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Remarks on regularity criteria for the Navier–Stokes equations with axisymmetric data

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Abstract. We consider the axisymmetric Navier–Stokes equations with non-zero swirl component. By invoking the Hardy–Sobolev interpolation inequality, Hardy inequality and the theory of A_β $(1<\beta<\infty)$ weights, we establish regularity criteria involving u^r, ω^z or ω^θ in some weighted Lebesgue spaces. This improves many previous results.

1. Introduction. The three-dimensional Navier–Stokes equations read

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta \boldsymbol{u} + \nabla\pi = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = \boldsymbol{0}, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases} \tag{1.1}$$

where

$$\boldsymbol{u} = (u^1, u^2, u^3) = u^1\boldsymbol{e}_1 + u^2\boldsymbol{e}_2 + u^3\boldsymbol{e}_3$$

is the fluid velocity field, π is a scalar pressure, and $\boldsymbol{u}_0$ is the prescribed initial data satisfying the compatibility condition $\nabla\cdot\boldsymbol{u}_0 = 0$ in the sense of distributions.

It is well-known that (1.1) has a global weak solution for initial data of finite energy [7, 13]. However, its regularity and uniqueness is an outstanding open problem in mathematical fluid dynamics. Pioneered by Serrin [17] and Prodi [16], there are many sufficient conditions to ensure the smoothness of the solution. In particular, we have the following regularity criterion (see [4, 16, 17] for example):

$$\boldsymbol{u}\in L^\alpha(0,T;L^\beta(\mathbb{R}^3)),\qquad \frac{2}{\alpha}+\frac{3}{\beta}=1,\ 3\le q\le\infty. \tag{1.2}$$

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In this paper, we shall concern ourselves with axisymmetric solutions of (1.1). A solution is *axisymmetric* if the velocity field can be represented as

$$\boldsymbol{u} = u^r(t,r,z)\boldsymbol{e}_r + u^z(t,r,z)\boldsymbol{e}_\theta + u^z(t,r,z)\boldsymbol{e}_z, \tag{1.3}$$

where

$$\boldsymbol{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right) = (\cos\theta, \sin\theta, 0), \quad \boldsymbol{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right) = (-\sin\theta, \cos\theta, 0),$$
$$\boldsymbol{e}_z = (0,0,1),$$

and u^r, u^θ and u^z are called the *radial*, *swirl* (or azimuthal) and *axial* components of $\boldsymbol{u}$ respectively. Thus the system (1.1) can be equivalently reformulated as

$$\begin{cases} \frac{\tilde{D}}{Dt}u^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^r - \frac{(u^\theta)^2}{r} + \partial_r \pi = 0, \\ \frac{\tilde{D}}{Dt}u^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^\theta + \frac{u^r u^\theta}{r} = 0, \\ \frac{\tilde{D}}{Dt}u^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)u^z + \partial_z \pi = 0, \\ \partial_r(ru^r) + \partial_z(ru^z) = 0, \\ (u^r, u^\theta, u^z)(0) = (u_0^r, u_0^\theta, u_0^z), \end{cases} \tag{1.4}$$

where

$$\frac{\tilde{D}}{Dt} = \partial_t + u^r\partial_r + u_z\partial_z \tag{1.5}$$

denotes the *convection derivative* (or *material derivative*).

If we take the curl of $(1.1)_1$ and denote

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \omega^r\boldsymbol{e}_r + \omega^\theta\boldsymbol{e}_\theta + \omega^z\boldsymbol{e}_z \tag{1.6}$$

with

$$\omega^r = -\partial_z u^\theta, \quad \omega^\theta = \partial_z u^r - \partial_r u^z, \quad \omega^z = \partial_r u^\theta + \frac{u^\theta}{r}, \tag{1.7}$$

then

$$\partial_t\boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{\omega} = \boldsymbol{0}, \tag{1.8}$$

which could be rewritten as

$$\begin{cases} \frac{\tilde{D}}{Dt}\omega^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^r - (\omega^r\partial_r + \omega^z\partial_z)u^r = 0, \\ \frac{\tilde{D}}{Dt}\omega^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^\theta - \frac{2u^\theta\partial_z u^\theta}{r} - \frac{u^r\omega^\theta}{r} = 0, \\ \frac{\tilde{D}}{Dt}\omega^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)\omega^z - (\omega^r\partial_r + \omega^z\partial_z)u^z = 0. \end{cases} \tag{1.9}$$

If the swirl component u_0^θ is zero, then (1.4) has a unique global regular solution [10, 12, 19]. However, if $u_0^\theta \neq 0$, then global regularity is open. Tremendous efforts have been devoted to the regularity problem, and interesting progress has been made: see [1, 3, 5, 8, 9, 11, 14, 15, 20] and references therein. Let us now list some regularity criteria which are relevant to our results:

(1) ([2, Theorem 1.1])

$$r^d u^\theta \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)), \qquad \frac{2}{\alpha}+\frac{3}{\beta}=1-d,\ \frac{3}{1-d}<\beta\leq\infty,\quad 0\leq d<1; \tag{1.10}$$

(2) ([2, Remark 1.3], under the assumption $ru_0^\theta \in L^\infty$)

$$r^d u^\theta \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)), \qquad \frac{2}{\alpha}+\frac{3}{\beta}=1-d,\ \frac{3}{1-d}<\beta\leq\infty,\quad -1\leq d<0; \tag{1.11}$$

(3) ([2, Theorem 1.4])

$$r^d u^z \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)), \qquad \frac{2}{\alpha}+\frac{3}{\beta}=1-d,\ \frac{3}{1-d}<\beta\leq\infty,\quad 0\leq d<1; \tag{1.12}$$

(4) ([2, Corollary 1.5], under the assumption $ru_0^\theta \in L^\infty$)

$$ru^z \in L^\infty(0,T;L^\infty(\mathbb{R}^3)); \tag{1.13}$$

(5) ([2, Theorem 1.3], under the assumption $ru_0^\theta \in L^\infty$)

$$\omega^z \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)), \qquad \frac{2}{\alpha}+\frac{3}{\beta}=2,\ \frac{3}{2}<\beta<\infty; \tag{1.14}$$

(6) ([1, Theorem 1])

$$\omega^\theta \in L^\alpha(0,T;L^\beta(\mathbb{R}^3)), \qquad \frac{2}{\alpha}+\frac{3}{\beta}=2,\ \frac{3}{2}<\beta<\infty, \tag{1.15}$$

with the limiting case $\omega^\theta \in L^1(0,T;L^\infty(\mathbb{R}^3))$ covered and extended to $\omega^\theta \in L^1(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3))$ in [3, Theorem 1.2].

By (1.10)–(1.11), we see that the regularity criterion involving $r^d u^\theta$ is complete. Notice that the case $d=1$ is a priori known, which could not be a regularity criterion (see Lemma 2.2). However, the smoothness criteria (1.14) and (1.15) are not complete. We shall extend them in the following theorem.

Our precise result reads:

THEOREM 1.1. *Let* $\boldsymbol{u}_0 \in H^2(\mathbb{R}^3)$ *be axially symmetric and divergence-free, and* $\boldsymbol{u} \in C([0,T); H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3))$ *be the unique axisymmetric classical solution of* (1.4). *If one of the following conditions holds:*

$$r^d u^r \in L^\alpha(0,T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1 - d, \ \frac{3}{1-d} < \beta \le \infty, \quad -1 \le d < 1; \tag{1.16}$$

$$r u_0^\theta \in L^\infty \text{ and } r^d \omega^z \in L^\alpha(0,T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2 - d, \ \frac{3}{2-d} < \beta < \infty, \quad -2 \le d < 2; \tag{1.17}$$

$$r^d \omega^\theta \in L^\alpha(0,T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2 - d, \ \frac{3}{2-d} < \beta < \infty, \quad 0 \le d < 2, \tag{1.18}$$

then the solution can be smoothly extended beyond T.

REMARK 1.2. In [9, 14], only some cases of (1.16) were treated. Also, (1.14) and (1.15) correspond to (1.17) and (1.18) with $d = 0$ respectively. Thus our results improve and extend previous results. One may refer to [9] for a weighted regularity criterion in terms of the negative part of u^r, where only the case $q = \infty$ was not treated in two cases.

REMARK 1.3. Regularity criteria involving $r^d u^z$ with $-1 \le d < 0$ (the case $0 \le d < 1$ was already established in [2, Theorem 1.4]) or $r^d \omega^r$ with $-2 \le d < 2$ (or the weaker condition $r^d \tilde{\nabla} u^\theta$ with $\tilde{\nabla} = (\partial_r, \partial_z)$) seem to be out of reach at this moment. This will be the subject of our future investigation.

The proof of Theorem 1.1 under conditions (1.16), (1.17) and (1.18) will be given in Sections 4, 5 and 6 respectively. Before doing that, we shall give some useful equalities and inequalities in Section 2, and establish a preliminary regularity criterion in Section 3.

2. Some useful equalities and inequalities for axisymmetric solutions. In the calculations later on, we shall often use the following fundamental relationships between the Cartesian and cylindrical coordinates. By the chain rule, we have

$$\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}, \tag{2.1}$$

and thus conversely,

$$\begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\end{pmatrix} = \begin{pmatrix}\cos\theta & -\frac{\sin\theta}{r} & 0\\ \sin\theta & \frac{\cos\theta}{r} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\partial_r\\ \partial_\theta\\ \partial_z\end{pmatrix}, \tag{2.2}$$

which can be abbreviated as

$$\nabla = \boldsymbol{e}_r\partial_r + \frac{1}{r}\boldsymbol{e}_\theta\partial_\theta + \boldsymbol{e}_z\partial_z, \tag{2.3}$$

and thus for a function $f(r,\theta,z)$,

$$\begin{aligned}\nabla f &= \partial_r f\boldsymbol{e}_r + \frac{1}{r}\partial_\theta f\boldsymbol{e}_\theta + \partial_z f\boldsymbol{e}_z, \qquad \text{so}\\ |\nabla f|^2 &= |\partial_r f|^2 + \left|\frac{\partial_\theta f}{r}\right|^2 + |\partial_z f|^2,\end{aligned} \tag{2.4}$$

and for an axisymmetric function $g(r,z)$,

$$|\tilde{\nabla} g|^2 = |\partial_r g|^2 + |\partial_z g|^2 = |\nabla g|^2 \tag{2.5}$$

with $\tilde{\nabla} = (\partial_1, \partial_2)$.

Moreover,

$$\begin{array}{lll}\partial_r\boldsymbol{e}_r = \boldsymbol{0}, & \partial_r\boldsymbol{e}_\theta = \boldsymbol{0}, & \partial_r\boldsymbol{e}_z = \boldsymbol{0},\\ \partial_\theta\boldsymbol{e}_r = \boldsymbol{e}_\theta, & \partial_\theta\boldsymbol{e}_\theta = -\boldsymbol{e}_r, & \partial_r\boldsymbol{e}_z = \boldsymbol{0},\\ \partial_z\boldsymbol{e}_r = \boldsymbol{0}, & \partial_z\boldsymbol{e}_\theta = \boldsymbol{0}, & \partial_z\boldsymbol{e}_z = \boldsymbol{0}.\end{array} \tag{2.6}$$

LEMMA 2.1. *Denote*

$$\tilde{\boldsymbol{u}} = u^r\boldsymbol{e}_r + u^z\boldsymbol{e}_z, \quad \tilde{\boldsymbol{\omega}} = \omega^r\boldsymbol{e}_r + \omega^z\boldsymbol{e}_z, \quad \tilde{\nabla} = (\partial_r, \partial_z). \tag{2.7}$$

Then

$$\nabla\cdot\tilde{\boldsymbol{u}} = 0, \qquad \nabla\times\tilde{\boldsymbol{u}} = \omega^\theta\boldsymbol{e}_\theta, \tag{2.8}$$

and thus for $1 < p < \infty$,

$$\|\nabla\tilde{\boldsymbol{u}}\|_{L^p} \le C(p)\|\omega^\theta\|_{L^p}; \tag{2.9}$$

moreover

$$|\nabla\tilde{\boldsymbol{u}}|^2 = \left|\frac{u^r}{r}\right|^2 + |\tilde{\nabla}u^r|^2 + |\tilde{\nabla}u^z|^2, \tag{2.10}$$

$$|\nabla\boldsymbol{u}|^2 = \left|\frac{u^r}{r}\right|^2 + \left|\frac{u^\theta}{r}\right|^2 + |\tilde{\nabla}u^r|^2 + |\tilde{\nabla}u^\theta|^2 + |\tilde{\nabla}u^z|^2. \tag{2.11}$$

Proof. Indeed, we have

$$\tilde{\boldsymbol{u}} = u^r\boldsymbol{e}_r + u^z\boldsymbol{e}_z = (u^r\cos\theta, u^r\sin\theta, u^z),$$

and thus

$$\begin{aligned}
\nabla\cdot\tilde{\boldsymbol{u}} &= \partial_1(u^r\cos\theta)+\partial_2(u^r\sin\theta)+\partial_z u^z\\
&= \left(\cos\theta\partial_r-\frac{\theta}{r}\partial_\theta\right)(u^r\cos\theta)+\left(\sin\theta\partial_r+\frac{\cos\theta}{r}\partial_\theta\right)(u^r\sin\theta)+\partial_z u^z\\
&= \partial_r u^r+\frac{u^r}{r}+\partial_z u^z=0 \qquad \text{(by (1.4))};\\
\nabla\times\tilde{\boldsymbol{u}} &= (\partial_2 u^z-\partial_3(u^r\sin\theta),\partial_3(u^r\cos\theta)-\partial_1 u^z,\partial_1(u^r\sin\theta)-\partial_2(u^r\cos\theta))\\
&= (-\sin\theta(\partial_z u^r-\partial_r u^z),\cos\theta(\partial_z u^r-\partial_r u^z),0)\\
&= (\partial_z u^r-\partial_r u^z)(-\sin\theta,\cos\theta,0)=\omega^\theta\boldsymbol{e}_\theta.
\end{aligned}$$

Consequently,

$$-\Delta\tilde{\boldsymbol{u}}=-\nabla(\nabla\cdot\tilde{\boldsymbol{u}})+\nabla\times(\nabla\times\tilde{\boldsymbol{u}})=\nabla\times(\nabla\times\tilde{\boldsymbol{u}}), \tag{2.12}$$

so $\partial_k\tilde{\boldsymbol{u}}=\mathcal{R}_k\boldsymbol{\mathcal{R}}\times(\omega^\theta\boldsymbol{e}_\theta)$ ($1\le k\le 3$, $\boldsymbol{\mathcal{R}}=(\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3)$ being the Riesz transformation). The boundedness of the Riesz transformation in L^p ($1<p<\infty$) then yields (2.9).

We now prove (2.11). First, we have

$$\begin{aligned}
\boldsymbol{u} &= u^1\boldsymbol{e}_1+u^2\boldsymbol{e}_2+u^3\boldsymbol{e}_3\\
&= u^r(t,r,z)\boldsymbol{e}_r+u^\theta(t,r,z)\boldsymbol{e}_\theta+u^z(t,r,z)\boldsymbol{e}_z\\
&= (u^r\cos\theta-u^\theta\sin\theta,u^r\sin\theta+u^\theta\cos\theta,u^z).
\end{aligned} \tag{2.13}$$

By (2.13) and (2.3), we obtain

$$\begin{aligned}
\nabla\boldsymbol{u} &= \left(\boldsymbol{e}_r\partial_r+\frac{1}{r}\boldsymbol{e}_\theta\partial_\theta+\boldsymbol{e}_z\partial_z\right)(u^r\cos\theta-u^\theta\sin\theta,u^r\sin\theta+u^\theta\cos\theta,u^z)\\
&= \begin{pmatrix} f\cos\theta+g\sin\theta & m\cos\theta+n\sin\theta & \partial_r u^z\cos\theta\\ f\sin\theta-g\cos\theta & m\sin\theta-n\cos\theta & \partial_r u^z\sin\theta\\ h & p & \partial_z u^z\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
f &= \partial_r u^r\cos\theta-\partial_r u^\theta\sin\theta, & g &= \frac{u^r\sin\theta+u^\theta\cos\theta}{r},\\
h &= \partial_z u^r\cos\theta-\partial_z u^\theta\sin\theta, & m &= \partial_r u^r\sin\theta+\partial_r u^\theta\cos\theta,\\
n &= -\frac{u^r\cos\theta-u^\theta\sin\theta}{r}, & p &= \partial_z u^r\sin\theta+\partial_z u^\theta\cos\theta.
\end{aligned}$$

Then direct computations show

$$\begin{aligned}|\nabla \boldsymbol{u}|^2 &= (f\cos\theta + g\sin\theta)^2 + (f\sin\theta - g\cos\theta)^2 + h^2\\ &\quad + (m\cos\theta + n\sin\theta)^2 + (m\sin\theta - n\cos\theta)^2 + p^2 + |\partial_r u^z|^2 + |\partial_z u^z|^2\\ &= f^2 + g^2 + h^2 + m^2 + n^2 + p^2 + |\partial_r u^z|^2 + |\partial_z u^z|^2\\ &= (f^2 + m^2) + (g^2 + n^2) + (h^2 + p^2) + |\partial_r u^z|^2 + |\partial_z u^z|^2\\ &= |\partial_r u^r|^2 + |\partial_r u^\theta|^2 + \left|\frac{u^r}{r}\right|^2 + \left|\frac{u^\theta}{r}\right|^2\\ &\quad + |\partial_z u^r|^2 + |\partial_z u^\theta|^2 + |\partial_r u^z|^2 + |\partial_z u^z|^2.\end{aligned}$$

This proves (2.11); and (2.10) can be shown in a similar fashion. ■

The next lemma concerns the a priori bound of ru^θ.

LEMMA 2.2 ([1, Proposition 1]). *Suppose $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ and $ru_0^\theta \in L^p$ for some $2 \le p \le \infty$. Then $ru^\theta \in L^\infty(0,T; L^p(\mathbb{R}^3))$.*

In our proof of Theorem 1.1, we shall invoke the following classical inequalities.

LEMMA 2.3 (Hardy–Sobolev interpolation inequality, [2, Lemma 2.4]). *For any $0 \le s < 2$, $2 \le p \le 2(3-s)$, $r = \sqrt{x_1^2 + x_2^2}$, we have*

$$\left\|\frac{f}{r^{s/p}}\right\|_{L^p} \le C\|f\|_{L^2}^{(3-s)/p-1/2}\|\nabla f\|_{L^2}^{3/2-(3-s)/p}. \tag{2.14}$$

LEMMA 2.4 (Hardy type inequality, [6, Theorem 330]). *If $1 < p < \infty$, $r \ne 1$, $f \ge 0$, and*

$$F(x) = \begin{cases}\int_0^x f(t)\,dt, & r > 1,\\ \int_x^\infty f(t)\,dt, & r < 1,\end{cases}$$

then

$$\int_0^\infty x^{-r}F^p\,dx \le \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xf)^p\,dx. \tag{2.15}$$

Proof. For completeness, we provide the proof. If $r > 1$, then integrating by parts and applying Hölder's inequality yields

$$\begin{aligned}\int_0^\infty x^{-r}F^p\,dx &= \frac{1}{1-r}\int_0^\infty F^p\,d(x^{1-r})\\ &= -\frac{1}{1-r}\int_0^\infty pF^{p-1}f\cdot x^{1-r}\,dx\\ &= \frac{p}{r-1}\int_0^\infty (x^{-r}F^p)^{(p-1)/p}\cdot[x^{-r}(xf)^p]^{1/p}\,dx\\ &\le \frac{p}{r-1}\Big(\int_0^\infty x^{-r}F^p\,dx\Big)^{(p-1)/p}\Big(\int_0^\infty (xf)^p\,dx\Big)^{1/p}.\end{aligned}$$

Absorbing the first term on the right-hand side into the left-hand side, we obtain the case $r > 1$. The case $r < 1$ can be shown in a similar way. ∎

Remark 2.5. The Hardy inequality requires the non-negativity of the function f, so that in the estimates above, the first term on the right-hand side could be absorbed into the left-hand side. Also, the definition of $F \ge 0$ (depending on the sign of $r-1$) realizes this process.

Finally, we introduce the definition of the class A_β $(1 < \beta < \infty)$ (see [18, pp. 194–217]), which was already utilized in the regularity theory of axisymmetric Navier–Stokes equations [8, 15].

Definition 2.6. Let $\beta \in (1,\infty)$. A real valued function $w(x)$ is said to be in the class A_β if it satisfies

$$\sup_{B\in\mathbb{R}^3}\Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)\cdot\Big(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,dx\Big)^{p/p'} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^3$, and p' is the Hölder conjugate of p, i.e. $1/p + 1/p' = 1$.

For $w \in A_p$, we can extend the Calderón–Zygmund inequality for singular integral operators to integrals with weight function w.

Lemma 2.7 ([18, p. 205]). *Let $\beta \in (1,\infty)$. Suppose T is a singular integral operator of convolution type, and $w \in A_\beta$. Then for $f \in L^\beta(\mathbb{R}^3)$, we have*

$$\int_{\mathbb{R}^3} |Tf(x)|^\beta w(x)\,dx \le C\int_{\mathbb{R}^3} |f(x)|^p w(x)\,dx.$$

Lemma 2.8. *Let r denote the distance of a point in $\mathbb{R}^3$ from the z-axis. Then $r^s \in A_\beta$ if $-2 < s < 2(\beta-1)$.*

Proof. Argue as in the proof of [1, Lemma 1]. ∎

An immediate consequence of Lemmas 2.1 and 2.8 is

LEMMA 2.9. *For any $1 < \beta < \infty$ and $-2/\beta < d < 2(\beta-1)/\beta$,*

$$\left\| r^d \left(\frac{|u^r|}{r} + |\tilde{\nabla} u^r| + |\tilde{\nabla} u^z| \right) \right\|_{L^\beta} \le C \| r^d \omega^\theta \|_{L^\beta}. \tag{2.16}$$

Proof. We have

$$\begin{aligned}
\int_{\mathbb{R}^3} \left(\left| \frac{u^r}{r} \right|^\beta + |\tilde{\nabla} u^r|^\beta + |\tilde{\nabla} u^z|^\beta \right) \cdot r^{d\beta}\, dx \\
&\le C \int_{\mathbb{R}^3} |\nabla \tilde{\boldsymbol{u}}|^\beta \cdot r^{d\beta}\, dx \quad \text{(by (2.10))} \\
&\le C \int_{\mathbb{R}^3} |\boldsymbol{\mathcal{R}}[\boldsymbol{\mathcal{R}} \times (\boldsymbol{\omega}^\theta \boldsymbol{e}_\theta)]|^\beta \cdot r^{d\beta}\, dx \quad \text{(by (2.12))} \\
&\le C \int_{\mathbb{R}^3} |\boldsymbol{\omega}^\theta \boldsymbol{e}_\theta|^\beta \cdot r^{d\beta}\, dx \quad \text{(by Lemmas 2.7 and 2.8)} \\
&\le C \int_{\mathbb{R}^3} |\boldsymbol{\omega}^\theta|^\beta \cdot r^{d\beta}\, dx. \ \blacksquare
\end{aligned}$$

3. A preliminary regularity criterion involving u^θ/r. The following proposition is precisely stated in [2, Lemma 2.5], which comes from the calculations in [8, 14, 20]. For the readers' convenience, we provide a formal proof. For a complete proof, one may argue as in [12, 20], where $(1.9)_2$ is multiplied with $\omega^\theta/r^{2-\varepsilon}$, and then after estimations, one lets $\varepsilon \to 0^+$.

PROPOSITION 3.1. *Let $\boldsymbol{u}_0 \in H^2(\mathbb{R}^3)$ be axially symmetric and divergence-free, and $\boldsymbol{u} \in C([0,T); H^2(\mathbb{R}^3)) \cap L^2(0,T; H^3(\mathbb{R}^3))$ be the unique axisymmetric classical solution of* (1.4). *If*

$$u^\theta/r \in L^4(0,T; L^4(\mathbb{R}^3)), \tag{3.1}$$

then the solution can be smoothly extended beyond T.

Proof. First, taking the inner product of $(1.9)_2$ and ω^θ/r^2 in $L^2(\mathbb{R}^3)$ formally, we find

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left\| \frac{\omega^\theta}{r} \right\|_{L^2}^2 + \left\| \tilde{\nabla} \left(\frac{\omega^\theta}{r} \right) \right\|_{L^2}^2 \\
&= 2 \int_{\mathbb{R}^3} \frac{u^\theta \partial_z u^\theta}{r} \cdot \frac{\omega^\theta}{r^2}\, dx = \int_{\mathbb{R}^3} \partial_z \left(\left| \frac{u^\theta}{r} \right|^2 \right) \cdot \frac{\omega^\theta}{r}\, dx = -\int_{\mathbb{R}^3} \left| \frac{u^\theta}{r} \right|^2 \cdot \partial_z \left(\frac{\omega^\theta}{r} \right) dx \\
&\le \frac{1}{2} \int_{\mathbb{R}^3} \left| \frac{u^\theta}{r} \right|^4 dx + \frac{1}{2} \left\| \partial_z \left(\frac{\omega^\theta}{r} \right) \right\|_{L^2}^2.
\end{aligned} \tag{3.2}$$

Integrating in time, we obtain

$$\left\|\frac{\omega^\theta}{r}\right\|_{L^\infty(0,T;L^2)} + \left\|\tilde{\nabla}\left(\frac{\omega^\theta}{r}\right)\right\|_{L^2(0,T;L^2)} \le C. \tag{3.3}$$

Then we bound ω^θ. Multiplying $(1.9)_2$ by ω^θ, and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega^\theta\|_{L^2}^2 + \|\tilde{\nabla}\omega^\theta\|_{L^2}^2 + \left\|\frac{\omega^\theta}{r}\right\|_{L^2}^2 &= \int_{\mathbb{R}^3} \frac{u^r}{r}|\omega^\theta|^2\,dx + 2\int_{\mathbb{R}^3} \frac{u^\theta\partial_z u^\theta}{r}\omega^\theta\,dx \\ &\equiv I_1 + I_2.\end{aligned} \tag{3.4}$$

For I_1, we have

$$\begin{aligned} I_1 &\le \|u^r\|_{L^\infty}\left\|\frac{\omega^\theta}{r}\right\|_{L^2}\|\omega^\theta\|_{L^2} && \text{(by the Hölder inequality)} \\ &\le C\|\tilde{\boldsymbol{u}}\|_{L^\infty}\|\omega^\theta\|_{L^2} && \text{(by (3.3))} \\ &\le C\|\tilde{\boldsymbol{u}}\|_{L^6}^{1/2}\|\nabla\tilde{\boldsymbol{u}}\|_{L^6}^{1/2}\cdot\|\omega^\theta\|_{L^2} && \text{(by the Gagliardo–Nirenberg inequality)} \\ &\le C\|\nabla\tilde{\boldsymbol{u}}\|_{L^2}^{1/2}\|\nabla^2\tilde{\boldsymbol{u}}\|_{L^2}^{1/2}\|\omega^\theta\|_{L^2} && \text{(by the Sobolev inequality).}\end{aligned}$$

By invoking (2.12) and its consequence

$$\begin{aligned}\|\partial_k\partial_l\tilde{\boldsymbol{u}}\|_{L^p} &= \|\mathcal{R}_k\boldsymbol{\mathcal{R}}\times[\partial_l(\omega^\theta\boldsymbol{e}_\theta)]\|_{L^p} \le C\|\nabla(\omega^\theta\boldsymbol{e}_\theta)\|_{L^p} \\ &\le C\|\nabla\omega^\theta\cdot\boldsymbol{e}_\theta\|_{L^p} + C\|\omega^\theta\nabla\boldsymbol{e}_\theta\|_{L^p} \\ &\le C\|\nabla\omega^\theta\|_{L^p} + C\left\|\frac{\omega^\theta}{r}\right\|_{L^p} \quad (|\nabla\boldsymbol{e}_\theta| = 1/r) \\ &\le C\|\nabla\omega^\theta\|_{L^p} \quad \text{(by (2.11), for } 1\le k,l\le 3,\ 1<p<\infty),\end{aligned} \tag{3.5}$$

it follows that

$$I_1 \le C\|\omega^\theta\|_{L^2}^{3/2}\|\nabla\omega^\theta\|_{L^2}^{1/2} \le C\|\omega^\theta\|_{L^2}^2 + \tfrac{1}{4}\|\nabla\omega^\theta\|_{L^2}^2. \tag{3.6}$$

By integration by parts and the Cauchy–Schwarz inequality,

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^3}\partial_z\frac{|u^\theta|^2}{r}\cdot\omega^\theta\,dx = -\int_{\mathbb{R}^3}\frac{|u^\theta|^2}{r}\cdot\partial_z\omega^\theta\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^3}\frac{|u^\theta|^4}{r^2}\,dx + \frac{1}{2}\int_{\mathbb{R}^3}|\partial_z\omega^\theta|^2\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^3}\left(\frac{|u^\theta|^4}{r^4}\right)^{3/4}|ru^\theta|\,dx + \frac{1}{2}\|\partial_z\omega^\theta\|_{L^2}^2 \\ &\le \frac{1}{2}\int_{\mathbb{R}^3}\left[\frac{3}{4}\cdot\frac{|u^\theta|^4}{r^4} + \frac{1}{4}\cdot|ru^\theta|^4\right]dx + \frac{1}{2}\|\partial_z\omega^\theta\|_{L^2}^2.\end{aligned} \tag{3.7}$$

Gathering (3.6)–(3.7) into (3.4), and applying the Gronwall inequality, we find by (3.1) and Lemma 2.2 that

$$\|\omega^\theta\|_{L^\infty(0,T;L^2)} + \|\nabla\omega^\theta\|_{L^2(0,T;L^2)} \le C. \tag{3.8}$$

Finally, multiplying $(1.9)_1$ by ω^r, and $(1.9)_3$ by ω^z, and adding the results, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\omega^r,\omega^z)\|_{L^2}^2 + \|\tilde\nabla(\omega^r,\omega^z)\|_{L^2}^2 + \left\|\frac{\omega^r}{r}\right\|_{L^2}^2 \qquad (3.9)\\
&= \int_{\mathbb{R}^3} [(\omega^r\partial_r u^r + \omega^z\partial_z u^r)\omega^r + (\omega^r\partial_r u^z + \omega^z\partial_z u^r)\omega^z]\,dx\\
&\le \int_{\mathbb{R}^3} |\nabla\tilde{\boldsymbol{u}}|\cdot|(\omega^r,\omega^z)|^2\,dx \quad \text{(by (2.10))}\\
&\le \|\nabla\tilde{\boldsymbol{u}}\|_{L^2}\|(\omega^r,\omega^z)\|_{L^4}^2 \quad \text{(by the Hölder inequality)}\\
&\le C\|\omega^\theta\|_{L^2}\cdot\|(\omega^r,\omega^z)\|_{L^2}^{1/2}\|\nabla(\omega^r,\omega^z)\|_{L^2}^{3/2}\\
&\qquad\qquad \text{(by Lemma 2.1 and the Gagliardo–Nirenberg inequality)}\\
&\le C\|\omega^\theta\|_{L^2}^4\|(\omega^r,\omega^z)\|_{L^2}^2 + \tfrac{1}{2}\|\nabla(\omega^r,\omega^z)\|_{L^2}^2 \quad \text{(by the Young inequality)}\\
&\le C\|(\omega^r,\omega^z)\|_{L^2}^2 + \tfrac{1}{2}\|\nabla(\omega^r,\omega^z)\|_{L^2}^2 \quad \text{(by (3.8))}.
\end{aligned}$$

Applying the Gronwall inequality, we deduce that

$$\|(\omega^r,\omega^z)\|_{L^\infty(0,T;L^2)} \le C.$$

Hence by (3.8), we see that $\|\boldsymbol{u}\|_{L^\infty(0,T;H^1)}$ is uniformly bounded in $[0,T)$. Standard higher-order energy estimates imply that $\|\boldsymbol{u}\|_{L^\infty(0,T;H^2)}$ is also uniformly bounded in $[0,T)$. This completes the proof of Proposition 3.1 (for details, one can refer to [21, pp. 3–4] for example). ∎

4. Proof of Theorem 1.1 under condition (1.16). In this section, we prove the conclusion of Theorem 1.1 under condition (1.16). We remark that our proof is more elegant and elementary than in [9].

By Proposition 3.1, we only need to bound u^θ/r in the $L^4(0,T;L^4(\mathbb{R}^3))$ norm. To this end, we multiply $(1.4)_2$ by $(u^\theta)^3/r^2$ and integrate over $\mathbb{R}^3$ to get

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\left\|\frac{|u^\theta|^2}{r}\right\|_{L^2}^2 + \frac{3}{4}\left\|\tilde\nabla\frac{|u^\theta|^2}{r}\right\|_{L^2}^2 + \frac{3}{4}\int_{\mathbb{R}^3}\left|\frac{u^\theta}{r}\right|^4 dx \qquad (4.1)\\
&\qquad = -\frac{3}{2}\int_{\mathbb{R}^3}\frac{u^r}{r}\left(\frac{|u^\theta|^2}{r}\right)^2 dx \equiv J.
\end{aligned}$$

We now estimate

$$(4.2)\qquad J = -\frac{3}{2}\int_{\mathbb{R}^3} r^d u^r \cdot \left(\frac{|u^\theta|^2/r}{r^{(1+d)/2}}\right)^2 dx$$

$$\le \frac{3}{2}\|r^d u^r\|_{L^\beta}\left\|\frac{|u^\theta|^2/r}{r^{(1+d)/2}}\right\|_{L^p}^2 \quad \left(\text{by the Hölder inequality with } \frac{1}{\beta}+\frac{2}{p}=1\right)$$

$$\le C\|r^d u^r\|_{L^\beta}\left\|\frac{|u^\theta|^2}{r}\right\|_{L^2}^{2(1-\theta)}\left\|\nabla\frac{|u^\theta|^2}{r}\right\|_{L^2}^{2\theta}$$

$$\left(\text{by Lemma 2.3 with } s=\frac{1+d}{2}p\in[0,2),\ \theta=\frac{3}{2}-\frac{3-s}{p}\in[0,1)\right)$$

$$\le C\|r^d u^\theta\|_{L^\beta}^{1/(1-\theta)}\left\|\frac{|u^\theta|^2}{r}\right\|_{L^2}^2+\frac{1}{2}\left\|\tilde{\nabla}\frac{|u^\theta|^2}{r}\right\|_{L^2}^2.$$

Direct computations show that

$$\frac{2}{1/(1-\theta)}+\frac{3}{\beta}=1-d.$$

Plugging (4.2) into (4.1), we may apply the Gronwall inequality to deduce that

$$\|u^\theta/r\|_{L^4(0,T;L^4)}\le C,$$

as desired.

5. Proof of Theorem 1.1 under condition (1.17). In this section, we prove the conclusion of Theorem 1.1 under condition (1.17).

We argue in a more flexible way than in [2]. By the Newton–Leibniz formula, we see by (1.7) that

$$ru^\theta(t,r,z)=\int_0^r \partial_s(su^\theta(t,s,z))\,dr=\int_0^r s\omega^z(t,s,z)\,ds,$$

and hence by Lemma 2.4,

$$\begin{aligned}\int_0^\infty |r^{d-1}u^\theta|^\beta\cdot r\,dr &= \int_0^\infty r^{-[(2-d)\beta-1]}|ru^\theta|^\beta\,dr\\ &\le \int_0^\infty r^{-[(2-d)\beta-1]}\left[\int_0^r |r\omega^z|\,ds\right]^\beta dr\\ &\le \left[\frac{\beta}{(2-d)\beta-2}\right]^\beta\int_0^\infty r^{-[(2-d)\beta-1]}(r\cdot|r\omega^z|)^\beta\,dr\\ &= \left[\frac{\beta}{(2-d)\beta-2}\right]^\beta\int_0^\infty |r^d\omega^z|^\beta\cdot r\,dr.\end{aligned}$$

Integrating in z and t, we find

$$\|r^{d-1}u^\theta\|_{L^\alpha(0,T;L^\beta)} \le C\|r^d\omega^z\|_{L^\alpha(0,T;L^\beta)},$$

and by the interpolation inequality and Lemma 2.2,

$$\begin{aligned}\|r^{d/2}u^\theta\|_{L^{2\alpha}(0,T;L^{2\beta})} &= \big\||r^{d-1}u^\theta|^{1/2}\cdot|ru^\theta|^{1/2}\big\|_{L^{2\alpha}(0,T;L^{2\beta})}\\ &\le \|r^{d-1}u^\theta\|^{1/2}_{L^\alpha(0,T;L^\beta)}\|ru^\theta\|^{1/2}_{L^\infty(0,T;L^\infty)}\\ &\le C\|r^d\omega^z\|^{1/2}_{L^\alpha(0,T;L^\beta)}.\end{aligned}$$

Invoking the regularity criterion (1.10)–(1.11), we complete the proof of Theorem 1.1 under condition (1.17).

6. Proof of Theorem 1.1 under condition (1.18). In this section, we prove the conclusion of Theorem 1.1 under condition (1.18). Taking the inner product of (1.9) with $\boldsymbol{\omega}$ in $L^2(\mathbb{R}^3)$ we find, by observing

$$\begin{aligned}(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} &= \Big(\omega^r\partial_r - \frac{1}{r}\omega^\theta\partial_\theta + \omega^z\partial_z\Big)(u^r\boldsymbol{e}_r + u^\theta\boldsymbol{e}_\theta + u^z\boldsymbol{e}_z) \quad \text{(by (2.3))}\\ &= \omega^r\partial_r u^r\boldsymbol{e}_r + \omega^r\partial_r u^\theta\boldsymbol{e}_\theta + \omega^r\partial_r u^z\boldsymbol{e}_z\\ &\quad - \frac{u^r}{r}\omega^\theta\boldsymbol{e}_\theta + \frac{u^\theta}{r}\omega^\theta\boldsymbol{e}_r\\ &\quad + \omega^z\partial_z u^r\boldsymbol{e}_r + \omega^z\partial_z u^\theta\boldsymbol{e}_\theta + \omega^z\partial_z u^z\boldsymbol{e}_z \quad \text{(by the Leibniz rule and (2.6))}\\ &= \Big(\omega^r\partial_r u^r + \frac{u^\theta}{r}\omega^\theta + \omega^z\partial_z u^r\Big)\boldsymbol{e}_r + \Big(\omega^r\partial_r u^\theta - \frac{u^r}{r}\omega^\theta + \omega^z\partial_z u^\theta\Big)\boldsymbol{e}_\theta\\ &\quad + (\omega^r\partial_r u^z + \omega^z\partial_z u^z)\boldsymbol{e}_z\end{aligned}$$

and its consequence

$$\begin{aligned}[(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{\omega} &= \Big(\omega^r\partial_r u^r + \frac{u^\theta}{r}\omega^\theta + \omega^z\partial_z u^r\Big)\omega^r\\ &\quad + \Big(\omega^r\partial_r u^\theta - \frac{u^r}{r}\omega^\theta + \omega^z\partial_z u^\theta\Big)\omega^\theta + (\omega^r\partial_r u^z + \omega^z\partial_z u^z)\omega^z,\end{aligned}$$

that

$$\begin{aligned}(6.1)\qquad &\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\omega}\|^2_{L^2} + \|\nabla\boldsymbol{\omega}\|^2_{L^2}\\ &= \int_{\mathbb{R}^3}\Big(\omega^r\partial_r u^r\omega^r + \omega^r\partial_r u^\theta\omega^\theta + \omega^r\partial_r u^z\omega^z - \frac{u^r}{r}\omega^\theta\omega^\theta\\ &\qquad + \frac{u^\theta}{r}\omega^\theta\omega^r + \omega^z\partial_z u^r\omega^r + \omega^z\partial_z u^\theta\omega^\theta + \omega^z\partial_z u^z\omega^z\Big)\,dx\\ &\equiv \sum_{i=1}^{8} K_i.\end{aligned}$$

For K_1, we have

$$
\begin{aligned}
(6.2)\qquad K_1 &= \int_{\mathbb{R}^3} r^d \partial_r u^r \cdot \left|\frac{\omega^r}{r^{d/2}}\right|^2 dx \\
&\le \|r^d \partial_r u^r\|_{L^\beta} \left\|\frac{\omega^r}{r^{d/2}}\right\|_{L^p}^2 \quad \left(\frac{1}{\beta}+\frac{2}{p}=1\right) \\
&\le C\|r^d\omega^\theta\|_{L^\beta}\|\omega^r\|_{L^2}^{1-\theta}\|\nabla\omega^r\|_{L^2}^{\theta} \\
&\quad \left(\text{by Lemmas 2.9, 2.3 with } s=\frac{d}{2}p\in[0,2),\ \theta=\frac{3}{2}-\frac{3-s}{p}\in[0,1)\right) \\
&\le C\|r^d\omega^\theta\|_{L^\beta}^{1/(1-\theta)}\|\boldsymbol{\omega}\|_{L^2}^2+\tfrac{1}{16}\|\nabla\boldsymbol{\omega}\|_{L^2}^2.
\end{aligned}
$$

The terms $K_3,\ K_6,\ K_8$ can be bounded similarly.

For K_2, we have

$$
\begin{aligned}
(6.3)\qquad K_2 &\le \|r^d\omega^\theta\|_{L^\beta}\left\|\frac{\omega^r}{r^{d/2}}\right\|_{L^p}\left\|\frac{\partial_r u^\theta}{r^{d/2}}\right\|_{L^p} \\
&\le C\|r^d\omega^\theta\|_{L^\beta}\|\omega^\theta\|_{L^2}^{1-\theta}\|\nabla\omega^r\|_{L^2}^{\theta}\cdot\|\partial_r u^\theta\|_{L^2}^{1-\theta}\|\nabla\partial_r u^\theta\|_{L^2}^{\theta} \\
&\le C\|r^d\omega^\theta\|_{L^\beta}\|\boldsymbol{\omega}\|_{L^2}^{2(1-\theta)}\|\nabla\boldsymbol{\omega}\|_{L^2}^{2\theta} \quad (\text{by } (2.11)) \\
&\le C\|r^d\omega^\theta\|_{L^\beta}^{1/(1-\theta)}\|\boldsymbol{\omega}\|_{L^2}^2+\tfrac{1}{16}\|\nabla\boldsymbol{\omega}\|_{L^2}^2.
\end{aligned}
$$

The terms $K_4,\ K_5,\ K_7$ can be dominated in the same fashion.

Gathering all the above estimates into (6.1), we find

$$\frac{d}{dt}\|\boldsymbol{\omega}\|_{L^2}^2+\|\nabla\boldsymbol{\omega}\|_{L^2}^2\le C\|r^d\omega^\theta\|_{L^\beta}^{1/(1-\theta)}\|\boldsymbol{\omega}\|_{L^2}^2,$$

where θ can be calculated from (6.2) so that

$$\frac{2}{1/(1-\theta)}+\frac{3}{\beta}=2-d.$$

Applying the Gronwall inequality, we deduce that $\|\boldsymbol{\omega}(t)\|_{L^2}$ is uniformly bounded in $[0,T]$. Arguing as in the proof of Proposition 3.1, we complete the proof of Theorem 1.1 under condition (1.18).

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