Baire classes of affine vector-valued functions

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Abstract. We investigate Baire classes of strongly affine mappings with values in Fréchet spaces. We show, in particular, that the validity of the vector-valued Mokobodzki result on affine functions of the first Baire class is related to the approximation property of the range space. We further extend several results known for scalar functions on Choquet simplices or on dual balls of L_1-preduals to the vector-valued case. This concerns, in particular, affine classes of strongly affine Baire mappings, the abstract Dirichlet problem and the weak Dirichlet problem for Baire mappings. Some of these results have weaker conclusions than their scalar versions. We also establish an affine version of the Jayne–Rogers selection theorem.

1. Introduction. Investigation of Baire hierarchy of affine functions on compact convex sets forms an important part of the Choquet theory. The abstract Dirichlet problem for Baire functions on Choquet simplices was studied for example in [25, 60, 61, 66, 63], and the descriptive properties of affine functions for example in [40, 39, 38, 67]. A large number of results valid for simplices were proved in a more general framework of L_1-preduals. Recall that a Banach space X is called an L_1-*predual* (or a *Lindenstrauss space*) if its dual X^* is isometric to $L^1(X, \mathcal{S}, \mu)$ for a measure space $(X, \mathcal{S}, \mu)$. There are two types of L_1-preduals: the real and complex ones. Unlike in most areas, there is a difference between the real and complex theories. More precisely, there are some results known in the real case and unknown in the complex case; moreover, proofs in the complex setting are often more involved. However, our results hold in both cases in the same form.

Real L_1-preduals were investigated in depth in [11, 15, 16, 2, 33, 44, 17, 45, 9, 21, 6, 40]. Complex L_1-preduals were studied for example in [74, 48, 75, 57, 53, 37, 54, 7, 72, 46, 10]. It has turned out that both real and complex L_1-preduals can be characterized by a "simplex-like" property of the dual

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unit ball B_{X^*} (see [34, 12]). This result was used to investigate variants of the abstract Dirichlet problem for L_1-preduals in [41, 39, 38].

In the present paper we study vector-valued affine functions, more precisely mappings with values in a Fréchet space. We show that some results remain valid in the same form, some results are true in a weaker form and some results cannot be transferred at all. Main results are contained in Section 2. In the rest of the introductory section we collect some definitions and basic facts on compact convex sets, Baire mappings and vector integration. In Section 3 we give auxiliary results needed to prove the main theorems. Some of them are known and we just collect them, some of them are up to our knowledge new and may be of independent interest. Several subsequent sections are devoted to the proofs of the main results. In the last section we discuss sharpness of our results and formulate related open problems.

1.1. Compact convex sets, simplices and L_1-preduals. We will deal both with real and complex spaces. To shorten notation we will use the symbol $\mathbb{F}$ to denote the appropriate field, $\mathbb{R}$ or $\mathbb{C}$.

If X is a compact Hausdorff space, we denote by $\mathcal{C}(X, \mathbb{F})$ the Banach space of all $\mathbb{F}$-valued continuous functions on X equipped with the sup-norm. The dual of $\mathcal{C}(X, \mathbb{F})$ will be identified (by the Riesz representation theorem) with $\mathcal{M}(X, \mathbb{F})$, the space of $\mathbb{F}$-valued Radon measures on X equipped with the total variation norm and the corresponding weak* topology.

Let X be a convex subset of a (real or complex) vector space E and F be another (real or complex) vector space. Recall that a mapping $f : X \to F$ is said to be *affine* if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ whenever $x, y \in X$ and $t \in [0,1]$. We stress that the notion of an affine function uses only the underlying structure of real vector spaces.

Let X be a compact convex set in a Hausdorff locally convex topological vector space. We write $\mathfrak{A}(X, \mathbb{F})$ for the space of all $\mathbb{F}$-valued continuous affine functions on X. This space is a closed subspace of $\mathcal{C}(X, \mathbb{F})$ and is equipped with the inherited sup-norm. Given a Radon probability measure μ on X, we write $r(\mu)$ for the *barycenter* of μ, i.e., the unique point $x \in X$ satisfying $a(x) = \int_X a \, d\mu$ for each affine continuous function on X (see [1, Proposition I.2.1] or [32, Chapter 7, §20]; note that it does not matter whether we consider real or complex affine functions). Conversely, for a point $x \in X$, we denote by $\mathcal{M}_x(X)$ the set of all Radon probability measures on X with barycenter x (i.e., of probabilities *representing* x).

The usual *dilation order* $\prec$ on the set $\mathcal{M}^1(X)$ of Radon probability measures on X is defined as $\mu \prec \nu$ if and only if $\mu(f) \leq \nu(f)$ for any real-valued convex continuous function f on X. (Recall that $\mu(f)$ is an abbreviation for $\int f \, d\mu$.) A measure $\mu \in \mathcal{M}^1(X)$ is said to be *maximal* if it is maximal with

respect to the dilation order. In case X is metrizable, the maximal measures are exactly the probabilities carried by the G_δ set ext X of extreme points of X (see, e.g., [1, p. 35] or [43, Corollary 3.62]). By the Choquet representation theorem, for any $x \in X$ there exists a maximal representing measure (see [32, p. 192, Corollary] or [1, Theorem I.4.8]). The set X is termed a *simplex* if this measure is uniquely determined for each $x \in X$ (see [32, §20, Theorem 3]). We write δ_x for this uniquely determined measure.

A measure $\mu \in \mathcal{M}(X, \mathbb{F})$ is called *boundary* if either $\mu = 0$ or the probability measure $|\mu|/\|\mu\|$ is maximal. If X is metrizable, the boundary measures are exactly the measures carried by ext X.

If X is a simplex, the space $\mathfrak{A}(X, \mathbb{F})$ is an example of an L_1-predual (see [18, Proposition 3.23] for the real case; the complex case follows from the real one and [32, §23, Theorem 6]). Moreover, L_1-preduals can be characterized using a simplex-like property of the dual unit ball:

FACT 1.1. *Let E be a Banach space over $\mathbb{F}$ and $X = (B_{E^*}, w^*)$. Then E is an L_1-predual if and only if for each $x^* \in X$ there is a unique $\mathbb{F}$-valued Radon measure μ on X with the properties:*

  (a) $\|\mu\| \leq 1$,
  (b) $\mu(\alpha A) = \overline{\alpha}\mu(A)$ *for any Borel set $A \subset X$ and any $\alpha \in \mathbb{F}$ with* $|\alpha| = 1$,
  (c) μ *is a boundary measure,*
  (d) $x^*(x) = \int_X y^*(x)\, d\mu(y^*)$ *for any $x \in E$.*

This is proved at the end of Section 3.4. It is a variant of the characterizations given in [34] for the real case and in [12] for the complex case. The unique measure μ provided by the previous fact will be denoted by $T(x^*)$.

The measures satisfying (b) are called *odd* in the real case and *anti-homogeneous* in the complex case (we will sometimes use the unified term $\mathbb{F}$-*anti-homogeneous*). Condition (d) means that μ represents x^* in a way. Notice that for a function $f : X \to \mathbb{F}$ the following assertions are equivalent:

  (i) There is $x \in E$ with $f(y^*) = y^*(x)$ for $y^* \in X$.
  (ii) f is affine, continuous and $f(\alpha y^*) = \alpha f(y^*)$ for each $y^* \in X$ and any $\alpha \in \mathbb{F}$ with $|\alpha| = 1$.

Functions with the third property in (ii) are called *odd* in the real case and *homogeneous* in the complex case (or $\mathbb{F}$-*homogeneous* in both cases).

1.2. Baire hierarchies of mappings. Given a set K, a topological space L and a family $\mathcal{F}$ of mappings from K to L, we define the *Baire classes* of mappings as follows. Let $(\mathcal{F})_0 = \mathcal{F}$. For $\alpha \in [1, \omega_1)$, assuming

that $(\mathcal{F})_\beta$ have already been defined for each $\beta < \alpha$, we set

$$(\mathcal{F})_\alpha = \Big\{ f : K \to L; \text{ there exists a sequence } (f_n) \text{ in } \bigcup_{\beta < \alpha} (\mathcal{F})_\beta$$
$$\text{such that } f_n \to f \text{ pointwise} \Big\}.$$

We will use several hierarchies matching this pattern:

- If K and L are topological spaces, we write $\mathcal{C}_\alpha(K, L)$ for $(\mathcal{C}(K, L))_\alpha$, where $\mathcal{C}(K, L)$ is the set of all continuous functions from K to L.
- If K is a compact convex set and L is a convex subset of a locally convex space, we write $\mathfrak{A}_\alpha(K, L)$ for $(\mathfrak{A}(K, L))_\alpha$, where $\mathfrak{A}(K, L)$ is the set of all affine continuous functions defined on K with values in L.
- If $K = (B_{E^*}, w^*)$, where E is a real Banach space and L is a convex symmetric subset of a locally convex space, we write $\mathfrak{A}_{\mathrm{odd},\alpha}(K, L)$ for $(\mathfrak{A}_{\mathrm{odd}}(K, L))_\alpha$, where $\mathfrak{A}_{\mathrm{odd}}(K, L)$ is the set of all odd affine continuous functions defined on K with values in L.
- If $K = (B_{E^*}, w^*)$, where E is a complex Banach space and L is an absolutely convex subset of a complex locally convex space, we write $\mathfrak{A}_{\mathrm{hom},\alpha}(K, L)$ for $(\mathfrak{A}_{\mathrm{hom}}(K, L))_\alpha$, where $\mathfrak{A}_{\mathrm{hom}}(K, L)$ is the set of all homogeneous affine continuous functions defined on K with values in L.

1.3. Vector integration and strongly affine mappings. To investigate vector-valued strongly affine mappings, we need some vector integral. We will use the Pettis approach. Our vector-valued mappings will mostly have values in Fréchet spaces, but sometimes in general Hausdorff locally convex spaces. Since we will deal only with Hausdorff spaces, by a *space* we mean always a Hausdorff space.

Let μ be an $\mathbb{F}$-valued σ-additive measure defined on an abstract measurable space $(X, \mathcal{A})$ (i.e., X is a set and $\mathcal{A}$ is a σ-algebra of subsets of X) and let F be a locally convex space over $\mathbb{F}$. (To avoid confusion we stress that we will consider only finite measures.) A mapping $f : X \to F$ is said to be *μ-measurable* if $f^{-1}(U)$ is μ-measurable for any open $U \subset F$, and *weakly μ-measurable* if $\tau \circ f$ is μ-measurable for each $\tau \in F^*$.

A mapping $f : X \to F$ is said to be *μ-integrable* over a μ-measurable set $A \subset X$ if

- $\tau \circ f \in L^1(|\mu|)$ for each $\tau \in F^*$,
- for each μ-measurable $B \subset A$ there exists $x_B \in F$ such that

$$\tau(x_B) = \int_B \tau \circ f \, d\mu, \quad \tau \in F^*.$$

It is clear that x_B is uniquely determined; we denote it $\int_B f \, d\mu$. If μ is clear, we say just that f is integrable.

Any μ-integrable mapping is necessarily weakly μ-measurable but not necessarily μ-measurable (cf. the discussion after Question 10.4 below).

If X is a compact convex set, a mapping $f : X \to F$ is called *strongly affine* if, for any $\mu \in \mathcal{M}^1(X)$, f is μ-integrable and $\int_X f\, d\mu = f(r(\mu))$. This is a strengthening of the notion of an affine mapping: indeed, f is affine if and only if the formula holds for any finitely supported probability μ.

The notion of a strongly affine mapping is also a straightforward generalization of the scalar case; in fact, the following easy observation shows a close connection with the scalar case.

FACT 1.2. *Let X be a compact convex set and F a locally convex space. A mapping $f : X \to F$ is strongly affine if and only if $\tau \circ f$ is strongly affine for each $\tau \in F^*$.*

Proof. The 'only if' part follows immediately from the definitions. Let us show the 'if' part. Suppose that $\tau \circ f$ is strongly affine for each $\tau \in F^*$. Given any $\mu \in \mathcal{M}^1(X)$, $A \subset X$ μ-measurable and $\tau \in F^*$, we have

$$\int_A \tau \circ f\, d\mu = \begin{cases} 0, & \mu(A) = 0, \\ \mu(A)\tau(f(r(\mu|_A/\mu(A)))), & \mu(A) > 0. \end{cases}$$

Hence f is μ-integrable and $\int_X f\, d\mu = f(r(\mu))$. ∎

2. Main results. In this section we collect our main results. We start from results on affine Baire-one maps on general compact convex sets. It turns out that the situation here is quite different from the scalar case. We continue with several positive results which are generalizations of theorems known in the scalar case. Sometimes we are only able to generalize weaker variants of those results. These theorems are formulated and proved in three versions: for simplices, for dual balls of real L_1-preduals and for dual balls of complex L_1-preduals.

Let us start by considering affine Baire-one mappings. If X is a compact convex set and $f : X \to \mathbb{F}$ is an affine function on X which is of the first Baire class (i.e., $f \in \mathcal{C}_1(X, \mathbb{F})$), then f is strongly affine by a result of Choquet (see, e.g., [1, Theorem I.2.6], [52, Section 14], [59] or [43, Corollary 4.22]), and moreover $f \in \mathfrak{A}_1(X, \mathbb{F})$ by a result of Mokobodzki (see, e.g., [55, Théorème 80] or [43, Theorem 4.24]).

If $\mathbb{F} = \mathbb{R}$, $X = (B_{E^*}, w^*)$ for a real Banach space E and f is odd, then $f \in \mathfrak{A}_{\mathrm{odd},1}(X, \mathbb{R})$. This follows easily from the general case since an affine function on X is odd if and only if it vanishes at the origin. If $\mathbb{F} = \mathbb{C}$, $X = (B_{E^*}, w^*)$ for a complex Banach space E, and f is homogeneous, then $f \in \mathfrak{A}_{\mathrm{hom},1}(X, \mathbb{C})$. This is a bit more difficult, and follows from Lemma 3.14(c) below.

The situation for vector-valued functions is different. Firstly, the following analogue of the Choquet result follows immediately from Fact 1.2.

THEOREM 2.1. *Let X be a compact convex set, F a locally convex space and $f : X \to F$ an affine mapping which is of the first Baire class. Then f is strongly affine.*

The vector version of the Mokobodzki theorem is not valid in general but it is valid under additional assumptions on the range space. This is illustrated by the following two results.

THEOREM 2.2. *Let X be a compact convex set and E a Banach space with the bounded approximation property. Then any affine $f \in \mathcal{C}_1(X, E)$ belongs to $\mathfrak{A}_1(X, E)$. If moreover $f(X) \subset B_E$ and E has the λ-bounded approximation property for some $\lambda \geq 1$, then $f \in \mathfrak{A}_1(X, \lambda B_E)$.*

EXAMPLE 2.3. *Let E be a separable reflexive Banach space which fails the compact approximation property. Let $X = (B_E, w)$ and let $f : X \to E$ be the identity embedding. Then f is affine, $f \in \mathcal{C}_1(X, E)$ and $f \notin \bigcup_{\alpha < \omega_1} \mathfrak{A}_\alpha(X, E)$.*

This example is a strengthening of [47, Example 2.22]. A Banach space satisfying the assumptions exists due to [5, Proposition 2.12]. Theorem 2.2 is a generalization and strengthening of [47, Theorem 2.12]. We point out that the proof of the latter theorem in [47] contains a gap. We provide a correct proof of a stronger version. The preceding two results are proved in Section 4 below, where also the definitions of approximation properties are recalled and the gap in the proof in [47] is commented on.

For affine functions of higher Baire classes the situation is different even in the scalar case. Firstly, an affine function of the second Baire class need not be strongly affine even if X is simplex (an example is due to Choquet, see, e.g., [1, Example I.2.10], [52, Section 14] or [43, Proposition 2.63]). Further, by [70] there is a compact convex set X and a strongly affine function $f : X \to \mathbb{R}$ of the second Baire class which does not belong to $\bigcup_{\alpha < \omega_1} \mathfrak{A}_\alpha(X, \mathbb{R})$. Nonetheless, some positive results hold for strongly affine functions on simplices and on dual balls of L_1-preduals.

We begin with the following theorem on the quality of the dilation mapping in the three cases. Let us explain the notation used in the theorem. We denote by $\mathcal{M}_{\mathrm{odd}}(X, \mathbb{R})$ the space of odd real-valued Radon measures on X, and by $\mathcal{M}_{\mathrm{ahom}}(X, \mathbb{C})$ the space of all anti-homogeneous complex Radon measures on X (see Section 1.1 for definitions). All the range spaces in the theorem are considered with the weak* topology. The operator T in cases (R) and (C) was defined in Section 1.1 using Fact 1.1.

THEOREM 2.4.

(S) *Let X be a metrizable simplex. Then the map $T : x \mapsto \delta_x$ belongs to $\mathfrak{A}_1(X, \mathcal{M}^1(X))$.*

(R) *Let E be a real separable L_1-predual and $X = (B_{E^*}, w^*)$. Then the map T belongs to $\mathfrak{A}_{\mathrm{odd},1}(X, B_{\mathcal{M}_{\mathrm{odd}}(X,\mathbb{R})})$.*

(C) *Let E be a complex separable L_1-predual and $X = (B_{E^*}, w^*)$. Then the map T belongs to $\mathfrak{A}_{\mathrm{hom},1}(X, B_{\mathcal{M}_{\mathrm{ahom}}(X,\mathbb{C})})$.*

The above theorem is proved in Section 5. Case (S) is essentially known (see [42, Theorem 6.6]). The formulation of the cited result is weaker, but the construction in fact gives (S). Let us point out that this result is formulated and proved only for metrizable X. However, in some special cases metrizability is not necessary, as formulated in the following remark which will also be discussed in Section 5.

REMARK 2.5. The following assertions hold even without the metrizability (separability) assumption:

- T is strongly affine.
- If ext X is closed, then T is continuous.

We also include Example 5.4 showing that T need not be a Baire mapping even if ext X is Lindelöf.

We continue with a result on affine Baire classes of strongly affine Baire mappings. The scalar version of assertion (S) is proved in [4, Theorem 2], the scalar version of (R) follows easily from [40, Theorem 1.4], and the scalar version of (C) is claimed to be unknown in [40]. The theorem is proved in Section 6.

THEOREM 2.6. *Let X be a compact convex set, F a Fréchet space, $1 \leq \alpha < \omega_1$ and $f \in \mathcal{C}_\alpha(X, F)$ strongly affine.*

(S) *If X is a simplex, then $f \in \mathfrak{A}_{1+\alpha}(X, F)$.*

(R) *If $X = (B_{E^*}, w^*)$, where E is a real L_1-predual, then $f \in \mathfrak{A}_{1+\alpha}(X, F)$. If f is moreover odd, then $f \in \mathfrak{A}_{\mathrm{odd},1+\alpha}(X, F)$.*

(C) *If $X = (B_{E^*}, w^*)$, where E is a complex L_1-predual, then $f \in \mathfrak{A}_{1+\alpha}(X, F)$. If F is moreover complex and f is homogeneous, then $f \in \mathfrak{A}_{\mathrm{hom},1+\alpha}(X, F)$.*

If moreover $\alpha = 1$, then $1 + \alpha$ can be replaced by α, that is, if f belongs to the class $\mathcal{C}_1$, it belongs to the class $\mathfrak{A}_1$.

In case ext X is an F_σ-set, $1 + \alpha$ can be replaced by α for each α.

The next theorem is devoted to the abstract Dirichlet problem for vector-valued Baire mappings. The scalar version of case (S) follows from [25], the scalar version of (R) follows from [41, Theorem 2.14], and the scalar version of (C) follows from [38, Theorem 2.22]. The scalar versions of all the three

cases also hold for $\alpha = 0$, but our proof of the vector version requires $\alpha \geq 1$. The Lindelöf property is a quite natural assumption. It surely cannot be dropped, as witnessed, for example, by the simplex from [1, Proposition I.4.15] (or [43, Example 3.82]). However, it is still an open problem whether the Lindelöf property is necessary for the validity of the scalar case for $\alpha = 0$ (i.e., for continuous functions, see Question 10.9 below).

THEOREM 2.7. *Let X be a compact convex set with* $\operatorname{ext} X$ *being Lindelöf,* $\alpha \geq 1$, F *a Fréchet space and* $f : \operatorname{ext} X \to F$ *a bounded mapping from* $\mathcal{C}_\alpha(\operatorname{ext} X, F)$.

- (S) *If X is a simplex, then f can be extended to a mapping in* $\mathfrak{A}_{1+\alpha}(X, F)$.
- (R) *If $X = (B_{E^*}, w^*)$, where E is a real L_1-predual and f is odd, then f can be extended to a mapping from* $\mathfrak{A}_{\mathrm{odd},1+\alpha}(X, F)$.
- (C) *If $X = (B_{E^*}, w^*)$, where E is a complex L_1-predual, F is complex and f is homogeneous, then f can be extended to a mapping from* $\mathfrak{A}_{\mathrm{hom},1+\alpha}(X, F)$.

If $\operatorname{ext} X$ is moreover F_σ, then $1 + \alpha$ can be replaced by α in all the cases.

If $\operatorname{ext} X$ is even closed and f is continuous, then we can find a continuous affine extension.

The next theorem is devoted to the so-called 'weak Dirichlet problem'. The scalar version of case (S) is known: for continuous functions it is proved in [1, Theorem II.3.12], and for Baire functions it is due to [62]. Up to our knowledge, cases (R) and (C) are new even in the scalar case. The result is proved in Section 8 using a simplified and generalized variant of the method of [62].

THEOREM 2.8. *Let X be a compact convex set, $K \subset \operatorname{ext} X$ a compact subset, F a Fréchet space and f a bounded mapping in $\mathcal{C}_\alpha(K, F)$.*

- (S) *If X is a simplex, then f can be extended to a mapping in* $\mathfrak{A}_\alpha(X, \overline{\operatorname{co}}\, f(K))$.
- (R) *If $X = (B_{E^*}, w^*)$, where E is a real L_1-predual, K is symmetric and f is odd, then f can be extended to a mapping in* $\mathfrak{A}_{\mathrm{odd},\alpha}(X, \overline{\operatorname{aco}}\, f(K))$.
- (C) *If $X = (B_{E^*}, w^*)$, where E is a complex L_1-predual, F is complex, K is homogeneous and f is homogeneous, then f can be extended to a mapping from* $\mathfrak{A}_{\mathrm{hom},\alpha}(X, \overline{\operatorname{aco}}\, f(K))$.

As a consequence, we get a result on extending Baire mappings from compact subsets of completely regular spaces (see Theorem 8.1).

Finally, the following result can be viewed as an affine version of the Jayne–Rogers selection theorem. Recall that a set-valued mapping Φ is said to be *upper semicontinuous* if $\{x \in X; \Phi(x) \subset U\}$ is open in X for any open

set $U \subset F$ (or equivalently $\{x \in X; \Phi(x) \cap H \neq \emptyset\}$ is closed in X for any closed set $H \subset F$).

THEOREM 2.9. *Let X be a compact convex set, F a Fréchet space and $\Phi : X \to F$ an upper semicontinuous set-valued mapping with nonempty closed values and bounded range.*

(S) *If X is a metrizable simplex and the graph of Φ is convex, then Φ has a selection in $\mathfrak{A}_2(X, F)$.*

(R) *If $X = (B_{E^*}, w^*)$, where E is a separable real L_1-predual and the graph of Φ is convex and symmetric, then Φ has a selection in $\mathfrak{A}_{\mathrm{odd},2}(X, F)$.*

(C) *If $X = (B_{E^*}, w^*)$, where E is a separable complex L_1-predual, F is complex and the graph of Φ is absolutely convex, then Φ has a selection in $\mathfrak{A}_{\mathrm{hom},2}(X, F)$.*

We point out that the Jayne–Rogers selection theorem provides a selection of the first class, while we obtain a selection of the second class. This is the best we can achieve, due to Example 9.2. In the same example we show that the metrizability assumption is essential.

3. Some auxiliary results. Below we collect auxiliary results which we will need to prove the main results. These results are divided into four sections. First we need the relationship between the Baire hierarchy of mappings and Baire measurability. These results are known; the only exception is Lemma 3.4 which is a generalization of a result of [47]. Further we establish some results on Pettis integration, especially a dominated convergence theorem. In the next section we collect properties of odd and homogeneous mappings and of the associated operators odd and hom. The results are either easy or vector-valued variants of the results from [38]. Finally, we investigate operators adjoint to odd and hom, and odd and antihomogeneous measures. We think that some of these results are new and of independent interest. In particular, we prove Fact 1.1 which is a simplex-like characterization of L_1-preduals.

3.1. Baire hierarchy of sets and Baire mappings. In this section we formulate the exact relationship between Baire mappings and mappings measurable with respect to the Baire σ-algebra.

If X is a topological space, a *zero set* in X is the inverse image of a closed set in $\mathbb{R}$ under a continuous function $f : X \to \mathbb{R}$. The complement of a zero set is a *cozero set*. If X is normal, it follows from Tietze's theorem that a closed set is a zero set if and only if it is also a G_δ set. We recall that *Baire sets* are members of the σ-algebra generated by the family of all cozero sets in X.

We will need a precise hierarchy of Baire sets. We define additive and multiplicative Baire classes of sets as follows: Let $\Sigma_1^b(X)$ be the family of all cozero sets and $\Pi_1^b(X)$ the family of all zero sets. For $\alpha \in (1, \omega_1)$, let

$\Sigma_\alpha^b(X) =$ the family of countable unions of sets in $\bigcup_{\beta < \alpha} \Pi_\beta^b(X)$,

$\Pi_\alpha^b(X) =$ the family of countable intersections of sets in $\bigcup_{\beta < \alpha} \Sigma_\beta^b(X)$.

The family $\Sigma_\alpha^b(X)$ is termed the *sets of Baire additive class α*, while $\Pi_\alpha^b(X)$ is called the *sets of Baire multiplicative class α*.

The following two lemmata collect some properties of Baire measurable mappings.

LEMMA 3.1. *Let X be a topological space and F a metrizable separable space. Let $f : X \to F$ be a Baire measurable mapping. Then:*

(a) *The σ-algebra of Baire subsets of X equals*

$$\bigcup_{\alpha < \omega_1} \Sigma_\alpha^b(X) = \bigcup_{\alpha < \omega_1} \Pi_\alpha^b(X).$$

(b) *There exists $\alpha < \omega_1$ such that f is $\Sigma_\alpha^b(X)$-measurable.*

(c) *If X is normal, F is a convex subset of a Fréchet space and $\alpha \in [0, \omega_1)$, then $f \in \mathcal{C}_\alpha(X, F)$ if and only if f is $\Sigma_{\alpha+1}^b(X)$-measurable.*

Proof. Assertion (a) is obvious.

(b) Since every separable metric space has a countable basis, the assertion easily follows.

(c) For $\alpha = 0$ the assertion is trivial. For $\alpha = 1$ it follows from [73, Theorem 3.7(i)].

The assertion for $\alpha > 1$ follows from [65, Theorem 2.7]. Indeed, if $\mathcal{F}$ denotes the algebra generated by the zero sets, then the families $\Sigma_\alpha(\mathcal{F})$ and $\Pi_\alpha(\mathcal{F})$ from [65] are exactly $\Sigma_\alpha^b(X)$ and $\Pi_\alpha^b(X)$ for $\alpha \geq 2$. Further, the family denoted by Φ_α in [65] is exactly $\mathcal{C}_\alpha(X, F)$ for $\alpha \geq 1$. For $\alpha = 1$ this follows from the previous paragraph, and for larger ordinals it follows from the definitions. ∎

An immediate consequence of assertion (c) above is the following.

COROLLARY 3.2. *Let X be a normal space, F a separable Fréchet space, $\alpha < \omega_1$ and $f \in \mathcal{C}_\alpha(X, F)$. Then $f \in \mathcal{C}_\alpha(X, \mathrm{co}\, f(X))$.*

LEMMA 3.3. *Let X be a Baire subset of a compact space and F a metrizable space. Let $f : X \to F$ be a Baire measurable mapping. Then:*

(a) *$f(X)$ is separable.*

(b) *There exists $\alpha < \omega_1$ such that f is $\Sigma_\alpha^b(X)$-measurable.*

(c) *If F is a convex subset of a Fréchet space and $\alpha \in [0, \omega_1)$, then $f \in \mathcal{C}_\alpha(X, F)$ if and only if f is $\Sigma_{\alpha+1}^b(X)$-measurable.*

Proof. Assertion (a) follows from [19, Theorem 1]. Indeed, the space X, being a Baire subset of a compact space, is K-analytic by [19, Proposition β, p. 1113] (spaces which are now called K-analytic are called analytic there). Thus X satisfies the assumption on the domain space in [19, Theorem 1]. Further, f is Baire measurable. Note that in [19] this notion has a different meaning, that the preimage of any Baire set is a Baire set. Since F is assumed to be metrizable, this coincides with our notion of Baire measurability. The conclusion is that $f(X)$ is K-analytic, hence Lindelöf (by [19, Proposition α, p. 1113]) and therefore separable (by metrizability).

(b) By (a) we can suppose that F is separable. Hence the assertion follows from Lemma 3.1(b).

(c) By (a) we can suppose that F is separable. Since X is normal (being regular and Lindelöf), the assertion follows from Lemma 3.1(c). ∎

Let us point out that assertion (a) of the previous lemma is not valid in general. It is valid (with the same proof) under the weaker assumption that X is K-analytic (but we do not need it), but it fails if X is, say, a general separable metric space. An example under Martin's axiom and negation of the continuum hypothesis is described in [29, Example 2.4(3)].

We continue with the following technical lemma which is a variant of [47, Proposition 2.8] that dealt with mappings with values in a Banach space. Our proof essentially follows that in [47] with necessary modifications due to the assumption that F is a Fréchet space.

LEMMA 3.4. *Let X be a compact space and F a Fréchet space over $\mathbb{F}$. Let $f_{n,m}, f_n, f : X \to F$ $(m, n \in \mathbb{N})$ satisfy the following conditions:*

(i) *$f_{n,m}$ is continuous on X for all $n, m \in \mathbb{N}$;*
(ii) *$f_{n,m}(x) \xrightarrow{m} f_n(x)$ weakly in F for all $n \in \mathbb{N}$ and $x \in X$;*
(iii) *$f_n(x) \xrightarrow{n} f(x)$ weakly in F for each $x \in X$;*
(iv) *$(f_{n,m})$ is uniformly bounded;*
(v) *$f \in \mathcal{C}_1(X, F)$.*

Then there is a sequence (g_k) of convex combinations of functions $f_{n,m}$, $n, m \in \mathbb{N}$, such that $g_k(x) \to f(x)$ in F for each $x \in X$.

Proof. Fix a closed absolutely convex bounded set $L \subset F$ containing the ranges of all the functions $f_{n,m}$, $n, m \in \mathbb{N}$. Since closed convex sets are weakly closed, L also contains the ranges of f_n, $n \in \mathbb{N}$, and of f. Hence by Corollary 3.2 we have $f \in \mathcal{C}_1(X, L)$ (note that L can be chosen to be separable). Fix a sequence (h_k) in $\mathcal{C}(X, L)$ pointwise converging to f.

Since F is a Fréchet space, its topology is generated by a sequence (p_k) of seminorms. We can suppose that $p_1 \leq p_2 \leq \cdots$. For each $k \in \mathbb{N}$ denote by K_k the polar of the set $\{y \in F; p_k(y) < 1\}$. By the Alaoglu theorem, K_k is weak* compact.

Given any function $u : X \to F$ define $\widehat{u} : X \times F^* \to \mathbb{F}$ by

$$\widehat{u}(x, x^*) = x^*(u(x)), \quad (x, x^*) \in X \times F^*.$$

It is easy to observe that:

- $\widehat{u_n} \to \widehat{u}$ pointwise whenever $u_n(x) \to u(x)$ weakly in F for each $x \in X$.
- If u is continuous, then $\widehat{u}$ is continuous on $X \times K_k$ for each $k \in \mathbb{N}$ (K_k is equipped with the weak* topology).

Indeed, the first assertion is obvious. To show the second, suppose u is continuous and $k \in \mathbb{N}$. Fix $(x, x^*) \in X \times K_k$ and $\varepsilon > 0$. For $(y, y^*) \in X \times K_k$ we have

$$\begin{aligned}
|\widehat{u}(y, y^*) - \widehat{u}(x, x^*)| &= |y^*(u(y)) - x^*(u(x))| \\
&\leq |y^*(u(y) - u(x))| + |y^*(u(x)) - x^*(u(x))| \\
&\leq p_k(u(y) - u(x)) + |y^*(u(x)) - x^*(u(x))| < \varepsilon
\end{aligned}$$

whenever

$$p_k(u(y) - u(x)) < \varepsilon/2 \quad \text{and} \quad |y^*(u(x)) - x^*(u(x))| < \varepsilon/2,$$

which defines a neighborhood of (x, x^*) in $X \times K_k$.

Fix $k \in \mathbb{N}$. Then $\widehat{h_n} \to \widehat{f}$ pointwise and hence $\widehat{f}|_{X \times K_k} \in \mathcal{C}_1(X \times K_k, \mathbb{F})$. Moreover, $\widehat{f_n} \to \widehat{f}$ and $\widehat{f_{n,m}} \xrightarrow{m} \widehat{f_n}$ pointwise on $X \times K_k$ for each $n \in \mathbb{N}$. Since $(\widehat{f_{n,m}}|_{X \times K_k})$ is a uniformly bounded family of continuous functions, [47, Lemma 2.5] yields a sequence $(u_n^k)_n$ in the convex hull of $(f_{n,m})$ such that $\widehat{u_n^k} \xrightarrow{n} \widehat{f}$ pointwise on $X \times K_k$. Then $\widehat{u_n^k} - \widehat{h_n} \xrightarrow{n} 0$ pointwise on $X \times K_k$. Since this sequence is uniformly bounded in $\mathcal{C}(X \times K_k, \mathbb{F})$, it converges weakly to zero in $\mathcal{C}(X \times K_k, \mathbb{F})$. By the Mazur theorem there is $v_k \in \mathrm{co}\{u_n^k - h_n; \, n \geq k\}$ such that $\|\widehat{v_k}|_{X \times K_k}\| < 1/k$. The function v_k can be expressed as $v_k = g_k - w_k$, where $g_k \in \mathrm{co}\{f_{n,m}; \, n, m \in \mathbb{N}\}$ and $w_k \in \mathrm{co}\{h_n; \, n \geq k\}$.

We claim that $g_n(x) \to f(x)$ for each $x \in X$. So, fix $x \in X$ and $k \in \mathbb{N}$. We will show that $p_k(f(x) - g_n(x)) \xrightarrow{n} 0$. Let $\varepsilon > 0$. Fix $n_0 \in \mathbb{N}$ such that $n_0 \geq k$, $1/n_0 < \varepsilon/2$ and $p_k(f(x) - h_n(x)) < \varepsilon/2$ for $n \geq n_0$. Fix $n \geq n_0$. Then $p_k(f(x) - w_n(x)) < \varepsilon/2$ (as $w_n(x) \in \mathrm{co}\{h_j(x); \, j \geq n_0\}$). Hence

$$\begin{aligned}
p_k(f(x) - g_n(x)) &\leq p_k(f(x) - w_n(x)) + p_k(w_n(x) - g_n(x)) \leq \varepsilon/2 + p_n(v_n(x)) \\
&= \varepsilon/2 + \sup\{|x^*(v_n(x))|; \, x^* \in K_k\} = \varepsilon/2 + \|\widehat{v_n}|_{X \times K_k}\| \\
&\leq \varepsilon/2 + \|\widehat{v_n}|_{X \times K_n}\| < \varepsilon/2 + 1/n < \varepsilon. \quad \blacksquare
\end{aligned}$$

An immediate consequence of the previous lemma is the following.

COROLLARY 3.5. *Let X be a compact convex space and F a bounded convex subset of a Fréchet space. If $f \in \mathcal{C}_1(X, F)$ and $f \in \mathfrak{A}_2(X, F)$, then $f \in \mathfrak{A}_1(X, F)$.*

3.2. Integrable vector-valued functions. We now collect several results on vector integration. We start by two lemmata on the relationship between measurability and weak measurability.

LEMMA 3.6. *Let F be a separable metrizable locally convex space. Then each open subset of F is F_σ in the weak topology.*

Proof. Let U be an open subset of F. For each $x \in U$ there is a convex open neighborhood V_x of zero such that $x + \overline{V_x} \subset U$. Since F is metrizable and separable, there is a countable set $C \subset U$ such that the sets $x + V_x$, $x \in C$, cover U. Then $U = \bigcup_{x \in C}(x + \overline{V_x})$. Since closed convex sets are weakly closed, the proof is complete. ∎

LEMMA 3.7. *Let μ be an $\mathbb{F}$-valued measure defined on a measurable space $(X, \mathcal{A})$ and let F be a separable metrizable locally convex space. Then any weakly μ-measurable function $f : X \to F$ is μ-measurable.*

Proof. Suppose that f is weakly measurable, i.e., $\tau \circ f$ is μ-measurable for any $\tau \in F^*$. It follows that $f^{-1}(U)$ is μ-measurable for any U from the canonical basis of the weak topology on F. Since F is separable and metrizable, the weak topology is hereditarily Lindelöf, thus $f^{-1}(U)$ is μ-measurable for any weakly open U. By Lemma 3.6, $f^{-1}(U)$ is μ-measurable for any U open in the original topology of F. ∎

LEMMA 3.8. *Let μ be an $\mathbb{F}$-valued measure defined on a measurable space $(X, \mathcal{A})$ and let F be a Fréchet space over $\mathbb{F}$. Suppose that $f : X \to F$ is a bounded weakly μ-measurable mapping with (essentially) separable range. Then:*

(a) *f is μ-integrable.*
(b) *If μ is moreover a probability and $L \subset F$ is a closed convex set such that $f(X) \subset L$, then $\mu(f) \in L$.*
(c) *If $\|\mu\| \le 1$ and $L \subset F$ is a closed absolutely convex set such that $f(X) \subset L$, then $\mu(f) \in L$.*
(d) *If ρ is any continuous seminorm on F, then $\rho \circ f$ is μ-integrable and $\rho(\int_X f \, d\mu) \le \int_X \rho \circ f \, d|\mu|$.*

Proof. Without loss of generality we can suppose that F is separable.

Assertion (a) for nonnegative measures then follows immediately from [71, Corollary 3.1]. The general case is an easy consequence.

(b) Assuming $\mu(f) \notin L$, by the Hahn–Banach separation argument we can find $\tau \in F^*$ and $c \in \mathbb{R}$ such that $\operatorname{Re} \tau(\mu(f)) > c > \sup\{\operatorname{Re} \tau(l); \, l \in L\}$. Then

$$c < \operatorname{Re} \tau(\mu(f)) = \int\limits_X \operatorname{Re} \tau(f(x)) \, d\mu(x) < \int\limits_X c \, d\mu(x) = c,$$

a contradiction.

(c) We proceed as in the proof of (b). If L is absolutely convex, we get

$$\sup\{\operatorname{Re}\tau(l); l \in L\} = \sup\{|\tau(l)|; l \in L\}$$

and

$$c < \operatorname{Re}\tau(\mu(f)) = \operatorname{Re}\int_X \tau(f(x))\, d\mu(x) \le \int_X |\tau(f(x))|\, d|\mu|(x) \le c.$$

To show (d), let ρ be any continuous seminorm on F. Since f is μ-measurable by Lemma 3.7, it is clear that $\rho\circ f$ is μ-measurable and bounded, hence μ-integrable. Set $V = \{x \in F; \rho(x) \le 1\}$ and let

$$V^0 = \{\tau \in F^*; |\tau(x)| \le 1 \text{ for } x \in V\}$$

denote the absolute polar of V. Then the Bipolar Theorem implies that

$$\rho(x) = \sup\{|\tau(x)|; \tau \in V^0\}, \quad x \in F.$$

Hence

$$\rho\Big(\int_X f\, d\mu\Big) = \sup\Big\{\Big|\tau\Big(\int_X f\, d\mu\Big)\Big|; \tau \in V^0\Big\} = \sup\Big\{\Big|\int_X \tau\circ f\, d\mu\Big|; \tau \in V^0\Big\}$$

$$\le \sup\Big\{\int_X |\tau\circ f|\, d|\mu|; \tau \in V^0\Big\} \le \int_X \rho\circ f\, d|\mu|. \quad \blacksquare$$

An important class of integrable functions is formed by Baire measurable functions.

LEMMA 3.9. *Let X be a compact space, μ an $\mathbb{F}$-valued Radon measure on X and $f : X \to F$ a bounded Baire measurable mapping from X to a Fréchet space F over $\mathbb{F}$. Then f is μ-integrable.*

Proof. By Lemma 3.3(a) the image $f(X)$ is separable. Hence the conclusion follows from Lemma 3.8(a). $\blacksquare$

We will also use the following version of the Dominated Convergence Theorem.

THEOREM 3.10 (Dominated Convergence Theorem). *Let μ be an $\mathbb{F}$-valued measure defined on a measurable space $(X, \mathcal{A})$ and let F be a Fréchet space over $\mathbb{F}$. Let $f_n, f : X \to F$ be mappings such that*

- *each f_n is weakly μ-measurable and has separable range,*
- *$\{f_n\}$ is bounded in F (i.e., $\bigcup_{n=1}^\infty f_n(X)$ is bounded in F),*
- *$f_n(x) \to f(x)$ in F for $x \in X$.*

Then f is bounded and μ-measurable. Moreover, all the functions involved are μ-integrable and $\int_X f_n\, d\mu \to \int_X f\, d\mu$ in F.

Proof. Set $L = \overline{\{f_n(x); n \in \mathbb{N}, x \in X\}}$. Then L is a separable closed bounded set and clearly $f(X) \subset L$. Thus f is bounded and has separable

range. Moreover, it is weakly μ-measurable as measurability is preserved by pointwise limits of sequences. By Lemma 3.7, all the mappings involved are even μ-measurable. Their μ-integrability now follows from Lemma 3.8(a). It remains to prove the convergence of the integrals.

To do so, let ρ be any continuous seminorm on F. By Lemma 3.8(c),

$$\rho\Big(\int_X f_n\,d\mu - \int_X f\,d\mu\Big) = \rho\Big(\int_X (f_n - f)\,d\mu\Big) \le \int_X \rho \circ (f_n - f)\,d|\mu|$$

for each $n \in \mathbb{N}$. Since $f_n \to f$ pointwise, $\rho \circ (f_n - f) \to 0$ pointwise. Clearly $\rho(f_n(x) - f(x)) \le 2\sup\{\rho(v); v \in L\}$ for each $x \in K$, thus the classical Dominated Convergence Theorem gives

$$\lim_{n\to\infty} \int_X \rho \circ (f_n - f)\,d|\mu| = 0.$$

Since ρ is an arbitrary continuous seminorm, the convergence of the integrals follows. ∎

3.3. Odd and homogeneous mappings. Let E be a vector space over $\mathbb{F}$. A set $A \subset E$ is called $\mathbb{F}$-*homogeneous* if $\alpha x \in A$ whenever $x \in A$ and $\alpha \in \mathbb{F}$ satisfies $|\alpha| = 1$. $\mathbb{R}$-homogeneous sets are called *symmetric*, and $\mathbb{C}$-homogeneous ones just *homogeneous*.

If $A \subset E$ is $\mathbb{F}$-homogeneous and F is another vector space over $\mathbb{F}$, then a mapping $f : A \to F$ is called $\mathbb{F}$-*homogeneous* if $f(\alpha x) = \alpha f(x)$ whenever $x \in A$ and $\alpha \in \mathbb{F}$ satisfies $|\alpha| = 1$. $\mathbb{R}$-homogeneous maps are called *odd*, and $\mathbb{C}$-homogeneous ones just *homogeneous*.

The following lemma shows that there is a close connection between $\mathbb{F}$-homogeneous affine maps and linear operators.

LEMMA 3.11. *Let E be a Banach space over $\mathbb{F}$ and let $f : B_E \to F$ be an $\mathbb{F}$-homogeneous affine mapping from B_E to a vector space F over $\mathbb{F}$. Then there exists a unique linear operator $L : E \to F$ extending f.*

Proof. We define $L : E \to F$ by

$$Lx = \begin{cases} \|x\| f(x/\|x\|), & \|x\| > 1, \\ f(x), & \|x\| \le 1. \end{cases}$$

It is a well defined linear extension of f which is obviously unique. ∎

We are now going to describe a procedure of "homogenization" of functions. This procedure is easier in the real case.

Let E, F be real vector spaces and $B \subset E$ a symmetric set. For any function $f : B \to F$ we define a function $\mathsf{odd}\, f$ by

$$(\mathsf{odd}\, f)(x) = \tfrac{1}{2}(f(x) - f(-x)), \quad x \in B.$$

It is clear that $\mathsf{odd}\, f$ is an odd function and that f is odd if and only if $f = \mathsf{odd}\, f$.

The complex procedure is a bit more involved. Again, we suppose that E and F are complex vector spaces, $B \subset E$ is a homogeneous set and $f : B \to F$ is a mapping. We would like to define a function $\mathsf{hom}\, f$ by

$$(\mathsf{hom}\, f)(x) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-it} f(e^{it} x)\, dt, \quad x \in B.$$

The difference from the real case is that a vector integral is involved and it need not be meaningful. Therefore we restrict our assumptions: we will assume that F is locally convex space and that the relevant integral is well defined (in the Pettis sense defined in Section 1.3) for each $x \in B$. These assumptions are satisfied, in particular, if E is a locally convex space, F is a Fréchet space and f is a bounded Borel function. Indeed, then for each $x \in B$ the mapping $t \mapsto e^{-it} f(e^{it} x)$ is a bounded Borel mapping from $[0, 2\pi]$ to F. Since for compact metric spaces the Baire and Borel σ-algebras coincide, we can conclude by Lemma 3.9. It is clear that $\mathsf{hom}\, f$ is homogeneous whenever it makes sense.

The following lemma sums up the basic properties of the operator odd.

LEMMA 3.12. *Let E, F be real locally convex spaces and $A \subset E$, $B \subset F$ be symmetric sets. Then:*

(a) *The function $\mathsf{odd}\, f$ is continuous for each $f : A \to F$ continuous.*
(b) *If $f \in \mathcal{C}_\alpha(A, B)$ for some $\alpha < \omega_1$, then $\mathsf{odd}\, f \in \mathcal{C}_{\mathrm{odd},\alpha}(A, B)$ where $\mathcal{C}_{\mathrm{odd},\alpha}(A, B) = (\mathcal{C}_{\mathrm{odd}}(A, B))_\alpha$ and $\mathcal{C}_{\mathrm{odd}}(A, B)$ is the space of all odd continuous mappings of A into B.*
(c) *If A and B are moreover convex and $f \in \mathfrak{A}_\alpha(A, B)$ for some $\alpha < \omega_1$, then $\mathsf{odd}\, f \in \mathfrak{A}_{\mathrm{odd},\alpha}(A, B)$.*
(d) *If A and B are moreover convex, then each odd function in $\mathfrak{A}_\alpha(A, B)$ belongs to $\mathfrak{A}_{\mathrm{odd},\alpha}(A, B)$.*

Proof. Assertion (a) is obvious. Assertion (b) follows from (a) by transfinite induction. Assertion (c) is obvious for $\alpha = 0$; the general case follows by transfinite induction. Finally, (d) is an immediate consequence of (c). ∎

We continue with the basic properties of the operator hom. The scalar version is proved in [38, Lemma 2.2].

LEMMA 3.13. *Let E be a complex locally convex space, F a complex Fréchet space, $A \subset E$ a homogeneous set and $B \subset F$ a closed absolutely convex bounded set. Then:*

(a) *The function $\mathsf{hom}\, f$ is continuous for each $f : A \to B$ continuous.*
(b) *Let $f_n : A \to B$ be a Borel function for each $n \in \mathbb{N}$. Suppose that the sequence (f_n) pointwise converges to a function f. Then $(\mathsf{hom}\, f_n)$ pointwise converges to $\mathsf{hom}\, f$.*

(c) *If $f \in \mathcal{C}_\alpha(A, B)$ for some $\alpha < \omega_1$, then $\mathsf{hom}\, f \in \mathcal{C}_{\mathrm{hom},\alpha}(A, B)$ where $\mathcal{C}_{\mathrm{hom},\alpha}(A, B) = (\mathcal{C}_{\mathrm{hom}}(A, B))_\alpha$ and $\mathcal{C}_{\mathrm{hom}}(A, B)$ is the space of all homogeneous continuous mappings of A into B.*

Proof. (a) Let $x \in A$. Fix any continuous seminorm ρ on F and any $\varepsilon > 0$. We will find a neighborhood U of zero in E such that for any $y \in (x + U) \cap A$ one has $\rho(\mathsf{hom}\, f(y) - \mathsf{hom}\, f(x)) \le \varepsilon$.

To do so, first observe that the mapping $h : A \times \mathbb{R} \to B$ defined by $h(y, t) = e^{-it} f(e^{it} y)$ is continuous. Therefore for any $t \in \mathbb{R}$ there is V_t, an absolutely convex neighborhood of zero in E, and $\delta_t > 0$ such that whenever $s \in (t - \delta_t, t + \delta_t)$ and $y \in (x + V_t) \cap A$, then $\rho(h(y, s) - h(x, t)) < \varepsilon/2$. Fix a finite set $J \subset [0, 2\pi]$ such that the intervals $(t - \delta_t, t + \delta_t)$, $t \in J$, cover $[0, 2\pi]$. Let $U = \bigcap_{t \in J} V_t$. Given $s \in [0, 2\pi]$, choose $t \in J$ with $s \in (t - \delta_t, t + \delta_t)$. Then for each $y \in (x + U) \cap A$ we have

$$\rho(h(y, s) - h(x, s)) \le \rho(h(y, s) - h(x, t)) + \rho(h(x, t) - h(x, s)) < \varepsilon.$$

Hence, for each $y \in (x + U) \cap A$,

$$\rho(\mathsf{hom}\, f(y) - \mathsf{hom}\, f(x)) = \rho\left(\frac{1}{2\pi} \int_0^{2\pi} (h(y, s) - h(x, s))\, ds \right)$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} \rho(h(y, s) - h(x, s))\, ds \le \varepsilon.$$

Assertion (b) follows immediately from Theorem 3.10; and (c) follows from (a) by using (b) and transfinite induction. ∎

In case f is affine, $\mathsf{hom}\, f$ is always well defined and no further measurability assumptions on f are needed. Indeed, given $x \in E$, the set $\{e^{it} x;\ t \in [0, 2\pi]\}$ is contained in a finite-dimensional subspace of E (more precisely, in a subspace of complex dimension one, hence of real dimension two). Moreover, f is continuous on this finite-dimensional space and maps it into a finite-dimensional subspace of F. The properties of the "homogenization" of affine mappings are summed up in the following lemma.

LEMMA 3.14. *Let E, F be complex locally convex spaces, $A \subset E$ an absolutely convex set and $B \subset F$ a closed absolutely convex set.*

(a) *If $f : A \to F$ is affine, then*

$$\mathsf{hom}\, f(x) = \frac{1}{2}(f(x) - i f(ix)) - \frac{1 - i}{2} f(0), \quad x \in A.$$

Moreover, there are unique homogeneous affine functions $u, v : A \to F$ such that $f(x) = f(0) + u(x) + \overline{v(x)}$ for $x \in A$. In this formula $u = \mathsf{hom}\, f$.

(b) *If $f \in \mathfrak{A}_\alpha(A, B)$ for some $\alpha < \omega_1$, then $\mathsf{hom}\, f \in \mathfrak{A}_{\mathrm{hom},\alpha}(A, B)$.*

(c) *Each homogeneous function from $\mathfrak{A}_\alpha(A, B)$ belongs to $\mathfrak{A}_{\mathrm{hom},\alpha}(A, B)$.*

Proof. (a) Set $h = f - f(0)$. Then h is odd, hence $h(\alpha x + \beta y) = \alpha h(x) + \beta h(y)$ whenever $\alpha, \beta \in \mathbb{R}$ and $x, y, \alpha x + \beta y \in A$. Indeed, this is clear in case $\beta = 0$. If $\beta \neq 0$, we have

$$
h(\alpha x + \beta y) = 2(|\alpha| + |\beta|)h\left(\frac{\alpha x + \beta y}{2(|\alpha| + |\beta|)}\right)
$$

$$
= (|\alpha| + |\beta|)\left(h\left(\frac{\alpha}{|\alpha| + |\beta|}x\right) + h\left(\frac{\beta}{|\alpha| + |\beta|}y\right)\right)
$$

$$
= \alpha h(x) + \beta h(y).
$$

Therefore

$$
\mathsf{hom}\, f(x) = \mathsf{hom}\, h(x) = \frac{1}{2\pi}\int\limits_0^{2\pi} e^{-it}h(e^{it}x)\, dt
$$

$$
= \frac{1}{2\pi}\int\limits_0^{2\pi} e^{-it}h(x\cos t + ix\sin t)\, dt
$$

$$
= \frac{1}{2\pi}\int\limits_0^{2\pi} (\cos^2 t - i\cos t\sin t)(h(x))\, dt
$$

$$
+ \frac{1}{2\pi}\int\limits_0^{2\pi} (\cos t\sin t - i\sin^2 t)h(ix)\, dt
$$

$$
= \frac{1}{2}(h(x) - ih(ix)) = \frac{1}{2}(f(x) - if(ix)) - \frac{1-i}{2}f(0).
$$

Set $u = \mathsf{hom}\, f$ and $v(x) = \overline{f(x) - f(0) - u(x)}$ for $x \in A$. Then u is a homogeneous affine function. Moreover, by the above formula we get

$$
v(x) = \tfrac{1}{2}\overline{(f(x) - f(0)) + i(f(ix) - f(0))}, \quad x \in A.
$$

Hence $v(ix) = iv(x)$ for $x \in A$, so v is homogeneous. The uniqueness is clear.

Assertion (b) follows from (a) by transfinite induction; and (c) follows immediately from (b). ∎

3.4. Odd and anti-homogeneous measures. Let E be a Banach space over $\mathbb{F}$ and $X = (B_{E^*}, w^*)$. Then X is a compact convex set. Odd and anti-homogeneous measures on X were defined in Section 1.1. In this section we will elaborate these notions, and in particular provide a proof of Fact 1.1.

There are two points of view on a Radon measure on X: we can view it as a set function or as a functional on $\mathcal{C}(X, \mathbb{F})$. In the previous section we defined the operators odd and hom on $\mathcal{C}(X, \mathbb{F})$. We use the same symbols to denote the adjoint operators on $\mathcal{C}(X, \mathbb{F})^*$. Let μ be an $\mathbb{F}$-valued Radon measure on X.

- If $\mathbb{F} = \mathbb{R}$, we define $\mathsf{odd}\,\mu \in \mathcal{C}(X,\mathbb{R})^*$ by $(\mathsf{odd}\,\mu)(f) = \mu(\mathsf{odd}\,f)$ for $f \in \mathcal{C}(X,\mathbb{R})$.
- If $\mathbb{F} = \mathbb{C}$, we define $\mathsf{hom}\,\mu \in \mathcal{C}(X,\mathbb{C})^*$ by $(\mathsf{hom}\,\mu)(f) = \mu(\mathsf{hom}\,f)$ for $f \in \mathcal{C}(X,\mathbb{C})$.

The following lemma sums up the basic properties of the operator odd on measures.

LEMMA 3.15. *Let E be a real Banach space, $X = (B_{E^*}, w^*)$ and μ a signed Radon measure on X. Then:*

(a) $\mathsf{odd}\,\mu(A) = \frac{1}{2}(\mu(A) - \mu(-A))$ *for any Borel set $A \subset X$.*
(b) $\mathsf{odd}\,\mu$ *is an odd measure.*
(c) μ *is odd if and only if $\mu = \mathsf{odd}\,\mu$.*
(d) *Let $B \subset X$ be a symmetric μ-measurable set and $f : B \to \mathbb{R}$ a bounded Borel function. Then $\int_B f\,d\,\mathsf{odd}\,\mu = \int_B \mathsf{odd}\,f\,d\mu$.*
(e) *If μ is a boundary measure, then so is $\mathsf{odd}\,\mu$.*

Proof. (a) Define $\nu(A) = \frac{1}{2}(\mu(A) - \mu(-A))$ for $A \subset X$ Borel. This is a well-defined measure. Moreover, for any $f \in \mathcal{C}(X,\mathbb{R})$ we have $\nu(f) = \mu(\mathsf{odd}\,f)$. Indeed, set $\sigma(x^*) = -x^*$ for $x^* \in X$. Then, given $f \in \mathcal{C}(X,\mathbb{R})$,

$$\nu(f) = \frac{1}{2}\Big(\int_X f\,d\mu - \int_X f\,d\sigma(\mu) \Big) = \frac{1}{2}\Big(\int_X f\,d\mu - \int_X f \circ \sigma^{-1}\,d\mu \Big)$$
$$= \mu(\mathsf{odd}\,f).$$

Assertions (b) and (c) follow immediately from (a), and assertion (d) follows from (a) by repeating the above computation. Assertion (e) follows easily from the characterization of boundary measures in [1, pp. 34–35]. ∎

We continue with the following lemma on odd measures and their relationship to probabilities.

LEMMA 3.16. *Let E be a real Banach space, $X = (B_{E^*}, w^*)$ and μ an odd Radon measure on X with $\|\mu\| \leq 1$. Then:*

(a) *There is a Borel set $A \subset X$ with $A \cap (-A) = \emptyset$ such that μ^+ is supported by A and μ^- is supported by $-A$.*
(b) *There is a unique $x^* \in X$ such that for each $x \in E$ one has $x^*(x) = \int y^*(x)\,d\mu(y^*)$.*
(c) *There is a probability ν on X such that $\mathsf{odd}\,\nu = \mu$ and $r(\nu) = x^*$ (where x^* is provided by (b)). If μ is a boundary measure, then ν can be chosen maximal.*

Proof. (a) Fix disjoint Borel sets A^+ and A^- such that μ^+ is supported by A^+ and μ^- is supported by A^-. It is enough to take $A = A^+ \setminus (-A^+)$.

To show (b) it is enough to observe that $x \mapsto \int y^*(x)\,d\mu(y^*)$ defines a linear functional on E of norm not greater than $\|\mu\|$.

(c) If $\mu = 0$, take $\nu = \frac{1}{2}(\varepsilon_{y^*} + \varepsilon_{-y^*})$ where y^* is an extreme point of X. If $\mu \neq 0$, one can take

$$\nu = 2\mu^+ + \frac{1 - 2\|\mu^+\|}{\|\mu\|}|\mu|.$$

Indeed, let A be as in (a). Then $\|\mu^+\| = \mu^+(A) = \mu(A) = -\mu(-A) = \mu^-(-A) = \|\mu^-\|$, thus $\|\mu^+\| \leq 1/2$. It follows that ν is a probability measure. Moreover, given $B \subset X$ Borel, we have

$$\mathsf{odd}\,\nu(B) = \frac{1}{2}(\nu(B) - \nu(-B))$$

$$= \mu^+(B) - \mu^+(-B) + \frac{1 - 2\|\mu^+\|}{2\|\mu\|}(|\mu|(B) - |\mu|(-B))$$

$$= \mu(B \cap A) - \mu((-B) \cap A)$$

$$+ \frac{1 - 2\|\mu^+\|}{2\|\mu\|}\big(\mu(B \cap A) - \mu(B \cap (-A))$$

$$- \mu((-B) \cap A) + \mu((-B) \cap (-A))\big)$$

$$= \mu(B \cap A) + \mu(B \cap (-A)) + 0 = \mu(B),$$

thus $\mathsf{odd}\,\nu = \mu$.

Let $f : X \to \mathbb{R}$ be a continuous affine function. Then $f - f(0)$ is odd, thus there is $x \in E$ such that $f(x^*) - f(0) = x^*(x)$ for each $x^* \in X$. Hence

$$\nu(f) = \nu(f(0)) + \nu(f - f(0)) = f(0) + \mu(f - f(0)) = f(0) + x^*(x) = f(x^*),$$

therefore ν represents x^*.

Finally, if μ is boundary, then both $|\mu|$ and μ^+ are boundary, hence ν is maximal. ∎

The next lemma sums up the properties of the operator hom on complex measures. It is analogous to Lemma 3.15 but the proof is more involved due to the more complicated definition in the complex case.

LEMMA 3.17. *Let E be a complex Banach space, $X = (B_{E^*}, w^*)$ and μ a complex Radon measure on X. Then:*

(a) $\mathsf{hom}\,\mu(A) = (2\pi)^{-1}\int_0^{2\pi} e^{-it}\mu(e^{-it}A)\,dt$ *for any Borel set $A \subset X$.*
(b) $\mathsf{hom}\,\mu$ *is an anti-homogeneous measure.*
(c) *μ is anti-homogeneous if and only if $\mu = \mathsf{hom}\,\mu$.*
(d) *Let $B \subset X$ be a homogeneous μ-measurable set and $f : B \to \mathbb{C}$ a bounded Borel function. Then $\int_B f\,d\,\mathsf{hom}\,\mu = \int_B \mathsf{hom}\,f\,d\mu$.*
(e) *If μ is a boundary measure, then so is $\mathsf{hom}\,\mu$.*

Proof. (a) Let λ denote the normalized Lebesgue measure on $[0, 2\pi]$ and $\lambda \times \mu$ the product measure on $[0, 2\pi] \times X$. Set $h(t, x) = e^{-it}$. Then h is a bounded continuous function on $[0, 2\pi] \times X$, hence we can set $\nu_1 = h \cdot (\lambda \times \mu)$,

a measure on $[0, 2\pi] \times X$ with density h with respect to $\lambda \times \mu$. Further, consider $\varphi : [0, 2\pi] \times X \to X$ defined by $\varphi(t, x) = e^{it}x$. It is clearly continuous, thus $\nu_2 = \varphi(\nu_1)$ is a well-defined Radon measure on X. Then for any $A \subset X$ Borel we have

$$\nu_2(A) = \nu_1(\varphi^{-1}(A)) = \nu_1(\{(t, x); \, e^{it}x \in A\})$$

$$= \nu_1(\{(t, e^{-it}x); \, t \in [0, 2\pi], \, x \in A\})$$

$$= \int\limits_{\{(t, e^{-it}x); \, t \in [0, 2\pi], x \in A\}} e^{-it} \, d(\lambda \times \mu) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-it} \mu(e^{-it}A) \, dt.$$

This shows that the formula in (a) defines the measure ν_2. To show $\nu_2 =$ $\mathsf{hom}\,\mu$ it is enough to prove that $\nu_2(f) = \mu(\mathsf{hom}\,f)$ for any $f \in \mathcal{C}(X, \mathbb{C})$. Indeed, fix $f \in \mathcal{C}(X, \mathbb{C})$. Then

$$\mu(\mathsf{hom}\,f) = \int\limits_X \left(\frac{1}{2\pi} \int\limits_0^{2\pi} e^{-it} f(e^{it}x) \, dt \right) d\mu$$

$$= \int\limits_{[0,2\pi] \times X} e^{-it} f(e^{it}x) \, d(\lambda \times \mu)(t, x) = \int\limits_{[0,2\pi] \times X} f(e^{it}x) \, d\nu_1(t, x)$$

$$= \int\limits_{[0,2\pi] \times X} f(\varphi(t, x)) \, d\nu_1(t, x) = \int\limits_X f \, d\nu_2.$$

Assertions (b) and (c) follow immediately from (a). So does (d), since the computation of $\mu(\mathsf{hom}\,f)$ can be repeated for any bounded Borel function f.

(e) For the proof see [12, Lemma 4.2] or [32, §23, Lemma 10]. ∎

We continue with the complex analogue of Lemma 3.16. The proof is again more involved than in the real case.

LEMMA 3.18. *Let E be a complex Banach space, $X = (B_{E^*}, w^*)$ and μ an anti-homogeneous Radon measure on X with $\|\mu\| \leq 1$. Then:*

(a) *$\operatorname{Re}\mu$ and $\operatorname{Im}\mu$ are odd measures.*
(b) *$\operatorname{Im}\mu(A) = \operatorname{Re}\mu(iA)$ for $A \subset X$ Borel.*
(c) *There is a unique $x^* \in X$ such that $x^*(x) = \int y^*(x) \, d\mu(y^*)$ for each $x \in E$.*
(d) *There is a Radon probability ν on X such that $\mathsf{hom}\,\nu = \mu$ and $r(\nu) = x^*$. If μ is a boundary measure, then ν can be chosen maximal.*

Proof. Assertions (a) and (b) are obvious. To show (c) it is enough to observe that $x \mapsto \int y^*(x) \, d\mu(y^*)$ defines a linear functional on E of norm at most $\|\mu\|$.

(d) If $\mu = 0$ take $\nu = \frac{1}{2}(\varepsilon_{y^*} + \varepsilon_{-y^*})$ where y^* is an extreme point of X.

Let $\mu \neq 0$. Then the absolute variation $|\mu|$ is invariant with respect to rotations, i.e., $|\mu|(\alpha A) = |\mu|(A)$ for any $A \subset X$ Borel and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$.

Let h_0 be a Borel function on X such that $\mu = h_0|\mu|$ (i.e., h_0 is the density of μ with respect to $|\mu|$). Then $|h_0| = 1$ $|\mu|$-almost everywhere. We may suppose that h_0 is Baire measurable. Indeed, by the Luzin theorem there is a sequence (f_n) of continuous functions converging $|\mu|$-almost everywhere to h_0. Then the function h_1 defined by

$$h_1(y^*) = \begin{cases} \lim f_n(y^*) & \text{if the limit exists,} \\ 0 & \text{otherwise,} \end{cases}$$

is a Baire measurable function which equals h_0 $|\mu|$-almost everywhere. Further, define

$$h(y^*) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-it} h_1(e^{-it}y^*)\, dt, \quad y^* \in X.$$

Then h is also Baire measurable (by Lemmata 3.3 and 3.13(c)). Moreover, $h(\alpha y^*) = \bar{\alpha} h(y^*)$ for any $y^* \in X$ and any complex unit α. Finally, $h = h_1$ $|\mu|$-almost everywhere since h is also a density of μ with respect to $|\mu|$. Indeed, for any Borel set $A \subset X$ we have

$$\mu(A) = \mathsf{hom}\, \mu(A) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-it} \mu(e^{-it}A)\, dt$$

$$= \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-it} \int\limits_{e^{-it}A} h_1(y^*)\, d|\mu|(y^*)\, dt$$

$$= \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-it} \int\limits_A h_1(e^{-it}y^*)\, d|\mu|(y^*)\, dt$$

$$= \int\limits_A \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-it} h_1(e^{-it}y^*)\, dt\, d|\mu|(y^*) = \int\limits_A h(y^*)\, d|\mu|(y^*).$$

Set $X_1 = \{y^* \in X;\, |h(y^*)| = 1\}$ and $X_0 = X \setminus X_1$. Then X_0 and X_1 are Baire subsets of X, $0 \in X_0$ (since $h(0) = 0$) and X_0 has $|\mu|$-measure zero. Further, set $P = \{y^* \in X;\, h(y^*) = 1\}$. Then P is a Baire subset of X and the mapping $\Phi : [0, 2\pi) \times P \to X_1$ defined by $\Phi(t, y^*) = e^{it}y^*$ is a continuous bijection. Since $Z = [0, 2\pi) \times P$ is a Baire subset of the compact space $[0, 2\pi] \times X$, Φ maps Baire sets in Z to Baire sets in X_1. (Indeed, let $A \subset Z$ be a Baire set. Then both A and $Z \setminus A$ are K-analytic, hence $\Phi(A)$ and $\Phi(Z \setminus A)$ are disjoint K-analytic subsets of X_1 covering X_1, hence they are Baire sets, by [20, Theorems 4.10 and 5.8].)

Therefore we can define $\nu_1 = \Phi^{-1}(|\mu|)$, a positive measure on the Baire σ-algebra of Z. Further, define $\nu_2 = \pi_2(\nu_1)$, where π_2 is the projection of $Z = [0, 2\pi) \times P$ onto the second coordinate. Then ν_2 is a positive measure on the Baire σ-algebra of P. Since P is a Baire subset of X, we can consider ν_2

to be defined on the Baire σ-algebra of X (and supported by P). Therefore ν_2 generates a positive functional on $\mathcal{C}(X, \mathbb{C})$. Since

$$\|\nu_2\| = \nu_2(1) = \nu_2(P) = \nu_1(Z) = |\mu|(X_1) = \|\mu\|,$$

we have $\|\nu_2\| \le 1$. Moreover, if $f \in \mathcal{C}(X, \mathbb{C})$ is homogeneous, then

$$\int_X f(y^*) \, d\nu_2(y^*) = \int_P f(y^*) \, d\nu_2(y^*) = \int_Z f(y^*) \, d\nu_1(t, y^*)$$
$$= \int_Z e^{-it} f(e^{it} y^*) \, d\nu_1(t, y^*) = \int_Z h(e^{it} y^*) f(e^{it} y^*) \, d\nu_1(t, y^*)$$
$$= \int_{X_1} h(z^*) f(z^*) \, d|\mu|(z^*) = \int_X h(z^*) f(z^*) \, d|\mu|(z^*)$$
$$= \int_X f(z^*) \, d\mu(z^*).$$

The first equality holds because ν_2 is supported by P, the second follows from the fact that $\nu_2 = \pi_2(\nu_1)$, and the third is valid since f is homogeneous. In the fourth one we have used the properties of h: $h(e^{it} y^*) = e^{-it} h(y^*) = e^{-it}$ for $y^* \in P$. The fifth inequality follows from the fact that $\nu_1 = \Phi^{-1}(|\mu|)$, in the sixth one we have used the fact that $|\mu|$ is supported by X_1, and the last one follows from the choice of h. Hence, for any $f \in \mathcal{C}(X, \mathbb{C})$ we have

$$\mathsf{hom}\, \nu_2(f) = \nu_2(\mathsf{hom}\, f) = \mu(\mathsf{hom}\, f) = \mu(f),$$

hence $\mathsf{hom}\, \nu_2 = \mu$.

Set $\nu = \nu_2 + \frac{1 - \|\nu_2\|}{\|\mu\|} |\mu|$. Then ν is a probability measure and $\mathsf{hom}\, \nu = \mu$ (since $\mathsf{hom}\, |\mu| = 0$). We continue by showing that ν represents x^*. Let f be an affine continuous function on X. By Lemma 3.14, there are homogeneous affine continuous functions $u, v : X \to \mathbb{C}$ such that $f = f(0) + u + \overline{v}$. Since u and v are given by evaluation at some points of E, we get

$$\nu(f) = \nu(f(0) + u + \overline{v}) = f(0) + \nu(u) + \overline{\nu(v)} = f(0) + u(x^*) + \overline{v(x^*)} = f(x^*).$$

It remains to show that ν is maximal whenever μ is boundary. Since ν is a probability and $|\mu|$ is a boundary measure, it is enough to check that ν_2 is boundary. We will do that by testing against convex continuous functions. For a real-valued continuous convex function f on X set

$$\hat{f}(y^*) = \sup\{\sigma(f); \, \sigma \in \mathcal{M}^1(X), \, r(\sigma) = y^*\}, \quad y^* \in X,$$

and

$$B(f) = \{y^* \in X; \, f(y^*) = \hat{f}(y^*)\}.$$

Then the boundary measures are exactly the measures supported by each $B(f)$ (see [1, pp. 34–35]). So, let f be any real-valued continuous convex

function on X. Set

$$g(y^*) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(e^{it}y^*)\,dt, \quad y^* \in X.$$

Then g is again a continuous convex function, and moreover $B(g) \subset B(f)$ by [12, Lemma 4.1]. Hence $|\mu|(X \setminus B(g)) = 0$. Since $X \setminus B(g)$ is homogeneous, by the definition of ν_2 we get $\nu_2(X \setminus B(g)) = |\mu|(X \setminus B(g)) = 0$. Hence ν_2 is supported by $B(g)$ and, a fortiori, by $B(f)$. ∎

Proof of Fact 1.1. The real case. By [32, §21, Theorem 7], E is an L_1-predual if and only if for any two maximal probability measures μ, ν on X with the same barycenter we have $\mathsf{odd}\,\mu = \mathsf{odd}\,\nu$.

Thus the sufficiency is clear: If μ and ν are two maximal probability measures with the same barycenter x^*, then $\mathsf{odd}\,\mu$ and $\mathsf{odd}\,\nu$ satisfy conditions (a)–(d) (by Lemma 3.15). So, by the uniqueness assumption we have $\mathsf{odd}\,\mu = \mathsf{odd}\,\nu$.

To show the necessity suppose that E is an L^1-predual. Let $x^* \in X$. Then there is a maximal probability ν representing x^*. Set $\mu = \mathsf{odd}\,\nu$. Then μ satisfies (a)–(d) by Lemma 3.15. To prove the uniqueness suppose that μ_1 is any measure satisfying (a)–(d). By Lemma 3.16 there is a maximal probability measure ν_1 with barycenter x^* such that $\mathsf{odd}\,\nu_1 = \mu_1$. Thus $\mu = \mu_1$.

The complex case is completely analogous. By [12, Theorem 4.3] or [32, §23, Theorem 5], E is an L_1-predual if and only if for any two maximal probability measures μ, ν on X with the same barycenter we have $\mathsf{hom}\,\mu = \mathsf{hom}\,\nu$. Thus the sufficiency is again clear and the necessity can be proved in the same way, only using Lemma 3.18. ∎

4. Affine maps of the first class and approximation properties. The aim of this section is to prove Theorem 2.2 and Example 2.3. To do so, we first make the following observation on strongly affine maps.

LEMMA 4.1. *Let $f : K \to F$ be a strongly affine mapping from a compact convex set K to a locally convex space F over $\mathbb{F}$. Then f is bounded.*

Proof. Let $\tau \in F^*$. By Fact 1.2 the mapping $\tau \circ f$ is strongly affine. By [1, Theorem I.2.6] (or [52, Section 14], [58] or [43, Lemma 4.5]), $\tau \circ f$ is bounded. Hence f is bounded by [22, Theorem 4, p. 151]. ∎

We continue by recalling definitions of some approximation properties of Banach spaces.

A Banach space E is said to have

- the *approximation property* if for every $\varepsilon > 0$ and every compact set $K \subset E$ there exists a finite-rank operator L on E such that $\sup_{x \in K} \|Lx - x\| \le \varepsilon$;

- the *compact approximation property* if for every $\varepsilon > 0$ and every compact set $K \subset E$ there is a compact operator L on E such that $\sup_{x \in K} \|Lx - x\| \le \varepsilon$.

In other words, E has the (compact) approximation property if and only if the identity operator on E is in the closure of finite-rank (compact) operators in the topology of uniform convergence on compact subsets of E. If the identity can be approximated by operators of the respective type with norm at most λ, then E is said to have the λ-*bounded approximation property* (resp. λ-*bounded compact approximation property*). Further, E is said to have the *bounded approximation property* if it has the λ-bounded approximation property for some $\lambda \ge 1$.

Now we give the proof of Example 2.3. It is a strengthening of [47, Example 2.22] which follows from [47, Proposition 2.21]. In [4] the authors prove that $f \notin \mathfrak{A}_1(X, E)$; we prove that it is not in any affine Baire class.

Proof of Example 2.3. Since E is reflexive, the unit ball B_E is weakly compact. By the Namioka theorem (see e.g. [14, Corollary 14.4]), the function f (which is the identity from $X = (B_E, w)$ to E equipped with the norm) has a dense (in fact residual) set of continuity points. The same is true for the restriction of f to any closed subset of X. Since E is separable, X is moreover metrizable, thus f is F_σ-measurable by [30, §31, X, Theorem 2]. This implies that $f \in \mathcal{C}_1(X, B_E)$ by Lemma 3.3(c).

Suppose that $f \in \bigcup_{\alpha < \omega_1} \mathfrak{A}_\alpha(X, E)$. Since f is odd [homogeneous], we get $f \in \bigcup_{\alpha < \omega_1} \mathfrak{A}_{\mathrm{odd}, \alpha}(X, E)$ by Lemma 3.12 [$f \in \bigcup_{\alpha < \omega_1} \mathfrak{A}_{\mathrm{hom}, \alpha}(X, E)$ by Lemma 3.13]. Let I denote the identity operator on E. Using Lemma 3.11 we then get $I \in \bigcup_{\alpha < \omega_1} (K(E))_\alpha$, where the notation $(K(E))_\alpha$ follows the pattern from Section 1.2. (Indeed, a linear operator $L : E \to E$ is weak-to-norm continuous on B_E if and only if it is compact.) To finish the argument it is enough to check that all the operators from $\bigcup_{\alpha < \omega_1} (K(E))_\alpha$ are bounded and are contained in the closure of $K(E)$ in the topology of uniform convergence on norm compact subsets of E. This will be done by transfinite induction. For $\alpha = 0$ it is clear. Let $\alpha > 0$ and suppose that the assertion is valid for each operator in $\bigcup_{\beta < \alpha} (K(E))_\beta$. Fix $L \in (K(E))_\alpha$. Then there is a sequence (L_n) in $\bigcup_{\beta < \alpha} (K(E))_\beta$ pointwise converging to L. By the induction hypothesis the operators L_n are bounded, hence the uniform boundedness principle shows that the sequence (L_n) is uniformly bounded. Thus L is bounded and $L_n \to L$ uniformly on compact sets. ∎

To prove Theorem 2.2 we need some lemmata. Let us first introduce some notation.

Let E, F be Banach spaces such that F has dimension $n \in \mathbb{N}$. Fix a basis $y_1, \ldots, y_n$ of F and denote by $y_1^*, \ldots, y_n^*$ the dual basis of F^*. For an n-tuple $\boldsymbol{x}^{**} = (x_1^{**}, \ldots, x_n^{**}) \in (E^{**})^n$ define operators $\Phi(\boldsymbol{x}^{**}) \in L(E^*, F)$

and $\Psi(\boldsymbol{x}^{**}) \in L(F^*, E^{**})$ by

$$\Phi(\boldsymbol{x}^{**})(x^*) = \sum_{i=1}^{n} x_i^{**}(x^*)y_i, \quad x^* \in E^*,$$

$$\Psi(\boldsymbol{x}^{**})(y^*) = \sum_{i=1}^{n} y^*(y_i)x_i^{**}, \quad x^* \in F^*.$$

It is clear that Φ is an isomorphism of $(E^{**})^n$ onto $L(E^*, F)$ and Ψ is an isomorphism of $(E^{**})^n$ onto $L(F^*, E^{**})$.

LEMMA 4.2. *Under the above notation:*

(i) $\|\Phi(\boldsymbol{x}^{**})\| = \|\Psi(\boldsymbol{x}^{**})\|$ *for each* $\boldsymbol{x}^{**} \in (E^{**})^n$.
(ii) Φ *restricted to* E^n *is an isomorphism of* E^n *onto the subspace of* $L(E^*, F)$ *formed by weak* continuous operators.*
(iii) Ψ *restricted to* E^n *is an isomorphism of* E^n *onto* $L(F^*, E)$.
(iv) *If we define on* E^n *a norm* $\|\cdot\|$ *by* $\|\boldsymbol{x}\| = \|\Phi(\boldsymbol{x})\|$, *then the bidual norm on* $(E^{**})^n$ *coming from the canonical duality of* E^n, $(E^*)^n$ *and* $(E^{**})^n$ *is given by* $\|\boldsymbol{x}^{**}\| = \|\Phi(\boldsymbol{x}^{**})\|$.

Proof. (i) Let $\boldsymbol{x}^{**} \in (E^{**})^n$. Then

$$\begin{aligned}\|\Phi(\boldsymbol{x}^{**})\| &= \sup\{\|\Phi(\boldsymbol{x}^{**})(x^*)\|_F;\ x^* \in B_{E^*}\}\\ &= \sup\{|y^*(\Phi(\boldsymbol{x}^{**})(x^*))|;\ x^* \in B_{E^*},\ y^* \in B_{F^*}\}\\ &= \sup\Big\{\Big|\sum_{i=1}^{n} x_i^{**}(x^*)y^*(y_i)\Big|;\ x^* \in B_{E^*},\ y^* \in B_{F^*}\Big\}\\ &= \sup\{|\Psi(\boldsymbol{x}^{**})(y^*)(x^*)|;\ x^* \in B_{E^*},\ y^* \in B_{F^*}\}\\ &= \sup\{\|\Psi(\boldsymbol{x}^{**})(y^*)\|_{E^{**}};\ y^* \in B_{F^*}\} = \|\Psi(\boldsymbol{x}^{**})\|.\end{aligned}$$

Assertion (ii) follows from the fact that the weak* continuous functionals on E^* are exactly those which come from E. Assertion (iii) is obvious.

(iv) Due to (i) it is enough to prove this assertion with Ψ instead of Φ. But this is proved for example in [8, Section 3]. ∎

LEMMA 4.3. *Let X be a compact convex set, F a finite-dimensional Banach space and $f : X \to F$ an affine function which belongs to $\mathcal{C}_1(X, F)$ and satisfies $\|f(x)\|_F \leq 1$ for each $x \in X$. Then $f \in \mathfrak{A}_1(X, B_F)$.*

Proof. Without loss of generality we may suppose that F is a real space. Since F has finite dimension, it easily follows from the Mokobodzki theorem that $f \in \mathfrak{A}_1(X, F)$. Let $E = \mathfrak{A}(X)$. For any $x \in X$ let $\varepsilon_x \in E^*$ be the corresponding evaluation functional. By [26, Lemma 2.2] there is a unique linear operator $L_f : E^* \to F$ such that $L_f(\varepsilon_x) = f(x)$ for each $x \in X$. It follows from [26, Lemma 2.3] that $\|L_f\| \leq 1$ and $L_f|_{B_{E^*}} \in \mathfrak{A}_1((B_{E^*}, w^*), F)$. The Banach–Dieudonné theorem easily implies that $L_f \in \mathfrak{A}_1((E^*, w^*), F)$.

Let H denote the subspace of $L(E^*, F)$ formed by the weak* continuous operators. By Lemma 4.2(iv) the bidual H^{**} can be canonically identified with $L(E^*, F)$. Hence, in this identification we have $L_f \in H^{**}$, and moreover L_f is the weak* limit of a sequence from H. Since $\|L_f\| \leq 1$, it follows from [49, Remark, p. 379] that L_f is the weak* limit of a sequence from B_H. But this means that $L_f \in \mathfrak{A}_1((E^*, w^*), B_F)$, hence $f \in \mathfrak{A}_1(X, B_F)$. ∎

Proof of Theorem 2.2. It is enough to prove the 'moreover' part since any affine mapping $f \in \mathcal{C}_1(X, E)$ is bounded by Theorem 2.1 and Lemma 4.1.

Suppose that E has the λ-bounded approximation property for some $\lambda \geq 1$. Let $f : X \to E$ be an affine mapping of the first Baire class such that $f(X) \subset B_E$. Observe that $f(X)$ is separable and fix a countable dense subset $C \subset f(X)$. The λ-bounded approximation property yields a sequence (L_n) of finite-rank operators pointwise converging to the identity on C such that $\|L_n\| \leq \lambda$ for each n. It follows that L_n converges to the identity also pointwise on $f(X)$. Then $L_n \circ f$ is also affine and of the first Baire class. Moreover, since the range has finite dimension and $(L_n \circ f)(X) \subset \lambda B_E$, Lemma 4.3 yields $L_n \circ f \in \mathfrak{A}_1(X, \lambda B_E)$. As $L_n \circ f \to f$ pointwise, we get $f \in \mathfrak{A}_2(X, \lambda B_E)$. Finally, Corollary 3.5 yields $f \in \mathfrak{A}_1(X, \lambda B_E)$. ∎

REMARK 4.4. Similar problems are investigated in [47]. In particular, Theorem 2.12 there is a weaker variant of our Theorem 2.2. However, the proof given in [47] contains a gap. The authors confuse a projection on a one-dimensional space with a coordinate functional. Therefore, it is not clear why the operators $T_{n,m}$ defined on p. 25 should be uniformly bounded. The correct estimate is $\|T_{n,m}\| \leq 2Kn\|T\|$, which is not a uniform bound. The same type of gap is in Remark 2.18.1 and in the proof of Theorem 2.19. These results can be repaired and improved using our Lemma 4.3.

5. The affine class of the dilation mapping. The aim of this section is to prove Theorem 2.4 and Remark 2.5. We start by proving the latter. First we show that the dilation mapping is always strongly affine.

LEMMA 5.1.

(S) *Let X be a simplex. Then the mapping $T : x \mapsto \delta_x$ is strongly affine.*

(R,C) *Let $X = (B_{E^*}, w^*)$ for an L^1-predual E. Then T is strongly affine.*

Proof. (S) It is clear that T is affine. (Indeed, if $x, y \in X$ and $t \in [0, 1]$, then $t\delta_x + (1 - t)\delta_y$ is a probability measure with barycenter $tx + (1 - t)y$. Since the maximal probabilities form a convex set (by [43, Theorem 3.70]), necessarily $t\delta_x + (1 - t)\delta_y = \delta_{tx+(1-t)y}$.) To prove that T is strongly affine, by Fact 1.2 it is enough to prove that $x \mapsto \delta_x(f)$ is strongly affine for each $f : X \to \mathbb{R}$ continuous. Since differences of convex continuous functions are

norm-dense in $\mathcal{C}(X, \mathbb{R})$ by the Stone–Weierstrass theorem, it is enough to prove the above for convex continuous functions. But if f is convex and continuous, then $x \mapsto \delta_x(f)$ is upper semicontinuous by [1, Theorem II.3.7], hence strongly affine by [43, Proposition A.122 and Theorem 4.21].

(R) The proof is completely analogous to case (S); we only need to use [32, Chapter 7, §21, Theorem 7] to show that Tf is a difference of two upper semicontinuous functions whenever f is continuous and convex.

(C) The assertion for complex L^1-preduals is proved in [51, Lemma 4.12]. ∎

LEMMA 5.2. *If* $\operatorname{ext} X$ *is closed, then* T *is continuous.*

Proof. (S) The mapping $r : \mathcal{M}^1(X) \to X$ which assigns to each $\mu \in \mathcal{M}^1(X)$ its barycenter $r(\mu)$ is clearly continuous (see, e.g., [43, Proposition 3.40]). Since $\operatorname{ext} X$ is compact, $\mathcal{M}^1(\operatorname{ext} X)$ is a compact subset of $\mathcal{M}^1(X)$. Moreover, the maximal probabilities are exactly those supported by $\operatorname{ext} X$ (see, e.g., [1, p. 35] or [43, Proposition 3.80]). By simpliciality the operator r restricted to $\mathcal{M}^1(\operatorname{ext} X)$ is one-to-one, hence it is a homeomorphism. The operator T is its inverse.

(R) The proof is similar. Let Y denote the set of all odd measures from $B_{\mathcal{M}(X, \mathbb{R})}$ and Y_b the set of all boundary measures from Y. For any $\mu \in Y$ let $u(\mu)$ denote the point of X provided by Lemma 3.16(b). It is clear that the operator u is continuous and, by Fact 1.1, the restriction of u to Y_b is one-to-one. The operator T is the inverse of $u|_{Y_b}$. To prove that T is continuous it is enough to check that Y_b is compact. But Y_b is the image of the set of all maximal probabilities on X under the continuous operator odd (by Lemma 3.16(c)) and the maximal probabilities on X are exactly the probabilities from the compact set $\mathcal{M}^1(\operatorname{ext} X)$ (by [43, Proposition 3.80]).

(C) The complex case is completely analogous to the real one. We just consider anti-homogeneous measures instead of odd ones and use Lemma 3.18 instead of Lemma 3.16. ∎

Now we proceed with the proof of Theorem 2.4. It is based on the following selection result, which can be viewed as an affine version of the Michael selection theorem. We recall that a set-valued mapping $\Phi : X \to F$ is said to be *lower semicontinuous* if $\{x \in X; \, \Phi(x) \cap U \neq \emptyset\}$ is open in X for any open set $U \subset F$.

THEOREM 5.3. *Let* X *be a compact convex set,* F *a Fréchet space and* $\Phi : X \to F$ *a lower semicontinuous set-valued mapping with nonempty closed values.*

(S) *If* X *is a simplex and the graph of* Φ *is convex, then* Φ *admits a continuous affine selection.*

(R) *If $X = (B_{E^*}, w^*)$, where E is a real L_1-predual and the graph of Φ is convex and symmetric, then Φ admits an odd continuous affine selection.*

(C) *If $X = (B_{E^*}, w^*)$, where E is a complex L_1-predual, F is complex and the graph of Φ is absolutely convex, then Φ admits a homogeneous continuous affine selection.*

Case (S) is due to [35, Theorem 3.1] (see also [43, Theorem 11.6]), case (R) is proved in [36, Theorem 2.2] or [32, Chapter 7, §22, Theorem 2], and case (C) in [50, Theorem 4.2]. More precisely, the assumptions of the cited results are formulated in a slightly different way. Instead of assuming that the graph of Φ is convex, it is assumed that Φ is affine (sometimes called convex), i.e., $\lambda\Phi(x_1) + (1 - \lambda)\Phi(x_2) \subset \Phi(\lambda x_1 + (1 - \lambda)x_2)$ whenever $x_1, x_2 \in X$ and $\lambda \in [0, 1]$. But these two assumptions are indeed equivalent.

Proof of Theorem 2.4. (S) This case is essentially due to [42, Theorem 6.6] (or [43, Theorem 11.26]). However, the formulation of those results is weaker and therefore we give a complete proof. Since X is metrizable, $\mathfrak{A}(X, \mathbb{R})$ is separable, hence we can choose a countable set $\{e_n; n \in \mathbb{N}\}$ dense in $\mathfrak{A}(X, \mathbb{R})$.

Fix $n \in \mathbb{N}$. Define a multivalued mapping $\Gamma_n : X \to \mathcal{M}^1(X)$ by

$$\Gamma_n(x) = \{\mu \in \mathcal{M}^1(X); |\mu(e_i) - e_i(x)| < 1/n, \, i = 1, \ldots, n\}, \quad x \in X.$$

We claim that Γ_n is a nonempty-valued lower semicontinuous mapping with convex graph. First, $\Gamma_n(x) \neq \emptyset$ as $\varepsilon_x \in \Gamma_n(x)$. Further, the graph of Γ_n is

$$\{(x, \mu) \in X \times \mathcal{M}^1(X); |\mu(e_i) - e_i(x)| < 1/n, \, i = 1, \ldots, n\},$$

hence it is convex.

We continue by showing that Γ_n is lower semicontinuous. Let $V \subset \mathcal{M}^1(X)$ be a nonempty open set and suppose $\Gamma_n(x) \cap V \neq \emptyset$ for some $x \in X$. We select $\mu \in V$ satisfying $|\mu(e_i) - e_i(x)| < 1/n$ for each $i = 1, \ldots, n$. By the continuity of e_i, $i = 1, \ldots, n$, there exists an open neighborhood U of x satisfying

$$|\mu(e_i) - e_i(y)| < 1/n, \quad y \in U, \, i = 1, \ldots, n.$$

Hence $\mu \in \Gamma_n(y) \cap V$ for $y \in U$, and thus the set $\{x \in X; \Gamma_n(x) \cap V \neq \emptyset\}$ is open. This implies that Γ_n is lower semicontinuous.

Next we modify Γ_n to have closed values. To this end we define $\overline{\Gamma}_n : X \to \mathcal{M}^1(X)$ by

$$\overline{\Gamma}_n(x) = \overline{\Gamma_n(x)}, \quad x \in X.$$

Since, for an open set $V \subset \mathcal{M}^1(X)$ and $x \in X$, $\overline{\Gamma}_n(x)$ intersects V if and

only if $\Gamma_n(x)$ intersects V, $\overline{\Gamma}_n$ is also lower semicontinuous. Moreover, it is clear that the graph of $\overline{\Gamma}_n$ convex.

Now we want to use the selection result of Theorem 5.3. This is possible since $\mathcal{M}^1(X)$, being a compact convex metrizable set, is affinely homeomorphic to a subset of ℓ_2. Hence there exists a continuous affine selection T_n from $\overline{\Gamma}_n$.

We continue by showing that $T_n \to T$ on $\operatorname{ext} X$. Fix $x \in \operatorname{ext} X$. Let μ be any cluster point of the sequence $(T_n(x))$ in $\mathcal{M}^1(X)$. By the very definition of Γ_n and $\overline{\Gamma}_n$, $\mu(e_i) = e_i(x)$ for $i \in \mathbb{N}$. By the density of (e_i) in $\mathfrak{A}(X, \mathbb{R})$ we infer that $\mu(e) = e(x)$ for all $e \in \mathfrak{A}(X, \mathbb{R})$, i.e., $r(\mu) = x$. Since x is an extreme point of X, necessarily $\mu = \varepsilon_x = \delta_x$. Thus $T_n(x) \to \delta_x$.

Finally, we will prove that $T_n \to T$ on X. Fix $x \in X$ and $f \in C(X, \mathbb{R})$. Then

$$\delta_x(f) = \int_X \delta_y(f)\, d\delta_x(y) = \lim_{n\to\infty} \int_X T_n(y)(f)\, d\delta_x(y) = \lim_{n\to\infty} T_n(x)(f).$$

The first equality follows from the strong affinity of T (see Lemma 5.1). To verify the second, we use the fact that δ_x is maximal, hence supported by $\operatorname{ext} X$, the already proved convergence $T_n \to T$ on $\operatorname{ext} X$, and Theorem 3.10. The last equality uses the fact that T_n is affine and continuous.

(R) The construction of the sequence (T_n) is analogous; we indicate the differences. First, $\{e_n;\, n \in \mathbb{N}\}$ is a dense subset of E. Further, $\mathcal{M}^1(X)$ is everywhere replaced by $Y = \{\mu \in \mathcal{M}(X);\, \mu \text{ is odd and } \|\mu\| \leq 1\}$, a compact convex symmetric set. The mappings Γ_n and $\overline{\Gamma}_n$ are defined in the same way. They are lower semicontinuous for the same reason, and their graphs are convex and symmetric. Using Theorem 5.3 we obtain a continuous odd affine selection T_n from $\overline{\Gamma}_n$.

We continue by showing that $T_n \to T$ on $\operatorname{ext} X$. Fix $x \in \operatorname{ext} X$. Let μ be any cluster point of $(T_n(x))$ in Y. By the definition of Γ_n and $\overline{\Gamma}_n$, $\mu(e_i) = x(e_i)$ for $i \in \mathbb{N}$. By density of (e_i) in E we infer that $\mu(e) = x(e)$ for all $e \in E$. It follows from Lemma 3.16 that there is a probability measure ν representing x such that $\operatorname{odd} \nu = \mu$. Since x is an extreme point of X, necessarily $\nu = \varepsilon_x$ and ν is maximal, thus $\mu = T(x)$. It follows that $T_n(x) \to T(x)$.

Finally, we will prove that $T_n \to T$ on X. Fix $x \in X$ and $f \in C(X, \mathbb{R})$. Let σ be a maximal probability representing x. Then

$$T(x)(f) = \int_X T(y)(f)\, d\sigma(y) = \lim_{n\to\infty} \int_X T_n(y)(f)\, d\sigma(y) = \lim_{n\to\infty} T_n(x)(f).$$

The first equality follows by the strong affinity of T (see Lemma 5.1). To verify the second, we use the fact that σ is maximal, hence supported by $\operatorname{ext} X$, the already proved fact that $T_n \to T$ on $\operatorname{ext} X$, and Theorem 3.10. The last equality uses the fact that T_n is affine and continuous.

(C) The proof in the complex case is completely analogous. Instead of the set of odd measures we consider the set of anti-homogeneous measures. If we define Γ_n and $\overline{\Gamma}_n$ by the same formula, their graphs are clearly absolutely convex and we can find a homogeneous affine continuous selection T_n of $\overline{\Gamma}_n$. The proof that $T_n \to T$ is analogous; we just use Lemma 3.18. ∎

We remark that the metrizability assumption was used in the previous proof in an essential way. First, we used the existence of a countable dense set in $\mathfrak{A}(X)$ (or in E), and secondly, we applied the selection Theorem 5.3 which works for mappings with values in a Fréchet space, hence we need the metrizability of the relevant set of measures. However, as we know by Lemma 5.2, if ext X is moreover closed, then T is even continuous and no metrizability assumption is needed. So, it is natural to ask how far one can go in this direction. It follows from Lemma 6.1(iii) below that if T is of class $\mathfrak{A}_1$, then necessarily Tf is Baire-one for any scalar continuous function f on X. This is the case if ext X is Lindelöf (by [25, 41, 38]) but not only then (see [27, Theorem 4]). The following example shows, in particular, that the Lindelöf property of ext X is not sufficient for T to be of class $\mathfrak{A}_1$.

EXAMPLE 5.4. *There are simplices X_1 and X_2 with the following properties:*

(a) *ext X_1 is Lindelöf and ext X_2 is an uncountable discrete set.*
(b) *The function $x \mapsto \delta_x(f)$ is Baire-one for $f : X_i \to \mathbb{R}$ continuous ($i = 1, 2$).*
(c) *The mapping $T : x \mapsto \delta_x$ is not in $\bigcup_{\alpha < \omega_1} \mathcal{C}_\alpha(X_i, \mathcal{M}^1(X_i))$ ($i = 1, 2$).*

Proof. We will use the well-known construction of 'porcupine simplices', used for example in [27]. Let $A \subset [0, 1]$ be an uncountable set. Let

$$K = ([0, 1] \times \{0\}) \cup (A \times \{-1, 1\})$$

be equipped with the following topology. The points of $A \times \{-1, 1\}$ are isolated, and a basis of neighborhoods of a point $(x, 0) \in [0, 1] \times \{0\}$ is formed by the sets of the form

$$(U \times \{-1, 0, 1\}) \cap K \setminus \{(x, 1), (x, -1)\},$$

where U is a standard neighborhood of x in $[0, 1]$. Then K is a compact space. Let

$$\mathcal{A} = \left\{ f \in \mathcal{C}(K, \mathbb{R}); \, f(x, 0) = \tfrac{1}{2}(f(x, -1) + f(x, 1)) \text{ for each } x \in A \right\},$$

and let $X = \{\xi \in \mathcal{A}^*; \, \|\xi\| = 1 \, \& \, \xi(1) = 1\}$ be equipped with the weak* topology. Then X is a simplex. Moreover, K canonically homeomorphically embeds into X (as evaluation mappings); in this way ext X is identified with $((K \setminus A) \times \{0\}) \cup (A \times \{-1, 1\})$.

Assertion (b) is valid for any such X by [27, Theorem 1]. If we take $A = [0, 1]$, then $\operatorname{ext} X$ is uncountable discrete (see [27, Theorem 4]); and there is an uncountable A such that $\operatorname{ext} X$ is Lindelöf (by [27, Theorem 2], A containing no uncountable compact subset is enough, cf. [27, p. 69]).

Finally, we will prove that (c) is valid for any X of the described form. We consider K canonically embedded in X. For any $a \in A$ set $f_a = \chi_{\{(a,1)\}} - \chi_{\{(a,-1)\}}$. Then $f_a \in \mathcal{A}$. Define

$$U = \{\xi \in X; \exists a \in A : \xi(f_a) > 1/2\}, \quad H = \{\xi \in X; \forall a \in A : \xi(f_a) \geq -1/2\}.$$

Then U is open and H is closed. Moreover, clearly $A \times \{1\} \subset U$ and $H \cap (A \times \{-1\}) = \emptyset$. Further, $U \subset H$. Indeed, pick any $\xi \in U$ and fix $a \in A$ such that $\xi(f_a) > 1/2$. Fix any $b \in A$. We will show that $\xi(f_b) \geq -1/2$. This is obvious if $b = a$. So, suppose $b \neq a$. Since the function $f_a - f_b$ belongs to $\mathcal{A}$ and has norm one, we have

$$1 \geq \xi(f_a - f_b) = \xi(f_a) - \xi(f_b) > 1/2 - \xi(f_b),$$

and the inequality follows.

Now consider the system of mappings

$$\mathcal{F} = \big\{S \colon X \to \mathcal{M}^1(X); \{a \in A; S(a,-1)(H) < S(a,0)(U)\} \text{ is countable}\big\}.$$

First observe that $\mathcal{F}$ contains all continuous mappings. Indeed, suppose that S is continuous. Fix any $q \in \mathbb{Q}$ and set

$$M_q = \{a \in A; S(a,-1)(H) < q < S(a,0)(U)\}.$$

Given $a \in M_q$, we have $S(a,0)(U) > q$. Since U is open, the mapping $\mu \mapsto \mu(U)$ is lower semicontinuous on $\mathcal{M}^1(X)$, hence $\{\mu \in \mathcal{M}^1(X); \mu(U) > q\}$ is open. Therefore there is a neighborhood W of a in $[0, 1]$ such that for any $x \in (W \times \{-1, 0, 1\}) \cap K \setminus \{(a,-1), (a,1)\}$ we have $S(x)(U) > q$. In particular, $W \cap M_q = \{a\}$ (recall that $H \supset U$ and so $S(b,-1)(H) > q$ for $b \in W \setminus \{a\}$). It follows that each point of M_q is isolated, hence M_q is countable. Therefore $\bigcup_{q \in \mathbb{Q}} M_q$ is countable as well, hence $S \in \mathcal{F}$.

Further observe that $\mathcal{F}$ is closed with respect to pointwise limits of sequences. Indeed, let (S_n) be a sequence in $\mathcal{F}$ pointwise converging to a mapping S. By the definition of $\mathcal{F}$ there is a countable set $C \subset A$ such that $S_n(a,-1)(H) \geq S_n(a,0)(U)$ for all $n \in \mathbb{N}$ and $a \in A \setminus C$.

We will show that $S(a,-1)(H) \geq S(a,0)(U)$ for $a \in A \setminus C$ as well. So, fix $a \in A \setminus C$ and suppose that $S(a,-1)(H) < S(a,0)(U)$. Fix a number q such that $S(a,-1)(H) < q < S(a,0)(U)$. Since $S_n(a,0) \to S(a,0)$ and U is open, we can find $n_0 \in \mathbb{N}$ such that $S_n(a,0)(U) > q$ for each $n \geq n_0$. Since $a \in A \setminus C$, we get $S_n(a,-1)(H) \geq S_n(a,0) > q$ for $n \geq n_0$. Since H is closed and $S_n(a,-1) \to S(a,-1)$, we get $S(a,-1)(H) \geq q$, a contradiction.

We conclude that $\bigcup_{\alpha < \omega_1} \mathcal{C}_\alpha(X, \mathcal{M}^1(X)) \subset \mathcal{F}$. Finally, the mapping T does not belong to $\mathcal{F}$, since for any $a \in A$ we have

$$T(a, 0)(U) = \delta_{(a,0)}(U) = \tfrac{1}{2}(\varepsilon_{(a,1)}(U) + \varepsilon_{(a,-1)}(U)) = 1/2,$$
$$T(a, -1)(H) = \delta_{(a,-1)}(H) = \varepsilon_{(a,-1)}(H) = 0. \ \blacksquare$$

6. Strongly affine Baire mappings. The aim of this section is to prove Theorem 2.6. The proof will be in two steps. First, we handle the case of X metrizable with the use of Theorem 2.4. Secondly, we reduce the general case to the metrizable case.

In the proof of the metrizable case we will need the following notation. Let X be a compact convex set, F a Fréchet space over $\mathbb{F}$, $U : X \to B_{\mathcal{M}(X,\mathbb{F})}$ a mapping and $f : X \to F$ a bounded Baire mapping. Then we define $Uf : X \to F$ by

$$Uf(x) = \int_X f \, dU(x), \quad x \in X.$$

The mapping Uf is well defined due to Lemma 3.9.

Lemma 6.1. *In the above notation:*

 (i) *If U and f are continuous, then so is Uf.*
 (ii) *If U is strongly affine, then so is Uf.*
 (iii) *If $U \in \mathfrak{A}_\alpha(X, B_{\mathcal{M}(X,\mathbb{F})})$ and $f \in \mathcal{C}_\beta(X, F)$ is a bounded mapping, then $Uf \in \mathfrak{A}_{\alpha+\beta}(X, \overline{\mathrm{aco}}\, f(X))$.*

Proof. (i) Since f is continuous, $f(X)$ is a compact subset of F. Thus $L = \overline{\mathrm{aco}}\, f(X)$ is an absolutely convex compact subset of F (see, e.g., [22, Proposition 6.7.2]). By Lemma 3.8(c), $Uf(X) \subset L$. We need to show that Uf is continuous. To this end, let $\tau \in F^*$. Then

$$\tau(Uf(x)) = \tau\Big(\int_X f \, dU(x)\Big) = \int_X \tau \circ f \, dU(x) = U(x)(\tau \circ f), \quad x \in X.$$

Since U is continuous and $\tau \circ f \in \mathcal{C}(X, \mathbb{F})$, the mapping $x \mapsto U(x)(\tau \circ f)$ is continuous on X. Thus $Uf : X \to (L, \mathrm{weak})$ is continuous. Since L is compact, the original topology of F coincides on L with the weak topology. Hence $Uf \in \mathcal{C}(X, L)$.

(ii) First suppose that f is continuous. Let μ be any Radon probability on X. Then for any $\tau \in F^*$ we have

$$\int_X \tau(Uf(x)) \, d\mu(x) = \int_X \tau\Big(\int_X f \, dU(x)\Big) d\mu(x) = \int_X \int_X \tau \circ f \, dU(x) \, d\mu(x)$$
$$= \int_X U(x)(\tau \circ f) \, d\mu(x) = U(r(\mu))(\tau \circ f)$$
$$= \int_X \tau \circ f \, dU(r(\mu)) = \tau(Uf(r(\mu))).$$

Thus $\tau \circ Uf$ is strongly affine for each $\tau \in F^*$, so Uf is strongly affine by Fact 1.2.

The general case follows by transfinite induction on class using Corollary 3.2 and the following observation: If (f_n) is a bounded sequence of Baire mappings pointwise converging to a mapping f, then for each $x \in X$ and each $\tau \in F^*$ we have

$$\tau(Uf_n(x)) = \int_X \tau \circ f_n \, dU(x) \to \int_X \tau \circ f \, dU(x) = \tau(Uf(x)).$$

(iii) Set $L = \overline{\mathrm{aco}}\, f(X)$. By Lemma 3.8(c) we have $Uf(X) \subset L$.

Assume first that $\beta = 0$, i.e., f is continuous. The case $\alpha = 0$ follows from (i) and (ii). We continue by transfinite induction. It is enough to show that $U_n \to U$ pointwise on X implies $U_n f \to Uf$ pointwise on X. So, suppose that $U_n \to U$ pointwise on X. Fix $x \in X$. Then $U_n(x) \to U(x)$. For any $\tau \in F^*$ we have

$$\tau(U_n f(x)) = U_n(x)(\tau \circ f) \to U(x)(\tau \circ f) = \tau(Uf(x)).$$

Thus $U_n f(x) \to Uf(x)$ weakly in F. But since the sequence is contained in the compact set L, we deduce that $U_n f(x) \to Uf(x)$ in F. This completes the proof for $\beta = 0$.

Suppose now that $\gamma > 0$ is such that the assertion is valid for any $\beta < \gamma$. Let $f \in \mathcal{C}_\gamma(X, F)$ be bounded. Then $L = \overline{\mathrm{aco}}(F)$ is bounded and $f \in \mathcal{C}_\gamma(X, L)$ due to Corollary 3.2. So, fix a sequence (f_n) in $\bigcup_{\beta < \gamma} \mathcal{C}_\beta(X, L)$ pointwise converging to f. Then $Uf_n \in \bigcup_{\beta < \gamma} \mathfrak{A}_{\alpha+\beta}(X, L)$ by the induction hypothesis. Further, $Uf_n \to Uf$ pointwise by Theorem 3.10, hence $Uf \in \mathfrak{A}_{\alpha+\gamma}(X, L)$. ∎

Now we are ready to complete the first step:

Proof of Theorem 2.6 for X metrizable. (S) Let f be strongly affine and $f \in \mathcal{C}_\alpha(X, F)$. Then $f(x) = \delta_x(f)$ for each $x \in X$. Since the mapping $x \mapsto \delta_x$ belongs to $\mathfrak{A}_1(X, \mathcal{M}_1(X))$ by Theorem 2.4, from Lemma 6.1(iii) we conclude that $f \in \mathfrak{A}_{1+\alpha}(X, F)$.

(R) If f is odd and strongly affine, then $f = Tf$. Since T belongs to $\mathfrak{A}_1(X, B_{\mathcal{M}_{\mathrm{odd}}(X, \mathbb{R})})$ by Theorem 2.4, we conclude by using Lemmata 6.1(iii) and 3.12(c). If f is not odd, then $f = f(0) + (f - f(0))$. Since $f - f(0)$ is odd, we get $f - f(0) \in \mathfrak{A}_{\mathrm{odd},1+\alpha}(X, F)$, thus $f \in \mathfrak{A}_{1+\alpha}(X, F)$.

(C) If F is complex and f is homogeneous and strongly affine, then $f = Tf$. Since $T \in \mathfrak{A}_1(X, B_{\mathcal{M}_{\mathrm{ahom}}(X, \mathbb{C})})$ by Theorem 2.4, we conclude by applying Lemmata 6.1(iii) and 3.14(b). If F is complex and f is not homogeneous, then we can write

$$f(x) = f(0) + u(x) + \overline{v(x)}, \quad x \in X,$$

where u and v are homogeneous and strongly affine, and moreover $u = \mathsf{hom}\, f$

(see Lemma 3.14(a)). Then $u, v \in \mathcal{C}_\alpha(X, F)$ by Lemma 3.13(c), hence $u, v \in \mathfrak{A}_{\mathrm{hom},1+\alpha}(X, F)$. It follows that $f \in \mathfrak{A}_{1+\alpha}(X, F)$. Finally, if F is a real Fréchet space, denote by $F_{\mathbb{C}}$ its complexification. Then F is a real-linear subspace of $F_{\mathbb{C}}$, thus $f \in \mathcal{C}_\alpha(X, F_{\mathbb{C}})$, so $f \in \mathfrak{A}_{1+\alpha}(X, F_{\mathbb{C}})$. Since the canonical projection of $F_{\mathbb{C}}$ onto F is continuous and real-linear (hence affine), it is clear that $f \in \mathfrak{A}_{1+\alpha}(X, F)$.

Finally, suppose $\alpha = 1$. In all the cases we deduce as above by using Lemma 6.1(iii) that $f \in \mathfrak{A}_2(X, \overline{\mathrm{aco}\, f(X)})$. Hence, Corollary 3.5 yields $f \in \mathfrak{A}_1(X, \overline{\mathrm{aco}\, f(X)})$. ∎

To prove the general statement we need a reduction to the metrizable case. The first step is the following lemma.

LEMMA 6.2. *Let E be a Banach space, $X \subset B_{E^*}$ a weak* compact convex set and L a convex subset of a Fréchet space F. Let $f : X \to L$ be a mapping of type $\mathcal{C}_\alpha(X, L)$ for some $\alpha \in [0, \omega_1)$. Then there exist a closed separable subspace $E_1 \subset E$ and $g : \pi(X) \to L$ such that $g \in \mathcal{C}_\alpha(\pi(X), L)$ and $f = g \circ \pi$. (Here $\pi \colon E^* \to E_1^*$ denotes the restriction mapping.)*

Proof. Assume first that $\alpha = 0$, i.e., f is continuous. Then $f(X)$ is a compact subset of F, hence a compact metrizable space. It follows that there is a homeomorphic injection $p : f(X) \to \mathbb{R}^{\mathbb{N}}$. Further, since X is equipped with the weak* topology, it can be canonically embedded into the Cartesian product $\mathbb{F}^E$. The continuous mapping $p \circ f : X \to \mathbb{R}^{\mathbb{N}}$ can be extended to a continuous mapping $g : \mathbb{F}^E \to \mathbb{R}^{\mathbb{N}}$ (by [13, Theorem 3.1.7]). Further, by [56, Theorem 4] (see also [13, Problem 2.7.12(d)]) there is a countable set $C \subset E$ such that
$$u, v \in \mathbb{F}^E, u|_C = v|_C \ \Rightarrow \ g(u) = g(v).$$
Hence we can take E_1 to be the closed linear span of C. Then E_1 is a closed separable subspace of E; moreover, $\pi(x_1^*) = \pi(x_2^*)$ for some $x_1^*, x_2^* \in X$ implies $(p \circ f)(x_1^*) = (p \circ f)(x_2^*)$, and so $f(x_1^*) = f(x_2^*)$. Thus there is a mapping $g : \pi(X) \to L$ with $f = g \circ \pi$. Since f is continuous and π is a closed continuous mapping, g is continuous.

Assume now that $\alpha \in (0, \omega_1)$ and $f : X \to L$ is of type $\mathcal{C}_\alpha(X, L)$. We select a countable family $\mathcal{F} = \{f_n; \, n \in \mathbb{N}\}$ in $\mathcal{C}(X, L)$ such that $f \in \mathcal{F}_\alpha$. For each $n \in \mathbb{N}$ we find using the previous step a countable set $C_n \subset E$ such that for any $x_1^*, x_2^* \in X$ we have
$$x_1^*|_{C_n} = x_2^*|_{C_n} \ \Rightarrow \ f_n(x_1^*) = f_n(x_2^*).$$
Let E_1 be the closed linear span of $\bigcup_n C_n$. Then there are mappings $g : \pi(X) \to L$ and $g_n : \pi(X) \to L$ for $n \in \mathbb{N}$ such that $f = g \circ \pi$ and $f_n = g_n \circ \pi$. As above, the g_n are continuous, and it is easy to check by transfinite induction that $g \in (\{g_n : n \in \mathbb{N}\})_\alpha$, thus $g \in \mathcal{C}_\alpha(\pi(X), L)$. ∎

The next step is the following lemma on cofinality.

LEMMA 6.3.

(S) *Let X be a simplex and $f : X \to X_1$ an affine continuous sur-
jection of X onto a metrizable compact convex set. Then there
is a metrizable simplex X_2 and affine continuous surjections $f_1 :
X_2 \to X_1$ and $f_2 : X \to X_2$ such that $f = f_1 \circ f_2$.*

(R,C) *Let E be an L_1-predual over $\mathbb{F}$ and let E_1 be its separable sub-
space. Then there exists a separable L_1-predual E_2 satisfying
$E_1 \subset E_2 \subset E$.*

Proof. (S) In this proof we will denote by $\mathfrak{A}(X)$ the space $\mathfrak{A}(X, \mathbb{R})$, and
similarly for other compact convex sets. Let $f^* : \mathfrak{A}(X_1) \to \mathfrak{A}(X)$ denote the
canonical isometric embedding defined by $f^*(u) = u \circ f$ for $u \in \mathfrak{A}(X_1)$. Let E
be a (for a while arbitrary) closed subspace of $\mathfrak{A}(X)$ containing $f^*(\mathfrak{A}(X_1))$.
Denote by $\pi_2 : \mathfrak{A}(X)^* \to E^*$ the canonical restriction map. Define $\pi_1 :
E^* \to \mathfrak{A}(X_1)^*$ by $\pi_1(x^*)(u) = x^*(f^*(u))$, $x^* \in E^*$, $u \in \mathfrak{A}(X_1)$. Further, let
$\kappa_X : X \to \mathfrak{A}(X)^*$ be the canonical evaluation mapping, i.e., $\kappa_X(x)(u) = u(x)$
for $x \in X$ and $u \in \mathfrak{A}(X)$. Similarly, κ_{X_1} denotes the analogous mapping
for X_1. Set $X_2 = \pi_2(\kappa_X(X))$, $f_2 = \pi_2 \circ \kappa_X$ and $f_1 = (\kappa_{X_1})^{-1} \circ \pi_1|_{X_2}$. Then
X_2 is a compact convex set, and $f_1 : X_2 \to X_1$ and $f_2 : X \to X_2$ are affine
continuous surjections satisfying $f = f_1 \circ f_2$. So, to complete the proof it is
enough to choose E in such a way that X_2 is a metrizable simplex.

Observe that E is canonically isometric to $\mathfrak{A}(X_2)$. More precisely, if we
consider the isometric embedding $f_2^* : \mathfrak{A}(X_2) \to \mathfrak{A}(X)$ defined by $f_2^*(u) =
u \circ f_2$ for $u \in \mathfrak{A}(X_2)$, then $E = f_2^*(\mathfrak{A}(X_2))$. Indeed, if $v \in E$, we define
$u \in \mathfrak{A}(X_2)$ by $u(x^*) = x^*(v)$ for $x^* \in X_2$. Then for each $x \in X$ we have

$$f_2^*(u)(x) = u(f_2(x)) = f_2(x)(v) = \pi_2(\kappa_X(x))(v) = \kappa_X(x)(v) = v(x),$$

thus $v \in f_2^*(\mathfrak{A}(X_2))$. Conversely, let $u \in \mathfrak{A}(X_2)$. Then $f_2^*(u) = u \circ f_2 \in \mathfrak{A}(X)$.
If $u \circ f_2 \notin E$, by the Hahn–Banach separation theorem there is $x^* \in \mathfrak{A}(X)^*$
such that $x^*|_E = 0$ and $x^*(u \circ f_2) \neq 0$. By [43, Proposition 4.31(a,b)] we have
$x^* = c_1 \kappa_X(x_1) - c_2 \kappa_X(x_2)$ for some $x_1, x_2 \in X$ and c_1, c_2 nonnegative real
numbers. Since E contains the constant functions, we get $c_1 = c_2$. Hence
without loss of generality $c_1 = c_2 = 1$. Then $v(x_1) = v(x_2)$ for each $v \in E$.
It follows that $\pi_2 \circ \kappa_X(x_1) = \pi_2 \circ \kappa_X(x_2)$, thus $f_2(x_1) = f_2(x_2)$. So,

$$x^*(u \circ f_2) = (u \circ f_2)(x_1) - (u \circ f_2)(x_2) = 0,$$

a contradiction.

Hence, X_2 is metrizable provided E is separable, and X_2 is a simplex
provided E satisfies the weak Riesz interpolation property (see [1, Corollary
II.3.11]). Recall that E has the *weak Riesz interpolation property* if, whenever
$u_1, u_2, v_1, v_2 \in E$ are such that $\max\{u_1, u_2\} < \min\{v_1, v_2\}$, then there is
$w \in E$ with $\max\{u_1, u_2\} < w < \min\{v_1, v_2\}$. The inequalities are considered

pointwise. Further, it is clear that it is enough to check this property for u_1, u_2, v_1, v_2 in a norm-dense subset of E.

Thus we can construct E by a standard inductive procedure. We construct countable sets $A_0 \subset B_1 \subset A_1 \subset B_2 \subset \cdots \subset \mathfrak{A}(X)$ as follows:

- A_0 is a countable dense subset of $f^*(\mathfrak{A}(X_1))$.
- If $u_1, u_2, v_1, v_2 \in A_{n-1}$ are such that $\max\{u_1, u_2\} < \min\{v_1, v_2\}$, then there is $w \in B_n$ with $\max\{u_1, u_2\} < w < \min\{v_1, v_2\}$.
- A_n is the $\mathbb{Q}$-linear span of B_n.

Then we can set $E = \overline{\bigcup_n A_n}$.

(R,C) This is proved in [32, §23, Lemma 1]. ∎

The final ingredient is the following lemma.

LEMMA 6.4. *Let $\pi \colon K \to L$ be an affine continuous surjection of a compact convex set K onto a compact convex set L. Let $g : L \to F$ be a universally measurable mapping from L to a Fréchet space F. Then g is strongly affine if and only if $g \circ \pi$ is strongly affine.*

Proof. For $F = \mathbb{F}$ this is proved in [43, Proposition 5.29]. The vector-valued case then follows immediately from Fact 1.2. ∎

Now we can complete the proof:

Proof of Theorem 2.6 in the general case. (S) Let $f \in \mathcal{C}_\alpha(X, F)$ be strongly affine. Set $E = \mathfrak{A}(X, \mathbb{R})$ and $\kappa_X : X \to E^*$ be the canonical embedding. If we apply Lemma 6.2 to $\kappa_X(X)$ in place of X and $f \circ \kappa_X^{-1}$ in place of f, we get a separable space $E_1 \subset E$ and $g \in \mathcal{C}_\alpha(\pi(\kappa_X(X)), F)$ with $f \circ \kappa_X^{-1} = g \circ \pi$ (where $\pi : E^* \to E_1^*$ is the restriction mapping). Set $X_1 = \pi(\kappa_X(X))$ and $h = \pi \circ \kappa_X$. Then X_1 is a metrizable compact convex set and h is an affine continuous surjection of X onto X_1. By Lemma 6.3 there is a metrizable simplex X_2 and affine continuous surjections $h_1 : X_2 \to X_1$ and $h_2 : X \to X_2$ with $h = h_1 \circ h_2$. Then $g \circ h_1 \in \mathcal{C}_\alpha(X_2, F)$ and $g \circ h_1$ is strongly affine by Lemma 6.4. Thus by the metrizable case, $g \circ h_1 \in \mathfrak{A}_{1+\alpha}(X_2, F)$. It follows that $f = g \circ h_1 \circ h_2 \in \mathfrak{A}_{1+\alpha}(X, F)$.

(R), (C) Given f as in the premise, we use Lemma 6.2 to find a separable subspace $E_1 \subset E$ and $g : B_{E_1^*} \to F$ in $\mathcal{C}_\alpha(B_{E_1^*}, F)$ satisfying $f = g \circ \pi$ ($\pi : E^* \to E_1^*$ is again the restriction mapping). By Lemma 6.3 we can find a separable L_1-predual E_2 satisfying $E_1 \subset E_2 \subset E$. Denote by $\pi_1 : E_2^* \to E_1^*$ and $\pi_2 : E^* \to E_2^*$ the restriction maps. Then $g \circ \pi_1 \in \mathcal{C}_\alpha(B_{E_2^*}, F)$. Moreover, $g \circ \pi_1$ is strongly affine by Lemma 6.4. Hence by the metrizable case we get $g \circ \pi_1 \in \mathfrak{A}_{1+\alpha}(B_{E_2^*}, F)$ (or $g \circ \pi_1 \in \mathfrak{A}_{\mathrm{odd},1+\alpha}(B_{E_2^*}, F)$ or $g \circ \pi_1 \in \mathfrak{A}_{\mathrm{hom},1+\alpha}(B_{E_2^*}, F)$ in the special cases). Since $f = g \circ \pi_1 \circ \pi_2$, the proof is complete.

If $\alpha = 1$ we obtain $f \in \mathfrak{A}_1(X, F)$ by the same procedure. ∎

Finally, let us settle the remaining special case:

Proof of Theorem 2.6 if ext X *is an* F_σ-*set.* If ext X is F_σ, the result follows from Theorem 2.7 proved below. More precisely:

(S) Let $f \in \mathcal{C}_\alpha(X, F)$ be strongly affine. Then $f|_{\text{ext}\,X} \in \mathcal{C}_\alpha(\text{ext}\,X, F)$, hence by Theorem 2.7 this function can be extended to a $g \in \mathfrak{A}_\alpha(X, F)$. Since $f = g$ on ext X and both functions are strongly affine, and each maximal measure is supported by ext X, we conclude that $f = g$ on X, i.e., $f \in \mathfrak{A}_\alpha(X, F)$.

(R) Let $f \in \mathcal{C}_\alpha(X, F)$ be strongly affine. Then $f = f(0) + (f - f(0))$, and $f - f(0)$ is odd, strongly affine and belongs to $\mathcal{C}_\alpha(X, F)$. Using Theorem 2.7 as in case (S) we get $f - f(0) \in \mathfrak{A}_\alpha(X, F)$, thus $f \in \mathfrak{A}_\alpha(X, F)$.

(C) Let $f \in \mathcal{C}_\alpha(X, F)$ be strongly affine. Then $f = f(0) + u + \overline{v}$, where $u = \mathsf{hom}\, f$ and v are homogeneous, affine and belong to $\mathcal{C}_\alpha(X, F)$ (see Lemma 3.14(a)). Using Theorem 2.7 as in case (S) we get $u, v \in \mathfrak{A}_\alpha(X, F)$, thus $f \in \mathfrak{A}_\alpha(X, F)$. ∎

7. Extensions of Baire mappings. In this section we will prove Theorem 2.7. We will proceed in several steps, imitating, generalizing and simplifying the approach of [41]. The strategy is the following:

- Given a bounded Baire function $f : \text{ext}\,X \to F$, we extend it to a bounded Baire function $h : X \to F$. (We do not require affinity and we do not control the class of h.)
- We prove that the function $Th(x) = \int_X h \, dT(x)$, $x \in X$, used in the previous section, is the unique strongly affine extension of f.
- We show that Th is of the right affine class.

The first step is made in the following lemma which is a vector-valued variant of [41, Lemma 2.8] with a simplified proof.

LEMMA 7.1. *Let X be a compact convex set with* ext X *Lindelöf. Let F be a Fréchet space over* $\mathbb{F}$, *and f be a bounded function in* $\mathcal{C}_\alpha(\text{ext}\,X, F)$ *for some* $\alpha < \omega_1$. *Let $L = \overline{\text{co}}\, f(\text{ext}\,X)$. Then there exists a Baire measurable function $h \colon X \to L$ extending f. If $f \in \mathcal{C}_1(\text{ext}\,X, F)$, then h may be chosen from* $\mathcal{C}_1(X, L)$.

Proof. We use transfinite induction on α. Suppose first $\alpha = 1$, i.e., $f \in \mathcal{C}_1(\text{ext}\,X, F)$. Since L is separable and completely metrizable, by [28, Theorem 30 and Proposition 28] there is an extension $h : X \to L$ which is $\Sigma_2^b(X)$-measurable. Lemma 3.1 now implies $h \in \mathcal{C}_1(X, L)$.

Assume now that $\alpha > 1$ and the assertion is valid for all $\beta < \alpha$. Suppose that $f \in \mathcal{C}_\alpha(\text{ext}\,X, F)$ is a bounded mapping and let L be as above. Then $f \in \mathcal{C}_\alpha(\text{ext}\,X, L)$ by Corollary 3.2 (note that ext X is normal, being Lindelöf

and regular), and thus there exist $f_n \in \bigcup_{\beta < \alpha} \mathcal{C}_\beta(\mathrm{ext}\, X, L)$ converging pointwise to f on $\mathrm{ext}\, X$. Let $h_n \colon X \to L$ be their Baire measurable extensions and let

$$C = \{x \in X;\ (h_n(x))\ \text{converges}\}.$$

Let ρ be a compatible complete metric on F. Then

$$C = \{x \in X;\ \forall k \in \mathbb{N}\ \exists l \in \mathbb{N}\ \forall m_1, m_2 \geq l \colon \rho(h_{m_1}(x), h_{m_2}(x)) < 1/k\},$$

clearly a Baire subset of X. Let $z \in L$. Then the function

$$h(x) = \begin{cases} \lim_{n \to \infty} h_n(x), & x \in C, \\ z, & x \in X \setminus C, \end{cases}$$

is the required extension. ∎

The next lemma enables us to deduce the vector-valued version from the scalar one.

LEMMA 7.2. *Let K be a compact space and L a separable convex subset of a Fréchet space F. Let $f : K \to L$ satisfy $\tau \circ f \in \mathcal{C}_\alpha(K, L)$ for each $\tau \in F^*$. Then $f \in \mathcal{C}_{\alpha+1}(K, L)$.*

Proof. Without loss of generality we can suppose that F is separable. By Lemma 3.1, $\tau \circ f$ is $\Sigma_{\alpha+1}^b(K)$-measurable for each $\tau \in F^*$. Since $(\tau \circ f)^{-1}(U) = f^{-1}(\tau^{-1}(U))$ for any $U \subset \mathbb{F}$ and F, being separable, is hereditarily Lindelöf in the weak topology, we find that f is $\Sigma_{\alpha+1}^b$-measurable as a mapping from K to the weak topology of F. It follows from Lemma 3.6 that f is $\Sigma_{\alpha+2}^b(K)$-measurable as a mapping from K to the original topology of F. Thus $f \in \mathcal{C}_{\alpha+1}(K, L)$ by Lemma 3.1. ∎

LEMMA 7.3. *Let X be a compact convex set with $\mathrm{ext}\, X$ Lindelöf, F a Fréchet space and $h \colon X \to F$ a bounded Baire mapping. Suppose moreover that one of the following conditions is satisfied:*

(S) *X is a simplex.*
(R) *$X = (B_{E^*}, w^*)$, where E is a real L_1-predual and $h|_{\mathrm{ext}\, X}$ is odd.*
(C) *$X = (B_{E^*}, w^*)$, where E is a complex L_1-predual and $h|_{\mathrm{ext}\, X}$ is homogeneous.*

Then Th is the unique strongly affine mapping which coincides with h on $\mathrm{ext}\, X$. Moreover, Th is a Baire mapping and it is odd in case (R) and homogeneous in case (C).

Proof. Let us first show the uniqueness. Suppose that g_1, g_2 are two strongly affine mappings which coincide with h on $\mathrm{ext}\, X$. Fix $\tau \in F^*$. Then for each $i = 1, 2$ the function $\tau \circ g_i$ is strongly affine and $\tau \circ g_i|_{\mathrm{ext}\, X}$ is a Baire mapping (as it coincides with $h|_{\mathrm{ext}\, X}$), hence $\tau \circ g_i$ is a Baire mapping by [40, Theorem 5.2]. In particular, the set $\{x \in X;\ \tau(g_1(x)) = \tau(g_2(x))\}$ is a

Baire set containing ext X. Given any $x \in X$ let μ be a maximal probability representing x. Then μ is carried by any Baire set containing ext X, hence $\tau(g_1(x)) = \mu(\tau \circ g_1) = \mu(\tau \circ g_2) = \tau(g_2(x))$. Hence $\tau \circ g_1 = \tau \circ g_2$. Since $\tau \in F^*$ is arbitrary, we conclude $g_1 = g_2$.

Further, any strongly affine mapping coinciding with h on ext X is a Baire mapping. Indeed, let g be such a mapping and suppose that $h \in \mathcal{C}_\alpha(X, F)$. Using [40, Theorem 5.2] as in the previous paragraph we get $\tau \circ g \in \mathcal{C}_{1+\alpha}(X, \mathbb{F})$ for each $\tau \in F^*$, hence $g \in \mathcal{C}_{1+\alpha+1}(X, F)$ by Lemma 7.2. In particular, g is a Baire mapping.

Finally, Th is strongly affine by Lemmata 5.1 and 6.1(ii). To finish the proof it is enough to show that Th coincides with h on ext X and that it enjoys the appropriate symmetry in cases (R) and (C). The fact that Th is a Baire mapping then follows from the previous paragraph. Hence we distinguish the three cases:

(S) For $x \in \text{ext } X$ we have $Th(x) = \delta_x(h) = h(x)$ since δ_x is the Dirac measure supported at x.

(R) To show that g is odd it is enough to observe that T is odd (if $x^* \in X$, then $-T(x^*)$ satisfies conditions (a)–(d) from Fact 1.1 for $-x^*$). Further, for any $x^* \in \text{ext } X$ we have $T(x^*) = \text{odd } \varepsilon_{x^*}$, hence

$$Th(x^*) = \int_X h \, d\, \text{odd } \varepsilon_{x^*} = \int_X \text{odd } h \, d\varepsilon_{x^*} = \text{odd } h(x^*) = h(x^*).$$

(C) This case is completely analogous to case (R). ∎

The final ingredient is the following easy lemma:

LEMMA 7.4. *Let X be a compact convex set, F a Fréchet space and f, f_n, $n \in \mathbb{N}$, strongly affine Baire mappings defined on X with values in F. If the sequence (f_n) is uniformly bounded and converges to f pointwise on ext X, then it converges to f pointwise on X.*

Proof. Let $A = \{x \in X; \, f_n(x) \to f(x)\}$. By the assumption we have $A \supset \text{ext } X$ and A is a Baire set (as the functions in question are Baire functions, cf. the proof of Lemma 7.1). Therefore any maximal probability measure is carried by A. Fix any $x \in X$ and a maximal probability μ representing x. Using Theorem 3.10 we have

$$f(x) = \mu(f) = \lim \mu(f_n) = \lim f_n(x). \quad \blacksquare$$

Now we are prepared to complete the proof:

Proof of Theorem 2.7. Let first $\alpha = 1$, so suppose that $f \in \mathcal{C}_1(\text{ext } X, F)$ is a bounded mapping. Let $L = \overline{\text{aco}} \, f(\text{ext } X)$. By Lemma 7.1 there is $h \in \mathcal{C}_1(X, L)$ extending f. Set $g = Th$. By Lemma 7.3, g is a strongly affine Baire mapping extending f. Now let us distinguish the cases:

(S) As in the proof of Theorem 2.6, we find (using Lemmata 6.2 and 6.3) a metrizable simplex Y, an affine continuous surjection $\pi\colon X \to Y$, $\tilde{h} \in \mathcal{C}_1(Y, L)$ and a Baire mapping $\tilde{g}: Y \to L$ such that $h = \tilde{h} \circ \pi$ and $g = \tilde{g} \circ \pi$. By Lemma 6.4 the mapping $\tilde{g}$ is strongly affine. Moreover, since $\pi(\operatorname{ext} X) \supset \operatorname{ext} Y$, $\tilde{g}$ coincides with $\tilde{h}$ on $\operatorname{ext} Y$. Lemma 7.3 then yields $\tilde{g} = T\tilde{h}$. By Theorem 2.4 and Lemma 6.1(iii) we get $\tilde{g} \in \mathfrak{A}_2(Y, L)$. Hence $g \in \mathfrak{A}_2(X, L)$.

Cases (R) and (C) are analogous.

Let us continue with the general case. Lemmata 7.1 and 7.3 imply that for any (odd, homogeneous) bounded F-valued map f on $\operatorname{ext} X$ there is a unique strongly affine map Sf extending f. By Lemma 7.4 we know that $Sf_n \to Sf$ pointwise whenever (f_n) is a bounded sequence converging to f pointwise on $\operatorname{ext} X$. The already proved case $\alpha = 1$ shows that $Sf \in \mathfrak{A}_2(X, F)$ whenever $f \in \mathcal{C}_1(\operatorname{ext} X, F)$. Hence the result follows by transfinite induction.

The case of $\operatorname{ext} X$ being F_σ is proved in the next section using Theorem 2.8.

Finally, suppose that $\operatorname{ext} X$ is closed. For each bounded $f \in \mathcal{C}_\alpha(\operatorname{ext} X, F)$ (odd, homogeneous) we construct the extension g by the same method as above. We will prove that $g \in \mathfrak{A}_\alpha(\operatorname{ext} X, F)$. Since boundary measures are supported by $\operatorname{ext} X$, we have

$$g(x) = \int_{\operatorname{ext} X} f \, dT(x), \quad x \in X.$$

Suppose first that $\alpha = 0$, i.e., f is continuous. Then we can prove that g is continuous by a minor modification of the proof of Lemma 6.1(i) with the help of Lemma 5.2. The general case then follows by transfinite induction using Lemma 7.4. ∎

8. The weak Dirichlet problem for Baire mappings.
In this section we will prove Theorem 2.8. The proof will be a simplified and generalized version of the proof of the main result of [62].

Proof of Theorem 2.8. (S) Set $H = \overline{\operatorname{co}}\, K$ and $L = \overline{\operatorname{co}}\, f(K)$. By Corollary 3.2 we have $f \in \mathcal{C}_\alpha(K, L)$. Fix a countable family $\mathcal{F} = \{f_n; n \in \mathbb{N}\} \subset \mathcal{C}(K, L)$ such that $f \in (\mathcal{F})_\alpha$.

We define $\varphi : K \to F^\mathbb{N}$ by

$$\varphi(x) = (f_n(x))_{n=1}^\infty, \quad x \in K.$$

Then φ is a continuous mapping of K into $F^\mathbb{N}$. Further we set

$$\Gamma(x) = \begin{cases} \{\varphi(x)\}, & x \in K, \\ \overline{\operatorname{co}}\, \varphi(K), & x \in X \setminus K. \end{cases}$$

Then Γ is a lower semicontinuous mapping with closed values. Moreover,

the graph of Γ is convex. By Theorem 5.3 there exists a continuous affine selection $\gamma : X \to F^{\mathbb{N}}$ of Γ.

Let $h_n = \pi_n \circ \gamma$, $n \in \mathbb{N}$, where $\pi_n : F^{\mathbb{N}} \to F$ is the nth projection mapping. Then h_n is a continuous affine mapping of X into F such that

$$h_n(x) = \pi_n(\gamma(x)) = \pi_n(\varphi(x)) = f_n(x), \quad x \in K.$$

Further, for each $x \in X$ we choose some $r(x) \in H$ such that $h_n(x) = h_n(r(x))$ for each $n \in \mathbb{N}$. If $x \in H$ we set $r(x) = x$. If $x \in X \setminus H$, then $\gamma(x) \in \overline{\mathrm{co}}\,\varphi(K) = \overline{\mathrm{co}}\,\gamma(K) = \gamma(H)$, thus there exists an element $r(x) \in H$ such that $\gamma(x) = \gamma(r(x))$.

Finally, we can construct the required extension by

$$(8.1) \qquad \hat{f}(x) = \begin{cases} \delta_x(f), & x \in H, \\ \hat{f}(r(x)), & x \in X \setminus H. \end{cases}$$

First, if $x \in H$, then δ_x is supported by K, thus $\hat{f}$ is well defined on H. Next, $\hat{f}$ is extended to X using the mapping r fixed above. It is clear that $\hat{f}$ is an extension of f. Moreover,

$$(8.2) \qquad \hat{f} \in (\{h_n;\, n \in \mathbb{N}\})_\alpha \subset \mathfrak{A}_\alpha(X, L).$$

Indeed, (8.1) enables us to assign to each Borel function $g : K \to L$ its extension $\hat{g} : X \to L$. Note that

$$\hat{f}_n = h_n \quad \text{for } n \in \mathbb{N}.$$

Further, it follows from Theorem 3.10 that

$$g_k \to g \text{ pointwise on } K \;\Rightarrow\; \hat{g}_k \to \hat{g} \text{ pointwise on } X.$$

Hence it is easy to prove by transfinite induction on β that

$$g \in (\{f_n;\, n \in \mathbb{N}\})_\beta \;\Rightarrow\; \hat{g} \in (\{h_n;\, n \in \mathbb{N}\})_\beta,$$

so in particular (8.2) holds.

(R) We assume that K is moreover symmetric and f is odd. We proceed in the same way. The sets H and L are convex and symmetric. The family $\mathcal{F}$ may consist of odd functions by Lemma 3.12. Then φ is moreover odd and the graph of Γ is convex and symmetric. Thus the selection γ may be chosen to be moreover odd and the functions h_n are odd as well. The formula for the extension is similar:

$$\hat{f}(x) = \begin{cases} T(x)(f), & x \in H, \\ \hat{f}(r(x)), & x \in X \setminus H. \end{cases}$$

The rest of the proof is the same; we use the fact that $T(x)$ is supported by K whenever $x \in H$.

(C) The proof is completely analogous to the real case. ∎

Now we give the proof of the missing part of Theorem 2.7.

Proof of Theorem 2.7 if ext X *is* F_σ. (S) It is enough to prove that $x \mapsto \delta_x(f)$ is in $\mathfrak{A}_1(X, F)$ for any bounded $f \in \mathcal{C}_1(\text{ext } X, F)$. Choose such an f. Fix a bounded sequence of continuous functions $f_n : \text{ext } X \to F$ pointwise converging to f (this can be done by Corollary 3.2). Let $\text{ext } X = \bigcup_n K_n$, where (K_n) is an increasing sequence of compact sets. By Theorem 2.8 there are affine continuous maps h_n extending $f_n|_{K_n}$. Then h_n converges to f pointwise on $\text{ext } X$, thus for each $x \in X$,

$$\delta_x(f) = \lim_{n \to \infty} \delta_x(h_n) = \lim_{n \to \infty} h_n(x).$$

(R) The proof is analogous. Assume that f is moreover odd. We can choose f_n to be odd (Lemma 3.12) and K_n to be symmetric. In the final computation we use $T(x)$ instead of δ_x.

(C) If f is homogeneous, by Lemma 3.13 we can choose f_n to be homogeneous. Moreover, K_n can be chosen homogeneous as well. Indeed, if K is compact, then $\bigcup \{\alpha K; \alpha \in \mathbb{C}, |\alpha| = 1\}$ is compact as a continuous image of the compact set $K \times \{\alpha \in \mathbb{C}; |\alpha| = 1\}$. ∎

Another consequence of Theorem 2.8 is the following extension theorem.

THEOREM 8.1. *Let K be a compact subset of a completely regular space Z, F a Fréchet space and $f : K \to F$ a bounded mapping in $\mathcal{C}_\alpha(K, F)$. Then there exists a mapping $h \colon Z \to F$ in $\mathcal{C}_\alpha(Z, F)$ extending f such that $h(Z) \subset \overline{\text{co}}\, f(K)$.*

Proof. Let $L = \overline{\text{aco}}\, f(K)$. Let βZ be the Čech–Stone compactification of Z and $X = \mathcal{M}^1(\beta Z)$. Then X is a simplex and $\text{ext } X$ is canonically identified with βZ. Hence K is a compact subset of $\text{ext } X$. Therefore the result follows from Theorem 2.8. ∎

9. Affine version of the Jayne–Rogers selection theorem. The aim of this section is to prove Theorem 2.9. To this end we need the following lemma on 'measure-convexity':

LEMMA 9.1. *Let X be a compact convex set, F a Fréchet space over $\mathbb{F}$ and $\Gamma \colon X \to F$ an upper or lower semicontinuous mapping with closed values and convex graph such that $\Gamma(X)$ is bounded. Let $f : X \to F$ be a Baire measurable selection from Γ. If $\mu \in \mathcal{M}^1(X)$, then*

$$\mu(f) \in \Gamma(r(\mu)).$$

Proof. Since f is Baire measurable, $\mu(f)$ is well defined by Lemma 3.9. Set $x = r(\mu)$. Assuming $\mu(f) \notin \Gamma(x)$, $\Gamma(x)$, as a convex closed set, can be separated from $\mu(f)$ by some element from F^*, i.e., there exist $\tau \in F^*$ and $c \in \mathbb{R}$ such that

$$(\text{Re}\,\tau)(\mu(f)) > c > \sup(\text{Re}\,\tau)(\Gamma(x)).$$

Let
$$\varphi(y) = \sup(\operatorname{Re}\tau)(\Gamma(y)), \quad y \in X.$$
Since the graph of Γ is convex, φ is a concave function on X. Moreover, if Γ is upper semicontinuous, then φ is upper semicontinuous; and if Γ is lower semicontinuous, then φ is lower semicontinuous. In both cases, by [43, Proposition 4.7],
$$c < (\operatorname{Re}\tau)(\mu(f)) = \int_X (\operatorname{Re}\tau)(f(y))\,d\mu(y) \le \int_X \varphi(y)\,d\mu(y) \le \varphi(x) < c.$$
This contradiction finishes the proof. ∎

Proof of Theorem 2.9. (S) By [24, Theorem] (which is an improved version of [23, Theorem 2]) and Lemma 3.3(c) there is a selection f from Γ which belongs to $\mathcal{C}_1(X,F)$. Set $g(x) = \delta_x(f)$ for $x \in X$. By Theorem 2.4 and Lemma 6.1(iii) we have $g \in \mathfrak{A}_2(X,F)$ and by Lemma 9.1, g is a selection of Γ.

(R) We proceed in the same way. We find a selection f in $\mathcal{C}_1(X,F)$ and set $f_1 = \operatorname{odd} f$. Then f_1 is still a selection from Γ. Indeed, if $x \in X$, then $(x, f(x))$ and $(-x, f(-x))$ belong to the graph of Γ. Since the graph of Γ is symmetric, $(x, -f(-x))$ belongs there, too. By convexity we conclude that $(x, f_1(x)) = (x, \operatorname{odd} f(x))$ belongs there as well. Further, set $g = Tf_1$. By Theorem 2.4 and Lemma 6.1(iii) we have $g \in \mathfrak{A}_2(X,F)$. Let $x \in X$. Fix a maximal representing measure μ of x. Then
$$g(x) = T(x)(f_1) = \operatorname{odd}\mu(f_1) = \mu(f_1) \in \Gamma(x)$$
by Lemmata 9.1 and 3.15(d), hence g is a selection from Γ. It remains to observe that g is odd since T is odd.

(C) This case is analogous to (R). We find a selection f in $\mathcal{C}_1(X,F)$ and we define $f_1 = \operatorname{hom} f$. Then f_1 is still a selection of Γ. Indeed, if $x \in X$ and $t \in [0, 2\pi]$, then $(e^{it}x, f(e^{it}x))$ belongs to the graph of Γ. Since the graph is homogeneous, it also contains $(x, e^{-it}f(e^{it}x))$, hence $e^{-it}f(e^{it}x) \in \Gamma(x)$. Since $\Gamma(x)$ is closed and convex, we can deduce that $f_1(x) = \operatorname{hom} f(x) \in \Gamma(x)$. The rest of the proof is the same as in case (R). ∎

It is natural to ask whether we can obtain a nice affine selection even if X is not metrizable. We stress that the Jayne–Rogers theorem requires metrizability both of the domain space and of the range space. However, if F is separable, it easily follows from the Kuratowski–Ryll-Nardzewski selection theorem (see [31] or [68, Theorem 5.2.1]) that F admits a Borel measurable selection, in fact a selection of the first Borel class in the sense of [65]. However, an affine Borel selection need not exist even in the scalar case. This is illustrated by the following example, where we show also that the class of the affine selection in the metrizable case cannot be improved.

EXAMPLE 9.2. *There are simplices X_1, X_2 and upper semicontinuous mappings $\Gamma_i \colon X_i \to \mathbb{R}$ with closed values, bounded range and convex graph for $i = 1, 2$ such that:*

(i) *X_1 is metrizable and Γ_1 admits no affine Baire-one selection.*
(ii) *X_2 is nonmetrizable and Γ_2 admits no affine Borel selection.*

Proof. Let $A \subset [0, 1]$ be any subset. Let K, $\mathcal{A}$ and X be as in Example 5.4. We again consider K canonically embedded into X. Fix $B \subset A$ and define two real-valued functions on K by $u_0 = \chi_{B \times \{1\}}$ and $v_0 = \chi_{(B \times \{1\}) \cup ([0,1] \times \{0\})}$. Then $u_0 \leq v_0$. We further define two functions on X by

$$u(x) = \sup\{h(x); \; h \in \mathfrak{A}(X, \mathbb{R}), \; h \leq u_0 \text{ on } K\},$$
$$v(x) = \inf\{h(x); \; h \in \mathfrak{A}(X, \mathbb{R}), \; h \geq v_0 \text{ on } K\}.$$

Then u is lower semicontinuous and convex, while v is upper semicontinuous and concave. Moreover, $u \leq v$ (if $h_1, h_2 \in \mathfrak{A}(X, \mathbb{R})$ with $h_1 \leq u_0$ and $h_2 \geq v_0$ on K, then $h_1 \leq h_2$ on $\operatorname{ext} X$, hence $h_1 \leq h_2$ on X). Therefore $\Gamma(x) = [u(x), v(x)]$, $x \in X$, is a bounded upper semicontinuous mapping with nonempty closed values and convex graph.

We claim that $u|_K = u_0$ and $v|_K = v_0$. This follows from abstract results [43, Propositions 3.48 and 3.55], but it can also be seen directly:

Since $u_0 \geq 0$, it follows that $u(x) = 0 = u_0(x)$ for $x \in K \setminus (B \times \{1\})$. If $x = (b, 1)$ for some $b \in B$, set $f_b = \chi_{(b,1)} - \chi_{(b,-1)}$. Then $f_b \in \mathcal{A}$ and $f_b \leq u_0$. Since f_b defines a function in $\mathfrak{A}(X, \mathbb{R})$, we conclude that $u(x) = 1 = u_0(x)$.

Since $v_0 \leq 1$, we get $v(x) = 1 = v_0(x)$ for $x \in (B \times \{1\}) \cup ([0, 1] \times \{0\})$. If $x = (a, -1)$ for some $a \in A$, then $1 + f_a \geq v_0$, thus $v(x) = 0 = v_0(x)$. If $x = (a, 1)$ for some $a \in A \setminus B$, then $1 - f_a \geq v_0$, thus $v(x) = 0 = v_0(x)$.

In particular, $u = v$ on $A \times \{-1, 1\}$. Let g be any affine selection from Γ. Then

$$f(a, 0) = \begin{cases} 1/2, & a \in B, \\ 0, & a \in A \setminus B. \end{cases}$$

Therefore, to construct X_1 it is enough to take A to be a countable dense subset of $[0, 1]$ and $B \subset A$ such that both B and $A \setminus B$ are dense. To construct X_2 it is enough to take A to be an uncountable Borel set and $B \subset A$ a non-Borel subset. ∎

10. Sharpness of results and open problems. In this final section we collect several open questions and discuss which of the results are sharp. We start by pointing out a gap between Theorem 2.2 and Example 2.3.

QUESTION 10.1. *Let X be a compact convex set and E a Banach space having the approximation property (or the compact approximation property). Does any affine function $f \in \mathcal{C}_1(X, E)$ belong to $\mathfrak{A}_1(X, E)$?*

The next question concerns optimality of constants in Theorem 2.2.

QUESTION 10.2. *Let X be a compact convex set and E a Banach space with the bounded approximation property. Does any affine function $f \in \mathcal{C}_1(X, B_E)$ belong to $\mathfrak{A}_1(X, B_E)$?*

We continue with problems concerning Theorem 2.4. For a metrizable simplex we have proved that the mapping T is of class $\mathfrak{A}_1$ when the target space is $\mathcal{M}^1(X)$. The same proof gives a slightly stronger conclusion that T is of class $\mathfrak{A}_1$ when the target space is $\mathcal{M}^1(\overline{\text{ext } X})$, and a similar statement in cases (R) and (C). However the following question seems to be open.

QUESTION 10.3. *Let X be a metrizable simplex. Is it true that $T \in \mathfrak{A}_1(X, \mathcal{M}^1(\text{ext } X))$? Is the analogous statement valid for L_1-preduals?*

As remarked above, the metrizability assumption was used in the proof of Theorem 2.4 in an essential way. Further, this assumption cannot be omitted or just weakened to the Lindelöf property of ext X by Example 5.4. However, the general question when the operator T belongs to the class $\mathcal{C}_1$ or even $\mathfrak{A}_1$ seems to be hard. A sufficient condition is metrizability of X or closedness of ext X. But these conditions are not necessary. It is easy to construct a simplex X when T is of class $\mathfrak{A}_1$ even though X is not metrizable and ext X is not closed: for example, the simplex constructed by the method used in [27, Theorem 4] starting with $K = [0, \omega_1]$ and $A = \{\omega\}$. (This construction is based upon the well-known constructions from [69] and [3].) It seems that the following question is natural. (By an (S, R, C) *compact set* we mean a compact convex set which is either a simplex or the dual unit ball of a real or complex L_1-predual.)

QUESTION 10.4. *Let X be an (S,R,C) compact set. Is T of class $\mathcal{C}_1$ or even $\mathfrak{A}_1$ if one of the conditions below is satisfied?*

- ext X *is K-analytic.*
- ext X *is a Baire set.*
- ext X *is F_σ.*

Let us further remark that the operator T cannot be 'too nonmeasurable' since it is strongly affine due to Lemma 5.1. More precisely, it follows from the proof of that lemma that in cases (S) and (R), Tf is a uniform limit of a sequence of differences of lower semicontinuous functions for any $f \in \mathcal{C}(X, \mathbb{R})$. In case (C) the same property is enjoyed by $\text{Re } Tf$ and $\text{Im } Tf$ for each $f \in \mathcal{C}(X, \mathbb{C})$ by [51, proof of Lemma 4.12]. Hence, in all three cases, Tf is Borel measurable for any scalar continuous function f. (In fact, in this case Tf is of the first Borel class in the sense of [65].) It easily follows that $T^{-1}(U)$ is a Borel set (of the first additive class in the sense of [65]) for any cozero set U in the relevant set of measures (but it need not be Borel

measurable, as the spaces from Example 5.4 easily show). However, Tf need not be of the first Baire class—an example is given in [27, Theorem 4]. In fact, in the cited example T is not in $\mathfrak{A}_\alpha$ for any $\alpha < \omega_1$ since Tf need not even be a Baire function (by the cited example and [27, Theorem 5]).

We continue by looking at Theorem 2.6. It is sharp even in the scalar case, i.e. the shift $\alpha \to 1 + \alpha$ may occur. In case (S) it was shown in [64] for $\alpha = 2$ and in [67] for general $\alpha \in [2, \omega)$. The same examples can be easily transferred to cases (R) and (C). What we still do not know is the following.

QUESTION 10.5. *Let X be an (S,R,C) compact set such that* ext X *is a Lindelöf resolvable set. Can one replace $1 + \alpha$ by α in Theorem 2.6?*

Recall that a subset of a X is said to be *resolvable* (or an *H-set*) if its characteristic function has the point of continuity property. For more details see [66]. Within metrizable spaces this class is defined in [30, §12, III] using an equivalent condition. Further, a subset of a completely metrizable space is resolvable if and only if it is simultaneously F_σ and G_δ. Thus, the answer is positive if X is metrizable. Further, the answer is also positive in the scalar case. Case (S) follows from [66], case (R) from [40, Theorem 1.4], and case (C) from [38, Theorem 2.23].

The conclusion of Theorem 2.7 is optimal since it follows from the optimality of Theorem 2.6 (the above-mentioned examples from [64] and [67] are metrizable). However, it is not clear whether it is optimal for $\alpha = 0$.

QUESTION 10.6. *Let X be an (S,R,C) compact set with* ext X *Lindelöf, F a Fréchet space and f : ext $X \to F$ a bounded continuous (odd, homogeneous) map. Can f be extended to an element of $\mathfrak{A}_1(X, F)$?*

In the scalar case the answer is positive (by [25, 41, 38]), but in the vector case our proof just gives an extension in $\mathfrak{A}_2(X, F)$. Moreover, if g is such an extension, the scalar version yields $\tau \circ g \in \mathfrak{A}_1(X, \mathbb{F})$ for each $\tau \in F^*$.

Another question is the following.

QUESTION 10.7. *Let X be an (S,R,C) compact set such that* ext X *is Lindelöf resolvable. Can one replace $1 + \alpha$ by α in Theorem 2.7?*

In the scalar case the answer is positive (by [66, 41, 38]). In the vector case we are able to prove it only if ext X is an F_σ set, hence the answer is positive if X is metrizable.

Theorem 2.7 deals with extending Baire mappings. It is natural to ask whether we can extend Baire measurable mappings. If the target space F is separable, then any Baire measurable mapping with values in F which is defined on a regular Lindelöf space is a Baire mapping of the respective class (by Lemma 3.1). Hence the question of extending Baire measurable mappings reduces to the question whether a Baire measurable mapping has separable range.

QUESTION 10.8. *Let X be a compact convex set with* $\operatorname{ext} X$ *Lindelöf. Let $f : \operatorname{ext} X \to F$ be a Baire measurable mapping with values in a Fréchet space. Is the range of f separable?*

The answer is positive if $\operatorname{ext} X$ is a Baire set (or, more generally, a K-analytic set)—see Lemma 3.3(a). For a general Lindelöf (even separable metric) space in place of $\operatorname{ext} X$ the answer is negative by [29, Example 2.4(3)]. But we do not know whether an example can be of the form $\operatorname{ext} X$.

Finally, one can ask whether the assumption that $\operatorname{ext} X$ is Lindelöf is necessary. It cannot be just omitted (even in the scalar case: see, e.g., [27, Theorem 4]). However, the following question asked already in [25] is still open.

QUESTION 10.9. *Let X be a simplex such that each bounded continuous real-valued function on* $\operatorname{ext} X$ *can be extended to an affine Baire function on X. Is necessarily* $\operatorname{ext} X$ *Lindelöf? Does the analogous implication hold for dual balls of L_1-preduals?*

Let us remark that the answer is positive within a special class of 'Stacey simplices' by [27, Theorems 2 and 6], but the general case is not clear.

Finally, Theorem 2.8 is clearly sharp and Theorem 2.9 is sharp due to Example 9.2.

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