aligned}
|\mu|(A) &= \int_A |\varphi(|\xi|)f_1(\xi) + \psi(|\xi|)f_2(\xi)|\, d\lambda(\xi) \\
&\le \int_A \varphi(|\xi|)|f_1(\xi)|\, d\lambda(\xi) + \int_A \psi(|\xi|)|f_2(\xi)|)\, d\lambda(\xi) \\
&\le 2\int_{r_0}^1 M_1(f_1, r)\varphi(r) r\, dr + 2\int_{r_0}^1 M_1(f_2, r)\psi(r) r\, dr \\
&\le C_1 \int_0^1 M_1(f_1, r)\varphi(r)\, dr + C_2\int_{r_0}^\infty M_1(f_2, r)\psi(r)\, dr \\
&\le \frac{1}{C}\|f\|_{B_1^{\varphi,\psi}(A)},
\end{aligned}$$

with $C = 1/\max(C_1, C_2)$. Therefore, $\|f\|_{B_1^{\varphi,\upsilon}(A)} \le C$ implies $|\mu|(A) \le 1$. The computation

$$\begin{aligned}
\int_A F^\xi(z)\, d\mu(\xi) &= \int_A (1-|\xi|)^2\Phi^{-1}\Big(\frac{1}{1-|\xi|}\Big)K_1(\xi, z)\, d\mu(\xi) \\
&\quad + \int_A \Big(1-\frac{r_0}{|\xi|}\Big)^2 \Phi^{-1}\Big(\frac{1}{1-r_0/|\xi|}\Big)\frac{|\xi|^2}{r_0}K_2(\xi, z)\, d\mu(\xi) \\
&= \int_A \frac{1}{\varphi(|\xi|)}K_1(\xi, z)\, d\mu(\xi) + \int_A \frac{1}{\psi(|\xi|)}K_2(\xi, z)\, d\mu(\xi) \\
&= \int_A K_1(\xi, z)f_1(\xi)\, d\lambda(\xi) + \int_A K_2(\xi, z) f_2(\xi)\, d\lambda(\xi) \\
&= f(z)
\end{aligned}$$

shows that condition (4.7) is satisfied. To prove (4.8), we show that $\|F_1^\xi\|_{H^\Phi(A)} \le C$ where C is a constant (independent of ξ). Using the same techniques, we can prove that $\|F_2^\xi\|_{H^\Phi(A)} \le C'$ and then, by Corollary 3.2, we obtain (4.8). From [10] we know that

$$I^\xi(z) = \frac{1}{(1-\bar{\xi}z)^{\gamma+2}} = \sum_{n=0}^{\infty}\big(B(n+1,\gamma+1)\big)^{-1}(\bar{\xi})^n z^n$$

satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} |I^\xi(re^{it})|^\alpha\,dt \le \kappa(1-|\xi|)^{1-(\gamma+2)\alpha} \tag{4.10}$$

with a constant κ. Notice that for some constants C_1, C_2, C_3 (which depend on r_0 and γ) and $n \ge 1$, we have

$$|B(n+1,\gamma+1) - B^1_{r_0}(n+1,\gamma+1)| \le C_1\frac{r_0^{2n+2}}{n+1}, \tag{4.11}$$

$$\frac{C_2}{(n+1)^{\gamma+1}} \le B^1_{r_0}(n+1,\gamma+1) \le B(n+1,\gamma+1), \tag{4.12}$$

$$B(n+1,\gamma+1) \le C_3 B^1_{r_0}(n+1,\gamma+1). \tag{4.13}$$

Only the first inequality in (4.12) and inequality (4.13) are nontrivial. Suppose that $\gamma \le 0$. Then $(1-u)^\gamma$ is a nondecreasing function on $[0,1)$, and we have

$$\frac{1}{r_0^2}\int_0^{r_0^2} u^n(1-u)^\gamma\,du \le \frac{1}{1-r_0^2}\int_{r_0^2}^1 u^n(1-u)^\gamma\,du.$$

Multiplying both sides by r_0^2 and adding $\int_{r_0^2}^1 u^n(1-u)^\gamma\,du$ we obtain (4.13) with $C_3 = r_0^2/(1-r_0^2)+1$. If $\gamma > 0$, then $(1-u)^\gamma$ is continuous on $[0,1]$ and positive on $[0,1)$. Hence there exists a constant c such that

$$\frac{1}{r_0^2}\int_0^{r_0^2} u^n(1-u)^\gamma\,du \le \frac{c}{1-r_0^2}\int_{r_0^2}^1 u^n(1-u)^\gamma\,du.$$

Using the same argument as in the previous case we obtain (4.13). Recall that

$$B(n+1,\gamma+1) = \Gamma(\gamma+1)\frac{\Gamma(n+1)}{\Gamma(n+1+\gamma+1)}.$$

Using the fact that $\Gamma(z+k)/(k^z\Gamma(k)) \to 1$ as $k\to\infty$, with $z=\gamma+1$ and $k=n+1$ we get (4.12).

From (4.11) and (4.13) we have

$$\begin{aligned}
&\Big|I^\xi(re^{it}) - \sum_{n=0}^{\infty}\big(B^1_{r_0}(n+1,\gamma+1)\big)^{-1}(\bar{\xi})^n r^n e^{int}\Big| \qquad (4.14)\\
&\quad= \Big|\sum_{n=0}^{\infty}\frac{B(n+1,\gamma+1)-B^1_{r_0}(n+1,\gamma+1)}{B(n+1,\gamma+1)B^1_{r_0}(n+1,\gamma+1)}(\bar{\xi})^n r^n e^{int}\Big| \le c_1 < \infty,
\end{aligned}$$

so by (4.10) we can assume that

$$\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}(B^1_{r_0}(n+1,\gamma+1))^{-1}(\overline{\xi})^n e^{int}\Big|^{\alpha}\,dt \le \kappa'(1-|\xi|)^{1-(\gamma+2)\alpha}. \tag{4.15}$$

To finish the proof, consider the functions

$$h(z) = (1-|\xi|)^{\gamma+2}\sum_{n=0}^{\infty}\big(B^1_{r_0}(n+1,\gamma+1)\big)^{-1}(\overline{\xi})^n z^n,$$
$$h_1(z) = (1-|\xi|)^{\gamma+2} I^{\xi}(z).$$

Inequality (4.14) implies that $h_1(z) \approx h(z)$, and since $|h_1(z)| \le 1$, we get $c_2|h(z)| \le 1$, where c_2 is a constant. Now setting $t = \Phi^{-1}(1/(1-|\xi|))$ and applying the approximation $1-|\xi| \approx 1-|\xi|^2$, we have $|F^{\xi}_1(z)| \le c_3 t|h(z)|$ and

$$\Phi(|F^{\xi}_1(z)|) \le c_4\Phi(t|h(z)|) \le \kappa''|h(z)|^{\alpha}(1-|\xi|)^{-1}.$$

Integrating the last inequality we get

$$\rho_{\Phi}(F^{\xi}_1) \le \kappa''\|h\|^{\alpha}_{\alpha}(1-|\xi|)^{-1},$$

where ρ_{Φ} denotes the modular. From (4.15) we deduce that $\|h\|^{\alpha}_{\alpha} \le \kappa'(1-|\xi|)$. Finally, $\rho_{\Phi}(F^{\xi}_1) \le \kappa'\kappa''$, which proves (4.8). ∎

As a consequence of the preceding theorem we obtain a description of the dual of $H^{\Phi}(A)$.

Theorem 4.7. *Let $\Phi \in \overline{\Delta(\alpha,\beta)}$, $\beta < 1$. Then $H^{\Phi}(A)^* \cong B^{\omega,\upsilon}_{\infty}(A)$, where*

$$\omega(|z|) = (1-|z|)^{\gamma+2}\Phi^{-1}\Big(\frac{1}{1-|z|}\Big),$$
$$\upsilon(|z|) = \Big(1-\frac{r_0}{|z|}\Big)^{\gamma+2}\Phi^{-1}\Big(\frac{1}{1-r_0/|z|}\Big), \qquad \gamma > 1/\alpha - 2.$$

More precisely, if $f \in B^{\omega,\upsilon}_{\infty}(A)$ and

$$\lambda_f(g) = \int_A g(z)f(\overline{z})\varphi(|z|)\psi(|z|)\omega(|z|)\upsilon(|z|)\,dz, \qquad g \in H^{\Phi}(A),$$

then $\lambda_f \in B^{\varphi,\psi}_1(A)^*$. Conversely, given $\lambda \in H^{\Phi}(A)^*$ there exists a unique $f \in B^{\omega,\upsilon}_{\infty}(A)$ such that $\lambda = \lambda_f$.

Proof. By Theorem 4.6 we have $H^{\Phi}(A)^* = B^{\varphi,\psi}_1(A)^*$. It is easy to see that $\lambda_f \in B^{\varphi,\psi}_1(A)$. Suppose $\lambda \in B^{\varphi,\psi}_1(A)^*$. By Theorem 4.5, there exist unique $\lambda_1 \in B^{\varphi}_1(U)^*$ and $\lambda_2 \in B^{\psi}_{1,0}(E)^*$ such that $\lambda = \lambda_1 + \lambda_2$. From the description of the duals of $B^{\varphi}_1(U)$ and $B^{\psi}_{1,0}(E)$ we deduce the existence of $f_1 \in B^{\omega}_{\infty}(U)$ and $f_2 \in B^{\upsilon}_{\infty,0}(E)$ which correspond to λ_1 and λ_2, respectively. Using again Theorem 4.5, we conclude that $f = f_1 + f_2 \in B^{\omega,\upsilon}_{\infty}(A)$ and $\lambda = \lambda_f$. ∎

Acknowledgements. We thank the referees for helpful comments that improved the presentation of the paper.

The author was supported by the Foundation for Polish Science (FNP).

References

[1] D. M. Boyd, *The metric space $H^p(A)$, $0 < p < 1$, and its containing Banach space*, Colloq. Math. 40 (1978/79), 119–129.

[2] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[3] P. L. Duren, B. W. Romberg and A. L. Shields, *Linear functionals on H^p spaces with* $0 < p < 1$, J. Reine Angew. Math. 238 (1969), 32–60.

[4] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, *Composition operators on Hardy–Orlicz spaces*, Mem. Amer. Math. Soc. 207 (2010), no. 974.

[5] M. Mastyło and P. Mleczko, *Solid hulls of quasi-Banach spaces of analytic functions and interpolation*, Nonlinear Anal. 73 (2010), 84–98.

[6] A. Michalak and M. Nawrocki, *Banach envelopes of vector valued H^p spaces*, Indag. Math. (N.S.) 13 (2002), 185–195.

[7] M. Pavlović, *On the Banach envelope of Hardy–Orlicz spaces*, Funct. Approx. Comment. Math. 20 (1992), 9–19.

[8] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Pure Appl. Math. 146, Dekker, 1991.

[9] M. Rosenblum and J. Rovnyak, *Topics in Hardy Classes and Univalent Functions*, Birkhäuser, Basel, 1994.

[10] J. H. Shapiro, *Mackey topologies, reproducing kernels and diagonal maps on the Hardy and Bergman spaces*, Duke Math. J. 43 (1976), 187–202.

[11] A. L. Shields and D. L. Williams, *Bounded projections, duality and multipliers in spaces of analytic functions*, Trans. Amer. Math. Soc. 162 (1971), 287–305.

Michał Rzeczkowski
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
61-614 Poznań, Poland
E-mail: rzeczkow@amu.edu.pl