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Average r-rank Artin conjecture

by

LORENZO MENICI (Roma) and CIHAN PEHLIVAN (Istanbul)

1. Introduction. Artin's conjecture for primitive roots (1927) states that for any integer $a \neq 0, \pm 1$ which is not a perfect square there exist infinitely many prime numbers p for which a is a primitive root modulo p. In particular, Artin conjectured that the number of primes not exceeding x for which a is a primitive root, $N_a(x)$, asymptotically satisfies

$$N_a(x) \sim A(a)\,\mathrm{Li}(x) \qquad \text{as } x \to \infty,$$

where $\mathrm{Li}(x)$ is the logarithmic integral and the positive constant $A(a)$ depends on the integer a. A breakthrough in this area was achieved by Hooley's [8] who proved Artin's conjecture under the assumption of the Generalized Riemann Hypothesis (GRH) for the Dedekind zeta function over the Kummer extension $\mathbb{Q}(a^{1/k}, \zeta_k)$ for any positive square-free integer k. Several generalizations of Artin's original conjecture were studied by many authors during the following years (for an exhaustive survey see [10]). A first unconditional result on Artin's conjecture in the 3-rank case was found by Gupta and Ram Murty [5], improved a few years later by Heath-Brown [7].

In the case of rank $r = 1$, a first study of the average behavior of $N_a(x)$ was proposed by Stephens [14] in 1969: Stephens proved that if $T > \exp(4(\log x \log\log x)^{1/2})$, then

$$\frac{1}{T}\sum_{a \le T} N_a(x) = \sum_{p \le x} \frac{\varphi(p-1)}{p-1} + O\left(\frac{x}{(\log x)^D}\right) \tag{1}$$
$$= A\,\mathrm{Li}(x) + O\left(\frac{x}{(\log x)^D}\right),$$

where φ is the Euler totient function, $A := \prod_p \bigl(1 - \frac{1}{p(p-1)}\bigr)$ is Artin's constant

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and D is an arbitrary constant greater than 1. Stephens also proved that if $T > \exp(6(\log x \log\log x)^{1/2})$, then

$$\frac{1}{T}\sum_{a\le T}\{N_a(x) - A\,\mathrm{Li}(x)\}^2 \ll \frac{x^2}{(\log x)^{D'}} \tag{2}$$

for any constant $D' > 2$. In 1976, Stephens refined his results with different methods [15], getting both the asymptotic bounds (1) and (2) under the weaker assumption $T > \exp(C(\log x)^{1/2})$ with C a positive constant.

For any $a \in \mathbb{N}\setminus\{0,\pm 1\}$ and $m \in \mathbb{N}$, let $N_{a,m}(x)$ be the number of primes $p \equiv 1 \pmod m$ not exceeding x such that $[\mathbb{F}_p^* : \langle a \pmod p\rangle] = m$. For $T > \exp(4(\log x \log\log x)^{1/2})$ Moree [11] showed that

$$\frac{1}{T}\sum_{a\le T} N_{a,m}(x) = \sum_{\substack{p\le x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} \frac{\varphi((p-1)/m)}{p-1} + O\biggl(\frac{x}{(\log x)^E}\biggr) \tag{3}$$

for any constant $E > 1$.

In the present work, we will discuss the average version of the *r-rank Artin quasi primitive root conjecture*, adapting the methods used in [14] by Stephens to the case of rank r. Let $\Gamma \subset \mathbb{Q}^*$ be a multiplicative subgroup of finite rank r. For almost all primes, namely those primes p such that for all $g \in \Gamma$ the p-adic valuation $v_p(g)$ is 0, one can consider the reduction group

$$\Gamma_p = \{g \pmod p : g \in \Gamma\},$$

which is a well defined subgroup of the multiplicative group $\mathbb{F}_p^*$. We denote by $N_{\Gamma,m}(x)$ the number of primes $p \equiv 1 \pmod m$ not exceeding x for which $[\mathbb{F}_p^* : \Gamma_p] = m$. It was proven by Cangelmi, Pappalardi and Susa [12, 2, 13], assuming the GRH for $\mathbb{Q}(\zeta_k, \Gamma^{1/k})$ for any natural number k, that for any $\varepsilon > 0$, if $m \le x^{\frac{r-1}{(r+1)(4r+2)}-\varepsilon}$, then

$$N_{\Gamma,m}(x) = \biggl(\delta_\Gamma^m + O\biggl(\frac{1}{\varphi(m^{r+1})(\log x)^r}\biggr)\biggr)\mathrm{Li}(x) \quad \text{as } x \to \infty,$$

where δ_Γ^m is a rational multiple of

$$C_r = \sum_{n\ge 1}\frac{\mu(n)}{n^r\varphi(n)} = \prod_p \biggl(1 - \frac{1}{p^r(p-1)}\biggr).$$

Here we restrict ourselves to studying subgroups $\Gamma = \langle a_1, \dots, a_r\rangle$, with $a_i \in \mathbb{Z}$ for all $i = 1, \dots, r$, and we prove the following theorems:

THEOREM 1. *Suppose*

$$T^* := \min\{T_i : i = 1, \dots, r\} > \exp(4(\log x \log\log x)^{1/2})$$

and $m \leq (\log x)^D$ *for an arbitrary positive constant* D. *Then*

$$\frac{1}{T_1 \cdots T_r} \sum_{\substack{a_i \in \mathbb{Z} \\ 0<a_i \leq T_i \\ 1<i\leq r}} N_{\langle a_1, \ldots, a_r \rangle, m}(x) = C_{r,m} \operatorname{Li}(x) + O\biggl(\frac{x}{(\log x)^M}\biggr),$$

where

$$C_{r,m} := \sum_{n \geq 1} \frac{\mu(n)}{(nm)^r \varphi(nm)}$$

and $M > 1$ *is arbitrarily large.*

Theorem 2. *Suppose* $T^* > \exp(6(\log x \log\log x)^{1/2})$ *and* $m \leq (\log x)^D$ *for an arbitrary positive constant* D. *Then*

$$\frac{1}{T_1 \cdots T_r} \sum_{\substack{a_i \in \mathbb{Z} \\ 0<a_i 1 \leq T_i \\ 1\leq i< r}} \{N_{\langle a_1, \ldots, a_r \rangle, m}(x) - C_{r,m} \operatorname{Li}(x)\}^2 \ll \frac{x^2}{(\log x)^{M'}}$$

where $M' > 2$ *is arbitrarily large.*

Now since $\varphi(mn) = \varphi(m)\varphi(n)\gcd(m,n)/\varphi(\gcd(m,n))$ and $\gcd(m,n)$ is a multiplicative function of n for any fixed integer m, we also have the following Euler product expansion:

$$\begin{aligned} C_{r,m} &= \frac{1}{m^r \varphi(m)} \sum_{n \geq 1} \frac{\mu(n)}{n^r \varphi(n)} \prod_{p \mid \gcd(m,n)} \biggl(1 - \frac{1}{p}\biggr) \\ &= \frac{1}{m^r \varphi(m)} \prod_{p \mid m} \biggl[1 - \frac{1}{p^r(p-1)}\biggl(1 - \frac{1}{p}\biggr)\biggr] \prod_{p \nmid m} \biggl(1 - \frac{1}{p^r(p-1)}\biggr) \\ &= \frac{1}{m^{r+1}} \prod_{p \mid m} \biggl(1 - \frac{p^r}{p^{r+1}-1}\biggr)^{-1} C_r. \end{aligned}$$

The results found in the present paper (see in particular (8) and Lemma 2) will lead as a side product to the asymptotic identity

$$\frac{1}{T_1 \cdots T_r} \sum_{\substack{a_i \in \mathbb{Z} \\ 0<a_i \leq T_i \\ 1\leq i\leq r}} N_{\langle a_1, \ldots, a_r \rangle, m}(x) = \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{m}}} \frac{J_r((p-1)/m)}{(p-1)^r} + O\biggl(\frac{x}{(\log x)^M}\biggr)$$

if $T_i > \exp(4(\log x \log\log x)^{1/2})$ for all $i = 1, \ldots, r$, $m \leq (\log x)^D$ and $M > 1$ is an arbitrary constant, where

$$J_r(n) = n^r \prod_{\substack{\ell \mid n \\ \ell \text{ prime}}} \biggl(1 - \frac{1}{\ell^r}\biggr)$$

is *Jordan's totient function*. This provides a natural generalization of Moree's result in [11].

Theorem 2 leads to the following corollary:

COROLLARY 1. *For any* $\epsilon > 0$, *let*

$$\mathcal{H} := \{(a_1, \dots, a_r) \in \mathbb{Z}^r : 0 < a_i \le T_i,\ i \in \{1, \dots, r\},\\ |N_{\langle a_1,\dots,a_r\rangle,m}(x) - C_{r,m}\operatorname{Li}(x)| > \epsilon \operatorname{Li}(x)\}.$$

Then, supposing $T^* > \exp(6(\log x \log\log x)^{1/2})$, *we have*

$$\#\mathcal{H} \le KT_1 \dots T_r/\epsilon^2(\log x)^F$$

for every positive constant F, *where* K *is an absolute positive constant.*

Proof. The proof is a trivial generalization of that in [14, Corollary, p. 187]. ■

2. Notation and conventions. In order to simplify the formulas, we introduce the following notation. Underlined letters stand for general r-tuples defined within some set, e.g. $\underline{a} = (a_1, \dots, a_r) \in (\mathbb{F}_p^*)^r$ or $\underline{T} = (T_1, \dots, T_r) \in (\mathbb{R}^{>0})^r$; moreover, given two r-tuples, $\underline{a}$ and $\underline{n}$, their scalar product is $\underline{a} \cdot \underline{n} = a_1 n_1 + \dots + a_r n_r$, and we write e.g. $\underline{a} \le \underline{n}$ if $a_i \le n_i$ for all i. The null vector is $\underline{0} = \{0, \dots, 0\}$. Similarly, if $\underline{\chi} = (\chi_1, \dots, \chi_r)$ is an r-tuple of Dirichlet characters and $\underline{a} \in \mathbb{Z}^r$, then we denote $\underline{\chi}(\underline{a}) = \chi_1(a_1) \cdots \chi_r(a_r) \in \mathbb{C}$.

In addition, $(q, \underline{a}) := (q, a_1, \dots, a_r) = \gcd(q, a_1, \dots, a_r)$; otherwise, to avoid possible misinterpretations, we will write explicitly $\gcd(n_1, \dots, n_r)$ instead of $(\underline{n})$. Given any r-tuple $\underline{a} \in \mathbb{Z}^r$, we indicate with

$$\langle \underline{a} \rangle_p := \langle a_1 \pmod p, \dots, a_r \pmod p \rangle$$

the reduction modulo p of the subgroup $\langle \underline{a} \rangle = \langle a_1, \dots, a_r \rangle \subset \mathbb{Q}$; if $\Gamma = \langle a_1, \dots, a_r \rangle$, then $\Gamma_p = \langle \underline{a} \rangle_p$.

In the whole paper, ℓ and p will always indicate prime numbers. For a finite field $\mathbb{F}_p$, we set $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ and $\widehat{\mathbb{F}_p^*}$ will denote its relative dual group (or character group). Finally, given an integer a, $v_p(a)$ is its p-adic valuation.

3. Lemmata. Let $q > 1$ be an integer and let $\underline{n} \in \mathbb{Z}^r$. We define the *multiple Ramanujan sum* as

$$c_q(\underline{n}) := \sum_{\substack{\underline{a} \in (\mathbb{Z}/q\mathbb{Z})^r \\ (q,\underline{a})=1}} e^{2\pi i \underline{a}\cdot\underline{n}/q}.$$

It is well known (see [6, Theorem 272]) that, for any integer n,

$$c_q(n) = \mu\left(\frac{q}{(q,n)}\right)\frac{\varphi(q)}{\varphi(q/(q,n))}. \tag{4}$$

In the following lemma, we generalize this result to r-rank.

LEMMA 1. *Let*

$$J_r(m) := m^r \prod_{\ell \mid m} \left(1 - \frac{1}{\ell^r}\right)$$

be Jordan's totient function. Then

$$c_q(\underline{n}) = \mu\left(\frac{q}{(q, \underline{n})}\right) \frac{J_r(q)}{J_r(q/(q, \underline{n}))}.$$

Proof. Let us start by considering the case when $q = \ell$ is prime. Then

$$\begin{aligned} c_\ell(\underline{n}) &= \sum_{\underline{a} \in (\mathbb{Z}/\ell\mathbb{Z})^r \setminus \{\underline{0}\}} e^{2\pi i \underline{a} \cdot \underline{n}/\ell} \\ &= -1 + \prod_{j=1}^{r} \sum_{a_j=1}^{\ell} e^{2\pi i a_j n_j/\ell} = \begin{cases} -1 & \text{if } \ell \nmid \gcd(n_1, \ldots, n_r), \\ \ell^r - 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Next we consider the case when $q = \ell^k$ with $k \geq 2$ and ℓ prime. We need to show that

$$c_{\ell^k}(\underline{n}) = \begin{cases} 0 & \text{if } \ell^{k-1} \nmid \gcd(n_1, \ldots, n_r), \\ -\ell^{r(k-1)} & \text{if } \ell^{k-1} \| \gcd(n_1, \ldots, n_r), \\ \ell^{rk}(1 - 1/\ell^r) & \text{if } \ell^k \mid \gcd(n_1, \ldots, n_r). \end{cases}$$

To do so, we write

$$\begin{aligned} c_{\ell^k}(\underline{n}) &= \sum_{\substack{\underline{a} \in (\mathbb{Z}/\ell^k\mathbb{Z})^r \\ (\ell, \underline{a})=1}} e^{2\pi i \underline{a} \cdot \underline{n}/\ell^k} \\ &= c_{\ell^k}(n_1) \prod_{j=2}^{r} \sum_{a_j=1}^{\ell^k} e^{2\pi i a_j n_j/\ell^k} + c_{\ell^k}(n_2, \ldots, n_r) \sum_{j=1}^{k} \sum_{\substack{a_1 \in \mathbb{Z}/\ell^k\mathbb{Z} \\ (a_1, \ell^k) = \ell^j}} e^{2\pi i a_1 n_1/\ell^k} \\ &= c_{\ell^k}(n_1) \prod_{j=2}^{r} \sum_{a_j=1}^{\ell^k} e^{2\pi i a_j n_j/\ell^k} + c_{\ell^k}(n_2, \ldots, n_r) \sum_{j=1}^{k} c_{\ell^{k-j}}(n_1). \end{aligned}$$

If we apply (4), we obtain

$$\begin{aligned} c_{\ell^k}(n_1, \ldots, n_r) &= \mu\left(\frac{\ell^k}{(\ell^k, n_1)}\right) \frac{\varphi(\ell^k)}{\varphi(\ell^k/(\ell^k, n_1))} \prod_{j=2}^{r} \sum_{a_j=1}^{\ell^k} e^{2\pi i a_j n_j/\ell^k} \\ &\quad + c_{\ell^k}(n_2, \ldots, n_r) \sum_{j=1}^{k} \mu\left(\frac{\ell^{k-j}}{(\ell^{k-j}, n_1)}\right) \frac{\varphi(\ell^{k-j})}{\varphi(\ell^{k-j}/(\ell^{k-j}, n_1))}. \end{aligned}$$

Now, for $k \geq 2$, we distinguish two cases:

(1) $\ell^{k-1} \nmid \gcd(n_1, \dots, n_r)$,
(2) $\ell^{k-1} \mid \gcd(n_1, \dots, n_r)$.

In the first case we can assume, without loss of generality, that $\ell^{k-1} \nmid n_1$. Hence $\mu(\ell^k/(\ell^k, n_1)) = 0$ and if $k_1 = v_\ell(n_1) < k-1$, then

$$\mu\left(\frac{\ell^{k-j}}{(\ell^{k-j}, n_1)}\right) = \mu(\ell^{\max\{0,k-k_1-j\}}) = \begin{cases} 0 & \text{if } 1 \le j \le k-k_1-2, \\ -1 & \text{if } j = k-k_1-1, \\ 1 & \text{if } j \ge k-k_1. \end{cases}$$

Hence

$$\sum_{j=1}^{k} \mu\left(\frac{\ell^{k-j}}{(\ell^{k-j}, n_1)}\right) \frac{\varphi(\ell^{k-j})}{\varphi(\ell^{k-j}/(\ell^{k-j}, n_1))} = -\ell^{k_1} + \sum_{j=k-k_1}^{k} \varphi(\ell^{k-j}) = 0.$$

In the second case, from the definition of $c_q(\underline{n})$ we find

$$\begin{aligned} c_{\ell^k}(\underline{n}) &= \ell^{r(k-1)} c_\ell\left(\frac{n_1}{\ell^{k-1}}, \dots, \frac{n_r}{\ell^{k-1}}\right) \\ &= \begin{cases} \ell^{rk}(1-1/\ell^r) & \text{if } \ell^k \mid \gcd(n_1, \dots, n_r), \\ -\ell^{r(k-1)} & \text{if } \ell^{k-1} \| \gcd(n_1, \dots, n_r). \end{cases} \end{aligned}$$

So, the formula holds for $q = \ell^k$.

Finally, we claim that if $q', q'' \in \mathbb{N}$ are such that $\gcd(q', q'') = 1$, then

$$c_{q'q''}(\underline{n}) = c_{q'}(\underline{n})\, c_{q''}(\underline{n});$$

this amounts to saying that the multiple Ramanujan sum is multiplicative in q. Indeed,

$$\begin{aligned} &\sum_{\substack{\underline{a} \in (\mathbb{Z}/q'\mathbb{Z})^r \\ (q', \underline{a})=1}} e^{2\pi i \underline{a}\cdot\underline{n}/q'} \sum_{\substack{\underline{b} \in (\mathbb{Z}/q''\mathbb{Z})^r \\ (q'', \underline{b})=1}} e^{2\pi i \underline{b}\cdot\underline{n}/q''} \\ &\qquad = \sum_{\substack{\underline{a} \in (\mathbb{Z}/q'\mathbb{Z})^r \\ \underline{b} \in (\mathbb{Z}/q''\mathbb{Z})^r \\ \gcd(q', \underline{a})=1 \\ \gcd(q'', \underline{b})=1}} e^{2\pi i [n_1(q''a_1+q'b_1)+\dots+n_r(q''a_r+q'b_r)]/(q'q'')}, \end{aligned}$$

and the result follows from the remark that since $\gcd(q', q'') = 1$,

- for all $j = 1, \dots, r$, if a_j runs through a complete set of residues modulo q', and b_j runs through a complete set of residues modulo q'', then $q''a_j + q'b_j$ runs through a complete set of residues modulo $q'q''$;
- for all $\underline{a} \in (\mathbb{Z}/q'\mathbb{Z})^r$ and $\underline{b} \in (\mathbb{Z}/q''\mathbb{Z})^r$,

$$\begin{aligned} &\gcd(q', \underline{a}) = 1 \text{ and } \gcd(q'', \underline{b}) = 1 \\ &\qquad \Leftrightarrow \gcd(q'q'', q'b_1 + q''a_1, \dots, q'b_r + q''a_r) = 1. \end{aligned}$$

The lemma now follows from the multiplicativity of μ and of J_r. ∎

From the previous lemma we deduce the following corollary:

Corollary 2. *Let p be an odd prime, and let $m \in \mathbb{N}$ be a divisor of $p-1$. Given an r-tuple $\underline{\chi} = (\chi_1, \ldots, \chi_r)$ of Dirichlet characters modulo p, set*

$$c_m(\underline{\chi}) := \frac{1}{(p-1)^r} \sum_{\substack{\underline{\alpha} \in (\mathbb{F}_p^*)^r \\ [\mathbb{F}_p^* : \langle \underline{\alpha} \rangle_p] = m}} \underline{\chi}(\underline{\alpha}).$$

Then

$$c_m(\underline{\chi}) = \frac{1}{(p-1)^r} \mu\left(\frac{p-1}{m \gcd\left(\frac{p-1}{m}, \frac{p-1}{\mathrm{ord}(\chi_1)}, \ldots, \frac{p-1}{\mathrm{ord}(\chi_r)}\right)}\right) \times \frac{J_r((p-1)/m)}{J_r\left(\frac{p-1}{m \gcd\left(\frac{p-1}{m}, \frac{p-1}{\mathrm{ord}(\chi_1)}, \ldots, \frac{p-1}{\mathrm{ord}(\chi_r)}\right)}\right)}. \tag{5}$$

Proof. Fix a primitive root $g \in \mathbb{F}_p^*$. For each $j = 1, \ldots, r$, let $n_j \in \mathbb{Z}/(p-1)\mathbb{Z}$ be such that

$$\chi_j = \chi_j(g) = e^{2\pi i n_j/(p-1)}.$$

Write $\alpha_j = g^{a_j}$ for $j = 1, \ldots, r$. Then

$$[\mathbb{F}_p^* : \langle \underline{\alpha} \rangle_p] = m \ \Leftrightarrow\ (p-1, \underline{a}) = m.$$

Therefore, naming $t = (p-1)/m$, we have

$$\begin{aligned} c_m(\underline{\chi}) &= \frac{1}{(p-1)^r} \sum_{\substack{\underline{a} \in (\mathbb{F}_p^*)^r \\ (p-1, \underline{a}) = m}} \chi_1(g)^{a_1} \cdots \chi_r(g)^{a_r} \\ &= \frac{1}{(p-1)^r} \sum_{\substack{\underline{a}' \in (\mathbb{Z}/t\mathbb{Z})^r \\ (t, \underline{a}') = 1}} e^{2\pi i \underline{a}' \cdot \underline{n}/t} = \frac{1}{(p-1)^r} c_{(p-1)/m}(\underline{n}). \end{aligned} \tag{6}$$

By definition we have $\mathrm{ord}(\chi_j) = (p-1)/\gcd(n_j, p-1)$, so

$$\frac{p-1}{m \gcd\left(\frac{p-1}{m}, \underline{n}\right)} = \frac{p-1}{m \gcd\left(\frac{p-1}{m}, \frac{p-1}{\mathrm{ord}(\chi_1)}, \ldots, \frac{p-1}{\mathrm{ord}(\chi_r)}\right)},$$

and this together with Lemma 1 concludes the proof. ∎

For a fixed rank r, define

$$R_p(m) := \#\{\underline{a} \in (\mathbb{Z}/(p-1)\mathbb{Z})^r : (\underline{a}, p-1) = m\}.$$

Then using the well-known properties of the Möbius function, we can write

$$R_p(m) = \sum_{\underline{a} \in (\mathbb{Z}/(p-1)\mathbb{Z})^r} \sum_{n \mid \left(\frac{\underline{a}}{m}, \frac{p-1}{m}\right)} \mu(n) = \sum_{n \mid \frac{p-1}{m}} \mu(n)[h_m(n)]^r,$$

where

$$h_m(n) = \#\left\{a \in \mathbb{Z}/(p-1)\mathbb{Z} : n \mid \frac{a}{m}\right\} = \frac{p-1}{nm},$$

so that

$$R_p(m) = \left(\frac{p-1}{m}\right)^r \sum_{n \mid \frac{p-1}{m}} \frac{\mu(n)}{n^r} = J_r\left(\frac{p-1}{m}\right). \tag{7}$$

Defining

$$\begin{aligned} S_m(x) &:= \frac{1}{m^r} \sum_{\substack{p \le x \\ p \equiv 1 \ (\mathrm{mod}\, m)}} \sum_{n \mid \frac{p-1}{m}} \frac{\mu(n)}{n^r} \\ &= \sum_{\substack{p \le x \\ p \equiv 1 \ (\mathrm{mod}\, m)}} \frac{1}{(p-1)^r} J_r\left(\frac{p-1}{m}\right), \end{aligned} \tag{8}$$

we have the following lemma.

Lemma 2. *If $m \le (\log x)^D$ with D an arbitrary positive constant, then for every constant $M > 1$,*

$$S_m(x) = C_{r,m} \operatorname{Li}(x) + O\left(\frac{x}{m^r (\log x)^M}\right),$$

where $C_{r,m} = \sum_{n \ge 1} \frac{\mu(n)}{(nm)^r \varphi(nm)}$.

Proof. We choose an arbitrary positive constant B, and for every co-prime integers a and b, we denote $\pi(x; a, b) = \#\{p \le x : p \equiv a \ (\mathrm{mod}\ b)\}$, then

$$\begin{aligned} S_m(x) &= \sum_{n \le x} \frac{\mu(n)}{(nm)^r} \pi(x; 1, nm) \\ &= \sum_{n \le (\log x)^B} \frac{\mu(n)}{(nm)^r} \pi(x; 1, nm) + O\left(\sum_{(\log x)^B < n \le x} \frac{1}{(nm)^r} \pi(x; 1, nm)\right). \end{aligned}$$

The sum in the error term is

$$\begin{aligned} \sum_{(\log x)^B < n \le x} \frac{1}{(nm)^r} \pi(x; 1, nm) &\le \frac{1}{m^r} \sum_{n > (\log x)^B} \frac{1}{n^r} \sum_{\substack{2 \le a \le x \\ a \equiv 1 \ (\mathrm{mod}\ mn)}} 1 \\ &\le \frac{1}{m^{r+1}} \sum_{n > (\log x)^B} \frac{x}{n^{r+1}} \\ &\ll \frac{x}{m^{r+1} (\log x)^{rB}}. \end{aligned}$$

For the main term we apply the Siegel–Walfisz Theorem [17], which states that for any positive constants B and C, if $a \le (\log x)^B$, then

$$\pi(x;1,a) = \frac{\mathrm{Li}(x)}{\varphi(a)} + O\Big(\frac{x}{(\log x)^C}\Big).$$

So, if we restrict $m \le (\log x)^D$ for any positive constant D, then

$$\begin{aligned} S_m(x) = &\sum_{n\le(\log x)^B} \frac{\mu(n)}{(nm)^r\varphi(mn)}\,\mathrm{Li}(x) + O\Big(\frac{x}{(\log x)^C}\sum_{n\le(\log x)^B}\frac{1}{(nm)^r}\Big)\\ &+ O\Big(\frac{x}{m^{r+1}(\log x)^{rB}}\Big)\\ = &\;C_{r,m}\,\mathrm{Li}(x) + O\Big(\sum_{n>(\log x)^B}\frac{\mathrm{Li}(x)}{(nm)^r\varphi(nm)}\Big) + O\Big(\frac{x\log\log x}{m^r(\log x)^C}\Big)\\ &+ O\Big(\frac{x}{m^{r+1}(\log x)^{rB}}\Big)\\ = &\;C_{r,m}\,\mathrm{Li}(x) + O\Big(\frac{1}{m^r\varphi(m)}\sum_{n>(\log x)^B}\frac{\mathrm{Li}(x)}{n^r\varphi(n)}\Big)\\ &+ O\Big(\frac{x\log\log x}{m^r(\log x)^C}\Big) + O\Big(\frac{x}{m^{r+1}(\log x)^{rB}}\Big), \end{aligned}$$

where we have used the elementary inequality $\varphi(mn) \ge \varphi(m)\varphi(n)$. Since, for every $n \ge 3$, we have (see [1, Theorem 8.8.7])

$$\frac{n}{\varphi(n)} < e^{\gamma}\log\log n + \frac{3}{\log\log n} \ll \log\log n, \tag{9}$$

it follows that

$$\sum_{n>(\log x)^B}\frac{1}{n^r\varphi(n)} \ll \sum_{n>(\log x)^B}\frac{\log\log n}{n^{r+1}} \ll \frac{\log\log\log x}{(\log x)^{rB}}.$$

Thus

$$\frac{1}{m^r\varphi(m)}\sum_{n>(\log x)^B}\frac{1}{n^r\varphi(n)}\,\mathrm{Li}(x) \ll \frac{x}{m^r\varphi(m)(\log x)^{rB}},$$

proving the lemma. ∎

The following lemma concerns the Titchmarsh Divisor Problem [16] in the case of primes $p \equiv 1 \pmod m$. Asymptotic results on this topic can be found in [3] and [4].

Lemma 3. *Let τ be the divisor function and $m \in \mathbb{N}$. If $m \leq (\log x)^D$ for an arbitrary positive constant D, then*

$$\sum_{\substack{p\leq x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} \tau\left(\frac{p-1}{m}\right) \leq \frac{8x}{m}.$$

Proof. Write $p-1 = mjk$ so that $jk \leq (x-1)/m$ and set $Q = \sqrt{(x-1)/m}$. We distinguish three cases:

- $j \leq Q$, $k > Q$,
- $j > Q$, $k \leq Q$,
- $j \leq Q$, $k \leq Q$.

So we have the identity

$$\begin{aligned}
\sum_{\substack{p\leq x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} \tau\left(\frac{p-1}{m}\right) &= \sum_{j\leq Q} \sum_{\substack{Q<k\leq Q^2/j\\ mjk+1 \text{ prime}}} 1 + \sum_{k\leq Q} \sum_{\substack{Q<j\leq Q^2/k\\ mjk+1 \text{ prime}}} 1 \\
&\quad + \sum_{j\leq Q} \sum_{\substack{k\leq Q\\ mjk+1 \text{ prime}}} 1 \\
&= 2\sum_{k\leq Q} \sum_{\substack{mkQ+1<p\leq x\\ p\equiv 1 \,(\mathrm{mod}\, km)}} 1 + \sum_{k\leq Q} \sum_{\substack{p\leq mkQ+1\\ p\equiv 1 \,(\mathrm{mod}\, km)}} 1 \\
&= 2\sum_{k\leq Q} \big(\pi(x;1,km) - \pi(mkQ+1;1,km)\big) \\
&\quad + \sum_{k\leq Q} \pi(mkQ+1;1,km) \\
&= 2\sum_{k\leq Q} \pi(x;1,km) - \sum_{k\leq Q} \pi(mkQ+1;1,km).
\end{aligned}$$

Using the Montgomery–Vaughan version of the Brun–Titchmarsh Theorem,

$$\pi(x;a,q) \leq \frac{2x}{\varphi(q)\log(x/q)},$$

for $m \leq (\log x)^D$ with D an arbitrary positive constant we obtain

$$\begin{aligned}
\sum_{\substack{p\leq x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} \tau\left(\frac{p-1}{m}\right) &\leq 2\sum_{k\leq Q} \frac{2x}{\varphi(km)\log(x/km)} \\
&\leq \frac{4x}{\log(x/mQ)} \sum_{k\leq Q} \frac{1}{\varphi(km)} \leq \frac{8x}{\log(x/m)} \sum_{k\leq Q} \frac{1}{\varphi(km)}.
\end{aligned}$$

Now, substitute the elementary inequality $\varphi(km) \geq m\varphi(k)$ and use a result of Montgomery [9],

$$\sum_{k\le Q}\frac{1}{\varphi(k)}=A\log Q+B+O\biggl(\frac{\log Q}{Q}\biggr),$$

where

$$A=\frac{\zeta(2)\zeta(3)}{\zeta(6)}=1.9436\ldots\quad\text{and}\quad B=A\gamma-\sum_{n=1}^{\infty}\frac{\mu^2(n)\log n}{n\varphi(n)}=-0.0606\ldots,$$

which in particular implies that, for Q large enough,

$$A\log Q-1\le\sum_{k\le Q}\frac{1}{\varphi(k)}\le A\log Q\le\log(x/m).$$

This yields the desired conclusion. ∎

Lemma 4. *Let p be an odd prime number and let $\chi\neq\chi_0$ be a non-principal Dirichlet character modulo p. Define*

$$d_{m,i}(\chi):=\sum_{\substack{\underline{\chi}\in(\widehat{\mathbb{F}_p^*})^r\\ \chi_i=\chi}}|c_m(\underline{\chi})|.$$

Then

$$d_{m,i}(\chi)\le\frac{1}{m}\prod_{\ell\mid\frac{p-1}{m}}\biggl(1+\frac{1}{\ell}\biggr).$$

Proof. From (6) and Lemma 1, we have

$$d_{m,i}(\chi)$$
$$=\frac{1}{(p-1)^r}\sum_{\substack{\underline{n}\in(\mathbb{Z}/(p-1)\mathbb{Z})^r\\ n_i=n}}\mu^2\biggl(\frac{(p-1)/m}{((p-1)/m,\underline{n})}\biggr)\frac{J_r((p-1)/m)}{J_r((p-1)/m/((p-1)/m,\underline{n}))},$$

where $\chi=e^{2\pi in/(p-1)}$ with $n\in\mathbb{Z}/(p-1)\mathbb{Z}\setminus\{0\}$; naming $t=(p-1)/m$ and $u=\gcd(t,n_i)$ we get

$$d_{m,i}(\chi)=\frac{1}{(p-1)^r}\sum_{d\mid t}\mu^2\biggl(\frac{t}{d}\biggr)\frac{J_r(t)}{J_r(t/d)}H(d),$$

where

$$H(d):=\#\{\underline{x}\in(\mathbb{Z}/(p-1)\mathbb{Z})^{r-1}:(u,\underline{x})=d\}=\biggl(\frac{p-1}{d}\biggr)^{r-1}\sum_{k\mid\frac{u}{d}}\frac{\mu(k)}{k^{r-1}}.$$

Then

$$d_{m,i}(\chi) = \frac{1}{p-1}\sum_{d|t}\mu^2\Big(\frac{t}{d}\Big)\frac{J_r(t)}{d^{r-1}J_r(t/d)}\sum_{k|\frac{u}{d}}\frac{\mu(k)}{k^{r-1}}$$
$$\le \frac{1}{p-1}\sum_{d|t}\mu^2\Big(\frac{t}{d}\Big)d = \frac{t}{p-1}\sum_{k|t}\frac{\mu^2(k)}{k} = \frac{1}{m}\prod_{\ell|\frac{p-1}{m}}\Big(1+\frac{1}{\ell}\Big). \blacksquare$$

4. Proof of Theorem 1. We follow the method of Stephens [14]. By exchanging the order of summation we obtain

$$\sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}} N_{\langle\underline{a}\rangle,m}(x) = \sum_{\substack{p\le x\\ p\equiv 1\ (\mathrm{mod}\, m)}} M_p^m(\underline{T}),$$

where $M_p^m(\underline{T})$ is the number of r-tuples $\underline{a}\in\mathbb{Z}^r$ with $0<a_i\le T_i$ and $v_p(a_i)=0$ for $i=1,\dots,r$ whose reduction modulo p satisfies $[\mathbb{F}_p^*:\langle\underline{a}\rangle_p]=m$. We can write

$$M_p^m(\underline{T}) = \sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}} t_{p,m}(\underline{a})$$

with

$$t_{p,m}(\underline{a}) = \begin{cases} 1 & \text{if } [\mathbb{F}_p^*:\langle\underline{a}\rangle_p]=m,\\ 0 & \text{otherwise.}\end{cases}$$

Given an r-tuple $\underline{\chi}$ of Dirichlet characters mod p, by orthogonality relations it is easy to verify that

$$t_{p,m}(\underline{a}) = \sum_{\underline{\chi}\in(\widehat{\mathbb{F}_p^*})^r} c_m(\underline{\chi})\underline{\chi}(\underline{a}), \tag{10}$$

so we have

$$\sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}} N_{\langle\underline{a}\rangle,m}(x) = \sum_{\substack{p\le x\\ p\equiv 1\ (\mathrm{mod}\, m)}}\ \sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}}\ \sum_{\underline{\chi}\in(\widehat{\mathbb{F}_p^*})^r} c_m(\underline{\chi})\underline{\chi}(\underline{a}). \tag{11}$$

Let $\underline{\chi}_0 := (\chi_0,\dots,\chi_0)$ be the r-tuple of principal characters. Then

$$c_m(\underline{\chi}_0) = \frac{1}{(p-1)^r}\sum_{\substack{\underline{a}\in(\mathbb{F}_p^*)^r\\ [\mathbb{F}_p^*:\langle\underline{a}\rangle_p]=m}} \underline{\chi}_0(\underline{a})$$
$$= \frac{1}{(p-1)^r}\#\{\underline{a}\in(\mathbb{Z}/(p-1)\mathbb{Z})^r : (\underline{a},p-1)=m\} = \frac{1}{(p-1)^r}R_p(m).$$

Denoting $|\underline{T}| := \prod_{i=1}^{r} T_i$ and $T^* := \min\{T_i : i = 1, \ldots, r\}$, through (8) and (7), we can write the main term in (11) as

$$\begin{aligned}
\frac{1}{|\underline{T}|} \sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\substack{\underline{a} \in \mathbb{Z}^r \\ \underline{0} < \underline{a} \le \underline{T}}} c_m(\underline{\chi}_0) \underline{\chi}_0(\underline{a}) \\
&= \frac{1}{|\underline{T}|} \sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, m)}} c_m(\underline{\chi}_0) \prod_{i=1}^{r} \{ \lfloor T_i \rfloor - \lfloor T_i/p \rfloor \} \\
&= \sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, m)}} c_m(\underline{\chi}_0) \biggl(\biggl(1 - \frac{1}{p}\biggr)^r + \sum_{i=1}^{r} O\biggl(\frac{1}{T_i}\biggr) \biggr) \\
&= \sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, m)}} c_m(\underline{\chi}_0) + O\biggl(\sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, m)}} \frac{1}{p} \biggr) + O\biggl(\frac{x}{T^* \log x} \biggr) \\
&= S_m(x) + O(\log\log x) + O\biggl(\frac{x}{T^* \log x} \biggr).
\end{aligned}$$

By hypothesis $m \le (\log x)^D$, $D > 0$, and $T^* > \exp(4(\log x \log\log x)^{1/2})$, so we can apply Lemma 2 to obtain

$$\frac{1}{|\underline{T}|} \sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\substack{\underline{a} \in \mathbb{Z}^r \\ \underline{0} < \underline{a} \le \underline{T}}} c_m(\underline{\chi}_0) \underline{\chi}_0(\underline{a}) = C_{r,m} \operatorname{Li}(x) + O\biggl(\frac{x}{m^r (\log x)^M} \biggr)$$

for any $M > 1$. For the error term we need to estimate the sum

$$\begin{aligned}
(12) \qquad E_{r,m}(x) &:= \frac{1}{|\underline{T}|} \sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\underline{\chi} \in (\widehat{\mathbb{F}_p^*})^r \setminus \{\underline{\chi}_0\}} \biggl| c_m(\underline{\chi}) \sum_{\substack{\underline{a} \in \mathbb{Z}^r \\ \underline{0} < \underline{a} \le \underline{T}}} \underline{\chi}(\underline{a}) \biggr| \\
&\ll \sum_{i=1}^{r} \frac{1}{T_i} \sum_{\substack{p \le x \\ p \equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\chi \in \widehat{\mathbb{F}_p^*} \setminus \{\chi_0\}} d_{m,i}(\chi) \biggl| \sum_{\substack{a \in \mathbb{Z} \\ 0 < a \le T_i}} \chi(a) \biggr|,
\end{aligned}$$

since the r main contributions to (12) come from the cases in which just one Dirichlet character in $\underline{\chi}$ is non-principal, say $\chi_i = \chi \ne \chi_0$, while for every $j \ne i$ we choose $\chi_j = \chi_0$, giving

$$\biggl| \sum_{\substack{\underline{a} \in \mathbb{Z}^r \\ \underline{0} < \underline{a} \le \underline{T}}} \underline{\chi}(\underline{a}) \biggr| = \biggl| \sum_{\substack{a \in \mathbb{Z} \\ 0 < a \le T_i}} \chi(a) \sum_{\substack{0 \le j \le r \\ j \ne i}} \sum_{\substack{a_j \in \mathbb{Z} \\ 0 < a_j \le T_j \\ p \nmid a_j}} 1 \biggr| \le \frac{|\underline{T}|}{T_i} \biggl| \sum_{\substack{a \in \mathbb{Z} \\ 0 < a \le T_i}} \chi(a) \biggr|.$$

Define

$$E^i_{r,m}(x) := \sum_{\substack{p\le x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}} d_{m,i}(\chi)\Big|\sum_{\substack{a\in\mathbb{Z}\\ 0<a\le T_i}} \chi(a)\Big|. \tag{13}$$

Then by Hölder's inequality

$$\{E^i_{r,m}(x)\}^{2s_i} \le \Bigg\{ \sum_{\substack{p\le x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}} \{d_{m,i}(\chi)\}^{\frac{2s_i}{2s_i-1}}\Bigg\}^{2s_i-1} \times \sum_{\substack{p\le x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}} \Big|\sum_{\substack{a\in\mathbb{Z}\\ 0<a\le T_i}} \chi(a)\Big|^{2s_i}. \tag{14}$$

As before, given a primitive root g modulo p, write $\chi_j(g) = e^{2\pi i n_j/(p-1)}$ for every $j = 1,\dots,r$ with $n_j \in \mathbb{Z}/(p-1)\mathbb{Z}$, so that by (6),

$$\sum_{\underline{\chi}\in(\widehat{\mathbb{F}_p^*})^r\setminus\{\underline{\chi}_0\}} c_m(\underline{\chi}) = \frac{1}{(p-1)^r} \sum_{\underline{n}\in(\mathbb{Z}/(p-1)\mathbb{Z})^r\setminus\{\underline{0}\}} c_{(p-1)/m}(\underline{n}).$$

Denoting again $t = (p-1)/m$, from Lemma 1 we derive

$$\begin{aligned}\sum_{\chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}} d_{m,i}(\chi) &\le \sum_{\underline{\chi}\in(\widehat{\mathbb{F}_p^*})^r\setminus\{\underline{\chi}_0\}} |c_m(\underline{\chi})| \\ &\le \sum_{d\mid t} \mu^2\Big(\frac{t}{d}\Big)\Big[\frac{J_r(t)}{(p-1)^r J_r(t/d)}\Big] \\ &\qquad\qquad \times \#\{\underline{n}\in(\mathbb{Z}/(p-1)\mathbb{Z})^r : (t,\underline{n}) = d\} \\ &= \sum_{d\mid t} \mu^2\Big(\frac{t}{d}\Big)\frac{J_r(t)}{d^r J_r(t/d)} \sum_{k\mid \frac{t}{d}} \frac{\mu(k)}{k^r} = \frac{J_r(t)}{t^r}\sum_{d\mid t}\mu^2\Big(\frac{t}{d}\Big) \\ &= \prod_{\ell\mid t}(1-1/\ell^r)2^{\omega(t)} \le 2^{\omega(t)}.\end{aligned}$$

Set $D_{m,i}(p) := \max\{d_{m,i}(\chi) : \chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}\}$. Then for every $s_i \ge 1$,

$$
\begin{aligned}
\sum_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,m)}} \sum_{\chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}} \{d_{m,i}(\chi)\}^{\frac{2s_i}{2s_i-1}}
&\le \sum_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,m)}} \sum_{\chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}} d_{m,i}(\chi)\{d_{m,i}(\chi)\}^{\frac{1}{2s_i-1}}\\
&\le \sum_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,m)}} \{D_{m,i}(p)\}^{\frac{1}{2s_i-1}} \sum_{\chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}} d_{m,i}(\chi)\\
&\le \sum_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,m)}} \{D_{m,i}(p)\}^{\frac{1}{2s_i-1}} 2^{\omega(\frac{p-1}{m})}\\
&\le m^{-\frac{1}{2s_i-1}} \sum_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,m)}} \prod_{\ell\mid\frac{p-1}{m}}\left(1+\frac{1}{\ell}\right) 2^{\omega(\frac{p-1}{m})}\\
&\ll m^{-\frac{1}{2s_i-1}} \sum_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,m)}} \prod_{\ell\mid\frac{p-1}{m}}\left(1-\frac{1}{\ell}\right)^{-1} 2^{\omega(\frac{p-1}{m})}\\
&\ll m^{-\frac{1}{2s_i-1}} \log\log x \sum_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,m)}} \tau\left(\frac{p-1}{m}\right) \ll m^{-\frac{2s_i}{2s_i-1}}\, x\log\log x,
\end{aligned}
$$

where we have used Lemmata 3 and 4 together with the simple observation

$$
\{mD_{m,i}(p)\}^{\frac{1}{2s_i-1}} \le \prod_{\ell\mid\frac{p-1}{m}}\left(1+\frac{1}{\ell}\right)^{\frac{1}{2s_i-1}} \le \prod_{\ell\mid\frac{p-1}{m}}\left(1+\frac{1}{\ell}\right).
$$

To estimate the other factor in (14) we use [14, Lemma 5]:

$$
\sum_{\substack{p\le x\\ p\equiv 1\,(\mathrm{mod}\,m)}} \sum_{\chi\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0\}} \Bigg|\sum_{\substack{a\in\mathbb{Z}\\ 0<a\le T_i}} \chi(a)\Bigg|^{2s_i} \ll (x^2+{T_i}^{s_i}){T_i}^{s_i}(\log(e{T_i}^{s_i-1}))^{{s_i}^2-1}.
$$

So, for every constant $M>1$, we find

$$
\begin{aligned}
\frac{1}{|\underline{T}|}\sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}} N_{\langle\underline{a}\rangle,m}(x) &= C_{r,m}\,\mathrm{Li}(x) + O\bigg(\frac{x}{m^r(\log x)^M}\bigg)\\
&\quad + O\bigg(\sum_{i=1}^{r}\frac{x}{T_i\log x}\bigg) + E_{r,m}(x)
\end{aligned}
$$

with

$$E_{r,m}(x) \ll \sum_{i=1}^{r} \frac{1}{T_i}\left[\left(\frac{x\log\log x}{m^{\frac{2s_i}{2s_i-1}}}\right)^{2s_i-1}(x^2+T_i{}^{s_i})T_i{}^{s_i}(\log(eT_i{}^{s_i-1}))^{s_i^2-1}\right]^{\frac{1}{2s_i}}.$$

If we choose $s_i = \left\lfloor \frac{2\log x}{\log T_i}\right\rfloor + 1$ for $i = 1,\dots,r$, then $T_i^{s_i-1} \le x^2 < T_i^{s_i}$ and

$$E_{r,m}(x) \ll \frac{1}{m}\sum_{i=1}^{r}(x\log\log x)^{1-\frac{1}{2s_i}}(\log(ex^2))^{\frac{s_i^2-1}{2s_i}}.$$

Now, if $T_i > x^2$ for all $i = 1,\dots,r$, then $s_1 = \dots = s_r = 1$ and

$$E_{r,m}(x) \ll \frac{1}{m}(x\log\log x)^{1/2};$$

in particular, $E_{r,m}(x) \ll x/(\log x)^M$ for every constant $M > 1$. Otherwise, if $T_j \le x^2$ for some $j \in \{1,\dots,r\}$, then $s_j \ge 2$ and the corresponding contribution to $E_{r,m}(x)$ will be

$$E^j_{r,m}(x) \ll \frac{1}{m}(x\log\log x)^{1-\frac{1}{2s_j}}(\log(ex^2))^{\frac{3\log x}{2\log T_j}}.$$

By hypothesis

$$T^* > \exp(4(\log x\log\log x)^{1/2}) \tag{15}$$

and, through computations similar to those in [14, p. 184], we can derive

$$E_{r,m}(x) \ll \frac{1}{m}x\log\log x\cdot(T^*)^{-1/16}.$$

Also in this case, using (15), we have $E_{r,m}(x) \ll x/(\log x)^M$ for every $M > 1$. This ends the proof of Theorem 1. ∎

5. Proof of Theorem 2. We now consider

$$H := \frac{1}{|\underline{T}|}\sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}}\{N_{\langle\underline{a}\rangle,m}(x) - C_{r,m}\operatorname{Li}(x)\}^2.$$

We start bounding H as follows:

$$\sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}}\left\{N_{\langle\underline{a}\rangle,m}(x) - C_{r,m}\operatorname{Li}(x)\right\}^2$$
$$\le \sum_{\substack{p,q\le x\\ p,q\equiv 1\ (\mathrm{mod}\ m)}} M^m_{p,q}(\underline{T}) - 2C_{r,m}\operatorname{Li}(x)\sum_{\substack{p\le x\\ p\equiv 1\ (\mathrm{mod}\ m)}} M^m_p(\underline{T}) + |\underline{T}|(C_{r,m})^2\operatorname{Li}^2(x),$$

where $M^m_{p,q}(\underline{T})$ denotes the number of r-tuples $\underline{a} \in \mathbb{Z}^r$ with $a_i \le T_i$ and $v_p(a_i) = v_q(a_i) = 0$ for each $i = 1,\dots,r$ whose reductions modulo prime numbers p and q satisfy $[\mathbb{F}_p^* : \langle\underline{a}\rangle_p] = [\mathbb{F}_q^* : \langle\underline{a}\rangle_q] = m$.

From Theorem 1 we obtain

$$H \le \frac{1}{|\underline{T}|} \sum_{\substack{p,q\le x \\ p,q\equiv 1 \,(\mathrm{mod}\, m)}} M^m_{p,q}(\underline{T}) - C^2_{r,m} \operatorname{Li}^2(x) + O\biggl(\frac{x^2}{(\log x)^{M'}}\biggr)$$

for every constant $M' > 2$. If we write

$$\sum_{\substack{p,q\le x \\ p,q\equiv 1 \,(\mathrm{mod}\, m)}} M^m_{p,q}(\underline{T}) = \sum_{\substack{p\le x \\ p\equiv 1 \,(\mathrm{mod}\, m)}} M^m_{p}(\underline{T}) + \sum_{\substack{p,q\le x \\ p,q\equiv 1 \,(\mathrm{mod}\, m) \\ p\ne q}} M^m_{p,q}(\underline{T}),$$

Theorem 1 gives, for arbitrary $M > 1$,

$$\sum_{\substack{p\le x \\ p\equiv 1 \,(\mathrm{mod}\, m)}} M^m_{p}(\underline{T}) = C_{r,m}|\underline{T}| \operatorname{Li}(x) + O\biggl(\frac{|\underline{T}|x}{(\log x)^{M}}\biggr).$$

In the same spirit as in the proof of Theorem 1, we use (10) to deal with the sum

$$\begin{aligned}(16)\qquad \sum_{\substack{p,q\le x \\ p,q\equiv 1 \,(\mathrm{mod}\, m) \\ p\ne q}} M^m_{p,q}(\underline{T}) &= \sum_{\substack{p,q\le x \\ p,q\equiv 1 \,(\mathrm{mod}\, m) \\ p\ne q}} \sum_{\substack{\underline{a}\in\mathbb{Z}^r \\ \underline{0}<\underline{a}\le\underline{T}}} t_{p,m}(\underline{a}) t_{q,m}(\underline{a}) \\ &= \sum_{\substack{p,q\le x \\ p,q\equiv 1 \,(\mathrm{mod}\, m) \\ p\ne q}} \sum_{\underline{\chi}_1\in(\widehat{\mathbb{F}_p})^r} \sum_{\underline{\chi}_2\in(\widehat{\mathbb{F}_q})^r} c_m(\underline{\chi}_1)c_m(\underline{\chi}_2) \sum_{\substack{\underline{a}\in\mathbb{Z}^r \\ \underline{0}<\underline{a}\le\underline{T}}} \underline{\chi}_1(\underline{a})\underline{\chi}_2(\underline{a}),\end{aligned}$$

where $\underline{\chi}_1$ and $\underline{\chi}_2$ denote r-tuples of Dirichlet characters modulo p, q respectively. Therefore

$$\sum_{\substack{p,q\le x \\ p,q\equiv 1 \,(\mathrm{mod}\, m)}} M^m_{p,q}(\underline{T}) = H_1 + 2H_2 + H_3 + O(|\underline{T}| \operatorname{Li}(x)),$$

where H_1, H_2, H_3 are the contributions to the sum (16) when, respectively: $\underline{\chi}_1 = \underline{\chi}_2 = \underline{\chi}_0$; only one of $\underline{\chi}_1$ and $\underline{\chi}_2$ is equal to $\underline{\chi}_0$; neither $\underline{\chi}_1$ nor $\underline{\chi}_2$ is $\underline{\chi}_0$. First we deal with the inner sum in H_1. To avoid confusion, we write $\underline{\chi}_0^{(p)}$ and $\underline{\chi}_0^{(q)}$ for the r-tuples all of whose entries are the principal characters modulo p and modulo q respectively, so that

$$\sum_{\substack{\underline{a}\in\mathbb{Z}^r \\ \underline{0}<\underline{a}\le\underline{T}}} \underline{\chi}_0^{(p)}(\underline{a})\underline{\chi}_0^{(q)}(\underline{a}) = \prod_{i=1}^{r} \biggl\{ \lfloor T_i \rfloor - \biggl\lfloor \frac{T_i}{p} \biggr\rfloor - \biggl\lfloor \frac{T_i}{q} \biggr\rfloor + \biggl\lfloor \frac{T_i}{pq} \biggr\rfloor \biggr\}.$$

Using Lemma 2, with $M' > 2$ arbitrary we have

$$\begin{aligned} H_1 &= \sum_{\substack{p,q\le x \\ p,q\equiv 1 \pmod m \\ p\ne q}} c_m(\underline{\chi}_0^{(p)})c_m(\underline{\chi}_0^{(q)}) \sum_{\substack{\underline{a}\in\mathbb{Z}^r \\ \underline{0}<\underline{a}\le\underline{T}}} \underline{\chi}_0^{(p)}(\underline{a})\underline{\chi}_0^{(q)}(\underline{a}) \\ &= |\underline{T}| \sum_{\substack{p,q\le x \\ p,q\equiv 1 \pmod m \\ p\ne q}} c_m(\underline{\chi}_0^{(p)})c_m(\underline{\chi}_0^{(q)}) \left(\left(1-\frac{1}{p}-\frac{1}{q}+\frac{1}{pq}\right)^r + \sum_{i=1}^r O\left(\frac{1}{T_i}\right)\right) \\ &= |\underline{T}| \left(\left(\sum_{\substack{p\le x \\ p\equiv 1 \,(\mathrm{mod}\, m)}} c_m(\underline{\chi}_0^{(p)}) \right)^2 - \sum_{\substack{p\le x \\ p\equiv 1 \,(\mathrm{mod}\, m)}} (c_m(\underline{\chi}_0^{(p)}))^2 \right) \left(1+O\left(\frac{1}{T^*}\right)\right) \\ &\quad + |\underline{T}| O\left(\frac{x\log\log x}{\log x}\right) \\ &= |\underline{T}| \left(S_m^2(x) + O\left(\frac{x^2}{T^*(\log x)^2}\right) + O\left(\frac{x\log\log x}{\log x}\right)\right) \\ &= |\underline{T}| \left(C_{r,m}^2 \operatorname{Li}^2(x) + O\left(\frac{x^2}{m^r(\log x)^{M'}}\right)\right). \end{aligned}$$

Focus now on H_2 and assume without loss of generality that $\underline{\chi}_1 = \underline{\chi}_0 \ne \underline{\chi}_2$:

$$\begin{aligned} H_2 &= \sum_{\substack{p,q\le x \\ p,q\equiv 1 \,(\mathrm{mod}\, m) \\ p\ne q}} \sum_{\underline{\chi}_2 \in (\widehat{\mathbb{F}_q^*})^r \setminus \{\underline{\chi}_0^{(q)}\}} c_m(\underline{\chi}_0^{(p)})c_m(\underline{\chi}_2) \sum_{\substack{\underline{a}\in\mathbb{Z}^r \\ \underline{0}<\underline{a}\le\underline{T}}} \underline{\chi}_0^{(p)}(\underline{a})\underline{\chi}_2(\underline{a}) \\ &= \sum_{\substack{p\le x \\ p\equiv 1 \,(\mathrm{mod}\, m)}} c_m(\underline{\chi}_0^{(p)}) \sum_{\substack{q\le x \\ q\equiv 1 \,(\mathrm{mod}\, m) \\ q\ne p}} \sum_{\underline{\chi}_2 \in (\widehat{\mathbb{F}_q^*})^r \setminus \{\underline{\chi}_0^{(q)}\}} c_m(\underline{\chi}_2) \sum_{\substack{\underline{a}\in\mathbb{Z}^r \\ \underline{0}<\underline{a}\le\underline{T} \\ p\nmid \prod_{i=1}^r a_i}} \underline{\chi}_2(\underline{a}). \end{aligned}$$

Just as in the proof of Theorem 1, the quantity

$$U_2 := \sum_{\substack{q\le x \\ q\equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\underline{\chi}_2 \in (\widehat{\mathbb{F}_q^*})^r \setminus \{\underline{\chi}_0^{q}\}} \left| c_m(\underline{\chi}_2) \sum_{\substack{\underline{a}\in\mathbb{Z}^r \\ \underline{0}<\underline{a}\le\underline{T}}} \underline{\chi}_2(\underline{a}) \right|$$

can be estimated through Hölder's inequality combined with the large sieve inequality, to get $U_2 \ll x/(\log x)^M$ for any constant $M > 1$. Moreover, Lemma 3 gives

$$\begin{aligned}
V_2 :=& \sum_{\substack{q\le x\\ q\equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\underline{\chi}_2\in(\widehat{\mathbb{F}_q^*})^r\setminus\{\underline{\chi}_0^{(q)}\}} \Big| c_m(\underline{\chi}_2) \sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}\\ p\mid\prod_{i=1}^r a_i}} \underline{\chi}_2(\underline{a})\Big| \\
&\ll \frac{|\underline{T}|}{p^r} \sum_{\substack{q\le x\\ q\equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\underline{\chi}_2\in(\widehat{\mathbb{F}_q^*})^r\setminus\{\underline{\chi}_0^{(q)}\}} |c_m(\underline{\chi}_2)| \\
&\ll \frac{|\underline{T}|}{p^r} \sum_{\substack{q\le x\\ q\equiv 1 \,(\mathrm{mod}\, m)}} \tau\left(\frac{q-1}{m}\right) \\
&\ll \frac{|\underline{T}|x}{p^r m}.
\end{aligned}$$

Thus, for every constant $M' > 2$,

$$H_2 \ll \sum_{\substack{p\le x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} (U_2+V_2) \ll \frac{|\underline{T}|x^2}{(\log x)^{M'}}.$$

Finally, assume $\chi_1 \in \widehat{\mathbb{F}_p^*}\setminus\{\chi_0^{(p)}\}$ and $\chi_2 \in \widehat{\mathbb{F}_q^*}\setminus\{\chi_0^{(q)}\}$ with $p\ne q$; then $\chi_1\chi_2$ is a primitive character modulo pq. To obtain an upper bound on

$$H_3 = \sum_{\substack{p,q\le x\\ p,q\equiv 1 \,(\mathrm{mod}\, m)\\ p\ne q}} \sum_{\underline{\chi}_1\in(\widehat{\mathbb{F}_p^*})^r\setminus\{\underline{\chi}_0^{(p)}\}} \sum_{\underline{\chi}_2\in(\widehat{\mathbb{F}_q^*})^r\setminus\{\underline{\chi}_0^{(q)}\}} c_m(\underline{\chi}_1)c_m(\underline{\chi}_2) \times \sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}} \underline{\chi}_1(\underline{a})\underline{\chi}_2(\underline{a}),$$

we will apply again Hölder's inequality and the large sieve [14, Lemma 5]). To do so, since the r-tuples $\underline{\chi}_1$ and $\underline{\chi}_2$ in H_3 are both non-principal, we denote by $\chi_{1,i}$ the ith component of the r-tuple $\underline{\chi}_1$ (and similarly for $\chi_{2,i}$). Then the contributions to H_3 have two possible sources: a "diagonal" term H_3^d (in which for a certain $i\in\{1,\dots,r\}$ both $\chi_{1,i}$ and $\chi_{2,i}$ are non-principal) and a "non-diagonal" term H_3^{nd} (in which for no $i\in\{1,\dots,r\}$ is it possible to have $\chi_{1,i}$ and $\chi_{2,i}$ both non-principal). Analogously to what was done for the error term (12) in the proof of Theorem 1, the main contributions to H_3^d and H_3^{nd} come from the cases in which, for a certain r-tuple of characters modulo p or q, just one character is non-principal and the other $r-1$ are all principal. Explicitly, $H_3^d = \sum_{i=1}^r H_{3,i}$, where

$$\begin{aligned}
H_{3,i} :=& \sum_{\substack{p,q\le x\\ p,q\equiv 1 \,(\mathrm{mod}\, m)\\ p\ne q}} \sum_{\substack{\underline{\chi}_1\in(\widehat{\mathbb{F}_p^*})^r\\ \chi_1\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0^{(p)}\}}} \sum_{\substack{\underline{\chi}_2\in(\widehat{\mathbb{F}_q^*})^r\\ \chi_2\in\widehat{\mathbb{F}_q^*}\setminus\{\chi_0^{(q)}\}}} c_m(\underline{\chi}_1)c_m(\underline{\chi}_2) \sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}} \underline{\chi}_1(\underline{a})\underline{\chi}_2(\underline{a})\\
&\ll \frac{|\underline{T}|}{T_i} \sum_{\substack{p,q\le x\\ p,q\equiv 1 \,(\mathrm{mod}\, m)\\ p\ne q}} \sum_{\chi_1\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0^{(p)}\}} \sum_{\chi_2\in\widehat{\mathbb{F}_q^*}\setminus\{\chi_0^{(q)}\}} d_{m.i}(\chi_1)d_{m,i}(\chi_2)\\
&\quad\times\Bigg|\sum_{\substack{a_i\in\mathbb{Z}\\ 0<a_i\le T_i}} \chi_1(a_i)\chi_2(a_i)\Bigg|
\end{aligned}$$

and $H_3^{nd} = \sum_{\substack{i,j=1\\ i\ne j}}^r H_{3,ij}$, with

$$\begin{aligned}
H_{3,ij} :=& \sum_{\substack{p,q\le x\\ p,q\equiv 1 \,(\mathrm{mod}\, m)\\ p\ne q}} \sum_{\substack{\underline{\chi}_1\in(\widehat{\mathbb{F}_p^*})^r\\ \chi_1\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0^{(p)}\}}} \sum_{\substack{\underline{\chi}_2\in(\widehat{\mathbb{F}_q^*})^r\\ \chi_2\in\widehat{\mathbb{F}_q^*}\setminus\{\chi_0^{(q)}\}}} c_m(\underline{\chi}_1)c_m(\underline{\chi}_2) \sum_{\substack{\underline{a}\in\mathbb{Z}^r\\ \underline{0}<\underline{a}\le\underline{T}}} \underline{\chi}_1(\underline{a})\underline{\chi}_2(\underline{a})\\
&\ll \frac{|\underline{T}|}{T_iT_j} \sum_{\substack{p,q\le x\\ p,q\equiv 1 \,(\mathrm{mod}\, m)\\ p\ne q}} \sum_{\chi_1\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0^{(p)}\}} \sum_{\chi_2\in\widehat{\mathbb{F}_q^*}\setminus\{\chi_0^{(q)}\}} d_{m,i}(\chi_1)d_{m,j}(\chi_2)\\
&\quad\times\Bigg|\sum_{\substack{a_i,a_j\in\mathbb{Z}\\ 0<a_i\le T_i\\ 0<a_j\le T_j}} \chi_1(a_i)\chi_2(a_j)\Bigg|.
\end{aligned}$$

Dealing first with $H_{3,i}$, we use again Hölder's inequality together with the large sieve to get

$$\begin{aligned}
\frac{H_{3,i}}{|\underline{T}|} \ll& \frac{1}{T_i}\Bigg\{ \sum_{\substack{p,q\le x\\ p,q\equiv 1 \,(\mathrm{mod}\, m)\\ p\ne q}} \sum_{\substack{\chi_1\in\widehat{\mathbb{F}_p^*}\setminus\{\chi_0^{(p)}\}\\ \chi_2\in\widehat{\mathbb{F}_q^*}\setminus\{\chi_0^{(q)}\}}} [d_{m,i}(\chi_1)d_{m,i}(\chi_2)]^{\frac{2s_i}{2s_i-1}} \Bigg\}^{\frac{2s_i-1}{2s_i}}\\
&\times\Bigg\{ \sum_{\substack{p,q\le x\\ p,q\equiv 1 \,(\mathrm{mod}\, m)\\ p\ne q}} \sum_{\eta \pmod{pq}} \Bigg|\sum_{\substack{a_i\in\mathbb{Z}\\ 0<a_i\le T_i}} \eta(a_i)\Bigg|^{2s_i}\Bigg\}^{1/2s_i}\\
&\ll \frac{1}{T_i}\Bigg\{ \left(\frac{x\log\log x}{m^2}\right)^{4s_i-2} (x^4+T_i^{s_i})T_i^{s_i}(\log(eT_i^{s_i-1}))^{s_i^2-1}\Bigg\}^{1/2s_i}.
\end{aligned}$$

We now choose $s_i = \lfloor \frac{4\log x}{\log T_i} \rfloor + 1$, so that $T_i^{s_i-1} \le x^4 \le T_i^{s_i}$ and

$$\frac{H_{3,i}}{|\underline{T}|} \ll \frac{1}{m^2} x^{2-1/s_i} (\log\log x)^2 (\log(ex^4))^{\frac{s_i^2-1}{2s_i}}.$$

If $T_i > x^4$ then $s_i = 1$ and $H_{3,i}/|\underline{T}| \ll x(\log\log x)^2$. Otherwise, if $T_i \le x^4$ then $s_i \ge 2$ and assuming $T_i > \exp(6(\log x \log\log x)^{1/2})$, similarly to what was done to prove Theorem 1 we get

$$\frac{H_{3,i}}{|\underline{T}|} \ll x^{2-1/s_i} (\log\log x)^2 (\log(ex^4))^{\frac{3\log x}{\log T_i}} \ll \frac{x^2}{(\log x)^D}$$

for any positive constant $D > 2$.

It remains to estimate $H_{3,ij}$ where $i \ne j$. In this case $H_{3,ij}$ can be factorized into two products, and by the same methods used for (13) we get

$$\begin{aligned}\frac{H_{3,ij}}{|\underline{T}|} &\ll \frac{1}{T_iT_j} \sum_{\substack{p\le x\\ p\equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\chi_1 \in \widehat{\mathbb{F}_p^*}\setminus\{\chi_0^{(p)}\}} d_{m,i}(\chi_1) \Big| \sum_{\substack{a_i\in\mathbb{Z}\\ 0<a_i\le T_i}} \chi_1(a_i)\Big| \\ &\quad\times \sum_{\substack{q\le x\\ q\equiv 1 \,(\mathrm{mod}\, m)}} \sum_{\chi_2 \in \widehat{\mathbb{F}_q^*}\setminus\{\chi_0^{(q)}\}} d_{m,j}(\chi_2) \Big| \sum_{\substack{a_j\in\mathbb{Z}\\ 0<a_j\le T_j}} \chi_2(a_j)\Big| \\ &\ll \frac{1}{T_i}\bigg\{\bigg(\frac{x\log\log x}{m^2}\bigg)^{2s_i-1} (x^2+T_i^{s_i})T_i^{s_i}(\log(eT_i^{s_i-1}))^{s_i^2-1}\bigg\}^{1/2s_i} \\ &\quad\times \frac{1}{T_j}\bigg\{\bigg(\frac{x\log\log x}{m^2}\bigg)^{2s_j-1} (x^2+T_j^{s_j})T_j^{s_j}(\log(eT_j^{s_j-1}))^{s_j^2-1}\bigg\}^{1/2s_j}.\end{aligned}$$

We choose $s_i = \lfloor \frac{2\log x}{\log T_i} \rfloor + 1$ and $s_j = \lfloor \frac{2\log x}{\log T_j} \rfloor + 1$, so that

$$\frac{H_{3,ij}}{|\underline{T}|} \ll \frac{x^2}{(\log x)^E}$$

for every constant $E > 2$.

Eventually, since $H_3 \ll H_3^d + H_3^{nd}$, summing the upper bounds for H_1, H_2 and H_3 we get the proof of Theorem 2. ∎

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Lorenzo Menici
Dipartimento di Matematica
Università Roma Tre
Largo S. L. Murialdo, 1
I-00146 Roma, Italy
E-mail: menici@mat.uniroma3.it

Cihan Pehlivan
Department of Mathematics
Koc University
Rumelifeneri Yolu
34450 Sarıyer-İstanbul, Turkey
E-mail: cpehlivan@ku.edu.tr