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On square classes in generalized Fibonacci sequences

by

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1. Introduction. Let P and Q be nonzero integers. The *generalized Fibonacci* and *Lucas sequences*, $(U_n(P,Q))$ and $(V_n(P,Q))$, are defined as follows:

$$U_0(P,Q)=0, \quad U_1(P,Q)=1,$$
$$U_{n+1}(P,Q)=PU_n(P,Q)+QU_{n-1}(P,Q) \quad \text{for } n\ge 1,$$

and

$$V_0(P,Q)=2, \quad V_1(P,Q)=P,$$
$$V_{n+1}(P,Q)=PV_n(P,Q)+QV_{n-1}(P,Q) \quad \text{for } n\ge 1,$$

respectively. $U_n(P,Q)$ and $V_n(P,Q)$ are called the nth *generalized Fibonacci number* and nth *generalized Lucas number*, respectively. Since

$$U_n(-P,Q)=(-1)^{n-1}U_n(P,Q) \quad \text{and} \quad V_n(-P,Q)=(-1)^nV_n(P,Q),$$

it will be assumed that $P\ge 1$. Moreover, we will assume that $P^2+4Q>0$. Sometimes, instead of $U_n(P,Q)$ and $V_n(P,Q)$, we write just U_n and V_n. For more information about these sequences one can consult [7].

For $P=Q=1$, we have the classical Fibonacci and Lucas sequences (F_n) and (L_n). In this paper, we determine all n and m such that $U_n=wU_mx^2$ or $U_nU_m=wx^2$ with $w=1,2,3$, or 6 under the following assumption:

$$P^2+4Q>0, \quad P\ge 1 \text{ and } Q \text{ are odd}, \quad (P,Q)=1. \tag{1.1}$$

Regarding this issue, Keskin and Yosma [2] showed that if $F_n=2F_mx^2$ for $m\ge 3$, then $(m,n)=(3,12)$ or $(6,12)$; if $F_n=3F_mx^2$ for $m\ge 3$, then $(m,n)=(4,12)$; and no F_n satisfies $F_n=6F_mx^2$ for $m\ge 1$. Moreover, Cohn [1] determined all n and m such that $U_nU_m=x^2$ and $U_nU_m=2x^2$ when P is odd and $Q=\pm 1$.

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Also, in this paper, we will solve each of the equations $U_n = kx^2$ and $U_n = 2kx^2$ when $k \mid P$, $k > 1$, under the assumption (1.1). As an application, we determine all n such that $U_n = 6x^2$. First of all, we will solve the equations $V_n^2 - 3(-Q)^n = wx^2$ and $V_n^2 - (-Q)^n = wx^2$ for $w \in \{1, 2, 3, 6\}$, which is used to solve $U_n = wU_m x^2$, $U_n = kx^2$, and $U_n = 2kx^2$.

On the other hand, Cohn [1] studied the equations $U_n = kx^2$ and $U_n = 2kx^2$ when $P \geq 1$ is odd and $Q = 1$, and he obtained the following results, with $r = \min\{n : n > 0 \text{ and } k \mid U_n\}$:

1. If $r \not\equiv 0 \pmod 3$, then $U_n = kx^2$ can occur only for $n = r$, and $U_n = 2kx^2$ is impossible for $n > 0$.
2. If $r \equiv 3 \pmod 6$, then $U_n = kx^2$ is impossible for $n > 0$, and similarly for $U_n = 2kx^2$.
3. If $r \equiv 0 \pmod 6$, and if $2^{2t+1} \parallel r$, then $U_n = kx^2$ is impossible except if $P = 5$, $k = 455$, $n = 12$; if $2^{2t} \parallel r$, then $U_n = 2kx^2$ is impossible for $n > 0$.

Moreover, Ribenboim and McDaniel [10] solved the equation $U_n = kx^2$ under the assumption (1.1) and that the Jacobi symbol $\left(\frac{-V_{2u}}{h}\right)$ is defined and equals 1 for each odd divisor h of k with $u \geq 1$. In particular, they solved $U_n = 3x^2$ and gave the solutions as $n = 1, 3, 4$, or 6 but they must have forgotten writing $n = 2$.

2. Preliminaries. In this paper, we assume that $P \geq 1$ is an odd integer unless indicated otherwise, and also Q is an odd integer such that $(P, Q) = 1$. Firstly, we will give a list of properties of generalized Fibonacci and Lucas numbers, which will be needed later. Throughout, the symbol $\square$ denotes a perfect square.

$$U_{-n} = -(-Q)^{-n} U_n, \tag{2.1}$$

$$V_{-n} = (-Q)^{-n} V_n, \tag{2.2}$$

$$U_{2n} = U_n V_n, \tag{2.3}$$

$$V_{2n} = V_n^2 - 2(-Q)^n, \tag{2.4}$$

$$U_{3n} = U_n((P^2 + 4Q)U_n^2 + 3(-Q)^n) = U_n(V_n^2 - (-Q)^n), \tag{2.5}$$

$$V_{3n} = V_n(V_n^2 - 3(-Q)^n), \tag{2.6}$$

$$(U_{2n+1}, P) = (U_{n+1}, Q) = 1 \text{ for } n \geq 0, \tag{2.7}$$

$$(V_{2n}, P) = (V_n, Q) = 1 \text{ for } n \geq 0, \tag{2.8}$$

$$2 \mid V_n \Leftrightarrow 2 \mid U_n \Leftrightarrow 3 \mid n, \tag{2.9}$$

$$\text{if } U_m \neq 1, \text{ then } U_m \mid U_n \Leftrightarrow m \mid n, \tag{2.10}$$

(2.11) if $V_m \neq 1$, then $V_m \mid V_n \Leftrightarrow m \mid n$ and n/m is odd,

(2.12) if $d = (m, n)$, then $(U_m, U_n) = U_d$,

(2.13) $(U_m, V_m) = 1$ or 2,

(2.14) $U_{2n} \equiv nPQ^{n-1} \pmod{P^2}$ and $U_{2n+1} \equiv Q^n \pmod{P^2}$,

(2.15) $V_{2n} \equiv 2Q^n \pmod{P^2}$ and $V_{2n+1} \equiv nPQ^n \pmod{P^2}$,

(2.16) $$\frac{U_{mn}(P,Q)}{U_m(P,Q)} = U_n(V_m, -(-Q)^m).$$

All the above identities except (2.14)–(2.16) can be found in [3, 9]; (2.16) is given in [8]; and (2.14) and (2.15) can be proved by induction on n. Moreover, when P is even, it is well known that

(2.17) U_n is even $\Leftrightarrow$ n is even,

(2.18) U_n is odd $\Leftrightarrow$ n is odd.

Now, we give some theorems and lemmas which will be used in the proofs of the main theorems. The following theorem is proved in [13].

THEOREM 2.1. *Let $n, m \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$. Then*

(2.19) $U_{2mn+r} \equiv (-(-Q)^m)^n U_r \pmod{V_m}$,

(2.20) $V_{2mn+r} \equiv (-(-Q)^m)^n V_r \pmod{V_m}$,

where we require $mn + r \geq 0$ if $Q \neq \pm 1$.

The proofs of the following two lemmas can be found in [9].

LEMMA 2.2. *Let m be an odd positive integer and $r \geq 1$.*

(a) *If $3 \mid m$, then $V_{2^r m} \equiv 2 \pmod 8$.*
(b) *If $3 \nmid m$, then*

$$V_{2^r m} \equiv \begin{cases} 3 \pmod 8 & \text{if } r = 1 \text{ and } Q \equiv 1 \pmod 8, \\ 7 \pmod 8 & \text{otherwise.} \end{cases}$$

LEMMA 2.3. *Let r be a positive integer. Then*

(i) $\left(\frac{-1}{V_{2^r}}\right) = -1$,

(ii) $\left(\frac{2}{V_{2^r}}\right) = \begin{cases} -\left(\frac{-1}{Q}\right) & \text{if } r = 1, \\ 1 & \text{if } r \geq 2, \end{cases}$

(iii) $\left(\frac{Q}{V_{2^r}}\right) = \left(\frac{-1}{Q}\right)$,

(iv) $\left(\frac{U_3}{V_{2^r}}\right) = \begin{cases} -\left(\frac{-1}{Q}\right) & \text{if } r = 1, \\ 1 & \text{if } r \geq 2, \end{cases}$

(v) $\left(\frac{P^2+3Q}{V_{2^r}}\right) = \begin{cases} \left(\frac{-1}{Q}\right) & \text{if } r = 1, \\ 1 & \text{if } r \geq 2. \end{cases}$

The following lemma can be proved by induction.

Lemma 2.4. *If* $3 \nmid P$, *then*

$$V_{2^r} \equiv \begin{cases} 0 \pmod 3 & \text{if } r = 1 \text{ and } Q \equiv 1 \pmod 3, \\ 1 \pmod 3 & \text{if } r \geq 1 \text{ and } Q \equiv 0 \pmod 3 \\ & \text{or } r = 2 \text{ and } Q \equiv 1 \pmod 3, \\ 2 \pmod 3 & \text{if } r = 1, 2 \text{ and } Q \equiv 2 \pmod 3 \\ & \text{or } r \geq 3 \text{ and } Q \equiv 1, 2 \pmod 3, \end{cases}$$

and if $3 \mid P$, *then* $V_{2^r} \equiv 2 \pmod 3$ *for* $r \geq 2$.

Using Lemmas 2.3 and 2.4, we can see that

$$\left(\frac{3}{V_{2^r}}\right) = 1 \quad \text{if } Q \equiv 2 \pmod 3,\ r \geq 2 \text{ or } Q \equiv 1 \pmod 3,\ r \geq 3. \tag{2.21}$$

We recall the following results from [9] and [14].

Lemma 2.5. *If* $V_n = x^2$, *then* $n = 1$, 3, *or* 5; *if* $V_3 = x^2$, *then* $Q \equiv 1 \pmod 4$ *and also* $P = \square$, $P^2 + 3Q = \square$ *or* $P = 3\square$, $P^2 + 3Q = 3\square$; *if* $V_5 = x^2$, *then* $Q \equiv 3 \pmod 8$, $P = 5\square$, *and* $P^4 + 5P^2Q + 5Q^2 = 5\square$.

Lemma 2.6. *If* $V_n = 2x^2$, *then* $n = 0$, 3, *or* 6; *if* $V_3 = 2x^2$, *then* $Q \equiv 5, 7 \pmod 8$, $P = 3\square$, *and* $P^2 + 3Q = 6\square$; *if* $V_6 = 2x^2$, *then* $Q \equiv 1 \pmod 4$, $P^2 + 2Q = 3\square$, *and* $(P^2 + 2Q)^2 - 3Q^2 = 6\square$.

Lemma 2.7. *If* $V_n = 3x^2$, *then* $n = 1$, 2, 3, *or* 5; $V_1 = 3x^2$ *iff* $P = 3\square$; $V_2 = 3x^2$ *iff* $P^2 + 2Q = 3\square$ *and* $Q \equiv 1 \pmod 3$; $V_3 = 3x^2$ *iff* $P = \square$, $P^2 + 3Q = 3\square$, *and* $Q \equiv 1 \pmod 4$; $V_5 = 3x^2$ *iff* $P = 15\square$, $P^4 + 5P^2Q + 5Q^2 = 5\square$, *and* $Q \equiv 3 \pmod 8$.

Lemma 2.8. *If* $V_n = 6x^2$, *then* $n = 3$; $V_3 = 6x^2$ *iff* $P = \square$, $P^2 + 3Q = 6\square$, *and* $Q \equiv 5, 7 \pmod 8$.

Theorem 2.9. *Let* $k > 1$ *and* $k \mid P$. *If* $V_n = kx^2$ *for some integer* x, *then* $n = 1, 3$, *or* 5; *if* $V_5 = x^2$, *then* $P = 5k\square$, $P^4 + 5P^2Q + 5Q^2 = 5\square$, *and* $Q \equiv 3 \pmod 8$.

Theorem 2.10. *Let* $k > 1$ *and* $k \mid P$. *If* $V_n = 2kx^2$ *for some integer* x, *then* $n = 3$.

The proofs of the following four theorems can be found in [9] and [10].

Theorem 2.11. $U_n = x^2$ *if and only if either* (i) $n = 0, 1, 2$, *or* 3, (ii) $n = 6$, $P = 3\square$, $P^2 + Q = 2\square$, *and* $P^2 + 3Q = 6\square$, *or* (iii) $n = 12$, $P = \square$, $P^2 + Q = 2\square$, $P^2 + 2Q = 3\square$, $P^2 + 3Q = \square$, *and* $(P^2 + 2Q)^2 - 3Q^2 = 6\square$.

Theorem 2.12. $U_n = 2x^2$ *if and only if either* (i) $n = 0$ *or* 3, *or* (ii) $n = 6$, $P = \square$, $P^2 + Q = 2\square$, *and* $P^2 + 3Q = \square$.

THEOREM 2.13. *$U_n = 3x^2$ if and only if either* (i) *$n = 0$ or 2, or* (ii) *$n = 3$, $P^2 + Q = 3\square$, and $3 \nmid P$, or* (iii) *$n = 4$, $P = \square$, $P^2 + 2Q = 3\square$, $Q \equiv 1 \pmod{12}$, and $3 \nmid P$, or* (iv) *$n = 6$, $P = \square$, $P^2 + Q = 2\square$, $P^2 + 3Q = 6\square$, and $3 \mid P$.*

THEOREM 2.14.

(i) *If $3 \mid P$, then $3 \mid U_n \Leftrightarrow n$ is even.*

(ii) *If $3 \nmid P$, then*

$$3 \mid U_n \Leftrightarrow \begin{cases} 12 \mid n \text{ and } Q \equiv 1, 2 \pmod 3, \text{ or} \\ 4 \mid n,\ 3 \nmid n, \text{ and } Q \equiv 1 \pmod 3, \text{ or} \\ 4 \nmid n,\ 3 \mid n, \text{ and } Q \equiv 2 \pmod 3. \end{cases}$$

The proof of the following lemma is given in [12].

LEMMA 2.15. *If $3 \mid P$, then $3 \mid V_n$ iff n is odd. If $3 \nmid P$, then $3 \mid V_n$ iff $n \equiv 2 \pmod 4$ and $Q \equiv 1 \pmod 3$.*

Lastly, we will require the following theorem given in [11].

THEOREM 2.16. *If P is even, $Q \equiv -1 \pmod 4$, $(P, Q) = 1$, and n is odd, then $U_n(P, Q) = \square$ only if $n = \square$.*

3. Auxiliary theorems. From now on, assume that n and m are positive integers.

The following lemma can be proved by induction and therefore we omit its proof.

LEMMA 3.1. *For $k \geq 1$,*

$$V_{2^{k+2}} \equiv -Q^{2^{k+1}} \pmod{V_{2^{k+1}} + Q^{2^k}} \quad \text{and} \quad V_{2^{k+2}} \equiv -Q^{2^{k+1}} \pmod{V_4 - Q^2}.$$

By Lemmas 2.3 and 3.1, we can see that

$$J = \left(\frac{V_4 - Q^2}{V_{2^{k+2}}} \right) = 1 \quad \text{for } k \geq 1. \tag{3.1}$$

LEMMA 3.2. *Let $w \in \{1, 2, 3, 6\}$ and $V_n^2 - 3(-Q)^n = wx^2$ for some integer x. Then $n = 1$ or $n = 2$.*

Proof. If n is odd, it has been shown in [12] that the equation $V_n^2 - 3(-Q)^n = wx^2$ has no solutions for $n > 1$. So let n be even. Thus, $V_{2n} - Q^n = wx^2$ by (2.4). It is obvious that $wx^2 = V_{2n} - Q^n \equiv 1$ or $6 \pmod 8$ by Lemma 2.2. When $w = 2$ or $w = 3$, we have a contradiction.

Now assume that $w = 1$ or $w = 6$. We can write $n = 2^r z$ for some odd positive integer z with $r \geq 1$.

If $z = 1$, then $n = 2^r$, where $r \neq 1$, i.e., $r \geq 2$ since $n > 2$. In this case, if $w = 1$, then

$$x^2 = V_{2n} - Q^n = V_{2 \cdot 2^r} - Q^{2^r} \equiv 7 - 1 \equiv 6 \pmod 8$$

by Lemma 2.2. This is impossible. If $w=6$, then

$$6x^2 = V_{2\cdot 2^r} - Q^{2^r} \equiv -Q^{2^r}V_0 - Q^{2^r} \equiv -3Q^{2^r} \pmod{V_{2^r}}$$

by (2.20). Consequently, $1 = \left(\frac{-2}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{2}{V_{2^r}}\right) = -1$ by Lemma 2.3, a contradiction.

Thus $z>1$. So, we can write $z = 4q \pm 1$ for some $q>0$. Hence $2n = 2(2^r z) = 2(2^{r+2}q \pm 2^r) = 2\cdot 2^{r+2}q \pm 2^{r+1}$.

Let q be odd. Using (2.2) and (2.20), we get

$$wx^2 = V_{2n} - Q^n \equiv$$
$$-Q^{2^{r+2}q}V_{2^{r+1}} - Q^{2^{r+2}q+2^r} \text{ or } -Q^{2^{r+2}q-2^{r+1}}V_{2^{r+1}} - Q^{2^{r+2}q-2^r} \pmod{V_{2^{r+2}}},$$

i.e.,

$$wx^2 \equiv -Q^{2^{r+2}q}(V_{2^{r+1}} + Q^{2^r}) \text{ or } -Q^{2^{r+2}q-2^{r+1}}(V_{2^{r+1}} + Q^{2^r}) \pmod{V_{2^{r+2}}}.$$

In both cases,

$$J = \left(\frac{-w(V_{2^{r+1}} + Q^{2^r})}{V_{2^{r+2}}}\right) = 1.$$

On the other hand, $V_{2^{r+1}} + Q^{2^r} \equiv 0 \pmod 8$ by Lemma 2.2. So, $V_{2^{r+1}} + Q^{2^r} = 2^s t$ for some odd t and $s \geq 3$. Hence, $V_{2^{r+2}} \equiv -Q^{2^{r+1}} \pmod t$ by Lemma 3.1. If $w=1$, then we get

$$\begin{aligned} J &= \left(\frac{-(V_{2^{r+1}} + Q^{2^r})}{V_{2^{r+2}}}\right) = -\left(\frac{V_{2^{r+1}} + Q^{2^r}}{V_{2^{r+2}}}\right) = -\left(\frac{2}{V_{2^{r+2}}}\right)^s\left(\frac{t}{V_{2^{r+2}}}\right) \\ &= -(-1)^{(t-1)/2}\left(\frac{V_{2^{r+2}}}{t}\right) = -(-1)^{(t-1)/2}\left(\frac{-1}{t}\right) \\ &= -(-1)^{(t-1)/2}(-1)^{(t-1)/2} = -1 \end{aligned}$$

by Lemma 2.3, contrary to $J=1$.

Now, let $w=6$. If $3 \mid Q$, from the equation $V_{2n} - Q^n = 6x^2$, we have $3 \mid V_{2n}$ and therefore $2n \equiv 2 \pmod 4$, i.e., $n \equiv 1 \pmod 2$ by Lemma 2.15. This contradicts n being even. If $3 \nmid Q$, then we obtain

$$\begin{aligned} J &= \left(\frac{-6(V_{2^{r+1}} + Q^{2^r})}{V_{2^{r+2}}}\right) = -\left(\frac{2}{V_{2^{r+2}}}\right)\left(\frac{3}{V_{2^{r+2}}}\right)\left(\frac{2}{V_{2^{r+2}}}\right)^s\left(\frac{t}{V_{2^{r+2}}}\right) \\ &= -(-1)^{(t-1)/2}\left(\frac{V_{2^{r+2}}}{t}\right) = -(-1)^{(t-1)/2}(-1)^{(t-1)/2} = -1 \end{aligned}$$

by Lemma 2.3 and (2.21), a contradiction again.

Now, let q be even. Then $2n = 2(2^r z) = 2(2^{r+2}q \pm 2^r) = 2\cdot 2^{r+k+2}b \pm 2^{r+1}$ with b odd and $k \geq 1$. Similarly, we can see that

$$wx^2 \equiv -Q^{2^{r+k+2}b}(V_{2^{r+1}} + Q^{2^r}) \text{ or } -Q^{2^{r+k+2}b-2^{r+1}}(V_{2^{r+1}} + Q^{2^r}) \pmod{V_{2^{r+k+2}}}$$

by (2.2) and (2.20). This shows that

$$J = \left(\frac{-w(V_{2^{r+1}} + Q^{2^r})}{V_{2^{r+k+2}}}\right) = 1.$$

A similar argument shows that this is impossible. ∎

LEMMA 3.3. *Let $n \geq 1$ be an integer, $w \in \{1,2,3,6\}$, and $V_n^2 - (-Q)^n = wx^2$ for some integer x. Then $n = 1, 2$ or 4. In particular, $V_n^2 - (-Q)^n = x^2$ has a solution only for $n = 1$; $V_n^2 - (-Q)^n = wx^2$, $w \in \{2,6\}$, has a solution only for $n = 1$ or 2; $V_n^2 - (-Q)^n = 3x^2$ has a solution for $n = 1, 2,$ or 4.*

Proof. We divide the proof into two cases.

CASE 1: n odd. If $n = 1$, it is obvious that $P^2 + Q = wx^2$ has a solution for $w \in \{1,2,3,6\}$. So, assume that $n > 1$. Since n is odd, we have $V_n^2 - (-Q)^n = V_{2n} - Q^n = wx^2$ by (2.4). We can write $2n = 2(2^r z \pm 1) = 2 \cdot 2^r z \pm 2$ for some odd positive integer z with $r \geq 2$. Thus,

$$wx^2 = V_{2n} - Q^n \equiv -Q^{2^r z} V_2 - Q^{2^r z+1} \text{ or } -Q^{2^r z-2} V_2 - Q^{2^r z-1} \pmod{V_{2^r}},$$

i.e.,

$$wx^2 \equiv -Q^{2^r z}(P^2 + 3Q) \text{ or } -Q^{2^r z-2}(P^2 + 3Q) \pmod{V_{2^r}}$$

by (2.20). Hence

$$\left(\frac{-w(P^2 + 3Q)}{V_{2^r}}\right) = 1.$$

If $w = 1$ or $w = 2$, then, using Lemma 2.3, it can be easily seen that $J = -1$. This is impossible.

Let $w = 3$ and $3 \mid Q$. Then $3 \mid V_n$ since $V_n^2 - (-Q)^n = 3x^2$. This implies $3 \mid P$ by Lemma 2.15, contradicting $(P,Q) = 1$. Thus $3 \nmid Q$ and therefore $3 \nmid V_n$. This shows that $Q \equiv 2 \pmod 3$. Consequently,

$$1 = \left(\frac{-3(P^2 + 3Q)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{P^2 + 3Q}{V_{2^r}}\right) = -1$$

by Lemma 2.3 and (2.21), which is impossible.

If $w = 6$, a similar argument shows that $3 \nmid Q$ and $Q \equiv 2 \pmod 3$, and therefore

$$1 = \left(\frac{-6(P^2 + 3Q)}{V_{2^r}}\right) = \left(\frac{-1}{V_{2^r}}\right)\left(\frac{2}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right)\left(\frac{P^2 + 3Q}{V_{2^r}}\right) = -1$$

by Lemma 2.3 and (2.21), a contradiction again.

CASE 2: n even. Then $V_n^2 - Q^n = wx^2$ and thus $V_{2n} + Q^n = wx^2$ by (2.4). If we write $2n = 2(2^r z)$ for some odd positive integer z with $r \geq 1$, then

$$wx^2 = V_{2n} + Q^n = V_{2(2^r z)} + Q^{2^r z} \equiv -Q^{2^r z} V_0 + Q^{2^r z} \equiv -Q^{2^r z} \pmod{V_{2^r}}$$

by (2.20). This shows that

$$J = \left(\frac{-w}{V_{2^r}}\right) = 1.$$

When $w = 1$ or $w = 2, r \geq 2$, it can be seen that $J = -1$ by Lemma 2.3. This is impossible.

If $w = 3$ or $w = 6$, it follows that $J = -1$ for $r \geq 3$ by Lemma 2.3 and (2.21) when $3 \nmid Q$, a contradiction.

If $w = 3$ or $w = 6$, it follows that $3 \mid V_n$ from the equation $V_n^2 - Q^n = wx^2$ when $3 \mid Q$. This implies $3 \nmid P$ since $(P, Q) = 1$, and therefore $Q \equiv 1 \pmod 3$ by Lemma 2.15, contradicting $3 \mid Q$.

Now we consider each of the cases $w = 2, r = 1$ and $w = 3$ or $6, 3 \nmid Q, r = 1$ or 2. Let $r = 1$. Then $n = 2z$. If $n = 2$, we have $(P^2 + Q)(P^2 + 3Q) = wx^2$. We can see that this equation has a solution for some values of P and Q when $w \in \{2, 3, 6\}$. Therefore assume that $n > 2$. Then we can write $n = 2z = 2(4q \pm 1) = 8q \pm 2$ for some $q > 0$. Assume that q is odd. Thus $wx^2 = V_{2 \cdot 8q \pm 4} + Q^{8q \pm 2} \equiv -Q^{8q} V_4 + Q^{8q+2}$ or $-Q^{8q-4} V_4 + Q^{8q-2} \pmod{V_8}$, i.e.,

$$wx^2 \equiv -Q^{8q}(V_4 - Q^2) \text{ or } -Q^{8q-4}(V_4 - Q^2) \pmod{V_8}$$

by (2.20). Hence,

$$J = \left(\frac{-w(V_4 - Q^2)}{V_8}\right) = 1.$$

On the other hand, since $3 \nmid Q$ for $w = 3$ or $w = 6$, it can be seen that $\left(\frac{3}{V_8}\right) = 1$ and $\left(\frac{6}{V_8}\right) = 1$ by Lemma 2.3 and (2.21). Thus when $w \in \{2, 3, 6\}$, we have

$$J = \left(\frac{-w(V_4 - Q^2)}{V_8}\right) = -\left(\frac{w}{V_8}\right)\left(\frac{V_4 - Q^2}{V_8}\right) = -1$$

by Lemma 2.3 and (3.1), a contradiction.

Now assume that q is even. Then we can write $q = 2^k s$ for some odd $s \geq 1$ with $k \geq 1$. Thus $n = 8q \pm 2 = 2^{k+3}s \pm 2$. Therefore

$$wx^2 \equiv -Q^{2^{k+3}s} V_4 + Q^{2^{k+3}s+2} \text{ or } -Q^{2^{k+3}s-4} V_4 + Q^{2^{k+3}s-2} \pmod{V_{2^{k+3}}},$$

i.e.,

$$wx^2 \equiv -Q^{2^{k+3}s}(V_4 - Q^2) \text{ or } -Q^{2^{k+3}s-4}(V_4 - Q^2) \pmod{V_{2^{k+3}}}$$

by (2.20). This shows that

$$\left(\frac{-w(V_4 - Q^2)}{V_{2^{k+3}}}\right) = 1.$$

On the other hand, since $3 \nmid Q$ for $w = 3$ or $w = 6$, it can be seen that $\left(\frac{3}{V_{2^{k+3}}}\right) = 1$ and $\left(\frac{6}{V_{2^{k+3}}}\right) = 1$ by Lemma 2.3 and (2.21). Thus when $w \in$

$\{2,3,6\}$, we have

$$1=\left(\frac{-w(V_4-Q^2)}{V_{2^{k+3}}}\right)=-\left(\frac{w}{V_{2^{k+3}}}\right)\left(\frac{V_4-Q^2}{V_{2^{k+3}}}\right)=-1$$

by Lemma 2.3 and (3.1), a contradiction.

Now let $r=2, w=3$ or 6, and $3\nmid Q$. Then $n=4z$. Assume that $z>1$. Then we can write $n=4z=4(4q\pm 1)=2\cdot 8q\pm 4$ for some odd positive integer q. A similar argument shows that $V_n^2-Q^n=wx^2$ has no solutions when q is odd or even. When $z=1$, the equation $V_4^2-Q^4=3x^2$ has a solution, at least for $P=Q=1$. But $V_4^2-Q^4=6x^2$ has no solutions. Indeed, by (2.4), it follows that $6x^2=V_4^2-Q^4=V_8+Q^4$ and thus

$$6x^2=V_8+Q^4\equiv -Q^4V_0+Q^4\equiv -Q^4 \pmod{V_4}$$

by (2.20). This shows that

$$1=J=\left(\frac{-6}{V_4}\right)=-\left(\frac{2}{V_4}\right)\left(\frac{3}{V_4}\right)=-\left(\frac{3}{V_4}\right)$$

by Lemma 2.3. On the other hand, if $3\mid P$, then $J=-1$ by Lemma 2.4. Therefore $3\nmid P$. Now, if $Q\equiv 2 \pmod 3$, then $J=-1$ by (2.21). This is impossible.

Thus $Q\equiv 1 \pmod 3$ since $3\nmid Q$. Moreover, the equation $V_4^2-Q^4=6x^2$ implies that

$$\left(\frac{V_4-Q^2}{6}\right)(V_4+Q^2)=x^2 \tag{3.2}$$

since $V_4-Q^2\equiv 6 \pmod 8$ and $3\mid (V_4-Q^2)$ by Lemmas 2.2 and 2.4. Thus, (3.2) implies

$$V_4+Q^2=(P^2+Q)(P^2+3Q)=\square \tag{3.3}$$

since $\left(\frac{V_4-Q^2}{6}, V_4+Q^2\right)=1$. Then (3.3) implies

$$P^2+Q=2\square \quad \text{and} \quad P^2+3Q=2\square \tag{3.4}$$

since $(P^2+Q, P^2+3Q)=2$. It can be easily shown that (3.4) is impossible, by reducing modulo 8. ∎

4. Main theorems

4.1. Solutions of $U_n=kx^2, U_n=2kx^2$ and $U_n=wU_mx^2$

Theorem 4.1. *Let $k>1$ be a square free positive divisor of P. If $U_n=kx^2$ for some integer x, then $n=2,6,$ or 12.*

Proof. Assume that $U_n=kx^2$ for some integer x and $k\mid P$ with $k>1$. Then n is even by (2.14). Let $n=2m$. Hence $kx^2=U_n=U_{2m}=U_mV_m$ by

(2.3) and this implies that

$$U_m = a\square \quad \text{and} \quad V_m = b\square \tag{4.1}$$

or

$$U_m = 2a\square \quad \text{and} \quad V_m = 2b\square \tag{4.2}$$

for some integers a and b with $ab = k$ since $(U_m, V_m) = 1$ or 2 by (2.13).

Assume that (4.1) is satisfied. By Theorem 2.9, we have $m = 1, 3$, or 5 if $b > 1$ since $b \mid P$. If $b = 1$, then $V_m = \square$ implies $m = 1, 3$, or 5 by Lemma 2.5. Consequently, $n = 2, 6$, or 10. But, if $n = 10$, the equation $U_{10} = U_5V_5 = kx^2$ implies $U_5 = \square$ by (2.13) and (2.14), which is impossible by Theorem 2.11.

Assume that (4.2) is satisfied. By Theorem 2.10, we have $m = 3$ if $b > 1$ since $b \mid P$. If $b = 1$, then $V_m = 2\square$ implies $m = 3$ or 6 by Lemma 2.6. Thus $n = 6$ or 12. ∎

Theorem 4.2. *Let $k > 1$ be a square free positive divisor of P. If $U_n = 2kx^2$ for some integer x, then $n = 6$ or 12.*

Proof. Assume that $k > 1$, $k \mid P$, and $U_n = 2kx^2$. Then n is even by (2.14). Let $n = 2m$. Since $2 \mid U_n$, it follows that $3 \mid n$ by (2.9), and therefore $3 \mid m$. Hence $kx^2 = U_n/2 = U_{2m}/2 = U_m(V_m/2)$ and this implies

$$U_m = a\square \quad \text{and} \quad V_m = 2b\square \tag{4.3}$$

or

$$U_m = 2a\square \quad \text{and} \quad V_m = b\square \tag{4.4}$$

for some integers a and b with $ab = k$ since $(U_m, V_m) = 1$ or 2 by (2.13). Moreover, it can be easily seen that $a = 1$, $b = k$ or $a = k$, $b = 1$ since either $(U_m, k) = 1$ or $(V_m/2, k) = 1$ by (2.7) and (2.8). Then (4.3) implies that $m = 3$ or $m = 6$ by Lemma 2.6 and Theorems 2.10, 2.11, and 4.1 since $3 \mid m$. Similarly, (4.4) implies that $m = 3$ by Lemma 2.5 and Theorems 2.9 and 2.12. Consequently, $n = 6$ or $n = 12$. ∎

Corollary 4.3. *If $U_n = 6x^2$ for some integer x, then $n = 3$ or $n = 6$. $U_3 = 6x^2$ if and only if $P^2 + Q = 6x^2$; $U_6 = 6x^2$ if and only if $P = \square$, $P^2 + Q = 2\square$, $P^2 + 3Q = 3\square$, and $Q \equiv 1 \pmod 8$ or $P = \square$, $P^2 + Q = \square$, $P^2 + 3Q = 6\square$, and $Q \equiv 7 \pmod 8$.*

Proof. Assume that $U_n = 6x^2$. We divide the proof into two cases.

Case 1: $3 \mid P$. Then, since $U_n = 2 \cdot 3x^2$, it follows that $n = 6$ or 12 by Theorem 4.2.

If $n = 6$, it can be seen from $U_6 = 6x^2$ that $V_3 = 3\square, U_3 = 2\square$ or $V_3 = 6\square, U_3 = \square$ by Theorem 2.14 and Lemma 2.15. Hence, $P = \square$, $P^2 + Q = 2\square$, $P^2 + 3Q = 3\square$, and $Q \equiv 1 \pmod 8$ or $P = \square$, $P^2 + Q = \square$, $P^2 + 3Q = 6\square$, and $Q \equiv 7 \pmod 8$), respectively, by Lemmas 2.7 and 2.8 and Theorems 2.11 and 2.12.

If $n = 12$, then $U_{12} = 6x^2$ implies $U_6 = 3\square$ and $V_6 = 2\square$ by Lemma 2.2, Theorem 2.14, and Lemma 2.15. This is impossible by Lemma 2.6 and Theorem 2.13.

Case 2: $3 \nmid P$. Since $2 \mid U_n$ and $3 \mid U_n$, it is seen that $12 \mid n$, $3 \nmid Q$ or $3 \mid n$, $4 \nmid n$, and $Q \equiv 2 \pmod 3$ by (2.9) and Theorem 2.14.

Firstly, assume that $12 \mid n$ and $3 \nmid Q$. Then $n = 12m$. Hence $6x^2 = U_n = U_{12m} = U_{6m}V_{6m}$, which implies

$$U_{6m} = \square \quad \text{and} \quad V_{6m} = 6\square, \tag{4.5}$$

$$U_{6m} = 2\square \quad \text{and} \quad V_{6m} = 3\square, \tag{4.6}$$

$$U_{6m} = 3\square \quad \text{and} \quad V_{6m} = 2\square, \tag{4.7}$$

or

$$U_{6m} = 6\square \quad \text{and} \quad V_{6m} = \square \tag{4.8}$$

by (2.13). The identities (4.5), (4.6), and (4.8) are impossible by Lemmas 2.5, 2.7 and 2.8, and Theorems 2.11 and 2.12. The identity (4.7) implies that $m = 1$ by Lemma 2.6 and Theorem 2.13. Then $U_6 = 3\square$ and therefore $3 \mid P$ by Theorem 2.13. This contradicts $3 \nmid P$.

Secondly, assume that $3|n$, $4 \nmid n$, and $Q \equiv 2 \pmod 3$. Then $n = 3m$. Hence,

$$2x^2 = \frac{U_n}{3} = \frac{U_{3m}}{3} = U_m\left(\frac{V_m^2 - (-Q)^m}{3}\right)$$

by (2.5). Since

$$\left(U_m, \frac{(P^2 + 4Q)U_m^2 + 3(-Q)^m}{3}\right) = 1$$

by (2.7), it follows that

$$U_m = \square \quad \text{and} \quad V_m^2 - (-Q)^m = 6\square, \tag{4.9}$$

or

$$U_m = 2\square \quad \text{and} \quad V_m^2 - (-Q)^m = 3\square. \tag{4.10}$$

Assume that (4.9) is satisfied. Then $m = 1$ or $m = 2$ by Theorem 2.11 and Lemma 3.3. Therefore $n = 3$ or $n = 6$. The identity (4.10) is impossible by Theorem 2.12 and Lemma 3.3. ∎

In the following four theorems, we assume that $U_m \neq 1$ for all m. When $U_m = 1$, we have $U_n = wx^2$ with $w \in \{1, 2, 3, 6\}$. In this case, the solutions of these equations are given in Theorems 2.11–2.13 and Corollary 4.3.

Theorem 4.4. *Assume that $m > 1$ and $U_n = U_m x^2$ for some integer x. Then $m = n$ or $(m, n) = (5, 10), (2, 12),$ or $(3, 6)$.*

Proof. Since $U_m \mid U_n$, we have $n = mr$ for some integer r by (2.10). Thus,

$$x^2 = \frac{U_n}{U_m} = \frac{U_{mr}}{U_m} = U_r(V_m, -(-Q)^m) \tag{4.11}$$

by (2.16). If $r = 1$, then $m = n$. So, assume that $r \neq 1$.

Let $3 \nmid m$. Then V_m is odd by (2.9) and also $(V_m, -(-Q)^m) = 1$ by (2.8). Hence, (4.11) implies that $r = 2, 3, 6$, or 12, and therefore

$$V_m = x^2 \quad \text{if } r = 2, \tag{4.12}$$

$$V_m^2 - (-Q)^m = x^2 \quad \text{if } r = 3, \tag{4.13}$$

$$V_m = 3\square,\ V_m^2 - (-Q)^m = 2\square,\ V_m^2 - 3(-Q)^m = 6\square \quad \text{if } r = 6, \tag{4.14}$$

$$\begin{aligned} &V_m = \square,\ V_m^2 - (-Q)^m = 2\square, \\ &V_m^2 - 2(-Q)^m = 3\square,\ V_m^2 - 3(-Q)^m = \square \quad \text{if } r = 12, \end{aligned} \tag{4.15}$$

by Theorem 2.11. Since $3 \nmid m$ and $m > 1$, (4.12) implies that $m = 5$ and so $n = 10$ by Lemma 2.5. The identity (4.13) is impossible by Lemma 3.3 since $m > 1$. The identity (4.14) implies $m = 2$ by Lemmas 2.7, 3.2 and 3.3. Therefore $n = 12$. Lastly, (4.15) is impossible by Lemmas 2.5 and 3.2 since $m > 1$.

Now let $3 \mid m$. If r is even, then $r = 2a$ and therefore $n = mr = 2ma$. Hence, using (2.3) we get

$$x^2 = \frac{U_n}{U_m} = \frac{U_{2ma}}{U_m} = \frac{U_{ma}}{U_m} V_{ma},$$

and this implies

$$U_{ma} = U_m\square \quad \text{and} \quad V_{ma} = \square \tag{4.16}$$

or

$$U_{ma} = 2U_m\square \quad \text{and} \quad V_{ma} = 2\square \tag{4.17}$$

since $(U_{ma}/U_m, V_{ma}) = 1$ or 2 by (2.13).

Assume that (4.16) is satisfied. Then $m = 3$, $a = 1$ by Lemma 2.5 since $3 \mid m$. Thus $n = 6$.

Assume that (4.17) is satisfied. Then $m = 3$, $a = 1$ or $m = 3$, $a = 2$, or $m = 6$, $a = 1$ by Lemma 2.6. It can be seen that neither $m = 3$, $a = 1$ nor $m = 6$, $a = 1$ is possible for the equation $U_{ma} = 2U_m\square$. If $m = 3$ and $a = 2$, then we get $V_6 = 2\square$ and $V_3 = 2\square$, which is impossible by Lemma 2.6.

Assume that r is odd. Since $3 \mid m$, we can write $m = 3s$.

If s is even, then $s = 2b$ and so $n = mr = 3sr = 6br$. Hence, using (2.3) and (2.11), we get

$$x^2 = \frac{U_n}{U_m} = \frac{U_{6br}}{U_{6b}} = \frac{U_{3br}}{U_{3b}} \frac{V_{3br}}{V_{3b}},$$

and this implies

$$U_{3br} = U_{3b}\square \quad \text{and} \quad V_{3br} = V_{3b}\square, \tag{4.18}$$

or

$$U_{3br} = 2U_{3b}\square \quad \text{and} \quad V_{3br} = 2V_{3b}\square \tag{4.19}$$

since $(U_{3br}/U_{3b}, V_{3br}/V_{3b}) = 1$ or 2 by (2.13). Each of (4.18) and (4.19) is impossible, by [12, Theorems 3.2 and 3.3] respectively.

Now let s be odd. Then m is odd too.

Let $r \equiv 1 \pmod 4$. Then writing $n = mr = m(r-1)+m = 2\cdot 2^k mz + m$ for some odd positive integer z with $k \geq 1$, we get

$$U_m x^2 = U_n = U_{2\cdot 2^k mz+m} \equiv -Q^{2^k mz} U_m \pmod{V_{2^k}}$$

by (2.19). Since $(U_m, V_{2^k}) = 1$ by (2.13), the above congruence yields

$$x^2 \equiv -Q^{2^k mz} \pmod{V_{2^k}}.$$

This shows that $\left(\frac{-1}{V_{2^k}}\right) = 1$, which is impossible by Lemma 2.3.

If $r \equiv -1 \pmod 4$, then $n = mr = m(r+1) - m = 2\cdot 2^k mz - m$ with z odd and $k \geq 1$. Thus

$$U_m x^2 = U_n = U_{2\cdot 2^k mz-m} \equiv -Q^{2^k mz-m} U_m \pmod{V_{2^k}}$$

by (2.1) and (2.19). Since $(U_m, V_{2^k}) = 1$ by (2.13), we obtain

$$x^2 \equiv -Q^{2^k mz-m} \pmod{V_{2^k}}.$$

This shows that $\left(\frac{-Q}{V_{2^k}}\right) = 1$. If $Q \equiv 1 \pmod 4$, then, by Lemma 2.3,

$$1 = \left(\frac{-Q}{V_{2^k}}\right) = \left(\frac{-1}{V_{2^k}}\right)\left(\frac{Q}{V_{2^k}}\right) = -\left(\frac{-1}{Q}\right) = -1,$$

a contradiction. If $Q \equiv -1 \pmod 4$, then (4.11) implies that r is a perfect square by Theorem 2.16, contrary to $r \equiv -1 \pmod 4$. ∎

THEOREM 4.5. *Assume that $m > 1$ and $U_n = 2U_m x^2$ for some integer x. Then $(m,n) = (2,6), (3,6), (3,12)$, or $(6,12)$.*

Proof. Since $U_m \mid U_n$, it follows that $n = mr$ for some positive integer r by (2.10). Thus,

$$2x^2 = \frac{U_n}{U_m} = \frac{U_{mr}}{U_m} = U_r(V_m, -(-Q)^m) \tag{4.20}$$

by (2.16).

Firstly, let $3 \nmid m$. Then V_m is odd by (2.9) and also $(V_m, -(-Q)^m) = 1$ by (2.8). Hence, $r = 3$ or 6 by Theorem 2.12. If $r = 3$, then we obtain $V_m^2 - (-Q)^m = 2x^2$ from (4.20). Thus $m = 2$, and therefore $n = 6$ by Lemma 3.3. If $r = 6$, then $V_m = \square$, $V_m^2 - (-Q)^m = 2\square$, and $V_m^2 - 3(-Q)^m = \square$ by Theorem 2.12. This is impossible by Lemmas 2.5, 3.2, and 3.3.

Secondly, let $3 \mid m$. Then V_m is even by (2.9). Thus, since $2 \mid U_r$, r is even by (2.17). Let $r = 2a$. Hence, using (2.3), we get

$$x^2 = \frac{U_n}{2U_m} = \frac{U_{2ma}}{2U_m} = \frac{U_{ma}}{U_m} \cdot \frac{V_{ma}}{2},$$

and this implies

$$U_{ma} = U_m\square \quad \text{and} \quad V_{ma} = 2\square, \tag{4.21}$$

or

$$U_{ma} = 2U_m\square \quad \text{and} \quad V_{ma} = \square \tag{4.22}$$

since $(U_{ma}/U_m, V_{ma}/2) = 1$ or 2 by (2.13).

Assume that (4.21) holds. Then $(m, a) = (3, 1), (6, 1)$, or $(3, 2)$ by Lemma 2.6 and Theorem 4.4. Thus $(m, n) = (3, 6), (6, 12)$, or $(3, 12)$.

Assume that (4.22) is satisfied. Then $m = 3$, $a = 1$ by Lemma 2.5 since $3 \mid m$. But these values are impossible for $U_{ma} = 2U_m\square$. ■

THEOREM 4.6. *Assume that* $m > 1$ *and* $U_n = 3U_m x^2$ *for some integer* x. *Then* $(m, n) = (2, 4), (2, 6), (3, 6), (4, 12)$, *or* $(5, 10)$.

Proof. Since $U_m \mid U_n$, we have $n = mr$ for some integer r by (2.10). Thus

$$3x^2 = \frac{U_n}{U_m} = \frac{U_{mr}}{U_m} = U_r(V_m, -(-Q)^m) \tag{4.23}$$

by (2.16).

Let $3 \nmid m$. Then V_m is odd by (2.9) and also $(V_m, -(-Q)^m) = 1$ by (2.8). Thus (4.23) implies $r = 2$, 3, 4, or 6. Therefore, by Theorem 2.13,

$$V_m = 3x^2 \quad \text{if } r = 2, \tag{4.24}$$

$$V_m^2 - (-Q)^m = 3x^2 \quad \text{if } r = 3, \tag{4.25}$$

$$V_m = \square \text{ and } V_m^2 - 2(-Q)^m = 3\square \quad \text{if } r = 4, \tag{4.26}$$

$$V_m = \square,\ V_m^2 - (-Q)^m = 2\square \text{ and } V_m^2 - 3(-Q)^m = 6\square \quad \text{if } r = 6. \tag{4.27}$$

The identity (4.24) implies that $m = 2$ or $m = 5$ by Lemma 2.7 since $3 \nmid m$ and $m > 1$. Thus $n = 4$ or $n = 10$. The identity (4.25) implies that $m = 2$ or $m = 4$ by Lemma 3.3 and therefore $n = 6$ or $n = 12$.

Assume that (4.26) is satisfied. Since $V_{2m} = V_m^2 - 2(-Q)^m$ by (2.4), we have $V_m = \square$ and $V_{2m} = 3\square$. This is impossible by Lemmas 2.5 and 2.7. The identity (4.27) is impossible by Lemmas 2.5, 3.2, and 3.3.

Now let $3 \mid m$. Firstly, assume that r is even. Then $r = 2a$ and thus $n = mr = 2ma$. Hence, using (2.3), we have

$$3x^2 = \frac{U_n}{U_m} = \frac{U_{2ma}}{U_m} = \frac{U_{ma}}{U_m} \cdot V_{ma},$$

and this implies that

$$(4.28)\qquad U_{ma}=U_m\square \quad\text{and}\quad V_{ma}=3\square,$$

$$(4.29)\qquad U_{ma}=3U_m\square \quad\text{and}\quad V_{ma}=\square,$$

$$(4.30)\qquad U_{ma}=2U_m\square \quad\text{and}\quad V_{ma}=6\square,$$

or

$$(4.31)\qquad U_{ma}=6U_m\square \quad\text{and}\quad V_{ma}=2\square.$$

The identity (4.28) implies that $m=3$, $a=1$ by Lemma 2.7 since $3\mid m$. Thus $n=6$. The identity (4.29) implies that $m=3$, $a=1$ by Lemma 2.5. But this is impossible for the equation $U_{ma}=3U_m\square$. It can be seen that (4.30) is impossible by Lemma 2.8 and Theorem 4.5. The identity (4.31) implies $(m,a)=(3,1),(6,1)$, or $(3,2)$ by Lemma 2.6. It can be seen that $(m,a)=(3,1)$ or $(6,1)$ is impossible. If $m=3$, $a=2$, then $V_6=2\square$ and $V_3=6\square$. This is impossible by Lemma 2.6 and Lemma 2.8.

Secondly, assume that r is odd. Then, since $3\mid U_r$ by (4.23), it follows that $3\mid r$ by Theorem 2.14. Let $r=3s$ for some positive integer s. Then $n=mr=3ms$ and thus

$$3x^2=\frac{U_n}{U_m}=\frac{U_{3ms}}{U_m}=\frac{U_{ms}}{U_m}\big(V_{ms}^2-(-Q)^{ms}\big)$$

by (2.5). Since

$$\left(\frac{U_{ms}}{U_m},(P^2+4Q)U_{ms}^2+3(-Q)^{ms}\right)=1,3$$

by (2.7), it follows that

$$(4.32)\qquad U_{ms}=U_m\square \quad\text{and}\quad V_{ms}^2-(-Q)^{ms}=3\square$$

or

$$(4.33)\qquad U_{ms}=3U_m\square \quad\text{and}\quad V_{ms}^2-(-Q)^{ms}=\square.$$

But (4.32) and (4.33) are impossible by Lemma 3.3 since $3\mid m$. ∎

THEOREM 4.7. *Assume that $m>1$ and $U_n=6U_mx^2$ for some integer x. Then $(m,n)=(2,6)$ or $(3,6)$.*

Proof. Since $U_m\mid U_n$, it follows that $n=mr$ for some integer r by (2.10). Hence,

$$(4.34)\qquad 6x^2=\frac{U_n}{U_m}=\frac{U_{mr}}{U_m}=U_r\big(V_m,-(-Q)^m\big)$$

by (2.16).

Firstly, let $3\nmid m$. Then V_m is odd by (2.9) and also $(V_m,-(-Q)^m)=1$ by (2.8). Thus (4.34) implies that $r=3$ or 6 by Corollary 4.3. If $r=3$, we have $V_m^2-(-Q)^m=6\square$. Thus $m=2$ by Lemma 3.3 since $m>1$. Therefore

$n = 6$. If $r = 6$, then

$$V_m = \square, \quad V_m^2 - (-Q)^m = 2\square, \quad V_m^2 - 3(-Q)^m = 3\square$$

or

$$V_m = \square, \quad V_m^2 - (-Q)^m = \square, \quad V_m^2 - 3(-Q)^m = 6\square$$

by Corollary 4.3. But both of these are impossible by Lemmas 2.5, 3.2, and 3.3.

Secondly, let $3 \mid m$. Then V_m is even by (2.9). Thus, since $2 \mid U_r$ by (4.34), r is even by (2.17). Let $r = 2s$. Hence, using (2.3), we have

$$3x^2 = \frac{U_n}{2U_m} = \frac{U_{mr}}{2U_m} = \frac{U_{2ms}}{2U_m} = \frac{U_{ms}}{U_m} \cdot \frac{V_{ms}}{2}$$

since $3 \mid m$. This implies

(4.35) $$U_{ms} = U_m\square \quad \text{and} \quad V_{ms} = 6\square,$$

(4.36) $$U_{ms} = 3U_m\square \quad \text{and} \quad V_{ms} = 2\square,$$

(4.37) $$U_{ms} = 2U_m\square \quad \text{and} \quad V_{ms} = 3\square,$$

or

(4.38) $$U_{ms} = 6U_m\square \quad \text{and} \quad V_{ms} = \square$$

since $(U_{ms}/U_m, V_{ms}/2) = 1$ or 2 by (2.13).

The identity (4.35) implies $m = 3$, $s = 1$ by Lemma 2.8 and so $n = 6$. The identity (4.36) implies $m = 3$, $s = 2$ by Lemma 2.6 and Theorem 4.6. But, in this case, we obtain $V_3 = 3\square$ and $V_6 = 2\square$. This is impossible by Lemmas 2.6 and 2.7. The identity (4.37) is impossible by Lemma 2.7 and Theorem 4.5. The identity (4.38) implies $m = 3$, $s = 1$ by Lemma 2.5. But this is impossible for the equation $U_{ms} = 6U_m\square$. ∎

4.2. On square classes in a generalized Fibonacci sequence. In [4, 5, 8], the authors defined $U_n \sim U_m$ iff there exist nonzero integers x and y such that $x^2U_n = y^2U_m$, or equivalently, $U_nU_m = \square$. If $U_n \sim U_m$, then U_n and U_m are said to be *in the same square class*, and a square class containing more than one term of the sequence (U_n) is called *non-trivial*.

Now we briefly summarize the relevant known facts. Ribenboim [5] has explicitly shown that if $m \neq 1, 2, 3, 6, 12$, then the square classes of F_m is trivial. That is, if $m \neq 1, 2, 3, 6, 12$ and $F_nF_m = \square$, then $m = n$. It should be pointed out that more generally, Cohn [1] determined the square classes of the sequence $(U_n(P, Q))$ when $Q = \pm 1$ and P is odd. Ribenboim [6] has determined the square classes of the sequences $U_n(Q + 1, Q)$. Moreover, when P and Q are nonzero relatively prime integers such that $P^2 + 4Q \neq 0$, Ribenboim and McDaniel [8] showed that each square class of the sequences (U_n) and (V_n) is finite, and its elements are effectively computable. Moreover, in [4] they showed that for all odd relatively prime integers P and Q

with $P > 0$ and $P^2 + 4Q > 0$, if $U_n U_m = \square$ for $1 \le m < n$, then either $(m,n) = (1,2), (1,6), (1,12), (2,3), (2,12), (3,6), (5,10)$, or $(10,15)$, or $n = 3m$, $3 \nmid m$, m odd. But $(m,n) = (10,15)$ is impossible: if $(m,n) = (10,15)$, then

$$U_{15}U_{10} = U_5^2(V_5^2 + Q^5)V_5 = \square$$

by (2.3) and (2.5) and this implies that

$$V_5 = \square \text{ and } V_5^2 + Q^5 = \square$$

since $(V_5^2 + Q^5, V_5) = 1$ by (2.8). The equation $V_5^2 + Q^5 = \square$ has no solutions by Lemma 3.3. Moreover, we will prove in Theorem 4.8 that m may only be 1, and therefore $n = 3$, in case $U_n U_m = \square$ for $n = 3m$, $3 \nmid m$, m odd. Lastly, Şiar [12] determined all n and m such that $V_n V_m = w\square$ with $w \in \{1, 2, 3, 6\}$.

Now let a and b be square-free positive integers such that $(a, b) = 1$. Then we define an equivalence relation as follows: $aU_n \sim bU_m$ iff there exist non-zero integers x and y such that $x^2 aU_n = y^2 bU_m$, or equivalently, $U_n U_m = ab\square$.

Now, we consider the equivalence relation $aU_n \sim bU_m$ when $ab \in \{1, 2, 3, 6\}$. In the following four theorems, we assume (1.1).

THEOREM 4.8. *Assume that $U_n U_m = x^2$ for $1 \le m \le n$. Then $m = n$ or $(m,n) = (1,2), (1,3), (1,6), (1,12), (2,3), (2,12), (3,6)$, or $(5,10)$.*

Proof. It is obvious that $m = n$ is a solution. So, let $m \ne n$. Let $d = (m,n)$. Then $(U_m, U_n) = U_d$ by (2.12) and therefore

$$\frac{U_n}{U_d}\frac{U_m}{U_d} = \left(\frac{x}{U_d}\right)^2.$$

Since $(U_n/U_d, U_m/U_d) = 1$, it follows that $U_n = U_d\square$ and $U_m = U_d\square$. Assume that $U_d \ne 1$. Then it is obvious that $d > 1$. Thus, by Theorem 4.4,

(4.39) $\quad n = d, \quad$ or $\quad (d,n) = (5,10), (2,12)$, or $(3,6)$,

(4.40) $\quad m = d, \quad$ or $\quad (d,m) = (5,10), (2,12)$, or $(3,6)$.

The identities (4.39) and (4.40) imply that $(m,n) = (5,10), (2,12)$, or $(3,6)$. If $U_d = 1$, then $U_n = \square$ and $U_m = \square$, and these imply that

$$(m,n) = (1,2), (1,3), (1,6), (1,12), (2,3), (2,6), \text{ or } (2,12)$$

by Theorem 2.11. But $(m,n) = (2,6)$ is impossible for $U_n U_m = x^2$. ∎

THEOREM 4.9. *Assume that $U_n U_m = 2x^2$ for $1 \le m \le n$. Then $(m,n) = (1,3), (1,6), (2,3), (2,6), (3,6), (3,12)$, or $(6,12)$.*

Proof. It is obvious that $m \ne n$. Let $d = (m,n)$. Then $(U_m, U_n) = U_d$ by (2.12) and therefore

$$\frac{U_n}{U_d}\frac{U_m}{U_d} = 2\left(\frac{x}{U_d}\right)^2.$$

Since $(U_n/U_d, U_m/U_d) = 1$, it follows that

$$U_n = U_d\square \quad \text{and} \quad U_m = 2U_d\square \tag{4.41}$$

or

$$U_n = 2U_d\square \quad \text{and} \quad U_m = U_d\square. \tag{4.42}$$

Assume that $U_d \neq 1$. Then it is obvious that $d > 1$. Thus (4.41) and (4.42) imply that $(m,n) = (6,12)$ and $(m,n) = (2,6),(3,6),(3,12)$, or $(6,12)$ by Theorems 4.4 and 4.5. If $U_d = 1$, then $U_n = \square$, $U_m = 2\square$ or $U_n = 2\square$, $U_m = \square$. From these equations, we can see that $(m,n) = (1,3),(1,6),(2,3)$, or $(2,6)$ by Theorems 2.11 and 2.12. ∎

Similarly, the following theorems can be proved using Theorems 2.11–2.13, Corollary 4.3, and Theorems 4.6 and 4.7. Therefore we omit their proofs.

THEOREM 4.10. *If $U_nU_m = 3x^2$ for $1 \leq m \leq n$, then* $(m,n) = (1,2)$, $(1,3),(1,4),(1,6)$, $(2,3),(2,4),(2,6),(3,6),(4,12)$, *or* $(5,10)$.

THEOREM 4.11. *If $U_nU_m = 6x^2$ for $1 \leq m \leq n$, then* $(m,n) = (1,3)$, $(1,6),(2,3)$, $(2,6),(3,4),(3,6)$, *or* $(4,6)$.

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