FUNDAMENTA
MATHEMATICAE
234 (2016)

On the set-theoretic strength of the n-compactness of generalized Cantor cubes

by

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Abstract. We investigate, in set theory without the Axiom of Choice **AC**, the set-theoretic strength of the statement

$Q(n)$: *For every infinite set X, the Tychonoff product 2^X, where $2 = \{0, 1\}$ has the discrete topology, is n-compact*,

where $n = 2, 3, 4, 5$ (definitions are given in Section 1).

We establish the following results:

(1) For $n = 3, 4, 5$, $Q(n)$ is, in **ZF** (Zermelo–Fraenkel set theory minus **AC**), equivalent to the Boolean Prime Ideal Theorem **BPI**, whereas
(2) $Q(2)$ is strictly weaker than **BPI** in **ZFA** set theory (Zermelo–Fraenkel set theory with the Axiom of Extensionality weakened in order to allow atoms).

This settles the open problem in Tachtsis (2012) on the relation of $Q(n)$, $n = 2, 3, 4, 5$, to **BPI**.

1. Introduction, terminology and known results. Let X be an infinite set. The collection $\mathcal{B}_X = \{[p] : p \in \mathrm{Fn}(X, 2)\}$, where $\mathrm{Fn}(X, 2)$ is the set of all finite partial functions from X into 2 and $[p] = \{f \in 2^X : p \subset f\}$, is the *standard open base* for the product topology on 2^X, where $2 = \{0, 1\}$ has the discrete topology. (In fact, for each $p \in \mathrm{Fn}(X, 2)$, $[p]$ is a clopen subset of 2^X, that is, $[p]$ is simultaneously closed and open in 2^X.) The set $\mathcal{D}_X = \{2^X \setminus [p] : p \in \mathrm{Fn}(X, 2)\}$ consisting of complements of standard open basic sets is called the *standard closed base* for the product topology. For every $n \in \mathbb{N}$ ($= \omega \setminus \{0\}$, where ω denotes, as usual, the set of all natural numbers), let $\mathcal{B}^n_X = \{[p] \in \mathcal{B}_X : |p| = n\}$ (i.e., for each $[p] \in \mathcal{B}^n_X$, there is a

2010 *Mathematics Subject Classification*: Primary 03E25; Secondary 03E35, 54B10, 54D30.
Key words and phrases: Axiom of Choice, Boolean prime ideal theorem, compactness and n-compactness of generalized Cantor cubes, Fraenkel–Mostowski (FM) permutation models.
Received 10 January 2014; revised 20 December 2015.
Published online 4 July 2016.

DOI: 10.4064/fm961-1-2016

bijection $f : p \to n$) and $\mathcal{D}_X^n = \{2^X \setminus [p] : [p] \in \mathcal{B}_X^n\}$. For $n \in \mathbb{N}$, elements of $\mathcal{B}_X^n$ are called *n-basic open sets* of 2^X and elements of $\mathcal{D}_X^n$ are called *n-basic closed sets*. Clearly, $\mathcal{B}_X = \bigcup\{\mathcal{B}_X^n : n \in \mathbb{N}\}$ and $\mathcal{D}_X = \bigcup\{\mathcal{D}_X^n : n \in \mathbb{N}\}$.

The product space 2^ω is known as the *Cantor cube*. Replacing ω with any infinite set X, we call the corresponding Tychonoff product 2^X a *generalized Cantor cube*.

The following extension of compactness for generalized Cantor cubes was introduced in [7]:

DEFINITION 1.1. For X an infinite set and for $n \in \mathbb{N}$, 2^X is called *n-compact* if every cover $\mathcal{U} \subseteq \mathcal{B}_X^n$ of 2^X has a finite subcover.

Recall the following well-known notion:

DEFINITION 1.2. A non-empty family $\mathcal{F}$ of subsets of a set X has the *finite intersection property*, which we shall abbreviate by fip, if $\bigcap \mathcal{G} \neq \emptyset$ for every (non-empty) finite subfamily $\mathcal{G}$ of $\mathcal{F}$.

The concept of n-compactness could equivalently be formulated in terms of n-basic closed sets:

FACT 1. *If X is an infinite set, then 2^X is n-compact if and only if every subset $\mathcal{D} \subseteq \mathcal{D}_X^n$ with the fip has a non-empty intersection.*

FACT 2 ([7]). *Assume that X is an infinite set. Then 2^X is n-compact if and only if for every collection $\mathcal{F}$ of sets of the form*

$$\bigcup\{[p] : p \in S\} \text{ where for some } Q \subset X \text{ such that } |Q| = n, S \subseteq 2^Q, \tag{1.1}$$

if $\mathcal{F}$ has the fip, then $\mathcal{F}$ has a non-empty intersection.

Sets of the form described in (1.1) above were introduced in Keremedis and Tachtsis [7] and studied there, in Morillon [8], in Tachtsis [12], and in Howard and Tachtsis [4] and [5]. For sets of the form (1.1), the authors of [7] used the term "restricted clopen sets", whereas the author of [8] called them "elementary closed sets". We call the reader's attention to the fact that for $n \in \mathbb{N}$, every n-basic closed set can be written in the form (1.1), while the converse is *not* necessarily true, that is, a closed set of the form (1.1) may not be an n-basic closed set.

In the interest of making our paper self-contained, we give an outline of the argument for Fact 2.

Proof of Fact 2. For a fixed $Q \subset X$ such that $|Q| = n$, the complement F' (in 2^X) of a set F of the form (1.1) can be written in the same form. Therefore, F is the complement of a finite union of n-basic open sets and hence F is the intersection of n-basic closed sets. Assuming that 2^X is n-compact and that $\mathcal{F}$ is a family of sets of the form given by (1.1) which has the fip, the family $\mathcal{F}' = \{F' \in \mathcal{D}_X^n : \exists F \in \mathcal{F} \text{ such that } F \subseteq F'\}$ is a family of subsets of $\mathcal{D}^n_X$ with the fip and by our observation at the beginning of the proof, $\bigcap \mathcal{F}' = \bigcap \mathcal{F}$. The assumption that 2^X is n-compact gives $\bigcap \mathcal{F}' \neq \emptyset$ and therefore $\bigcap \mathcal{F} \neq \emptyset$. ■

A related, and quite useful, fact is given by the following result.

FACT 3 ([7], [12]). *Let X be an infinite set and assume that 2^X is n-compact for some $n \in \mathbb{N}$. Then every cover $\mathcal{V} \subseteq \bigcup\{\mathcal{B}^m_X : m \leq n\}$ of 2^X has a finite subcover. Equivalently, every collection $\mathcal{W}$ of sets of the form*

$$\bigcup\{[p] : p \in S\} \text{ where for some } Q \subset X \text{ such that } |Q| \leq n, S \subseteq 2^Q, \tag{1.2}$$

with the fip, has a non-empty intersection. In particular, 2^X is m-compact for every positive integer $m < n$.

NOTATION 1. (1) For $n \in \mathbb{N}$, let (following the notation in [7] and [12]) $Q(n)$ stand for the following statement:

$Q(n)$: *For every infinite set X, 2^X is n-compact.*

(2) The Boolean Prime Ideal Theorem **BPI** is the principle: Every non-trivial Boolean algebra has a prime ideal. Equivalently, every proper filter of a non-trivial Boolean algebra is included in an ultrafilter (see [2] and [6]).

We conclude this section with a summary of what is known and not known about $Q(n)$, $n \in \mathbb{N}$.

$Q(1)$ *is* a theorem of **ZF** set theory (see [7]) and we have Mycielski's characterization of **BPI** in [9]:

FACT 4. *The following statements are equivalent in* **ZF***:*

(i) **BPI**,
(ii) *For every infinite set X, 2^X is compact.*

It follows that **BPI** implies $Q(n)$ for all $n \in \mathbb{N}$. Furthermore, Keremedis and Tachtsis [7] showed

FACT 5. *For every integer $n > 1$, $Q(n)$ implies $\mathbf{AC}_n$ (i.e., $\mathbf{AC}$ for families of n-element sets).*

On the other hand, Tachtsis [12] established

FACT 6. *For every integer $n \geq 6$, $Q(n)$ is equivalent to* **BPI**.

The set-theoretic strength of $Q(n)$, $n = 2, 3, 4, 5$, and in particular the question of whether $Q(n)$ is equivalent to **BPI** for $n = 2, 3, 4, 5$, is stated as an open problem in [12]. We settle this problem here. In particular, we establish in Theorem 3.1 below that, in **ZF**, $Q(3)$ is equivalent to **BPI**, hence, by Fact 3, $Q(n)$ is equivalent to **BPI** for every natural $n \geq 3$.

The situation with $Q(2)$ is *strikingly different!* In particular, we will prove in Lemma 3.2 that in the FM model $\mathcal{N}2^*(3)$ of [2], $Q(2)$ holds, whereas it is known (see [2] or [6]) that **BPI** fails in that model, hence we will infer in Theorem 3.4 that $Q(2)$ does not imply **BPI** in **ZFA**. We do not know whether or not **BPI** is strictly stronger than $Q(2)$ in the stronger theory **ZF**. So we pose the following:

QUESTION. Is there a model of **ZF** in which $Q(2)$ is true and **BPI** is false?

2. Diagram of known and new results on $Q(n)$. In the following diagram we summarize the known and new results which concern the set-theoretic strength of the principle $Q(n)$, $n \in \mathbb{N}$.

$Q(1)$ is a theorem of **ZF**

$Q(n)$ is, in **ZF**, equivalent to **BPI**, for every integer $n \geq 3$ (Theorem 3.1 and Fact 6)

In **ZF**, **BPI** $\to Q(2)$, but $Q(2) \nrightarrow$ **BPI** in **ZFA** (Lemma 3.2 and Theorem 3.4)

Diagram: The set-theoretic strength of $Q(n)$, $n \in \mathbb{N}$.

3. The two main results. We begin by establishing the equivalence between **BPI** and $Q(n)$ for $n = 3, 4, 5$. Prior to this, let us recall that if $(B, +, \cdot, 0_B, 1_B)$ is a Boolean algebra, then the binary relation $\leq$ defined on B by requiring for all $x, y \in B$, $x \leq y$ if and only if $x \cdot y = x$, is a partial order on B, so that $(B, \leq)$ is a complemented distributive lattice with smallest element 0_B and largest element 1_B. For $x, y \in B$, the supremum of $\{x, y\}$ is $\sup(\{x, y\}) = x + y + x \cdot y$ and the infimum of $\{x, y\}$ is $\inf(\{x, y\}) = x \cdot y$. The complement of an element $x \in B$ is the unique element $x' \in B$ such that $\sup(\{x, x'\}) = 1_B$ and $\inf(\{x, x'\}) = 0_B$. Note that for $x \in B$, $x' = x + 1$.

THEOREM 3.1. *In* **ZF**, *$Q(3)$ is equivalent to* **BPI**. *Hence, by Fact* 3, *for every integer $n \geq 3$, $Q(n)$ is equivalent to* **BPI**.

Proof. It suffices to show that $Q(3)$ implies **BPI**. Assuming $Q(3)$, we need to show that every proper filter of a non-trivial Boolean algebra is included in an ultrafilter. To this end, let $(B, +, \cdot, 0_B, 1_B)$ be a non-trivial Boolean algebra and let F be a proper filter of B. We will show that there exists an ultrafilter G of B which includes F. To this end, let L be a propositional language with propositional variables p_a, $a \in B$. The intended meaning of the variable p_a is that a belongs to the required ultrafilter. Let

$$\mathrm{Var} = \{p_a : a \in B\}.$$

Let $\mathcal{F}$ be the set of all formulas in the language L, and let Σ be the subset of $\mathcal{F}$ which consists of the following formulas:

(a) p_a for each $a \in F$.
(b) $p_a \to p_b$ for all $a, b \in B$ such that $a \leq b$.
(c) $p_a \wedge p_b \to p_{a \cdot b}$ for all $a, b \in B$.
(d) $p_a \vee p_{a+1}$ for all $a \in B$.

Consider the generalized Cantor cube 2^{Var}. We define the following clopen subsets of 2^{Var}:

For each $a \in F$, we let

$$K_a = [\{(p_a, 1)\}].$$

For all $a, b \in B$, we let

$$\begin{aligned} M_{(a,b)} = {} & [\{(p_a, 0), (p_b, 0), (p_{a\cdot b}, 0)\}] \\ & \cup [\{(p_a, 1), (p_b, 0), (p_{a\cdot b}, 0)\}] \\ & \cup [\{(p_a, 1), (p_b, 1), (p_{a\cdot b}, 1)\}] \\ & \cup [\{(p_a, 0), (p_b, 1), (p_{a\cdot b}, 0)\}] \end{aligned}$$

Note that $M_{(a,b)} = M_{(b,a)}$ for all $a, b \in B$. Further, if $a, b \in B$ with $a \leq b$, then $M_{(a,b)}$ obtains the following simpler form:

$$M_{(a,b)} = [\{(p_a, 1), (p_b, 1)\}] \cup [\{(p_a, 0), (p_b, 0)\}] \cup [\{(p_a, 0), (p_b, 1)\}].$$

Finally, for each $a \in B$, we let

$$N_a = [\{(p_a, 1), (p_{a+1}, 0)\}] \cup [\{(p_a, 0), (p_{a+1}, 1)\}].$$

Note that for each $a \in B$, $N_a = N_{a+1}$. Set

$$\mathcal{W} = \{K_a : a \in F\} \cup \{M_{(a,b)} : a, b \in B\} \cup \{N_a : a \in B\}.$$

Clearly, $\mathcal{W}$ is a collection of closed subsets of 2^{Var} of the form (1.2) in Fact 3; in our case here, $n = 3$. Furthermore, as every filter of a finite Boolean algebra A can be extended to an ultrafilter of A, it is reasonably easy to verify that $\mathcal{W}$ has the fip. Indeed, if $\mathcal{V} = \{W_1, \ldots, W_n\} \subseteq \mathcal{W}$, let S be the set of all $a \in B$ such that for some i, $1 \leq i \leq n$, a appears as a subscript in the notation of W_i as K_x or $M_{(x,y)}$ or N_x. Let A be the Boolean subalgebra of B which is generated by S. Since A is finite, we may define effectively (i.e., without using any form of choice) an ultrafilter G of A which extends the filter base $F \cap A$, which without loss of generality we assume to be non-empty. Let f be such that for each $a \in A$, $f(p_a) = 1$ if and only if $a \in G$. Via induction on the complexity of all formulas in $\mathcal{F}$ we may extend f to a valuation mapping $f' \in 2^{\mathcal{F}}$. Then $f' \restriction \mathrm{Var} \in \bigcap \mathcal{V}$ and $\mathcal{W}$ has the fip as asserted.

By our assumption, that is, by $Q(3)$, and using Fact 3, let $f \in \bigcap \mathcal{W}$ and let $f' \in 2^{\mathcal{F}}$ be the valuation mapping which extends f. By the definition of the members of $\mathcal{W}$, it easily follows that $f'(\phi) = 1$ for all $\phi \in \Sigma$. Let

$$G = \{a \in B : f'(p_a) = 1\}.$$

Then G is an ultrafilter of the Boolean algebra B which includes the filter F, since $f \in \bigcap\{K_a : a \in F\}$ and $f \subseteq f'$. This completes the proof of the theorem. ∎

We show next that $Q(2)$ does not imply **BPI** in **ZFA**, hence, $Q(2)$ is strictly weaker than **BPI** in **ZFA**. We need to prove first the following lemma which asserts that $Q(2)$ is valid in the FM model $\mathcal{N}2^*(3)$ of [2].

LEMMA 3.2. *In the FM model* $\mathcal{N}2^*(3)$ *of* [2], $Q(2)$ *is true.*

Proof. To construct the model $\mathcal{N}2^*(3)$, we begin with a model $\mathcal{M}$ of **ZFA** + **AC** which has a countable set A of atoms written as a disjoint union $\bigcup_{n\in\omega} T_n$ of triples $T_n = \{a_n, b_n, c_n\}$. Unless otherwise specified we will work in the model $\mathcal{M}$. Let G be the group generated by the following permutations ψ_n of A:

$$\psi_n{\restriction}T_n \text{ is the 3-cycle } (a_n, b_n, c_n) \text{ and } \psi_n(x) = x \text{ for all } x \in A \setminus T_n.$$

Note that G is commutative since the ψ_ns commute and that every non-identity element of G has order 3. For any finite $E \subseteq A$ we let $\mathrm{fix}_G(E) = \{\phi \in G : \forall e \in E,\ \phi(e) = e\}$. Let Γ be the (normal) finite support filter of subgroups of G generated by $\{\mathrm{fix}_G(E) : E \in [A]^{<\omega}\}$, where $[A]^{<\omega}$ is the set of all finite subsets of A. $\mathcal{N}2^*(3)$ is the permutation model determined by G and Γ.

For the remainder of the proof we will use $\mathcal{N}$ for $\mathcal{N}2^*(3)$. For each subgroup H of G and each element $x \in \mathcal{N}$ we let $\mathrm{Orb}_H(x)$ be the orbit of x under the action of the group H. That is, $\mathrm{Orb}_H(x) = \{\phi(x) : \phi \in H\}$. If E is a finite subset of A and $H = \mathrm{fix}_G(E)$ we will abbreviate $\mathrm{Orb}_H(x)$ by $\mathrm{Orb}_E(x)$.

We will need the following fact about $\mathcal{N}$ which follows from Lemma 4.2 of [3]:

(3.1) for all $x \in \mathcal{N}$ and for all subgroups H of G, $|\mathrm{Orb}_H(x)| = 3^k$ for some $k \in \omega$.

For the reader's convenience, we shall simplify here the notation for 1-basic open sets and 2-basic closed sets. In particular, let X be an element of $\mathcal{N}$, x an element of X and λ in $\{0, 1\}$. Then we let

$$\langle x, \lambda\rangle = \{f \in 2^X \cap \mathcal{M} : f(x) = \lambda\}$$

or

$$\langle x, \lambda\rangle = \{f \in 2^X : f(x) = \lambda\}$$

since we are working in $\mathcal{M}$, and we let

$$\langle x,\lambda\rangle_{\mathcal{N}} = \{f \in 2^X \cap \mathcal{N} : f(x) = \lambda\} = \langle x,\lambda\rangle \cap \mathcal{N}.$$

(Using the notation of Section 1, we could have denoted $\langle x,\lambda\rangle$ by $[\{(x,\lambda)\}]$, which might be more cumbersome due to technical details appearing in the proof.) With this new notation, a 2-basic closed subset of 2^X in $\mathcal{M}$ has the form

$$\langle x,\lambda\rangle \cup \langle y,\mu\rangle$$

and a 2-basic closed subset of 2^X in $\mathcal{N}$ has the form

$$\langle x,\lambda\rangle_{\mathcal{N}} \cup \langle y,\mu\rangle_{\mathcal{N}} = (\langle x,\lambda\rangle \cup \langle y,\mu\rangle) \cap \mathcal{N}$$

for some x and y in X and some λ and μ in $\{0,1\}$. We leave it to the reader to verify that the set of pairs $H = \{(\langle x,\lambda\rangle_{\mathcal{N}} \cup \langle y,\mu\rangle_{\mathcal{N}}, \langle x,\lambda\rangle \cup \langle y,\mu\rangle) : x,y \in X \text{ and } \lambda,\mu \in \{0,1\}\}$ is a one-to-one function from the 2-basic closed sets in $\mathcal{N}$ onto the 2-basic closed subsets in $\mathcal{M}$. (Some care is required when $x = y$ and $\lambda \neq \mu$, since for any $x \in X$ and distinct $\lambda,\mu \in \{0,1\}$, $\langle x,\lambda\rangle_{\mathcal{N}} \cup \langle x,\mu\rangle_{\mathcal{N}} = 2^X \cap \mathcal{N}$.) Since $\mathcal{N} \subseteq \mathcal{M}$ we have

$$\langle x,\lambda\rangle_{\mathcal{N}} \cup \langle y,\mu\rangle_{\mathcal{N}} \subseteq \langle x,\lambda\rangle \cup \langle y,\mu\rangle = H(\langle x,\lambda\rangle_{\mathcal{N}} \cup \langle y,\mu\rangle_{\mathcal{N}}).$$

In order to prove that $Q(2)$ is true in $\mathcal{N}$ we assume that $X \in \mathcal{N}$ and that $\mathcal{F}$ is a collection of 2-basic closed subsets of 2^X (in $\mathcal{N}$) with finite support E and with the fip (in $\mathcal{N}$). Let

$$\mathcal{F}' = \{\langle x,\lambda\rangle \cup \langle y,\mu\rangle : \langle x,\lambda\rangle_{\mathcal{N}} \cup \langle y,\mu\rangle_{\mathcal{N}} \in \mathcal{F}\}\ (= \{H(F) : F \in \mathcal{F}\}).$$

We first note that since $\mathcal{F}$ has support E,

$$\forall \langle x,\lambda\rangle \cup \langle y,\mu\rangle \in \mathcal{F}',\ \forall \phi \in \mathrm{fix}_G(E),\quad \langle \phi(x),\lambda\rangle \cup \langle \phi(y),\mu\rangle \in \mathcal{F}'. \tag{3.2}$$

Or in more compact form

$$\forall F' \in \mathcal{F}',\ \forall \phi \in \mathrm{fix}_G(E),\quad \phi(F') \in \mathcal{F}'. \tag{3.3}$$

Since $F \subseteq H(F)$ for every $F \in \mathcal{F}$, the set $\mathcal{F}'$ is a collection of 2-basic closed sets in $\mathcal{M}$ which has the fip. Since **AC** is true in $\mathcal{M}$ and $Q(2)$ follows from **AC**, there is a function $f_0 \in \bigcap \mathcal{F}'$. Our plan is to use f_0 to define a function f_1 which is in $\bigcap \mathcal{F}'$ and in $\mathcal{N}$. This will suffice since such an f_1 will be in $\bigcap \mathcal{F}$.

For any finite subset Y of X with an odd number of elements, we let $\mathrm{Maj}(Y,f_0)$ be the element λ of $\{0,1\}$ for which $|\{y \in Y : f_0(y) = \lambda\}|$ is largest. Since $\mathrm{Orb}_E(x)$ is finite and has an odd number of elements for every $x \in \mathcal{N}$ (by (3.1)), we may define $f_1 : X \to \{0,1\}$ by

$$f_1(x) = \mathrm{Maj}(\mathrm{Orb}_E(x), f_0). \tag{3.4}$$

Since f_1 is constant on $\mathrm{Orb}_E(x)$ for all $x \in X$, f_1 has support E and is therefore in $\mathcal{N}$.

Now we argue that $f_1 \in \bigcap \mathcal{F}'$. Assume that $F \in \mathcal{F}'$ and that $F = \langle x, \lambda\rangle \cup \langle y, \mu\rangle$. The easiest case to handle is when $x = y$ and $\lambda \neq \mu$ since in this case $F = 2^X$. For the remaining cases we note that, by (3.3), for every $J \in \mathrm{Orb}_E(F)$, $f_0 \in J$. We will use this fact to show that $f_1 \in F$.

Case 1: $x = y$ and $\lambda = \mu$. In this case $F = \langle x, \lambda\rangle$ for some $x \in X$ and $\lambda \in \{0, 1\}$. Assume that $t \in \mathrm{Orb}_E(x)$. Then $t = \phi(x)$ for some $\phi \in \mathrm{fix}_G(E)$. By the comments preceding Case 1, $f_0 \in \phi(F) = \langle \phi(x), \lambda\rangle = \{f \in 2^X : f(t) = \lambda\}$. So $f_0(t) = \lambda$ for every $t \in \mathrm{Orb}_E(x)$. By the definition of f_1, $f_1(x) = \lambda$ and therefore $f_1 \in F$.

Case 2: $x \neq y$, $\mathrm{Orb}_E(x) = \mathrm{Orb}_E(y)$ and $\lambda \neq \mu$. Since x and y are in the same orbit and f_1 is constant on orbits $f_1(x) = \lambda = f_1(y)$ or $f_1(x) = \mu = f_1(y)$. In either case, $f_1 \in F$.

Case 3: $x \neq y$, $\mathrm{Orb}_E(x) = \mathrm{Orb}_E(y)$ and $\lambda = \mu$. In this case $F = \langle x, \lambda\rangle \cup \langle y, \lambda\rangle$. Choose $\psi \in \mathrm{fix}_G(E)$ such that $\psi(x) = y$ and let $\psi^2(x) = z$. Since ψ has order 3, x, y and z are different elements of $\mathrm{Orb}_E(x)$ and $\psi(z) = x$.

For each $t \in \mathrm{Orb}_E(x)$ we let $C_t = \{t, \psi(t), \psi^2(t)\}$. Since ψ has order 3, we could also write

$$C_t = \{\psi^n(t) : n \in \mathbb{Z}\}. \tag{3.5}$$

We also claim the following:

1. (1) For $t \in \mathrm{Orb}_E(x)$ the set C_t has exactly three elements.
2. (2) The set $P = \{C_t : t \in \mathrm{Orb}_E(x)\}$ is a partition of $\mathrm{Orb}_E(x)$.
3. (3) The 2-basic closed sets $F_t = \langle t, \lambda\rangle \cup \langle \psi(t), \lambda\rangle$, $\psi(F_t) = \langle \psi(t), \lambda\rangle \cup \langle \psi^2(t), \lambda\rangle$ and $\psi^2(F_t) = \langle \psi^2(t), \lambda\rangle \cup \langle t, \lambda\rangle$ are in $\mathrm{Orb}_E(F)$, for all $t \in \mathrm{Orb}_E(x)$.

To prove item (1) we first note that $C_x = \{x, \psi(x), \psi^2(x)\} = \{x, y, z\}$ has three elements as we remarked above. If we choose an $\eta \in \mathrm{fix}_G(E)$ such that $\eta(x) = t$, then

$$\eta(C_x) = \{\eta(x), \eta(\psi(x)), \eta(\psi^2(x))\} = \{t, \psi(t), \psi^2(t)\} = C_t$$

(where the second to last equality has used the fact that the group G is commutative). Since C_x has three elements and η is an isomorphism of the model, $\eta(C_x) = C_t$ has three elements.

For item (2) we show that for any t and t' in $\mathrm{Orb}_E(x)$, if $C_t \cap C_{t'} \neq \emptyset$ then $C_t = C_{t'}$. Using the characterization of C_t given in (3.5), it follows from the assumption $C_t \cap C_{t'} \neq \emptyset$ that there are integers m and k such that $\psi^m(t) = \psi^k(t')$. Using (3.5) again, it follows that $C_t = C_{t'}$.

For the proof of item (3) we assume $t \in \mathrm{Orb}_E(x)$ and that $t = \eta(x)$ where $\eta \in \mathrm{fix}_G(E)$. Then $\eta(F) = \langle \eta(x), \lambda\rangle \cup \langle \eta(y), \lambda\rangle = \langle t, \lambda\rangle \cup \langle \eta(\psi(x)), \lambda\rangle = \langle t, \lambda\rangle \cup \langle \psi(t), \lambda\rangle = F_t$. (The second to last equality uses the fact that G

is commutative.) So $F_t \in \mathrm{Orb}_E(F)$, and since $\psi \in \mathrm{fix}_G(E)$, it follows that $\psi(F_t)$ and $\psi^2(F_t)$ are also in $\mathrm{Orb}_E(F)$.

It follows from item (3) and (3.3) that for each $t \in \mathrm{Orb}_E(x)$, the three sets $F_t, \psi(F_t)$ and $\psi^2(F_t)$ are in $\mathcal{F}'$. Since $f_0 \in \bigcap \mathcal{F}'$, $f_0 \in F_t \cap \psi(F_t) \cap \psi^2(F_t)$ from which it follows that $|\{s \in \{t, \psi(t), \psi^2(t)\} : f_0(s) = \lambda\}| \geq 2$. (If $f_0(s) = 1-\lambda$ for two or more elements of $\{t, \psi(t), \psi^2(t)\}$ then it fails to be in at least one of the three sets $F_t, \psi(F_t)$ or $\psi^2(F_t)$.) Since the sets $\{t, \psi(t), \psi^2(t)\}$ for $t \in \mathrm{Orb}_E(x)$ partition $\mathrm{Orb}_E(x)$ we conclude that $\mathrm{Maj}(\mathrm{Orb}_E(x), f_0) = \lambda$. Therefore $f_1(t) = \lambda$ for all $t \in \mathrm{Orb}_E(x)$. In particular $f_1(x) = \lambda$, so $f_1 \in F$.

Case 4: $x \neq y$, $\mathrm{Orb}_E(x) \neq \mathrm{Orb}_E(y)$. We first argue that

$$\mathrm{Maj}(\mathrm{Orb}_E(x)), f_0) = \lambda \quad \text{or} \quad \mathrm{Maj}(\mathrm{Orb}_E(y), f_0) = \mu \tag{3.6}$$

(where, recall, $F = \langle x, \lambda\rangle \cup \langle y, \mu\rangle$ and $f_0 \in \bigcap\{\phi(F) : \phi \in \mathrm{fix}_G(E)\}$).

Let

$$\mathrm{Orb}_y(x) = \{\phi(x) : \phi \in \mathrm{fix}_G(E) \text{ and } \phi(y) = y\},$$
$$\mathrm{Orb}_x(y) = \{\psi(y) : \psi \in \mathrm{fix}_G(E) \text{ and } \psi(x) = x\}.$$

Let also

$$\mathbb{P}_x = \mathrm{Orb}_E(\mathrm{Orb}_y(x)) = \{\eta(\mathrm{Orb}_y(x)) : \eta \in \mathrm{fix}_G(E)\},$$
$$\mathbb{P}_y = \mathrm{Orb}_E(\mathrm{Orb}_x(y)) = \{\eta(\mathrm{Orb}_x(y)) : \eta \in \mathrm{fix}_G(E)\}.$$

Lemma 3.3.

(1) *The sets $\mathbb{P}_x$ and $\mathbb{P}_y$ are finite and odd-sized partitions of* $\mathrm{Orb}_E(x)$ *and* $\mathrm{Orb}_E(y)$, *respectively.*
(2) *The binary relation $R = \{(\eta(\mathrm{Orb}_y(x)), \eta(\mathrm{Orb}_x(y))) : \eta \in \mathrm{fix}_G(E)\}$ is a one-to-one function from $\mathbb{P}_x$ onto $\mathbb{P}_y$.*
(3) *For all $Z_1, Z_2 \in \mathbb{P}_x$, $|Z_1| = |Z_2|$, and for all $W_1, W_2 \in \mathbb{P}_y$, $|W_1| = |W_2|$.*
(4) *For every $Z \in \mathbb{P}_x$, and for all $z \in Z$ and $w \in R(Z)$, the set $\langle z, \lambda\rangle \cup \langle w, \mu\rangle$ is in* $\mathrm{Orb}_E(F)$.

Proof. (1) We only prove (1) for $\mathbb{P}_x$. First note that by the definition of $\mathbb{P}_x$ and (3.1), it readily follows that $\mathbb{P}_x$ is a finite odd-sized set. Further, it is clear that $\bigcup \mathbb{P}_x = \mathrm{Orb}_E(x)$, hence in order to prove that $\mathbb{P}_x$ is a partition of $\mathrm{Orb}_E(x)$, it suffices to assume that $\eta_1(\mathrm{Orb}_y(x)) \cap \eta_2(\mathrm{Orb}_y(x)) \neq \emptyset$, where η_1 and η_2 are in $\mathrm{fix}_G(E)$, and prove that $\eta_1(\mathrm{Orb}_y(x)) = \eta_2(\mathrm{Orb}_y(x))$. By the assumption there is an element t in the intersection which can therefore be written as

$$t = \eta_1(\phi_1(x)) = \eta_2(\phi_2(x))$$

where ϕ_1 and ϕ_2 are in $\mathrm{fix}_G(E)$ and $\phi_1(y) = \phi_2(y) = y$. Solving the displayed equation and using the fact that G is commutative we obtain

$$x = \eta_1^{-1}\eta_2\phi_1^{-1}\phi_2(x). \tag{3.7}$$

Therefore if z is another element of $\eta_1(\mathrm{Orb}_y(x))$, $z = \eta_1(\phi_3(x))$ where $\phi_3 \in \mathrm{fix}_G(E)$ and $\phi_3(y) = y$, then by (3.7), we have $z = \eta_1(\phi_3(\eta_1^{-1}\eta_2\phi_1^{-1}\phi_2(x)))$ $= \eta_2\phi_3\phi_1^{-1}\phi_2(x)$ and therefore $z \in \eta_2(\mathrm{Orb}_y(x))$. Similarly every element of $\eta_2(\mathrm{Orb}_y(x))$ is in $\eta_1(\mathrm{Orb}_y(x))$.

(2) It is clear that every element of $\mathbb{P}_x$ is in the domain of R and every element of $\mathbb{P}_y$ is in the range of R. We will prove R is a function. The proof that R is one-to-one is similar and we take the liberty of omitting it. It suffices to prove that for all $\eta_1, \eta_2 \in \mathrm{fix}_G(E)$, if $\eta_1(\mathrm{Orb}_x(y)) \neq \eta_2(\mathrm{Orb}_x(y))$ then $\eta_1(\mathrm{Orb}_y(x)) \neq \eta_2(\mathrm{Orb}_y(x))$. Letting $\beta = \eta_2^{-1}\eta_1$ this is equivalent to showing that for all $\beta \in \mathrm{fix}_G(E)$, if $\beta(\mathrm{Orb}_x(y)) \neq \mathrm{Orb}_x(y)$ then $\beta(\mathrm{Orb}_y(x)) \neq \mathrm{Orb}_y(x)$. Assume the hypothesis holds and the conclusion is false. Then $\beta(y) \notin \mathrm{Orb}_x(y)$ (otherwise, $\beta(\mathrm{Orb}_x(y)) \cap \mathrm{Orb}_x(y) \neq \emptyset$, hence $\beta(\mathrm{Orb}_x(y)) = \mathrm{Orb}_x(y)$, since $\mathbb{P}_y$ is a partition, a contradiction) and therefore $\beta(x) \neq x$ (by the definition of $\mathrm{Orb}_x(y)$ if $\beta(x) = x$ then $\beta(\mathrm{Orb}_x(y)) = \mathrm{Orb}_x(y)$). Since $\beta(\mathrm{Orb}_y(x)) = \mathrm{Orb}_y(x)$, $\beta(x) \in \mathrm{Orb}_y(x)$ and therefore $\beta(x) = \phi(x)$ for some $\phi \in \mathrm{fix}_G(E)$ for which $\phi(y) = y$. But then $\phi^{-1}\beta(x) = x$ and therefore $\phi^{-1}\beta(y) \in \mathrm{Orb}_x(y)$. But $\phi^{-1}\beta(y) = \beta(y)$, contradicting our assumption that $\beta(y) \notin \mathrm{Orb}_x(y)$.

(3) Note that every $Z \in \mathbb{P}_x$ has the same cardinality as $\mathrm{Orb}_y(x)$ since for some $\eta \in \mathrm{fix}_G(E)$, $Z = \eta(\mathrm{Orb}_y(x))$ and η is an $\in$-isomorphism of the model $\mathcal{M}$. Similarly, any two elements of $\mathbb{P}_y$ have the same cardinal number.

(4) Let $Z \in \mathbb{P}_x$, $z \in Z$ and $w \in R(Z)$. Then there is an $\eta \in \mathrm{fix}_G(E)$ such that $Z = \eta(\mathrm{Orb}_y(x))$ and $R(Z) = \eta(\mathrm{Orb}_x(y))$. There are also permutations ϕ_1 and ϕ_2 in $\mathrm{fix}_G(E)$ such that $\eta(\phi_1(x)) = z$, $\phi_1(y) = y$, $\eta(\phi_2(y)) = w$ and $\phi_2(x) = x$. The set $\eta\phi_1\phi_2(F)$ is in $\mathrm{Orb}_E(F)$ and

$$\begin{aligned}\eta\phi_1\phi_2(F) &= \langle \eta\phi_1\phi_2(x), \lambda\rangle \cup \langle \eta\phi_1\phi_2(y), \mu\rangle \\ &= \langle \eta\phi_1(x), \lambda\rangle \cup \langle \eta\phi_2(y), \mu\rangle = \langle z, \lambda\rangle \cup \langle w, \mu\rangle.\end{aligned}$$

The conclusion of (4) follows. ■

By Lemma 3.3(4), we have

$$\forall Z \in \mathbb{P}_x, \forall z \in Z, \forall w \in R(Z), \qquad \langle z, \lambda\rangle \cup \langle w, \mu\rangle \in \mathcal{F}'.$$

Since $f_0 \in \bigcap \mathcal{F}'$ it follows that

$$\forall Z \in \mathbb{P}_x,\ \text{either } (\forall z \in Z, f_0(z) = \lambda) \text{ or } (\forall w \in R(Z), f_0(w) = \mu). \tag{3.8}$$

Let K_0 be the odd integer $|\mathbb{P}_x| = |\mathbb{P}_y| = |R|$. It follows from (3.8) that either

$$|\{Z \in \mathbb{P}_x : \forall z \in Z,\ f_0(z) = \lambda\}| > K_0/2 \quad \text{or} \tag{3.9}$$

$$|\{W \in \mathbb{P}_y : \forall w \in W,\ f_0(w) = \mu\}| > K_0/2. \tag{3.10}$$

(Recall that $\mathbb{P}_y$ is the image of $\mathbb{P}_x$ under R. If both of the above inequalities fail then the set of all pairs $(Z, R(Z)) \in R$ such that $(\forall z \in Z, f_0(z) = \lambda)$ or $(\forall w \in R(Z), f_0(w) = \mu)$ has cardinality smaller than $K_0 = |R|$. There would then be a pair $(Z, R(Z)) \in R$ for which both $(\forall z \in Z, f_0(z) = \lambda)$ and $(\forall w \in R(Z), f_0(w) = \mu)$ are false. This contradicts (3.8).)

By Lemma 3.3(3), all elements of $\mathbb{P}_x$ have the same cardinality. Therefore, if alternative (3.9) holds, then $\mathrm{Maj}(\mathrm{Orb}_E(x), f_0) = \lambda$. Similarly, if (3.10) holds, then $\mathrm{Maj}(\mathrm{Orb}_E(y), f_0) = \mu$. Therefore, either $f_1(x) = \lambda$ or $f_1(y) = \mu$ and in either case, $f_1 \in F$.

We have shown that $f_1 \in \bigcap \mathcal{F}'$, which, as remarked earlier, is sufficient to complete the proof. ∎

THEOREM 3.4. *In* **ZFA**, $Q(2)$ *does not imply* **BPI**, *hence by Theorem* 3.1, $Q(2)$ *does not imply* $Q(n)$, *for any integer* $n \geq 3$.

Proof. This follows from Lemma 3.2 and the known fact that **BPI** fails in the FM model $\mathcal{N}2^*(3)$ (see [2] or [6]). ∎

We note that many consequences of the Axiom of Choice are known to hold in $\mathcal{N}2^*(3)$. Of particular interest to us are:

> **The Axiom of Multiple Choice MC:** For every set X of non-empty sets there is a function f with domain X such that for each $y \in X$, $f(y)$ is a non-empty finite subset of y,

and its consequence (see [8, Corollary 2])

> **Rado's Selection Lemma RL** ([10]): Let $\mathfrak{F}$ be a family of finite sets and suppose that to every finite subset F of $\mathfrak{F}$ there corresponds a choice function ϕ_F whose domain is F such that $\phi_F(T) \in T$ for each $T \in F$. Then there is a choice function f whose domain is $\mathfrak{F}$ with the property that for every finite subset F of $\mathfrak{F}$, there is a finite subset F' of $\mathfrak{F}$ such that $F \subseteq F'$ and $f(T) = \phi_{F'}(T)$ for all $T \in F$.

(For an extensive study on Rado's selection lemma, the reader is referred to [1], [4], [8], [10], [11].)

On the other hand, a principle related to generalized Cantor cubes (which fails in $\mathcal{N}2^*(3)$, see Corollary 3.5 below), introduced and studied in [5], is the following:

> **MCP:** For every infinite set X, the generalized Cantor cube 2^X has the minimal cover property, i.e., for every open cover $\mathcal{U}$ of 2^X there is a subcover $\mathcal{V}$ of $\mathcal{U}$ with the property that for every $V \in \mathcal{V}$, $\mathcal{V} \setminus \{V\}$ does not cover 2^X.

The following is shown in [5]:

FACT 7. **MCP** *implies* $Q(2) + \mathbf{AC_{fin}}$, *where* $\mathbf{AC_{fin}}$ *is the Axiom of Choice for families of non-empty finite sets.*

Many finite choice axioms, for example $\mathbf{AC}_3^{\omega}$, the axiom of choice for countable sets of 3-element sets, are known to fail in $\mathcal{N}2^*(3)$ (see [2] or [6]). We also note that $\mathbf{AC}_2$, the axiom of choice for sets of 2-element sets, holds in $\mathcal{N}2^*(3)$ (see [2] or [6]). A fairly complete list of both kinds of forms can be found in [2]. As a consequence to the above discussion and results, we also have

Corollary 3.5. *In* **ZFA***,* $(Q(2) + \mathbf{MC}\ (+\mathbf{RL}))$ *does not imply* $\mathbf{AC}_3^{\omega}$*, hence does not imply* **MCP***.*

Acknowledgements. We would like to thank the anonymous referee for his review work and, in particular, for his suggestion of adding in the paper the open question of whether $Q(2)$ implies **BPI** in **ZF**.

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