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Locally convex algebras which determine a locally compact group

by

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Abstract. There are several algebras associated with a locally compact group $\mathcal{G}$ which determine $\mathcal{G}$ in the category of topological groups, such as $L^1(\mathcal{G})$, $M(\mathcal{G})$, and their second duals. In this article we add a fairly large family of locally convex algebras to this list. More precisely, we show that for two infinite locally compact groups $\mathcal{G}_1$ and $\mathcal{G}_2$, there are infinitely many locally convex topologies τ_1 and τ_2 on the measure algebras $M(\mathcal{G}_1)$ and $M(\mathcal{G}_2)$, respectively, such that $(M(\mathcal{G}_1), \tau_1)^{**}$ is isometrically isomorphic to $(M(\mathcal{G}_2), \tau_2)^{**}$ if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are topologically isomorphic. In particular, this leads to a new proof of Ghahramani–Lau's isometrical isomorphism theorem for compact groups, different from those of Ghahramani and J. P. McClure (2006) and Dales et al. (2012).

1. Introduction. A long-standing question in abstract harmonic analysis is: Which Banach algebras (or more generally locally convex algebras) associated to a locally compact group $\mathcal{G}$ determine $\mathcal{G}$ in the category of topological groups? The present paper is an effort towards the answer to this general question. The first publication towards this is by Helson [7], who showed that, for two locally compact abelian groups $\mathcal{G}_1$ and $\mathcal{G}_2$, if T is a contractive isomorphism from $L^1(\mathcal{G}_1)$ onto $L^1(\mathcal{G}_2)$, then T is isometric and $\mathcal{G}_1$ and $\mathcal{G}_2$ are topologically isomorphic. Wendel [17, 18] extended this to non-abelian groups; on his result, all later conclusions relied. In particular, Johnson [11] showed that the measure algebra $M(\mathcal{G})$ determines $\mathcal{G}$. In the past three decades research on the above problem has centred on second dual type algebras, after publication of [14] and [9]. In [14] the authors showed that if $LUC(\mathcal{G}_1)^*$ and $LUC(\mathcal{G}_2)^*$ are isometrically algebra isomorphic, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are topologically isomorphic. Lau and Losert [13] showed that, for locally compact abelian groups, $L^1(\mathcal{G})^{**}$ determines $\mathcal{G}$. This was extended to arbi-

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trary locally compact groups in [2] (see also [5]). Some other algebras which also determine $\mathcal{G}$ are $L_0^\infty(\mathcal{G})^*$ [15], and $VN(\mathcal{G})^*$ for discrete groups [3]. In [4], Ghahramani and Lau proved that for two locally compact groups $\mathcal{G}_1$ and $\mathcal{G}_2$ such that the algebras $M(\mathcal{G}_1)^{**}$ and $M(\mathcal{G}_2)^{**}$ are isometrically isomorphic, the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are algebraically isomorphic. Motivated by this result, they raised the following question for arbitrary locally compact groups.

Ghahramani–Lau's Problem. Does $M(\mathcal{G})^{**}$ determine $\mathcal{G}$ in the category of topological groups?

This question was answered positively for compact groups in [6] and recently for locally compact groups in [1, p. 87].

In all of the above mentioned results all algebras considered are Banach algebras and all the discussed isomorphisms are assumed to be or become isometric. But very little is known about the answer to the above-mentioned general question when the associated algebras are just locally convex. Our approach is different from earlier works on the second duals of measure algebras, and relies on the theory of generalized functions on topological semigroups, which was invented by Shreĭder [16] and Wong [19]. The second and third authors have recently shown in [10] that, for a locally compact group $\mathcal{G}$, there are infinitely many locally convex topologies τ on $M(\mathcal{G})$ under which $(M(\mathcal{G}),\tau)^*$ endowed with the strong topology can be identified with a closed subspace of $(M(\mathcal{G}),\mathtt{n}(\mathcal{G}))^*$. They have also shown that, except for the trivial case where $\mathcal{G}$ is finite, $M(\mathcal{G})$ has infinitely many locally convex topologies τ with the same strong dual.

In this paper, we use this result to show that $(M(\mathcal{G}),\tau)^*$ is an introverted subspace of $(M(\mathcal{G}),\mathtt{n}(\mathcal{G}))^*$. This enables us to introduce an Arens-type multiplication on $(M(\mathcal{G}),\beta(\mathcal{G}))^{**}$ and use it to show that $(M(\mathcal{G}),\tau)^{**}$ determines $\mathcal{G}$ for infinitely many locally convex topologies τ on $M(\mathcal{G})$. This result is not only of interest on its own, but it also paves the way for an alternative proof of Ghahramani–Lau's theorem: if $\mathcal{G}_1$ and $\mathcal{G}_2$ are two locally compact groups and T is an isometric isomorphism from $M(\mathcal{G}_1)^{**}$ onto $M(\mathcal{G}_2)^{**}$ which maps $(M(\mathcal{G}_1),\beta(\mathcal{G}_1))^{**}$ onto $(M(\mathcal{G}_2),\beta(\mathcal{G}_2))^{**}$, where β is the largest of the above topologies, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are topologically isomorphic.

2. Preliminaries. Throughout, $\mathcal{G}$ denotes a locally compact group with left Haar measure λ. Also $C_0(\mathcal{G})$, $M(\mathcal{G})$, $L^1(\mathcal{G})$ and $L^\infty(\mu)$ for $\mu\in M(\mathcal{G})$ have their usual meanings, as in [8]. We denote the norm topology of $M(\mathcal{G})$ by $\mathtt{n}(\mathcal{G})$.

An element $f=(f_\mu)_{\mu\in M(\mathcal{G})}$ of the product linear space $\prod\{L^\infty(|\mu|):\mu\in M(\mathcal{G})\}$ is called a *generalized function* if $f_\mu=f_\nu$ $|\mu|$-a.e. for any $\mu,\nu\in M(\mathcal{G})$ with $\mu\ll\nu$, where $\mu\ll\nu$ means that $|\mu|$ is absolutely continuous with respect to $|\nu|$. Note that this condition implies that for every generalized function $f=(f_\mu)_{\mu\in M(\mathcal{G})}$,

$$\sup\{\|f_\mu\|_{\mu,\infty} : \mu\in M(\mathcal{G})\}<\infty,$$

where $\|f_\mu\|_{\mu,\infty}$ is the essential supremum norm of f_μ in $L^\infty(|\mu|)$ ($\mu\in M(\mathcal{G})$). Indeed, otherwise there is a sequence (μ_n) in $M(\mathcal{G})$ for which $\|f_{\mu_n}\|_{\mu_n,\infty}\ge n$ for all $n\in\mathbb{N}$. Set $\mu=\sum_{n=1}^\infty 2^{-n}\|\mu_n\|^{-1}|\mu_n|$. Then $\mu_n\ll\mu$, and hence $\|f_\mu\|_{\mu,\infty}\ge\|f_{\mu_n}\|_{\mu_n,\infty}\ge n$ for all $n\in\mathbb{N}$, a contradiction. As in Wong [19], we denote by $GL(\mathcal{G})$ the commutative unital C*-algebra of all generalized functions endowed with coordinatewise operations, the involution $f\mapsto f^*$, where $f^*=(\bar f_\mu)_{\mu\in M(\mathcal{G})}$, and the norm

$$\|f\|_\infty=\sup\{\|f_\mu\|_{\mu,\infty} : \mu\in M(\mathcal{G})\}\qquad (f=(f_\mu)_{\mu\in M(\mathcal{G})}).$$

The identity element of $GL(\mathcal{G})$ is the generalized function $1=(1_\mu)_{\mu\in M(\mathcal{G})}$, where 1_μ is the identity element of $L^\infty(|\mu|)$. Moreover, any bounded Borel-measurable function f on $\mathcal{G}$ can be regarded as an element $(f_\mu)_{\mu\in M(\mathcal{G})}$ of $GL(\mathcal{G})$, where f_μ denotes the equivalence class of f in $L^\infty(|\mu|)$ for each $\mu\in M(\mathcal{G})$. For simplicity we denote this generalized function by f again. Also, if h is a Borel-measurable function on $\mathcal{G}$, then hf is again a generalized function for all generalized functions $f=(f_\mu)_{\mu\in M(\mathcal{G})}$. Indeed $(hf)_\mu=hf_\mu$ for $\mu\in M(\mathcal{G})$.

Wong [19], with an elegant use of the Radon–Nikodym Theorem, proved that for each $f=(f_\mu)_{\mu\in M(\mathcal{G})}$ in $GL(\mathcal{G})$, the equation

$$\langle\Psi(f),\zeta\rangle := \int_{\mathcal{G}} f_\zeta(x)\,d\zeta(x)\qquad(\zeta\in M(\mathcal{G}))$$

defines a continuous linear functional $\Psi(f)$ on $M(\mathcal{G})$. Moreover, the map $f\mapsto\Psi(f)$ is an isometric linear mapping from $GL(\mathcal{G})$ onto $(M(\mathcal{G}),\mathtt{n}(\mathcal{G}))^*$; see [19, Theorems 2.1 and 2.2] and [16] for the same result in the special case where $\mathcal{G}$ is a locally compact abelian group with countable basis, and [12] for some recent work on $GL(\mathcal{G})$. In particular, any $L\in(M(\mathcal{G}),\mathtt{n}(\mathcal{G}))^*$ can be considered as a generalized function $\Psi^{-1}(L)$, and we do not distinguish between a generalized function f and its unique corresponding linear functional $\Psi(f)$.

Following Wong [19, 20], for a bounded Borel-measurable function g on $\mathcal{G}$ and $\gamma\in M(\mathcal{G})$ define the left and right convolutions $l_\gamma g$ and $r_\gamma g$ by

$$l_\gamma g(x)=\int_{\mathcal{G}} g(yx)\,d\gamma(y),\qquad r_\gamma g(x)=\int_{\mathcal{G}} g(xy)\,d\gamma(y)\qquad(x\in\mathcal{G}).$$

Denote the right and left dual module actions of $\zeta\in M(\mathcal{G})$ on $f\in GL(\mathcal{G})$ by $f\zeta$ and ζf, that is,

$$\langle f\zeta,\mu\rangle=\langle f,\zeta*\mu\rangle,\quad \langle\zeta f,\mu\rangle=\langle f,\mu*\zeta\rangle\quad(\mu\in M(\mathcal{G})).$$

Now define $\zeta\circ f$ and $f\circ\zeta$ in $\prod\{L^\infty(|\mu|):\mu\in M(\mathcal{G})\}$ by

$$(\zeta\circ f)_\mu=l_\zeta f_{\zeta*\mu},\qquad (f\circ\zeta)_\mu=r_\zeta f_{\mu*\zeta}\qquad(\mu\in M(\mathcal{G})).$$

Then, as was shown in [19], $\zeta \circ f$ and $f \circ \zeta$ are equal to the generalized functions corresponding to the functionals $f\zeta$ and ζf, respectively. For a more detailed study of these concepts see [19] and [20].

Let $\mathcal{K}(\mathcal{G})$ be the set of all compact subsets of $\mathcal{G}$. For an increasing sequence (K_n) in $\mathcal{K}(\mathcal{G})$ and an increasing sequence (α_n) of positive real numbers such that $\alpha_n \nearrow \infty$, set

$$U((K_n),(\alpha_n)) = \{\mu \in M(\mathcal{G}) : |\mu|(K_n) \le \alpha_n \text{ for all } n \ge 1\}.$$

Then $U((K_n),(\alpha_n))$ is a convex, balanced, absorbing subset of $M(\mathcal{G})$. We denote by $\mathcal{U}(\mathcal{G})$ the family of all sets of the form $U((K_n),(\alpha_n))$. Then $\mathcal{U}(\mathcal{G})$ is a base of neighbourhoods of zero for a locally convex topology, called $\beta(\mathcal{G})$, on $M(\mathcal{G})$. Thus $\beta(\mathcal{G})$ is the topology generated by the family $\{p_U : U \in \mathcal{U}(\mathcal{G})\}$ of seminorms on $M(\mathcal{G})$, where

$$p_U(\mu) = \sup\{\alpha_n^{-1}|\mu|(K_n) : n \ge 1\}$$

for all $\mu \in M(\mathcal{G})$ and $U := U((K_n),(\alpha_n)) \in \mathcal{U}(\mathcal{G})$. Denote by $\tau_b(\mathcal{G})$ the strong topology on $(M(\mathcal{G}),\beta(\mathcal{G}))^*$. Thus $\tau_b(\mathcal{G})$ is the topology of uniform convergence on $\sigma(M(\mathcal{G}),(M(\mathcal{G}),\beta(\mathcal{G}))^*)$-bounded subsets of $M(\mathcal{G})$. Note that $\beta(\mathcal{G}) \le \mathtt{n}(\mathcal{G})$, and $\beta(\mathcal{G})$ coincides with $\mathtt{n}(\mathcal{G})$ if and only if $\mathcal{G}$ is compact. See [10] for more details.

The second and third authors [10] have recently considered the C^*-subalgebra $GL_0(\mathcal{G})$ of $GL(\mathcal{G})$ consisting of all generalized functions $f = (f_\mu)_{\mu\in M(\mathcal{G})} \in GL(\mathcal{G})$ *vanishing at infinity*: for any $\varepsilon > 0$, there is a compact subset K of $\mathcal{G}$ for which $\|f\chi_{\mathcal{G}\setminus K}\|_\infty < \varepsilon$. Let Ψ_0 denote the restriction of Ψ to $GL_0(\mathcal{G})$, and recall from [10, Theorem 3.2] that Ψ_0 is a continuous isomorphism from $GL_0(\mathcal{G})$ onto $((M(\mathcal{G}),\beta(\mathcal{G}))^*,\tau_b(\mathcal{G}))$. Also by [10, Proposition 2.2], the strong topology $\tau_b(\mathcal{G})$ and the relative norm topology τ_n of $(M(\mathcal{G}),\beta(\mathcal{G}))^*$ inherited from $(M(\mathcal{G}),\mathtt{n}(\mathcal{G}))^*$ coincide. From now on we equip $(M(\mathcal{G}),\beta(\mathcal{G}))^{**}$ with the dual norm. In other words, by $(M(\mathcal{G}),\beta(\mathcal{G}))^{**}$ we mean $((M(\mathcal{G}),\beta(\mathcal{G}))^*,\tau_n)^*$. Consequently, the adjoint of Ψ_0 identifies $(M(\mathcal{G}),\beta(\mathcal{G}))^{**}$ with $GL_0(\mathcal{G})^*$.

3. The main results. We begin with the following key result which enables us to define an Arens-type multiplication on the second dual of $(M(\mathcal{G}),\beta(\mathcal{G}))$. Part (ii) of the following proposition is the analogue of [19, Theorem 3.2] for $(M(\mathcal{G}),\beta(\mathcal{G}))$.

PROPOSITION 3.1. *Let $\mathcal{G}$ be a locally compact group.*

(i) *For $\zeta \in M(\mathcal{G})$ and $L \in (M(\mathcal{G}),\beta(\mathcal{G}))^*$, we have $L\zeta, \zeta L \in (M(\mathcal{G}),\beta(\mathcal{G}))^*$ where $L\zeta$ and ζL are defined by*

$$\langle L\zeta,\mu\rangle = \langle L,\zeta * \mu\rangle, \quad \langle \zeta L,\mu\rangle = \langle L,\mu * \zeta\rangle \quad (\mu \in M(\mathcal{G})).$$

(ii) *The isomorphism* Ψ_0^{-1} *commutes with convolutions, that is, for every* $\zeta \in M(\mathcal{G})$ *and* $L \in (M(\mathcal{G}), \beta(\mathcal{G}))^*$, *we have*

$$\Psi_0^{-1}(\zeta \circ L) = \zeta \circ \Psi_0^{-1}(L), \quad \Psi_0^{-1}(L \circ \zeta) = \Psi_0^{-1}(L) \circ \zeta.$$

(iii) *For any* $\mathfrak{m} \in (M(\mathcal{G}), \beta(\mathcal{G}))^{**}$, *we have* $\mathfrak{m}L \in (M(\mathcal{G}), \beta(\mathcal{G}))^*$, *where* $\mathfrak{m}L$ *is defined by*

$$\langle \mathfrak{m}L, \zeta \rangle = \langle \mathfrak{m}, L\zeta \rangle \quad (\zeta \in M(\mathcal{G})).$$

Proof. We prove parts (i) and (ii) simultaneously. Let $\zeta \in M(\mathcal{G})$ and $L \in (M(\mathcal{G}), \beta(\mathcal{G}))^*$. First we show that $\zeta \circ \Psi_0^{-1}(L) \in GL_0(\mathcal{G})$. To this end, without loss of generality, we may suppose that ζ is positive and $\Psi_0^{-1}(L) \neq 0$. By the regularity of ζ, we can choose a compact subset K_1 of $\mathcal{G}$ such that $\zeta(K_1) > 0$ and $0 < \zeta(\mathcal{G} \setminus K_1) < \epsilon/(2\|\Psi_0^{-1}(L)\|_\infty)$. Moreover, since $\Psi_0^{-1}(L)$ vanishes at infinity, there is a compact subset K_2 of $\mathcal{G}$ with $\|\Psi_0^{-1}(L) - \chi_{K_2}\Psi_0^{-1}(L)\|_\infty < \epsilon/(2\zeta(K_1))$. Therefore

$$\begin{aligned}&\|\zeta \circ \Psi_0^{-1}(L) - (\chi_{K_1}\zeta) \circ (\chi_{K_2}\Psi_0^{-1}(L))\|_\infty \\ &\quad \le \|\zeta - \chi_{K_1}\zeta\| \, \|\Psi_0^{-1}(L)\|_\infty + \|\chi_{K_1}\zeta\| \, \|\Psi_0^{-1}(L) - \chi_{K_2}\Psi_0^{-1}(L)\|_\infty \le \epsilon.\end{aligned}$$

On the other hand, if we set $\eta := (\chi_{K_1}\zeta) * \mu$, then

$$\begin{aligned}\big((\chi_{K_1}\zeta) \circ (\chi_{K_2}\Psi_0^{-1}(L))\big)_\mu(x) &= \int_{\mathcal{G}} \chi_{K_2}(yx)(\Psi_0^{-1}(L))_\eta(yx)\, d(\chi_{K_1}\zeta)(y) \\ &= \int_{K_1} \chi_{K_2}(yx)(\Psi_0^{-1}(L))_\eta(yx)\, d\zeta(y).\end{aligned}$$

For every $x \in \mathcal{G} \setminus K_1^{-1}K_2$, we get $K_1x \subseteq \mathcal{G} \setminus K_2$, and hence

$$\big((\chi_{K_1}\zeta) \circ (\chi_{K_2}\Psi_0^{-1}(L))\big)_\mu(x)\chi_{\mathcal{G}\setminus K_1^{-1}K_2}(x) = 0 \quad \mu\text{-a.e.} \quad (\mu \in M(\mathcal{G})).$$

Therefore $(\chi_{K_1}\zeta) \circ (\chi_{K_2}\Psi_0^{-1}(L))$ is in $GL_0(\mathcal{G})$, and so $\zeta \circ \Psi_0^{-1}(L) \in GL_0(\mathcal{G})$.

Now, for every $\mu \in M(\mathcal{G})$ and every Borel subset A of $\mathcal{G}$, we have $\chi_A\mu \ll \mu$, where $\chi_A\mu \in M(\mathcal{G})$ is defined by $\chi_A\mu(B) = \int_B \chi_A \, d\mu$. So

$$\begin{aligned}\int_{\mathcal{G}} \chi_A(\Psi^{-1}(L\zeta))_\mu \, d\mu &= \int_{\mathcal{G}} (\Psi^{-1}(L\zeta))_{\chi_A\mu} \, d(\chi_A\mu) = \langle L, \zeta * (\chi_A\mu) \rangle \\ &= \int_{\mathcal{G}} (\Psi_0^{-1}(L))_{\zeta*(\chi_A\mu)} \, d(\zeta * (\chi_A\mu)) \\ &= \int_{\mathcal{G}} l_\zeta(\Psi_0^{-1}(L))_{\zeta*(\chi_A\mu)} \, d(\chi_A\mu) \\ &= \int_{\mathcal{G}} \chi_A(\zeta \circ \Psi_0^{-1}(L))_\mu \, d\mu.\end{aligned}$$

Thus $\Psi^{-1}(L\zeta) = \zeta \circ \Psi_0^{-1}(L) \in GL_0(\mathcal{G})$, and hence $L\zeta \in (M(\mathcal{G}), \beta(\mathcal{G}))^*$. This gives the first identity of (i). The second one can be proved in a similar way.

Moreover

$$\Psi_0^{-1}(\zeta \circ L) = \Psi_0^{-1}(L\zeta) = \Psi^{-1}(L\zeta) = \zeta \circ \Psi_0^{-1}(L),$$

which is the first identity of (ii). The proof of the other is similar.

(iii) Let $\mathfrak{m} \in (M(\mathcal{G}), \beta(\mathcal{G}))^{**}$. Then $\mathfrak{m}L \in (M(\mathcal{G}), \mathtt{n}(\mathcal{G}))^*$, since for any $\mu \in M(\mathcal{G})$, we have

$$|\langle \mathfrak{m}L, \mu\rangle| = |\langle \Psi_0^*(\mathfrak{m}), \Psi_0^{-1}(L\mu)\rangle| \le \|\Psi_0^*(\mathfrak{m})\| \, \|L\| \, \|\mu\|.$$

So it remains to show that $\Psi^{-1}(\mathfrak{m}L)$ vanishes at infinity. Suppose that $\mathcal{K}(\mathcal{G})$ is directed by inclusion and, for each $K \in \mathcal{K}(\mathcal{G})$, $u_K \in C_0(\mathcal{G})$ is chosen so that $0 \le u_K \le 1$ and $u_K(x) = 1$ $(x \in K)$. Then (u_K) is a bounded approximate identity for $GL_0(\mathcal{G})$. Also, since $GL_0(\mathcal{G})$ and its dual are spanned by their positive elements, we can suppose that $\Psi_0^{-1}(L)$ and $\Psi_0^*(\mathfrak{m})$ are real and positive. Let σ denote the restriction of $\Psi_0^*(\mathfrak{m})$ to $C_0(\mathcal{G})$, and consider σ as an element of $M(\mathcal{G})$. Then, for given $\varepsilon > 0$, there is a set K in $\mathcal{K}(\mathcal{G})$ such that $\sigma(\mathcal{G} \setminus K) < \varepsilon/2$. Now let π be the continuous linear functional on $GL_0(\mathcal{G})$ defined by

$$\langle \pi, f\rangle = \langle \Psi_0^*(\mathfrak{m}), f\chi_{\mathcal{G}\setminus K}\rangle \quad (f \in GL_0(\mathcal{G})).$$

Since π is a positive, there exists $K_0 \in \mathcal{K}(\mathcal{G})$ such that $\|\pi\| - \varepsilon/2 \le \langle \pi, u_{K_0}\rangle$. Thus

$$\|\pi\| - \varepsilon/2 \le \|\pi|_{C_0(\mathcal{G})}\| = \sigma(\mathcal{G} \setminus K),$$

which shows that $\|\pi\| \le \varepsilon$.

Since $\Psi_0^{-1}(L) \in GL_0(\mathcal{G})$, there is a set B in $\mathcal{K}(\mathcal{G})$ with

$$|(\Psi_0^{-1}(L))_\mu(x)\chi_{\mathcal{G}\setminus B}(x)| < \varepsilon \quad \mu\text{-a.e.} \quad (\mu \in M(\mathcal{G})).$$

Moreover, if ν is an arbitrary probability measure in $M(\mathcal{G})$ and if we set $\zeta := (\chi_{\mathcal{G}\setminus BK^{-1}})\nu$, then $\operatorname{supp}(\zeta) \subseteq \mathcal{G} \setminus BK^{-1}$ and there is a compact subset D in $\mathcal{G}$ with $D \subseteq \mathcal{G} \setminus BK^{-1}$ and $|\zeta|(\mathcal{G} \setminus D) < \varepsilon$. So, for every $x \in \mathcal{G} \setminus D^{-1}B$, we see that $Dx \subseteq \mathcal{G} \setminus B$, and hence

$$\begin{aligned}\big|(\zeta \circ \Psi_0^{-1}(L))_\mu(x)\big| &= \Big|\int_{\mathcal{G}} (\Psi_0^{-1}(L))_{\zeta*\mu}(yx)\, d\zeta(y)\Big| \\ &\le \int_{\mathcal{G}\setminus D} |(\Psi_0^{-1}(L))_{\zeta*\mu}(yx)|\, d|\zeta|(y) \\ &\quad + \int_{D} |(\Psi_0^{-1}(L))_{\zeta*\mu}(yx)|\, d|\zeta|(y) \\ &\le \varepsilon(\|\Psi_0^{-1}(L)\|_\infty + 1).\end{aligned}$$

Thus,

$$|(\zeta \circ \Psi_0^{-1}(L))_\mu(x)\chi_{\mathcal{G}\setminus D^{-1}B}(x)| \le (\|\Psi_0^{-1}(L)\|_\infty + 1)\varepsilon \quad \mu\text{-a.e.} \quad (\mu \in M(\mathcal{G})).$$

This together with $D^{-1}B \cap K = \emptyset$ implies that

$$\|\Psi^{-1}(L\zeta)\chi_K\|_\infty = \sup_{\mu\in M(\mathcal{G})} \|(\zeta \circ \Psi_0^{-1}(L))_\mu \chi_K\|_\infty < \varepsilon(\|L\|+1).$$

Thus

$$\begin{aligned}\int\limits_{\mathcal{G}\setminus BK^{-1}} (\Psi^{-1}(\mathfrak{m}L))_\zeta \, d\zeta &= \langle \Psi_0^*(\mathfrak{m}), \Psi_0^{-1}(L\zeta)\chi_K\rangle + \langle \pi, \Psi_0^{-1}(L\zeta)\rangle \\ &\le \varepsilon(\|L\|+1)\|\Psi_0^*(\mathfrak{m})\| + \varepsilon\|L\|.\end{aligned}$$

On the other hand, since $\zeta \ll \nu$, we have

$$\begin{aligned}\int\limits_{\mathcal{G}\setminus BK^{-1}} (\Psi^{-1}(\mathfrak{m}L))_\zeta(x)\, d\zeta(x) &= \int\limits_{\mathcal{G}\setminus BK^{-1}} (\Psi^{-1}(\mathfrak{m}L))_\nu(x)\, d\zeta(x) \\ &= \int\limits_{\mathcal{G}\setminus BK^{-1}} (\Psi^{-1}(\mathfrak{m}L))_\nu(x)\, d\nu(x).\end{aligned}$$

This shows that if $\nu \in M(\mathcal{G})$, then

$$(\Psi^{-1}(\mathfrak{m}L))_\nu(x) \le \varepsilon[(\|L\|+1)\|\Psi_0^*(\mathfrak{m})\| + \|L\|]$$

for ν-almost all $x \in \mathcal{G} \setminus BK^{-1}$. Therefore $\Psi^{-1}(\mathfrak{m}L) \in GL_0(\mathcal{G})$. That is, $\mathfrak{m}L \in (M(\mathcal{G}), \beta(\mathcal{G}))^*$. ∎

THEOREM 3.2. *The space* $(M(\mathcal{G}), \beta(\mathcal{G}))^{**}$ *with the first Arens multiplication* $\diamond$ *is a Banach algebra, where* $m \diamond n$ *is defined by the equation*

$$\langle \mathfrak{m} \diamond \mathfrak{n}, L\rangle = \langle \mathfrak{m}, \mathfrak{n}L\rangle \quad (\mathfrak{m}, \mathfrak{n} \in (M(\mathcal{G}), \beta(\mathcal{G}))^{**},\ L \in (M(\mathcal{G}), \beta(\mathcal{G}))^*).$$

Proof. Using Proposition 3.1, we need to show only that $\mathfrak{m} \diamond \mathfrak{n} \in (M(\mathcal{G}), \beta(\mathcal{G}))^{**}$. Note that, for every $f \in GL_0(\mathcal{G})$, we have

$$\langle \mathfrak{m} \diamond \mathfrak{n}, \Psi_0(f)\rangle = \langle \mathfrak{m}, \mathfrak{n}\Psi_0(f)\rangle = \langle \Psi_0^*(\mathfrak{m}), \Psi_0^{-1}(\mathfrak{n}\Psi_0(f))\rangle.$$

Moreover it follows easily that

$$\|\Psi_0^{-1}(\mathfrak{n}\Psi_0(f))\| \le \|\Psi_0^*(\mathfrak{n})\|\, \|f\|_\infty.$$

So the linear functional Λ on $GL_0(\mathcal{G})$ defined by

$$\Lambda(f) = \langle \mathfrak{m} \diamond \mathfrak{n}, \Psi_0(f)\rangle \quad (f \in GL_0(\mathcal{G}))$$

is bounded by $\|\Psi_0^*(\mathfrak{m})\|\, \|\Psi_0^*(\mathfrak{n})\|$. In particular, $\Lambda \in GL_0(\mathcal{G})^*$ and hence $\mathfrak{m} \diamond \mathfrak{n} = \Psi_0^{*-1}(\Lambda) \in (M(\mathcal{G}), \beta(\mathcal{G}))^{**}$. Moreover $\|\mathfrak{m} \diamond \mathfrak{n}\| \le \|\mathfrak{m}\|\, \|\mathfrak{n}\|$. Finally, one can see easily that $\diamond$ is an associative product on $(M(\mathcal{G}), \beta(\mathcal{G}))^{**}$. ∎

Let $M_a(\mathcal{G})$ be the closed, two-sided ideal of all measures in $M(\mathcal{G})$ which are absolutely continuous with respect to the Haar measure of $\mathcal{G}$, and identify $M_a(\mathcal{G})$ with $L^1(\mathcal{G})$. It is well known that when $\mathcal{G}$ is compact, $L^1(\mathcal{G})$ is a two-sided ideal of $(M(\mathcal{G}), \mathtt{n}(\mathcal{G}))^{**}$ [4, Lemma 2.16]. Since the coincidence of the two topologies $\beta(\mathcal{G})$ and $\mathtt{n}(\mathcal{G})$ is equivalent to the compactness of $\mathcal{G}$ [10], the following proposition can be considered as an extension of

[4, Lemma 2.16]. Note that if $P : GL(\mathcal{G}) \to L^\infty(\mathcal{G})$ denotes the adjoint of the natural embedding from $M_a(\mathcal{G})$ into $M(\mathcal{G})$, then P is a surjective norm-decreasing map.

PROPOSITION 3.3. *The space $M_a(\mathcal{G})$ is a closed ideal in $(M(\mathcal{G}), \beta(\mathcal{G}))^{**}$.*

Proof. Since $\Psi_0^*(M_a(\mathcal{G}))$ is a closed subspace of $GL_0(\mathcal{G})^*$, it follows that $M_a(\mathcal{G})$ is a closed subspace of $(M(\mathcal{G}), \beta(\mathcal{G}))^{**}$. Now suppose that $\zeta \in M_a(\mathcal{G})$ and $\mathfrak{m} \in (M(\mathcal{G}), \beta(\mathcal{G}))^{**}$. We show only that $\mathfrak{m} \diamond \zeta \in M_a(\mathcal{G})$; with a similar argument one can show that $\zeta \diamond \mathfrak{m} \in M_a(\mathcal{G})$. To this end, suppose that σ denotes the restriction of $\Psi_0^*(\mathfrak{m})$ to $C_0(\mathcal{G})$. Then $\Psi_0^*(\mathfrak{m}) = \sigma + \mathfrak{n}$, where $\mathfrak{n} \in C_0(\mathcal{G})^\perp$. Since $M_a(\mathcal{G})$ is an ideal in $M(\mathcal{G})$, we have $\sigma * \zeta \in M_a(\mathcal{G})$. So it suffices to show that $\mathfrak{m} \diamond \zeta = \sigma * \zeta$.

First let us recall that $C_0(\mathcal{G})$ is a closed subspace of $GL_0(\mathcal{G})$ in a natural way, as observed in Section 1, and note that if $L \in (M(\mathcal{G}), \beta(\mathcal{G}))^*$, then, by Proposition 3.1, $\Psi_0^{-1}(\zeta L) = \Psi_0^{-1}(L) \circ \zeta \in GL_0(\mathcal{G})$, that is, $\Psi^{-1}(\zeta L)$ vanishes at infinity. Moreover, $\Psi_0^{-1}(L) \circ \zeta$ is the generalized function $h = (h_\mu)_{\mu \in M(\mathcal{G})}$ in $GL(\mathcal{G})$ for which

$$h_\mu(x) = (\Psi_0^{-1}(L) \circ \zeta)_\mu(x) = r_\zeta f_{\mu*\zeta}(x) = \int_{\mathcal{G}} (\Psi_0^{-1}(L))_{\mu*\zeta}(xy)\, d\zeta(y)$$

for all $\mu \in M(\mathcal{G})$. On the other hand, if P is as in the paragraph preceding this proposition, then, for an arbitrary μ in $M(\mathcal{G})$ and any Borel subset A of $\mathcal{G}$, we have

$$\begin{aligned}\int_{\mathcal{G}} \chi_A h_\mu\, d\mu &= \int_{\mathcal{G}} h_{\chi_A\mu}\, d(\chi_A\mu) = \int_{\mathcal{G}} (\Psi_0^{-1}(L))_{(\chi_A\mu)*\zeta}\, d((\chi_A\mu) * \zeta) \\ &= \langle P(\Psi_0^{-1}(L)), (\chi_A\mu) * \zeta\rangle \\ &= \int_{\mathcal{G}} \chi_A(x)\Big(\int_{\mathcal{G}} P(\Psi_0^{-1}(L))(xy)\, d\zeta(y)\Big)\, d\mu(x).\end{aligned}$$

Hence $(\Psi_0^{-1}(L) \circ \zeta)_\mu = g$ $|\mu|$-a.e., where $g(x) = \int_{\mathcal{G}} P(\Psi_0^{-1}(L))(xy)\, d\zeta(y)$ for all $\mu \in M(\mathcal{G})$. In particular, $g \in C_b(\mathcal{G})$: this follows from the known fact that $M_a(\mathcal{G})$ can be identified with those $\nu \in M(\mathcal{G})$ such that the maps $x \mapsto \delta_x * |\nu|$ and $x \mapsto |\nu| * \delta_x$ from $\mathcal{G}$ into $M(\mathcal{G})$ are norm-continuous [8, 19.27 and 20.31]. Therefore $\Psi_0^{-1}(\zeta L) \in C_0(\mathcal{G})$. Hence we have

$$\langle \mathfrak{m} \diamond \zeta, L\rangle = \langle \sigma, \Psi_0^{-1}(\zeta L)\rangle + \langle \mathfrak{n}, \Psi_0^{-1}(\zeta L)\rangle = \langle \sigma * \zeta, L\rangle,$$

whence $\mathfrak{m} \diamond \zeta = \sigma * \zeta \in M_a(\mathcal{G})$, as required. ■

We say that a functional $\mathfrak{m} \in GL(\mathcal{G})^*$ [respectively, $\mathfrak{m} \in GL_0(\mathcal{G})^*$] has *compact carrier* if there exists a compact set K such that $\langle \mathfrak{m}, f\rangle = \langle \mathfrak{m}, \chi_K f\rangle$ for all $f \in GL(\mathcal{G})$ [respectively, $f \in GL_0(\mathcal{G})$]. Such a compact set K is called a *compact carrier* for $\mathfrak{m}$. Now let $M_{\mathcal{G}}$ be the norm-closure of the set of functionals with compact carrier in $GL(\mathcal{G})^*$. Using the identification of $GL(\mathcal{G})$

[respectively, $GL_0(\mathcal{G})$] with $(M(\mathcal{G}),\mathtt{n}(\mathcal{G}))^*$ [respectively, $(M(\mathcal{G}),\beta(\mathcal{G}))^*$], we consider $(M(\mathcal{G}),\beta(\mathcal{G}))^{**}$ as a subspace of $(M(\mathcal{G}),\mathtt{n}(\mathcal{G}))^{**}$ by the following lemma, which is needed for the proof of our main result.

LEMMA 3.4. *The subspace $M_{\mathcal{G}}$ of $(M(\mathcal{G}),\mathtt{n}(\mathcal{G}))^{**}$ is isometrically isomorphic to $(M(\mathcal{G}),\beta(\mathcal{G}))^{**}$.*

Proof. Let (u_K) be the approximate identity of $GL_0(\mathcal{G})$ which is defined on page 202. Then an argument similar to the proof of [15, Proposition 2.6] shows that the space of all functionals in $GL_0(\mathcal{G})^*$ with compact carrier is norm-dense in $GL_0(\mathcal{G})^*$. So it is enough to show that the restriction map is an isometry from the space of all compact carrier elements of $GL(\mathcal{G})^*$ onto the space of compact carrier elements of $GL_0(\mathcal{G})^*$. To see this, suppose that $\mathfrak{m}\in GL_0(\mathcal{G})^*$ has compact carrier and $\mathfrak{n}$ is a Hahn–Banach extension of $\mathfrak{m}$ to $GL(\mathcal{G})$. Choose a compact set K in $\mathcal{G}$ such that $\langle\mathfrak{m},f\rangle=\langle\mathfrak{m},\chi_K f\rangle$ $(f\in GL_0(\mathcal{G}))$. Then $\mathfrak{n}'\in GL(\mathcal{G})^*$ defined by $\langle\mathfrak{n}',f\rangle=\langle\mathfrak{n},\chi_K f\rangle$ $(f\in GL(\mathcal{G}))$ is also an extension of $\mathfrak{m}$ to $GL(\mathcal{G})$ with norm $\|\mathfrak{n}\|$. Moreover, the restriction of every $\mathfrak{n}\in GL(\mathcal{G})^*$ with compact carrier to $GL(\mathcal{G})$ has the same norm as $\mathfrak{n}$. ∎

Recall that, for a locally compact space X, a subset $S\subseteq M(X)$ is called *solid with respect to absolute continuity* if $n\in S$ whenever $n\ll s$ for some $s\in S$, where $M(X)$ is the Banach space of all complex regular Borel measures on X with the total variation norm.

Now we can state our main result.

THEOREM 3.5. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two locally compact groups. Suppose that T is an isometric isomorphism from $(M(\mathcal{G}_1),\beta(\mathcal{G}_1))^{**}$ onto $(M(\mathcal{G}_2),\beta(\mathcal{G}_2))^{**}$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic in the category of topological groups.*

Proof. By Proposition 3.3, $M_a(\mathcal{G}_i)$ is a closed ideal in $(M(\mathcal{G}_i),\beta(\mathcal{G}_i))^{**}$ for $i=1,2$. So we can adapt the arguments of [6, Lemma 5 and Theorem 6], as compactness of G in those results was used only to conclude that $M_a(\mathcal{G})$ is an ideal in $(M(\mathcal{G}_1),\mathtt{n}(\mathcal{G}_1))^{**}$. Thus using those arguments we conclude that $M_a(\mathcal{G}_i)$ is the unique minimal proper closed subset of $(M(\mathcal{G}_i),\beta(\mathcal{G}_i))^{**}$ which is an algebraic ideal and a solid set in $(M(\mathcal{G}_i),\beta(\mathcal{G}_i))^{**}$ with respect to absolute continuity. Being an algebra isomorphism, T is an order-preserving map from $(M(\mathcal{G}_1),\beta(\mathcal{G}_1))^{**}=GL_0(\mathcal{G}_1)^*=M(X_1)$ onto $(M(\mathcal{G}_2),\beta(\mathcal{G}_2))^{**}=GL_0(\mathcal{G}_2)^*=M(X_2)$, where X_i, for $i=1,2$, are the character spaces of the C*-algebras $GL_0(\mathcal{G}_i)$. Hence T restricted to $M_a(\mathcal{G}_1)$ is an isometric isomorphism from $M_a(\mathcal{G}_1)$ onto $M_a(\mathcal{G}_2)$. Thus Wendel's Theorem [18] implies that $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic in the category of topological groups. ∎

Let $\sigma_0(\mathcal{G})$ be the weak topology $\sigma(M(\mathcal{G}),GL_0(\mathcal{G}))$. Then $\sigma_0(\mathcal{G})\le\beta(\mathcal{G})$ $\le\mathtt{n}(\mathcal{G})$. As was shown in [10, Theorem 2.5], except for the trivial case where $\mathcal{G}$ is finite, $M(\mathcal{G})$ has infinitely many locally convex topologies τ with $\sigma_0(\mathcal{G})\le\tau\le\beta(\mathcal{G})$, and therefore $(M(\mathcal{G}),\tau)^{**}$ can be identified with

$GL_0(\mathcal{G})^*$. The following theorem, which is our main result, is an immediate consequence of the above theorem.

THEOREM 3.6. *Let $\mathcal{G}$ be a locally compact group, and let τ be a locally convex topology on $M(\mathcal{G})$ with $\sigma_0(\mathcal{G}) \leq \tau \leq \beta(\mathcal{G})$. Then $(M(\mathcal{G}), \tau)^{**}$ determines $\mathcal{G}$ in the category of topological groups.*

If $\mathcal{G}$ is a compact group, then $GL_0(\mathcal{G}) = GL(\mathcal{G})$, and hence $(M(\mathcal{G}), \tau)^{**} = M(\mathcal{G})^{**}$. So we obtain the following corollary, which is the main result of [6] and was reproved in [1]. Note that our approach is different from those of [6] and [1].

COROLLARY 3.7. *Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are compact groups such that $M(\mathcal{G}_1)^{**}$ and $M(\mathcal{G}_2)^{**}$ are isometrically isomorphic. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are topologically isomorphic.*

For arbitrary locally compact groups, if the answer to the following problem is positive, then an alternative proof of Ghahramani–Lau's isometrical isomorphism theorem would be obtained.

PROBLEM 3.8. *Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are two locally compact groups and T is an isometric isomorphism from $M(\mathcal{G}_1)^{**}$ onto $M(\mathcal{G}_2)^{**}$. Can we say that*

$$T((M(\mathcal{G}_1), \beta(\mathcal{G}_1))^{**}) = (M(\mathcal{G}_2), \beta(\mathcal{G}_2))^{**} \ ?$$

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