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MATHEMATICAL LOGIC AND FOUNDATIONS

On Sequential Compactness and Related Notions of Compactness of Metric Spaces in **ZF**

by

Kyriakos KEREMEDIS

Presented by Czesław BESSAGA

Summary. We show that:

(i) If every sequentially compact metric space is countably compact then for every infinite set X, $[X]^{<\omega}$ is Dedekind-infinite. In particular, every infinite subset of $\mathbb{R}$ is Dedekind-infinite.

(ii) Every sequentially compact metric space is compact iff every sequentially compact metric space is separable. In addition, if every sequentially compact metric space is compact then: every infinite set is Dedekind-infinite, the product of a countable family of compact metric spaces is compact, and every compact metric space is separable.

(iii) The axiom of countable choice implies that every sequentially bounded metric space is totally bounded and separable, every sequentially compact metric space is compact, and every uncountable sequentially compact, metric space has size $|\mathbb{R}|$.

(iv) If every sequentially bounded metric space is totally bounded then every infinite set is Dedekind-infinite.

(v) The statement: "Every sequentially bounded metric space is bounded" implies the axiom of countable choice restricted to the real line.

(vi) The statement: "For every compact metric space $\mathbf{X}$ either $|X| \leq |\mathbb{R}|$, or $|\mathbb{R}| \leq |X|$" implies the axiom of countable choice restricted to families of finite sets.

(vii) It is consistent with **ZF** that there exists a sequentially bounded metric space whose completion is not sequentially bounded.

(viii) The notion of sequential boundedness of metric spaces is countably productive.

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1. Notation and terminology. Let $\mathbf{X} = (X, d)$ be a metric space and $A \subseteq X$. Boldface letters will denote metric spaces and lightface letters will denote the underlying sets. For every $x \in X$ and $\varepsilon > 0$, $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ will denote the open disc in $\mathbf{X}$ with center x and radius ε, $\delta(A) = \sup\{d(x, y) : x, y \in A\}$ will denote the *diameter* of A, and $\widetilde{A}$ will denote the *sequential closure* of A, i.e., the set of all points $y \in X$ for which there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of points of A converging to y.

A is said to be *bounded* iff $\delta(A) < \infty$.

$\mathbf{X}$ is *Heine–Borel compact* or simply *compact* iff every open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$. Equivalently, $\mathbf{X}$ is compact iff every family of closed subsets of $\mathbf{X}$ having the *finite intersection property*, fip for abbreviation, has a non-empty intersection.

$\mathbf{X}$ is *countably compact* iff every countable open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$.

$\mathbf{X}$ is *complete* iff every Cauchy sequence of points of X converges to some element of X.

$\mathbf{X}$ is *limit point compact* iff every infinite subset of X has a limit point.

$\mathbf{X}$ is *sequentially compact* iff every sequence has a convergent subsequence.

$\mathbf{X}$ is *totally bounded* iff for every real $\varepsilon > 0$, there exists an ε-net, i.e., a finite subset $\{x_i : i \leq n\}$ of X such that $\bigcup\{B(x_i, \varepsilon) : i \leq n\} = X$. Clearly, each totally bounded metric space is bounded, but the converse is not true in general. For example, every infinite set equipped with the discrete metric is bounded but not totally bounded.

$\mathbf{X}$ is *sequentially bounded* or *Cauchy-precompact* iff every sequence of points of X admits a Cauchy subsequence.

$\mathbf{X}$ is *selective* iff the family of all non-empty open sets has a choice set. Equivalently (see [7]), $\mathbf{X}$ is selective iff $\mathbf{X}$ has a well ordered dense subset.

A *completion* of $\mathbf{X}$ is a complete metric space (Y, ρ) together with an isometric map $T : \mathbf{X} \to \mathbf{Y}$ such that $\overline{T(X)} = Y$. It is a well known **ZF** result that for every $x_0 \in X$ the mapping

$$T : (X, d) \to (C_b(X, \mathbb{R}), \rho), \qquad T(x) = f_x,$$

is such an isometric map, where $C_b(X, \mathbb{R})$ is the family of all bounded continuous functions from X to $\mathbb{R}$, ρ is the sup metric ($\rho(f, g) = \sup\{|f(x)-g(x)| : x \in X\}$) and for every $x \in X$, $f_x : X \to \mathbb{R}$ is the function given by

$$f_x(t) = d(x, t) - d(x_0, t).$$

Thus, $\mathbf{Y} = (\overline{T(X)}, \rho)$ is a completion of $\mathbf{X}$.

We recall that if $\{\mathbf{X}_n = (X_n, d_n) : n \in \mathbb{N}\}$ is a family of metric spaces then the function $d : X \times X \to \mathbb{R}$, $X = \prod_{n\in\mathbb{N}} X_n$, given by

$$d(x,y) = \sum_{n\in\mathbb{N}} \frac{\rho_n(x_n, y_n)}{2^n} \tag{1}$$

where $\rho_n(a,b) = \min\{1, d_n(a,b)\}$ for all $n \in \mathbb{N}$, is a metric and the metric topology T_d it produces on X coincides with the product topology of the family of spaces $\{\mathbf{X}_n : n \in \mathbb{N}\}$. We shall always assume that whenever a family $\{\mathbf{X}_n = (X_n, d_n) : n \in \mathbb{N}\}$ of metric spaces is given then $\delta(X_n) \leq 1$ for all $n \in \mathbb{N}$ and the product $X = \prod_{n\in\mathbb{N}} X_n$ carries the metric d given by (1).

Let c, cc, tb, sc, lpc, sb, b, s and hs abbreviate the following properties of metric spaces: compact, countably compact, totally bounded, sequentially compact, limit point compact, sequentially bounded, bounded, separable and hereditarily separable respectively. For every $p, q \in \{c, cc, tb, sc, lpc, sb, b, s, hs\}$, consider the statement

$\mathbf{M}(p,q)$: Every metric space having property p has property q.

For example, $\mathbf{M}(sc, c)$ will stand for the statement: “Every sequentially compact metric space is compact”.

A set X is said to be *Dedekind-infinite*, denoted by $\mathbf{DI}(X)$, iff X contains a countably infinite set. Otherwise X is said to be *Dedekind-finite*. By universal quantifying over X, $\mathbf{DI}(X)$ gives rise to the choice principle **IDI**: $\forall X(X \text{ infinite} \to \mathbf{DI}(X))$, that is, “every infinite set is Dedekind-infinite” (Form 9 of [3]). We will also consider the following statements:

- **CAC** (Form 8 of [3]): For every countable family $\mathcal{A}$ of non-empty sets there exists a function f such that for all $x \in \mathcal{A}$, $f(x) \in x$.
- $\mathbf{CAC}_{\text{fin}}$ (Form 10 of [3]): **CAC** restricted to countable families of non-empty finite sets. Equivalently (see Form [10 O] in [3]), every infinite well ordered family of non-empty finite sets has a partial choice function. i.e., some infinite subfamily has a choice function.
- $\mathbf{CAC}(\mathbb{R})$ (Form 94 of [3]): **CAC** restricted to countable families of non-empty subsets of $\mathbb{R}$. Equivalently (see e.g. [2]), every countable family of disjoint non-empty subsets of the real line has an infinite subfamily with a choice function.
- $\mathbf{CAC}_\omega(\mathbb{R})$ (Form 5 of [3]): $\mathbf{CAC}(\mathbb{R})$ restricted to countable families of non-empty countable subsets of $\mathbb{R}$. Equivalently (see e.g. [2]), every countable family of disjoint non-empty countable subsets of the real line has an infinite subfamily with a choice function.
- $\mathbf{IDI}(\mathbb{R})$ (Form 13 of [3]): $\forall X \in \mathcal{P}(\mathbb{R})(X \text{ infinite} \to \mathbf{DI}(X))$.
- **BPI** (Form [14 A] in [3]): For every infinite set X, every proper filter over X can be extended to an ultrafilter. Equivalently (see form [14 J] in [3]), the Tychonoff product of compact T_2 spaces is compact.

2. Introduction and some preliminary results. The theoretic setting in this paper is the Zermelo–Fraenkel set theory **ZF** without the axiom of choice **AC**. We shall continue with the study of the implications between the various notions of compactness, launched in [5]. The following diagram from [5] summarizes the implications which hold in **ZF**.

$$\begin{tikzcd}
& \boxed{\text{compact}} \arrow[dl] \arrow[dr] & \\
\boxed{\text{limit point compact}} \arrow[d] & & \boxed{\begin{array}{l}\text{complete and}\\ \text{totally bounded}\end{array}} \arrow[d] \\
\boxed{\text{sequentially compact}} \arrow[rr, leftrightarrow] & & \boxed{\begin{array}{l}\text{complete and}\\ \text{sequentially bounded}\end{array}}
\end{tikzcd}$$

Diagram 1

No other implications between these notions hold in **ZF**. For the non-implications, counterexamples are supplied in [5].

In view of Diagram 1 we see that, in **ZF**, Heine–Borel compactness is the strongest notion of compactness of metric spaces and sequential compactness is the weakest one. We stress that the latter property is so weak that any Dedekind-finite set admits such a structure. (The discrete metric will do the job.)

Some of the propositions $\mathbf{M}(p,q)$ hold true in **ZF**, for example, $\mathbf{M}(c,cc)$, $\mathbf{M}(c,tb)$, $\mathbf{M}(c,b)$, $\mathbf{M}(c,sb)$, $\mathbf{M}(c,lpc)$, $\mathbf{M}(c,sc)$, $\mathbf{M}(lpc,sc)$, $\mathbf{M}(sc,sb)$. However, some others, including $\mathbf{M}(b,sb)$, $\mathbf{M}(b,tb)$, $\mathbf{M}(tb,sc)$, $\mathbf{M}(tb,c)$, $\mathbf{M}(sb,c)$ and $\mathbf{M}(sb,sc)$, fail in **ZFC**. ($\mathbb{R}$ endowed with the discrete metric is bounded but neither totally bounded nor sequentially bounded, and $(0,1)$ with the usual metric is totally and sequentially bounded but neither compact nor sequentially compact.

From Diagram 1 and the counterexamples given in [5] we deduce:

Proposition 1. *The negation of each one of* $\mathbf{M}(sc,c)$, $\mathbf{M}(lpc,c)$ *and* $\mathbf{M}(sc,lpc)$ *is consistent with* **ZF**.

In the basic Fraenkel Model $\mathcal{N}1$ in [3], which is a permutation model, the set of atoms A is infinite but its power set $\mathcal{P}(A)$ has no countably infinite set. So, if d is the discrete metric on A then (A,d) is a countably compact, sequentially bounded and sequentially compact metric space but it is not compact, limit point compact, separable or totally bounded. An application of the Jech–Sochor Embedding Theorem [4, Theorem 6.1] yields a **ZF** model $\mathcal{M}$ witnessing the failure of each of $\mathbf{M}(p,q)$, $p \in \{cc, sb, sc\}$, $q \in \{c, lpc, s, tb\}$. Hence, we have proved the following proposition.

Proposition 2. *The negation of each of* $\mathbf{M}(p,q)$*,* $p \in \{cc, sb, sc\}$*,* $q \in \{c, lpc, s, tb\}$*, is consistent with* **ZF***.*

It is well known that the set A of all added Cohen reals in the Basic Cohen Model $\mathcal{M}1$ in [3] with the usual metric, being Dedekind-finite and dense in $\mathbb{R}$, is sequentially compact and sequentially bounded but it is not bounded, separable, totally bounded or compact. Hence, the conclusion of the following proposition is justified.

Proposition 3. *The negation of each of* $\mathbf{M}(p,q)$*,* $p \in \{sc, sb\}$*,* $q \in \{b, s, tb, c, hs\}$*, is consistent with* **ZF***.*

The following well known theorem lists in **ZFC** (= **ZF** and **AC**) the most popular forms of compactness of metric spaces.

Theorem 4 ([9]). (**ZFC**) *Let* $\mathbf{X}$ *be a metric space. Then the following are equivalent:*

(i) $\mathbf{X}$ *is compact.*
(ii) $\mathbf{X}$ *is limit point compact.*
(iii) $\mathbf{X}$ *is sequentially compact.*
(iv) $\mathbf{X}$ *is complete and totally bounded.*
(v) $\mathbf{X}$ *is complete and sequentially bounded.*
(vi) $\mathbf{X}$ *is countably compact.*

Recall the following theorem from [5]:

Theorem 5 ([5]). (**ZF**) *Every selective, sequentially compact metric space* $\mathbf{X} = (X, d)$ *is compact. In particular, a separable metric space is sequentially compact iff it is compact.*

In view of Theorem 5, we see that Theorem 4 holds true in **ZF** when restricted to the class of separable metric spaces. (It is known—see e.g. [1, Example 6.4]—that the existence of compact non-separable metric spaces is consistent with **ZF**.) Thus, $\mathbf{M}(sc, s)$ implies $\mathbf{M}(sc, c)$ and the conclusion of Theorem 4 holds true in $\mathbf{ZF} + \mathbf{M}(sc, s)$. It is not hard to verify that the axiom of dependent choices **DC** implies $\mathbf{M}(sc, s)$. However, it is not known whether the strictly weaker consequence **CAC** of **DC** implies $\mathbf{M}(sc, s)$, nor whether $\mathbf{M}(sc, c)$ implies $\mathbf{M}(sc, s)$.

Remark 1. In contrast to Theorem 5, the negation of the statement: "Every second countable metric space is sequentially compact iff it is compact" is consistent with **ZF**. Indeed, the set A of all added Cohen reals in $\mathcal{M}1$ with the usual metric, being a Dedekind-finite subspace of $\mathbb{R}$, is second countable and sequentially compact but not compact.

The statement $\mathbf{M}(c, s)$ was studied in [7] where the following characterizations were derived:

Theorem 6 ([7]). *The following are equivalent:*

(i) $\mathbf{M}(c,s)$.
(ii) *For every compact metric space the family of all non-empty closed subsets has a choice function.*
(iii) *Every compact metric space has a well ordered dense subset.*
(iv) *Every compact metric space is second countable.*

Clearly, $\mathbf{M}(sc,s)$ implies $\mathbf{M}(c,s)$, and the following question pops up:

Question 1. Does $\mathbf{M}(c,s)$ imply $\mathbf{M}(sc,s)$?

In [13, Exercise 24B, p. 182] it is stated that:

(A): A metric space is totally bounded iff it is sequentially bounded.

From Diagram 1 and the counterexamples in [5] it follows that the negation of (A) is consistent with **ZF**. So, one may ask:

Question 2. What is the place of (A) in the deductive hierarchy of choice principles?

Theorem 7.

(i) [10] $(\mathbf{ZF}+\mathbf{CAC})$ *Let* $\mathbf{X}=(X,d)$ *be a totally bounded metric space. Then* $\mathbf{X}$ *is second countable and separable. In particular,* **CAC** *implies* $\mathbf{M}(tb,s)$.
(ii) [5] $(\mathbf{ZF})$ *Let* $\mathbf{X}=(X,d)$ *be a metric space and* D *be a dense subset of* $\mathbf{X}$. *Then* $\mathbf{X}$ *is totally bounded iff* $\mathbf{D}$ *is totally bounded. In particular,* $\mathbf{X}$ *is totally bounded iff its completion is totally bounded.*
(iii) $(\mathbf{ZF})$ *Let* $\mathbf{X}=(X,d)$ *be a metric space and* $\mathbf{D}$ *be a well orderable, dense, sequentially bounded (resp. a well ordered, dense, sequentially compact) subspace of* $\mathbf{X}$. *Then* $\mathbf{X}$ *is sequentially bounded (resp. sequentially compact and well orderable).*
(iv) $(\mathbf{ZF})$ *Let* $\mathbf{X}=(X,d)$ *be a metric space and* D *be a dense subset of* $\mathbf{X}$. *Then* $\mathbf{X}$ *is bounded iff* $\mathbf{D}$ *is bounded.*
(v) $\mathbf{M}(sb,tb)\rightarrow\mathbf{M}(sc,tb)$.
(vi) $\mathbf{M}(sb,b)\rightarrow\mathbf{M}(sc,b)$.
(vii) $\mathbf{M}(tb,sb)$ *holds true in* **ZF** *but the negation of* $\mathbf{M}(sb,tb)$ *is consistent with* **ZF**. *In particular,* (A) $\leftrightarrow$ $\mathbf{M}(sb,tb)$.
(viii) $(\mathbf{ZF})$ *A metric space with a well orderable dense subset is totally bounded iff it is sequentially bounded.*
(ix) $\mathbf{M}(sb,s)\rightarrow\mathbf{M}(sc,hs)$.
(x) [5] $(\mathbf{ZF})$ *The product* $\mathbf{X}=\prod_{i\in\omega}\mathbf{X}_i$ *of a family* $\{(X_i,d_i):i\in\omega\}$ *of totally bounded (resp. sequentially compact) metric spaces is totally bounded (resp. sequentially compact).*

(xi) (**ZF**) *The product* $\mathbf{X} = \prod_{i\in\omega} \mathbf{X}_i$ *of a family* $\{(X_i, d_i) : i \in \omega\}$ *of sequentially bounded metric spaces is sequentially bounded.*

(xii) [6] *Let* $\mathbf{X}$ *be a metric space and* $F \in \mathcal{P}(X)$*. If* F *has a well ordered dense subset then* $\widetilde{F} = \overline{F}$*.*

Proof. (iii) Fix a well ordered, dense, sequentially bounded subspace $\mathbf{D}$ of $\mathbf{X}$. We shall show that every sequence $(x_n)_{n\in\mathbb{N}}$ of points of X admits a Cauchy subsequence. For every $n \in \mathbb{N}$ pick $y_n \in D$ with $d(x_n, y_n) < 1/n$. By our hypothesis, $(y_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence $(y_{k_n})_{n\in\mathbb{N}}$. We show that $(x_{k_n})_{n\in\mathbb{N}}$ is a Cauchy subsequence of $(x_n)_{n\in\mathbb{N}}$. Fix $\varepsilon > 0$ and let $n_1 \in \mathbb{N}$ satisfy $1/n_1 < \varepsilon/4$. Since $(y_{k_n})_{n\in\mathbb{N}}$ is Cauchy, for the given ε there exists $n_2 \in \mathbb{N}$ such that $d(y_{k_n}, y_{k_m}) \le \varepsilon/2$ for all $n, m \ge n_2$. Let $n_0 = \max\{n_1, n_2\}$ and fix $n, m \ge n_0$. We have

$$d(x_{k_n}, x_{k_m}) \le d(x_{k_n}, y_{k_n}) + d(y_{k_n}, y_{k_m}) + d(y_{k_m}, x_{k_m}) < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon.$$

Hence, $(x_{k_n})_{n\in\mathbb{N}}$ is a Cauchy subsequence of $(x_n)_{n\in\mathbb{N}}$ as required.

Similarly, we can prove that if $\mathbf{D}$ is a well ordered, dense, sequentially compact subspace of $\mathbf{X}$ then $X = D$. Hence, $\mathbf{X}$ is sequentially compact and well orderable.

(iv) This is straightforward.

(v), (vi), (ix) Since a sequentially compact metric space is trivially sequentially bounded and sequential boundedness is hereditary, it follows that $\mathbf{M}(sb, tb) \to \mathbf{M}(sc, tb)$, $\mathbf{M}(sb, b) \to \mathbf{M}(sc, b)$ and $\mathbf{M}(sb, s) \to \mathbf{M}(sc, hs)$.

(vii) Fix a totally bounded metric space (X, d). By (ii), the completion $\mathbf{Y}$ of $\mathbf{X}$ is totally bounded. Hence, by Diagram 1, $\mathbf{Y}$ is sequentially bounded. Since sequential boundedness is hereditary, it follows that $\mathbf{X}$ is sequentially bounded.

The second assertion follows from the observation that the set A of all added Cohen reals in the basic model $\mathcal{M}1$ (see [3]) is sequentially bounded but not totally bounded.

(viii) It suffices, in view of (ii) and (vii), to show that if (X, d) is a metric space and $\mathbf{D}$ is a well orderable, sequentially bounded dense subspace of $\mathbf{X}$ then $\mathbf{D}$ is totally bounded. This is straightforward and we leave it as a warm up exercise for the reader.

(xi) Fix a family $\{(X_i, d_i) : i \in \omega\}$ of sequentially bounded metric spaces and let $\mathbf{X} = \prod_{i\in\omega} \mathbf{X}_i$. If $X = \emptyset$ then $\mathbf{X}$ is trivially sequentially bounded. So, assume $X \ne \emptyset$. Fix a sequence $(x_n)_{n\in\omega}$ of points of $\mathbf{X}$ and for every $i \in \omega$, let $Y_i = \{x_n(i) : n \in \omega\}$. Clearly, (Y_i, d_i) is a well ordered, sequentially bounded metric space. Hence, by (viii), it is totally bounded. Thus, by (x), the product $\mathbf{Y} = \prod_{i\in\omega} \mathbf{Y}_i$ is totally bounded, and so sequentially bounded by (vii). Hence, $(x_n)_{n\in\omega}$ admits a Cauchy subsequence and $\mathbf{X}$ is sequentially bounded as required. ∎

Corollary 8. (**ZF**) *Let* $\mathbf{X} = (X, d)$ *be a metric space and* D *be a well ordered dense subset of* $\mathbf{X}$. *Then* $\mathbf{X}$ *is sequentially bounded iff* $\mathbf{D}$ *is sequentially bounded.*

Proof. This follows at once from Theorem 7(iii) and the observation that subspaces of sequentially bounded metric spaces are sequentially bounded. ∎

Remark 2. We remark here that the requirement that the set D in Theorem 7(iii) and Corollary 8 be well ordered is crucial. The reason is that the negation of the statement: "Every metric space having a dense, sequentially bounded subset is sequentially bounded" is consistent with **ZF**. Indeed, the set A of all added Cohen reals in $\mathcal{M}1$ with the usual metric is dense, sequentially compact, hence also sequentially bounded, in $\mathbb{R}$, but $\mathbb{R}$ is not sequentially bounded.

Clearly, in view of Diagram 1 (a metric space is sequentially compact iff it is complete and sequentially bounded) the implications $\mathbf{M}(sb, tb) \to \mathbf{M}(sc, tb)$, $\mathbf{M}(sb, b) \to \mathbf{M}(sc, b)$ and $\mathbf{M}(sb, s) \to \mathbf{M}(sc, hs)$ are reversible under the assumption

(C): The completion of a sequentially bounded metric space is sequentially bounded.

So, one may ask:

Question 3. Is the negation of (C) consistent with **ZF**?

The following diagram summarizes the web of implications/nonimplications between the main principles which are obtained in what follows.

$$\begin{array}{ccccc}
\mathbf{CAC} & & & & \mathbf{M}(sc,c) \\
\downarrow & \searrow & & \nearrow\swarrow & \\
\mathbf{M}(sb,tb) & & \mathbf{M}(sc,s) & & \downarrow\not\uparrow \\
\downarrow\not\uparrow & \not\nearrow & & \searrow & \\
\mathbf{CAC}(\mathbb{R}) & & \not\leftrightarrows & & \mathbf{M}(c,s)
\end{array}$$

Diagram 2

In Theorem 16 we prove that $\mathbf{M}(sc, c)$ implies $\mathbf{M}(sc, s)$. In Theorem 10 we show that $\mathbf{M}(c, s)$ does not imply $\mathbf{CAC}(\mathbb{R})$ or $\mathbf{M}(sc, s)$, and $\mathbf{CAC}(\mathbb{R})$ does not imply $\mathbf{M}(c, s)$, $\mathbf{M}(sb, tb)$ or $\mathbf{M}(sc, s)$. In Theorem 9 we show that **CAC** can replace **AC** in Theorem 4, **CAC** implies $\mathbf{M}(sb, tb)$, and the negation of (C) is consistent with **ZF**, i.e., there exists a **ZF** model including a sequentially bounded metric space whose completion is not sequentially bounded. Finally, in Theorem 11 we show that $\mathbf{M}(sb, tb)$ lies between $\mathbf{CAC}(\mathbb{R})$ and **CAC**.

3. Boundedness, sequential boundedness and total boundedness

THEOREM 9.

(i) **CAC** *implies that every sequentially bounded metric space is totally bounded and separable. In particular, the conclusion of Theorem* 4 *and* $\mathbf{M}(sb, tb)$ *hold true under* **CAC**.

(ii) (C) (*"The completion of a sequentially bounded metric space is sequentially bounded"*) *implies* $\mathbf{CAC}_\omega(\mathbb{R})$. *In particular, the negation of* (C) *is consistent with* **ZF**.

Proof. (i) Fix a sequentially bounded metric space (X, d). Assume, aiming for a contradiction, that $\mathbf{X}$ is not totally bounded and fix $\ell \in \mathbb{N}$ such that X cannot be covered by finitely many open discs of radius $1/\ell$. Clearly, in view of our assumption, for every $n \in \mathbb{N}$,

$$X_n = \{f \in X^n : d(f(i), f(j)) \geq 1/\ell \text{ for all } i, j \in n,\ i \neq j\} \neq \emptyset.$$

Fix, by **CAC**, a choice set $\{f_n : n \in \mathbb{N}\}$ of $\{X_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, let $\overline{x}_n \in X^\omega$ be given by

$$\overline{x}_n | n = f_n \quad \text{and} \quad \overline{x}_n(m) = a \quad \text{for all } m \geq n,$$

where a is a fixed element of X. By Theorem 7, $\mathbf{X}^\omega$ is sequentially bounded. Hence, $(\overline{x}_n)_{n\in\mathbb{N}}$ admits a Cauchy subsequence, say $(\overline{x}_{k_n})_{n\in\mathbb{N}}$. Clearly, for every $i \in \omega$, $(\overline{x}_{k_n}(i))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbf{X}$. Hence, for $\varepsilon = 1/(2\ell)$, for every $i \in \omega$, there exists $m_i \in \omega$ such that

$$\forall u, v \geq m_i, \quad d(\overline{x}_{k_u}(i), \overline{x}_{k_v}(i)) < 1/(2\ell). \tag{2}$$

Use the well ordering of $\mathbb{N}$ to inductively define a strictly increasing sequence $(n_i)_{i\in\omega}$ such that every n_i satisfies (2). We claim that

$$d(\overline{x}_{k_{n_i}}(i), \overline{x}_{k_{n_j}}(j)) \geq 1/(2\ell) \quad \text{for all } i, j \in \omega,\ i \neq j.$$

To see this, fix $i, j \in \omega$ with $i < j$. From the triangle inequality we have

$$1/\ell \leq d(\overline{x}_{k_{n_j}}(i), \overline{x}_{k_{n_j}}(j)) \leq d(\overline{x}_{k_{n_i}}(i), \overline{x}_{k_{n_j}}(i)) + d(\overline{x}_{k_{n_i}}(i), \overline{x}_{k_{n_j}}(j)). \tag{3}$$

Since $n_j > n_i$ it follows from (2) that $d(\overline{x}_{k_{n_i}}(i), \overline{x}_{k_{n_j}}(i)) < 1/(2\ell)$. Hence, (3) yields $d(\overline{x}_{k_{n_i}}(i), \overline{x}_{k_{n_j}}(j)) \geq 1/\ell - 1/(2\ell) = 1/(2\ell)$ as required.

Clearly, the sequence $(\overline{x}_{k_{n_i}}(i))_{i\in\omega}$ has no Cauchy subsequence, contradicting the fact that $\mathbf{X}$ is sequentially bounded. Thus, $\mathbf{X}$ is totally bounded as claimed.

The separability of $\mathbf{X}$ follows from its total boundedness and Theorem 7.

The last assertion follows from the separability of $\mathbf{X}$ and Theorem 7.

(ii) Assume, aiming for a contradiction, that $\mathbf{CAC}_\omega(\mathbb{R})$ fails, and fix a disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty countable subsets of $\mathbb{R}$ without a partial choice function. Since for every $n \in \mathbb{N}$ a one-to-one and onto function $f_n : \mathbb{R} \to (2n, 2n+1)$ can be defined, we may assume that $A_n \subseteq (2n, 2n+1)$.

We claim that the subspace $\mathbf{X}$ of $\mathbb{R}$, where $X = \bigcup \mathcal{A}$, is sequentially bounded. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of points of $\mathbf{X}$, then for some $k \in \mathbb{N}$, A_k contains a subsequence $(x_{k_n})_{n\in\mathbb{N}}$. (Otherwise we can define a partial choice for $\mathcal{A}$.). Since $(x_{k_n})_{n\in\mathbb{N}} \subseteq (2k, 2k+1)$ is bounded, it follows that a further subsequence $(x_{k_{m_n}})_{n\in\mathbb{N}}$ converges to some x in $[2k, 2k+1]$. Hence, $(x_n)_{n\in\mathbb{N}}$ admits a Cauchy subsequence and $\mathbf{X}$ is sequentially bounded as claimed.

Clearly, the closure $\mathbf{Y}$ of X in $\mathbb{R}$ is a completion of $\mathbf{X}$. Hence, by hypothesis, $\mathbf{Y}$ is sequentially bounded. From Theorem 7(xii), it follows that for every $n \in \mathbb{N}$, $\overline{A}_n = \widetilde{A}_n \subseteq Y \cap [2n, 2n+1]$ and $c_n = \sup \overline{A}_n \in Y$. Thus, $(c_n)_{n\in\mathbb{N}}$ admits a Cauchy subsequence. However, for any distinct $n, m \in \mathbb{N}$, we have $|c_n - c_m| \geq 1$, contradiction.

The second assertion follows from the first part of (ii) and the fact that $\mathbf{CAC}_\omega(\mathbb{R})$ fails in Sageev's Model I (Model $\mathcal{M}6$ in [3]). ■

Remark 3. (i) Since countable products of sequentially bounded, metric spaces are sequentially bounded, and sequential boundedness is hereditary, we can easily verify that the conclusion of Theorem 9(i) holds if we restrict **CAC** to families of sequentially bounded metric spaces, i.e., if we replace **CAC** with

$\mathbf{CAC}_{sb}$: For every family $\{(A_i, d_i) : i \in \omega\}$ of non-empty sequentially bounded metric spaces, $\{A_i : i \in \omega\}$ has a choice set.

(ii) Working as in the proof of Theorem 9(ii) one can show that (C) implies something slightly stronger than $\mathbf{CAC}_\omega(\mathbb{R})$: "Every countable family of non-empty, well orderable subsets of $\mathbb{R}$ has a choice function".

Theorem 10.

(i) **BPI** *implies* $\mathbf{M}(c, s)$ *and "every compact metric space has size* $\leq |\mathbb{R}|$*".*
(ii) $\mathbf{M}(c, s)$ *does not imply* $\mathbf{CAC}(\mathbb{R})$*,* $\mathbf{M}(sc, s)$*,* $\mathbf{M}(sc, b)$*,* $\mathbf{M}(sc, tb)$ *or* $\mathbf{M}(sb, s)$*.*
(iii) $\mathbf{CAC}(\mathbb{R})$ *does not imply* $\mathbf{M}(c, s)$*,* $\mathbf{M}(sc, s)$*,* $\mathbf{M}(sc, b)$*,* $\mathbf{M}(sc, tb)$*,* $\mathbf{M}(sb, tb)$ *or* $\mathbf{M}(sb, s)$*.*

Proof. (i) Fix a compact metric space $\mathbf{X}$. We show that $\mathbf{X}$ is separable. To this end, in view of Theorem 6, it suffices to show that the family of all non-empty closed subsets of $\mathbf{X}$ has a choice function. Fix $x_0 \in X$ and let $\mathcal{G} = \{G_i : i \in I\}$ be the family of all closed non-empty subsets of $\mathbf{X}$ which avoid x_0. Set $X_i = G_i \cup \{x_0\}$ for every $i \in I$. Then each X_i is a compact subspace of $\mathbf{X}$. Hence, by **BPI**, the product $\mathbf{Y} = \prod_{i\in I} \mathbf{X}_i$ is compact. Since $\{\pi_i^{-1}(G_i) : i \in I\}$ is clearly a family of closed subsets of $\mathbf{Y}$ with the fip, it follows that $\bigcap\{\pi_i^{-1}(G_i) : i \in I\} \neq \emptyset$. It is easy to see that any element g of this intersection is a choice function for $\mathcal{G}$. In addition, g can be trivially extended to a choice function f for the family of all non-empty closed subsets of $\mathbf{X}$ by setting

$$f(G) = \begin{cases} g(G) & \text{if } x_0 \notin G, \\ x_0 & \text{if } x_0 \in G. \end{cases}$$

The second assertion is a straightforward consequence of the first one.

(ii) It is known (see e.g. [3]) that in the basic Cohen model $\mathcal{M}1$, **BPI** holds true but $\mathbf{CAC}(\mathbb{R})$ is false. Hence, by (i), $\mathbf{M}(c,s)$ also holds true. However, by the proof of Proposition 3, $\mathbf{M}(sc,s)$, $\mathbf{M}(sc,b)$, $\mathbf{M}(sc,tb)$ and $\mathbf{M}(sb,s)$ fail in $\mathcal{M}1$.

(iii) Let $\mathcal{N}$ be the model given in [5, Example 9]. It is a permutation model but its **ZF** version has been constructed in [11].

It is known that $\mathbf{CAC}(\mathbb{R})$ holds true in $\mathcal{N}$ and there exists a disjoint family $\{(X_n, d_n) : n \in \mathbb{N}\}$ of non-empty compact metric spaces such that $\delta(X_n) \leq 1/n$ for every $n \in \mathbb{N}$, and $\{X_n : n \in \mathbb{N}\}$ has no partial choice. Let $\infty \notin Y$, where $Y = \bigcup\{X_n : n \in \mathbb{N}\}$, and set $X = Y \cup \{\infty\}$. Clearly, the function $d : X \times X \to \mathbb{R}$ given by

$$d(x,y) = d(y,x) = \begin{cases} 1/n & \text{if } x = \infty \text{ and } y \in X_n \text{ for some } n \in \mathbb{N}, \\ d_n(x,y) & \text{if } x, y \in X_n \text{ for some } n \in \mathbb{N}, \\ 1/m & \text{if } x \in X_n,\ y \in X_m \text{ for some } n, m \in \mathbb{N},\ n > m, \end{cases}$$

is a metric on X such that $\mathbf{X}$ is compact and for every $n \in \mathbb{N}$, X_n is an open subset of $\mathbf{X}$. Since $\{X_n : n \in \mathbb{N}\}$ has no choice function, $\mathbf{X}$ cannot be separable. Hence, $\mathbf{M}(c,s)$ and $\mathbf{M}(sc,s)$ fail in $\mathcal{N}$.

Let $\rho : Y \times Y \to \mathbb{R}$ be given by

$$\rho(x,y) = \rho(y,x) = \begin{cases} d_n(x,y) & \text{if } x, y \in X_n \text{ for some } n \in \mathbb{N}, \\ |n-m| & \text{if } x \in X_n,\ y \in X_m \text{ for some } n, m \in \mathbb{N},\ n \neq m. \end{cases}$$

Clearly, $\mathbf{Y}$ is unbounded and consequently not totally bounded. Since $\{X_n : n \in \mathbb{N}\}$ has no partial choice, it follows that $\mathbf{Y}$ is sequentially compact and sequentially bounded but not separable. Hence, $\mathbf{M}(sc,s)$, $\mathbf{M}(sc,b)$, $\mathbf{M}(sc,tb)$, $\mathbf{M}(sb,tb)$ and $\mathbf{M}(sb,s)$ fail in $\mathcal{N}$. ∎

QUESTION 4. Does **BPI** imply "every sequentially compact metric space has size $\leq |\mathbb{R}|$"?

Since an infinite Dedekind-finite set endowed with the discrete metric is clearly sequentially compact, hence also sequentially bounded, but it is not compact, separable or totally bounded, it follows that each of $\mathbf{M}(sb,tb)$, $\mathbf{M}(sc,c)$, $\mathbf{M}(sc,s)$, $\mathbf{M}(sc,tb)$ and $\mathbf{M}(sb,s)$ implies **IDI** (every infinite set is Dedekind-infinite). Since **IDI** implies $\mathbf{CAC}_{\mathrm{fin}}$, they also imply $\mathbf{CAC}_{\mathrm{fin}}$. Thus, part (i) of the next theorem is proved and some extra examples of the sort van Douwen described in [12] as horrors are emerging. In addition, we show that $\mathbf{M}(sb,tb)$ as well as $\mathbf{M}(sb,b)$ and $\mathbf{M}(sb,s)$ lie between $\mathbf{CAC}(\mathbb{R})$ and $\mathbf{CAC}$.

THEOREM 11.

(i) *Each of* $\mathbf{M}(sb,tb)$, $\mathbf{M}(sc,c)$, $\mathbf{M}(sc,s)$, $\mathbf{M}(sc,tb)$ *and* $\mathbf{M}(sb,s)$ *implies* $\mathbf{IDI}$. *In particular, all of them imply* $\mathbf{CAC}_{\mathrm{fin}}$.

(ii) $\mathbf{M}(sb,b)$ *implies* $\mathbf{CAC}_{sb}$ *("For every family* $\{(A_i,d_i) : i \in \omega\}$ *of non-empty sequentially bounded metric spaces,* $\{A_i : i \in \omega\}$ *has a choice set"). In particular, each of* $\mathbf{M}(sb,b)$ *and* $\mathbf{M}(sb,tb)$ *implies* $\mathbf{CAC}(\mathbb{R})$.

(iii) $\mathbf{M}(sb,s)$ *implies* $\mathbf{CAC}(\mathbb{R})$.

(vi) *Neither* $\mathbf{CAC}(\mathbb{R})$ *nor* $\mathbf{CAC}_{\mathrm{fin}}$ *implies* $\mathbf{M}(sb,b)$, $\mathbf{M}(sb,tb)$ *or* $\mathbf{M}(sb,s)$.

Proof. (ii) Fix a family $\{(A_i,d_i) : i \in \omega\}$ of non-empty sequentially bounded metric spaces. Without loss of generality we may assume that $\delta(A_i) \le 1$ for every $i \in \omega$. We show that $\mathcal{A} = \{A_i : i \in \omega\}$ has a choice set. Assume on the contrary that $\mathcal{A}$ has no choice set. Since for every $i \in \omega$, the space (Y_i,ρ_i), where $Y_i = \prod_{j\le i} A_j$ and $\rho_i(x,y) = \max\{d_j(x(j),y(j)) : j \le i\}$, is clearly a sequentially bounded metric space and any partial choice for the family $\{Y_i : i \in \omega\}$ yields a choice set for $\mathcal{A}$, we may assume that $\mathcal{A}$ has no partial choice. Let d be the metric on $X = \bigcup \mathcal{A}$ given by

$$d(x,y) = d(y,x) = \begin{cases} d_i(x,y) & \text{if } x,y \in A_i \text{ for some } i \in \omega,\\ |i-j| & \text{if } x \in A_i, y \in A_j \text{ and } i \ne j.\end{cases}$$

Clearly, $\mathbf{X}$ is not bounded. Working as in the proof of Theorem 9(ii) we can show that $\mathbf{X}$ is sequentially bounded. Thus, by $\mathbf{M}(sb,b)$, $\mathbf{X}$ is bounded, a contradiction. Hence, $\mathcal{A}$ has a choice set as required.

To see that $\mathbf{M}(sb,b) \to \mathbf{CAC}(\mathbb{R})$, fix a disjoint family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty subsets of $\mathbb{R}$. Without loss of generality we may assume that $A_i \subseteq (i,i+1)$ for all $i \in \omega$. Since bounded subspaces of $\mathbb{R}$ with the usual metric are sequentially bounded, it follows by $\mathbf{CAC}_{sb}$ that $\mathcal{A}$ has a choice set.

Finally, the implication $\mathbf{M}(sb,tb) \to \mathbf{CAC}(\mathbb{R})$ follows from the observation $\mathbf{M}(sb,tb) \to \mathbf{M}(sb,b) \to \mathbf{CAC}(\mathbb{R})$.

(iii) From Theorem 7(viii) we infer that $\mathbf{M}(sb,s) \to \mathbf{M}(sb,tb)$. Hence, by (ii), we see that $\mathbf{CAC}(\mathbb{R})$ holds true.

(iv) Since $\mathbf{CAC}(\mathbb{R})$ and $\mathbf{CAC}_{\mathrm{fin}}$ are independent of each other, it follows that neither $\mathbf{CAC}(\mathbb{R})$ nor $\mathbf{CAC}_{\mathrm{fin}}$ implies $\mathbf{M}(sb,b)$, $\mathbf{M}(sb,tb)$ or $\mathbf{M}(sb,s)$. ∎

4. The cardinality of compact metric spaces. It is very well known that in $\mathbf{ZF}$ a separable metric space has size $\le |\mathbb{R}|$. The following theorem indicates that the cardinality of a compact metric space may be incomparable with $|\mathbb{R}|$ in **ZF**.

THEOREM 12. *The statement: "For every compact metric space* $\mathbf{X}$ *either* $|X| \leq |\mathbb{R}|$ *or* $|\mathbb{R}| \leq |X|$*" implies* $\mathbf{CAC}_{\mathrm{fin}}$.

Proof. Fix a disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of non-empty finite sets. We shall show that some infinite subfamily of $\mathcal{A}$ has a choice set. Consider the metric d on $X = \bigcup\{A_n : n \in \mathbb{N}\} \cup \{\infty\}$, where $\infty \notin \bigcup\{A_n : n \in \mathbb{N}\}$, given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{1/n, 1/m\} & \text{if } x \in A_n \text{ and } y \in A_m, \\ 1/n & \text{if } y \in A_n \text{ and } x = \infty. \end{cases}$$

It is straightforward to see that the complement of every open set including ∞ is finite. Hence, $\mathbf{X}$ is compact and by our hypothesis, $|X| \leq |\mathbb{R}|$ or $|\mathbb{R}| \leq |X|$. If $|X| \leq |\mathbb{R}|$ then we may view X as a subset of $\mathbb{R}$ and choose $\min A_n$ from A_n for every $n \in \mathbb{N}$. So, assume that $|\mathbb{R}| \leq |X|$ and $\mathbb{R} \subseteq X$. Since $\mathbb{R}$ is infinite, there are infinitely many $n \in \mathbb{N}$ such that $A_n \cap \mathbb{R} \neq \emptyset$. For every such $n \in \mathbb{N}$, pick $\min(A_n \cap \mathbb{R})$ from $A_n \cap \mathbb{R}$. Hence, an infinite subfamily of $\mathcal{A}$ has a choice set as required. ∎

In the second Cohen model (Model $\mathcal{M}7$ in [3]), there exists a disjoint family $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ of finite non-empty sets without a choice set. Hence, in $\mathcal{M}7$ there exists a compact metric space $\mathbf{X}$ whose size is incomparable with $|\mathbb{R}|$.

The next folklore result shows that an infinite separable compact metric is either countable or has the cardinality of the continuum.

THEOREM 13 (**ZF**). *Let* $\mathbf{X} = (X, d)$ *be an uncountable compact separable metric space. Then* $|X| = |\mathbb{R}|$.

Proof. Since separable metric spaces have size $\leq |\mathbb{R}|$, it suffices to show that $|\mathbb{R}| \leq |X|$ for every uncountable, compact, separable metric space $\mathbf{X}$. Fix such an $\mathbf{X}$ and let G be the set of all points of X such that $B(x, \varepsilon)$ is uncountable for every $\varepsilon > 0$. Since $\mathbf{X}$ is compact, it follows that $G \neq \emptyset$. Clearly, for every $x \in G^c$ there exists $\varepsilon > 0$ such that $|B(x, \varepsilon)| \leq \aleph_0$. Thus, $B(x, \varepsilon) \subseteq G^c$ and consequently G is closed, hence compact. Fix a countable dense subset $D = \{d_n : n \in \omega\}$ of $\mathbf{X}$. Clearly, $\mathcal{B} = \{B(d_n, 1/m) \cap G : n, m \in \mathbb{N}\}$ is a countable base for the topology of $\mathbf{G}$. Let $\{B_n : n \in \omega\}$ be an enumeration of $\mathcal{B}$. We inductively construct an upside down binary tree $T = \bigcup\{T_n : n \in \omega\} \subseteq \mathcal{B}$ as follows:

For $n = 0$ let $T_0 = \{B_0\}$, where B_0 is the first member of $\mathcal{B}$ of diameter $\delta(B_0) \leq 1$.

For $n = k+1$ and for every dyadic sequence $s \in 2^k$ we let $B_{s^\frown 0}$ and $B_{s^\frown 1}$ be the first and second member of $\mathcal{B}$ such that

$$B_{s^\frown 0}, B_{s^\frown 1} \subseteq B_s, \quad \delta(B_{s^\frown 0}) < 1/n, \quad \delta(B_{s^\frown 1}) < 1/n$$

and

$$\overline{B}_{s^\frown 0} \cap \overline{B}_{s^\frown 1} = \emptyset,$$

where $s^\frown 0$ (resp. $s^\frown 1$) denotes the concatenation of the dyadic sequence s and (0) (resp. of s and (1)). Set $T_n = \{B_{s^\frown 0},\ B_{s^\frown 1} : s \in 2^k\}$. Clearly, by the compactness of $\mathbf{G}$, and since $\bigcap\{\overline{B}_{f|n} : n \in \mathbb{N}\} = \{x_f\}$ is a singleton for every $f \in 2^\omega$, it follows that the mapping: $f \mapsto x_f$ from 2^ω into X is one-to-one. Hence, $|\mathbb{R}| = |2^\omega| \leq |X|$ as required. ∎

Corollary 14. *Assume* **CAC**. *Let* $\mathbf{X} = (X, d)$ *be an uncountable sequentially compact metric space. Then* $|X| = |\mathbb{R}|$.

Proof. Fix an uncountable sequentially compact metric space $\mathbf{X}$. By Theorem 9, $\mathbf{X}$ is separable, and by Theorem 13, $|X| = |\mathbb{R}|$. ∎

Theorem 15. *None of the statements* $\mathbf{M}(sc, s)$, $\mathbf{M}(sc, b)$, $\mathbf{M}(sc, tb)$, $\mathbf{M}(sb, tb)$ *and* $\mathbf{M}(sb, s)$ *implies* **CAC**.

Proof. We recall that the Jech–Levy–Pincus model $\mathcal{N}16$ of [3] is obtained by considering a set A of atoms of cardinality $\aleph_\omega$ and letting the ideal of supports be the set of all subsets of A of cardinality less than $\aleph_\omega$. Without loss of generality we may assume that the kernel of $\mathcal{N}16$ satisfies $|\mathcal{P}(\mathbb{R})| = \aleph_2$. It is known (see e.g. [3, p. 194] and [4, Theorem 8.9, p. 124]) that **CAC** fails but the statement

$W_{\aleph_\omega}$: For every $i \in \omega$, for every set X, either $|X| \leq \aleph_i$ or $\aleph_i \leq |X|$

holds true in $\mathcal{N}16$. Furthermore, it is known that this result transfers to **ZF**, i.e., there exists a **ZF** model satisfying $W_{\aleph_\omega}$ and the negation of **CAC**. Fix such a **ZF** model $\mathcal{M}$. We claim that every sequentially bounded metric space is well orderable in $\mathcal{M}$. To see this, assume otherwise and fix a non-well orderable sequentially bounded metric space $\mathbf{X} = (X, d)$ in $\mathcal{M}$. By $W_{\aleph_\omega}$ and our hypothesis, $\mathbf{X}$ has a subspace Y of size $\aleph_2$. By Theorem 7, $\overline{Y}$ is totally bounded and well orderable. Hence, by Theorem 7, the completion $\mathbf{Z}$ of $\overline{Y}$ is totally bounded. Hence, $\mathbf{Z}$ is separable. Thus, by Theorem 13, $\aleph_1 \geq |Z| \geq |Y| = \aleph_2$, a contradiction.

By the claim, every sequentially bounded metric space $\mathbf{X} = (X, d)$ in $\mathcal{M}$ is well orderable. Hence, $\mathcal{P}(X)\backslash\{\emptyset\}$ has a choice function f_X. Using f_X and arguing along the lines of the proof of Theorem 9, we can show that $\mathbf{X}$ is separable and totally bounded. Hence, $\mathbf{M}(sc, s)$, $\mathbf{M}(sc, tb)$, $\mathbf{M}(sc, b)$, $\mathbf{M}(sb, tb)$ and $\mathbf{M}(sb, s)$ hold true in $\mathcal{M}$. ∎

5. Sequentially and countably compact metric spaces. The following is a well known result of **ZF**:

$\mathbf{M}(cc, sc)$: Every countably compact metric space $\mathbf{X}$ is sequentially compact.

(If $\mathbf{X}$ is a countably compact metric space then for every sequence $(x_n)_{n\in\mathbb{N}}$ of points of X the family $\mathcal{A} = \{\overline{A}_n : n \in \mathbb{N}\}$, where $A_n = \{x_{n+k} : k \in \mathbb{N}\}$ for every $n \in \mathbb{N}$, has the fip. Hence, $\bigcap \mathcal{A} \neq \emptyset$. It is easy to verify that every member x of this intersection is a limit of some subsequence of $(x_n)_{n\in\mathbb{N}}$. Thus, $\mathbf{X}$ is sequentially compact.) However, in the Basic Cohen Model $\mathcal{M}1$ the set A of all added Cohen reals with the usual metric is sequentially compact but the open cover $\mathcal{U} = \{(-n, n) \cap A : n \in \mathbb{N}\}$ has no finite subcover. Hence, the statement

$\mathbf{M}(sc, cc)$: A sequentially compact metric space $\mathbf{X}$ is countably compact

fails in $\mathcal{M}1$. We show next that $\mathbf{M}(sc, cc)$ implies that every infinite subset of $\mathbb{R}$ is Dedekind-infinite, and $\mathbf{M}(sc, c)$ implies $\mathbf{M}(sc, s)$.

Theorem 16.

(i) $\mathbf{M}(sc, cc)$ *implies* $\mathbf{IDI}(\mathbb{R})$ (*"Every infinite subset of* $\mathbb{R}$ *is Dedekind-infinite").*

(ii) $\mathbf{M}(sc, c)$ *implies each one of the propositions:*

(a) **CPCMC**: *If* $\{\mathbf{X}_n = (X_n, d_n) : n \in \mathbb{N}\}$ *is a family of compact metric spaces then* $\mathbf{X} = \prod_{n\in\mathbb{N}} \mathbf{X}_n$ *is compact,*

(b) $\mathbf{CAC}_{CM}$: *For every metric space* $\mathbf{X} = (X, d)$, *every family* $\mathcal{A} = \{A_i : i \in \omega\}$ *of non-empty compact subsets of* $\mathbf{X}$ *has a choice function,*

(c) $\mathbf{M}(sc, s)$.

(iii) *Neither* **CPCMC** *nor* $\mathbf{CAC}_{CM}$ *implies* $\mathbf{M}(sc, s), \mathbf{M}(sc, b), \mathbf{M}(sb, s)$ *or* $\mathbf{M}(sb, b)$ *in* **ZF**.

Proof. (i) Fix an infinite subset X of the real line $\mathbb{R}$. Assume, aiming for a contradiction, that X has no countably infinite subset. For every $n \in \mathbb{N}$, define a one-to-one and onto function $f_n : \mathbb{R} \to (1/n + 1, 1/n)$ and let $X_n = f_n(X)$. Clearly, for every $n \in \mathbb{N}$, X_n is an infinite Dedekind-finite set. Without loss of generality assume that $k = \sup(X_1) \notin X_1$.

We claim that the subspace $\mathbf{Y}$ of $\mathbb{R}$ with $Y = \{0\} \cup \bigcup\{X_n : n \in \mathbb{N}\}$ is sequentially compact. Indeed, if $(y_n)_{n\in\mathbb{N}}$ is a sequence of points of Y such that each X_n contains finitely many terms of $(y_n)_{n\in\mathbb{N}}$ then 0 is a limit point of $(y_n)_{n\in\mathbb{N}}$ and consequently 0 is the limit of some subsequence of $(y_n)_{n\in\mathbb{N}}$. Otherwise, there exists $n \in \mathbb{N}$ such that X_n contains infinitely many terms of $(y_n)_{n\in\mathbb{N}}$ and since X_n has no countably infinite subsets, some term of

$(y_n)_{n\in\mathbb{N}}$ repeats infinitely many times. Thus, a convergent subsequence of $(y_n)_{n\in\mathbb{N}}$ can be defined and $\mathbf{Y}$ is sequentially compact as claimed.

We show next that $\mathbf{Y}$ is not countably compact. Since X_1 is an infinite Dedekind-finite set, it follows that $X_1 \cap \mathbb{Q}$ is finite. So, by deleting finitely many points from X_1, we may assume that $X_1 \subseteq \mathbb{Q}^c$. Via a straightforward induction we construct a strictly increasing sequence $(r_n)_{n\in\mathbb{N}}$ of rational numbers converging to k such that $(r_n, r_{n+1}) \cap X_1 \neq \emptyset$ for every $n \in \mathbb{N}$ and $X_1 = \bigcup\{(r_n, r_{n+1}) \cap X_1 : n \in \mathbb{N}\}$. Clearly,

$$\mathcal{U} = \{[0, 1/2) \cap Y\} \cup \{(r_n, r_{n+1}) \cap Y : n \in \mathbb{N}\}$$

is a countable open cover of $\mathbf{Y}$ without a finite subcover. Hence, $\mathbf{Y}$ is not countably compact, a contradiction.

(ii) (a) Fix a family $\{\mathbf{X}_n = (X_n, d_n) : n \in \mathbb{N}\}$ of compact metric spaces and let $\mathbf{X} = \prod_{n\in\mathbb{N}} \mathbf{X}_n$. Since every compact metric space is sequentially compact, it follows from Theorem 7 that $\mathbf{X}$ is sequentially compact. Hence, by $\mathbf{M}(sc, c)$, $\mathbf{X}$ is compact as required.

(b) Fix a metric space $\mathbf{Y} = (Y, d)$ with $\delta(Y) \leq 1$ and let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a family of non-empty compact subsets of $\mathbf{Y}$. Fix a point $\infty \notin \bigcup \mathcal{A}$ and for every $n \in \mathbb{N}$, let $X_n = A_n \cup \{\infty\}$. Clearly, $\mathbf{X}_n = (X_n, d_n)$, where d_n is given by

$$d_n(x, y) = d_n(y, x) = \begin{cases} 1 & \text{if } x = \infty \text{ and } y \in A_n, \\ d(x, y) & \text{if } x, y \in A_n \end{cases}$$

is a compact metric space. Hence, in view of $\mathbf{M}(sc, c) \to \mathbf{CPCMC}$, the product $\mathbf{X} = \prod_{n\in\mathbb{N}} \mathbf{X}_n$ is compact. Since $G_n = \pi_n^{-1}(A_n)$ is closed for every $n \in \mathbb{N}$, and $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ has the fip, it follows that $\bigcap \mathcal{G} \neq \emptyset$. Clearly, any $f \in \bigcap \mathcal{G}$ is a choice function for $\mathcal{A}$.

(c) Fix a sequentially compact metric space $\mathbf{X} = (X, d)$. We show that $\mathbf{X}$ is separable. By $\mathbf{M}(sc, c)$, $\mathbf{X}$ is compact and consequently totally bounded. Hence, for every $\varepsilon > 0$ there exist $x_i \in X$, $i = 1, \ldots, n$, such that $\bigcup\{\overline{B(x_i, \varepsilon)} : i \in n\} = X$. For every $k \in \mathbb{N}$, let m_k denote the least natural number m for which there exists a set $\{x_i : i \in m\} \subseteq X$ with $\bigcup\{\overline{B(x_i, 1/k)} : i \in m\} = X$. For every $k \in \mathbb{N}$, let $\mathbf{Y}_k = (X^{m_k}, \rho_k)$, where ρ_k is the uniform metric on X^{m_k}, i.e.,

$$\rho_k(f, g) = \max\{d(f(i), g(i)) : i \in m_k\}.$$

Since $\mathbf{X}$ is compact, it follows that $\mathbf{Y}_k$ is compact.

We claim that for every $k \in \mathbb{N}$, the set

$$G_k = \left\{ f \in X^{m_k} : \bigcup\{\overline{B(f(i), 1/k)} : i \in m_k\} = X \right\}$$

is a (non-empty) closed subset of $\mathbf{Y}_k$. Fix $q \in \mathbf{Y}_k \backslash G_k$ and let $x \in X$ satisfy

$x \notin \bigcup\{\overline{B(q(i), 1/k)} : i \in m_k\}$. Clearly,

$$r = \min\{d(x, q(i)) : i \in m_k\} > 1/k.$$

Let $\varepsilon = \frac{1}{2}(r - 1/k)$. We claim that $B(q, \varepsilon) \cap G_k = \emptyset$. Assume otherwise and let $f \in B(q, \varepsilon) \cap G_k$. Since $f \in G_k$, it follows from the definition of G_k that $x \in \overline{B(f(i), 1/k)}$ for some $i \in m_k$. We have

$$r \leq d(q(i), x) \leq d(f(i), q(i)) + d(f(i), x) < \varepsilon + 1/k \leq r/2 + 1/2k < r,$$

a contradiction. Hence, $B(q, \varepsilon) \cap G_k = \emptyset$ as claimed and G_k is closed. Since $\mathbf{Y}_k$ is compact, it follows that G_k is compact. Thus, by $\mathbf{CAC}_{CM}$, $G = \{G_k : k \in \mathbb{N}\}$ has a choice set, say $\{f_k : k \in \mathbb{N}\}$. Clearly,

$$D = \{f_k(i) : i \in m_k,\ k \in \mathbb{N}\},$$

being a countable union of well ordered finite sets, is countable. It is easy to verify that D is dense in $\mathbf{X}$. Hence $\mathbf{X}$ is separable as required, finishing the proof of (c).

(iii) Since **BPI** holds true in $\mathcal{M}1$ and **BPI** clearly implies **CPCMC** and $\mathbf{CAC}_{CM}$, it follows that **CPCMC** and $\mathbf{CAC}_{CM}$ hold true in $\mathcal{M}1$. However, by the proof of Proposition 3, $\mathbf{M}(sc, s)$, $\mathbf{M}(sc, b)$, $\mathbf{M}(sb, s)$ and $\mathbf{M}(sb, b)$ fail in $\mathcal{M}1$. ∎

Corollary 17. $\mathbf{M}(sc, c)$ (*"Every sequentially compact metric space is compact"*) *implies* $\mathbf{M}(sc, s)$ (*"Every sequentially compact metric is separable"*).

Remark 4. $\mathbf{M}(sc, cc)$ implies something stronger than $\mathbf{IDI}(\mathbb{R})$, namely,

(B): For every infinite set X, $[X]^{<\omega}$ is Dedekind-infinite.

To see this, assume, aiming for a contradiction, that X is an infinite set such that $[X]^{<\omega}$ is Dedekind-finite. Clearly, for every $n \in \mathbb{N}$, the set $Y_n = [X]^n$ of all n-element subsets of X is Dedekind-finite. Let d be the discrete metric on $Y = \bigcup\{Y_n : n \in \mathbb{N}\}$.

We claim that $\mathbf{Y}$ is sequentially compact. To see this fix a sequence $(x_n)_{n\in\mathbb{N}}$ of points of Y. If some term of $(x_n)_{n\in\mathbb{N}}$ repeats infinitely many times then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence. So, assume that $(x_n)_{n\in\mathbb{N}}$ is finite-to-one. Clearly in this case, $\{\bigcup A_m : m \in \mathbb{N}\}$, where for every $m \in \mathbb{N}$, A_m is the set of all terms of $(x_n)_{n\in\mathbb{N}}$ included in Y_m, is a countably infinite subset of $[X]^{<\omega}$, a contradiction. Therefore, $\mathbf{Y}$ is sequentially compact as claimed. Hence, by $\mathbf{M}(sc, cc)$, $\mathbf{Y}$ is countably compact and consequently the open cover $\{Y_n : n \in \mathbb{N}\}$ of $\mathbf{Y}$ has a finite subcover, a contradiction.

Clearly, **IDI** implies (B), and the conjunction of (B) and $\mathbf{CAC}_{\mathrm{fin}}$ implies **IDI**. Since $\mathbf{CAC}_{\mathrm{fin}}$ holds in Cohen's basic model $\mathcal{M}1$ but **IDI** fails, it follows that (B) fails in $\mathcal{M}1$. Hence, (B) and its negation are consistent with **ZF**.

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Kyriakos Keremedis
Department of Mathematics
University of the Aegean
Karlovassi, Samos 83200, Greece
E-mail: kker@aegean.gr