COLLOQUIUM MATHEMATICUM

VOL. 145 2016 NO. 2

APPROXIMATE BIPROJECTIVITY AND ϕ-BIFLATNESS OF CERTAIN BANACH ALGEBRAS

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Abstract. In the first part of the paper, we investigate the approximate biprojectivity of some Banach algebras related to the locally compact groups. We show that a Segal algebra $S(G)$ is approximate biprojective if and only if G is compact. Also for every continuous weight w, we show that $L^1(G,w)$ is approximate biprojective if and only if G is compact, provided that $w(g) \geq 1$ for every $g \in G$.

In the second part, we study ϕ-biflatness of some Banach algebras, where ϕ is a character. We show that if $S(G)$ is ϕ_0-biflat, then G is an amenable group, where ϕ_0 is the augmentation character on $S(G)$. Finally, we show that the ϕ-biflatness of $L^1(G)^{**}$ implies the amenability of G.

1. Introduction and preliminaries. B. E. Johnson [J] defined the class of amenable Banach algebras and showed that $L^1(G)$ is an amenable Banach algebra if and only if G is an amenable group. At about the same time A. Ya. Helemskii studied the class of biflat and biprojective Banach algebras. Like amenability, he showed that $L^1(G)$ is biprojective (biflat) if and only if G is a compact (amenable) group, respectively (see [H, Theorem IV.5.13]).

The present authors [SP1] have studied some generalization of Helemskii's theory. The concepts of ϕ-biflatness, ϕ-biprojectivity, ϕ-Johnson amenability were introduced and studied. It was shown that $L^1(G)$ is ϕ-biflat if and only if G is an amenable group, and the Fourier algebra $\mathcal{A}(G)$ is ϕ-biprojective if and only if G is a discrete group.

Other generalized notions of Helemskii's theory are approximate biprojectivity and approximate biflatness. These generalizations have been introduced by Zhang [Z] and Samei et al. [SSS], respectively. Samei et al. [SSS] studied the approximate biflatness of Segal algebras and Fourier algebras, and showed that a Segal algebra $S(G)$ is pseudo contractible if and only if G is compact. Note that the pseudo contractibility of Banach algebras implies

2010 *Mathematics Subject Classification*: Primary 43A07, 43A20; Secondary 46H05.
Key words and phrases: Beurling algebras, Segal algebras, approximate biprojectivity, ϕ-biflat.
Received 1 December 2014; revised 3 November 2015.
Published online 8 July 2016.

DOI: 10.4064/cm6459-11-2015

their approximate biprojectivity [GZ, Proposition 3.8] (for more details see [GZ]).

In this paper we show that a Segal algebra $S(G)$ is approximately biprojective if and only if G is compact, which is an extension of [SSS, Theorem 3.5] or [CGZ, Theorem 5.3].

Next, we show that the weighted group algebra $L^1(G, w)$ is approximately biprojective if and only if G is compact, for every continuous weight w on G with $w(g) \geq 1$ for every $g \in G$. This is an extension of [H, Theorem IV.5.13]. Finally, we show that if a Segal algebra $S(G)$ is ϕ_0-biflat, then G is amenable, where ϕ_0 is the augmentation character on $L^1(G)$, and if $L^1(G)^{**}$ is $\tilde{\phi}$-biflat, then G is amenable, where $\tilde{\phi}$ is an extension of a character ϕ on $L^1(G)$.

We recall some standard notation and definitions. Let A be a Banach algebra. If X is a Banach A-bimodule, then X^* is also a Banach A-bimodule via the following actions:

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in A,\, x \in X,\, f \in X^*).$$

Throughout, $\Delta(A)$ denotes the character space of A, that is, the set of all non-zero multiplicative linear functionals on A. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension $\tilde{\phi} \in \Delta(A^{**})$ defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$.

Let A and B be Banach algebras. The ℓ^1-direct sum $A \oplus_1 B$ is a Banach algebra with the usual product and with the norm $||(a, b)|| = ||a|| + ||b||$. It is easy to see that

$$\Delta(A \oplus_1 B) = (\Delta(A) \times \{0\}) \cup (\{0\} \times \Delta(B)).$$

Let A and B be Banach algebras. The projective tensor product $A \otimes_p B$ is a Banach A-bimodule via the following actions:

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

We recall that $\Delta(A \otimes_p B) = \{\phi \otimes \psi \mid \phi \in \Delta(A), \psi \in \Delta(B)\}$, where $\phi \otimes \psi(a \otimes b) = \phi(a)\psi(b)$ for all $a \in A$ and $b \in B$. The product morphism $\pi_A : A \otimes_p A \to A$ is specified by $\pi_A(a \otimes b) = ab$ for $a, b \in A$.

Let G be a locally compact group. The Fourier algebra on G is denoted by $\mathcal{A}(G)$. It is well-known that the character space $\Delta(\mathcal{A}(G))$ consists of all point evaluation maps $\phi_t : \mathcal{A}(G) \to \mathbb{C}$ such that $\phi_t(f) = f(t)$ for each $f \in \mathcal{A}(G)$ (see [E]).

We also recall some concepts of Banach homology. A Banach algebra A is called *biprojective* if there exists a bounded A-bimodule morphism $\rho : A \to A \otimes_p A$ such that ρ is a right inverse for π_A [H]. Moreover, A is *approximately biprojective* if there exists a net of bounded A-bimodule morphisms $\rho_\alpha : A \to A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) \to a$ for each $a \in A$ (see [Z]). A Banach algebra A is called *ϕ-biflat*, for $\phi \in \Delta(A)$, if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$

for every $a \in A$ [SP1]. Also, A is called *left ϕ-amenable* [*left ϕ-contractible*] if there exists $m \in A^{**}$ [$m \in A$] such that $am = \phi(a)m$ and $\tilde{\phi}(m) = 1$ [$\phi(m) = 1$] for every $a \in A$. For more details on left ϕ-amenability and left ϕ-contractibility see [KLP] and [NS], respectively.

The following theorems come from [SP2]. They characterize the approximate biprojectivity of some semigroup algebras. We apply these theorems in order to characterize the approximate biprojectivity of algebras related to locally compact groups.

THEOREM 1.1 ([SP2]). *Let A be an approximately biprojective Banach algebra with a left approximate identity* [*right approximate identity*] *and let $\phi \in \Delta(A)$. Then A is left ϕ-contractible* [*right ϕ-contractible*].

THEOREM 1.2 ([SP2]). *Let A be a Banach algebra with a left approximate identity and let $\Delta(A)$ be a non-empty set. Then the triangular Banach algebra $T = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ is not approximately biprojective.*

2. Approximate biprojectivity. We recall that, for a locally compact group G, a linear subspace $S(G)$ of $L^1(G)$ is said to be a *Segal algebra* on G if it satisfies the following conditions:

(i) $S(G)$ is dense in $L^1(G)$,
(ii) $S(G)$ with a norm $\|\cdot\|_{S(G)}$ is a Banach space and $\|f\|_{L^1(G)} \leq \|f\|_{S(G)}$ for every $f \in S(G)$,
(iii) for every $f \in S(G)$ and $y \in G$ we have $L_y f \in S(G)$ and the map $y \mapsto L_y f$ of G into $S(G)$ is continuous, where $L_y f(x) = f(y^{-1}x)$,
(iv) $\|L_y f\|_{S(G)} = \|f\|_{S(G)}$ for every $f \in S(G)$ and $y \in G$.

It is well-known that $S(G)$ has a left approximate identity. Also, every Segal algebra is an abstract Segal algebra with respect to $L^1(G)$. For more information on Segal algebras see [Re].

Note that $\Delta(S(G)) = \{\phi_{|S(G)} \mid \phi \in \Delta(L^1(G))\}$ and ϕ_0 (the augmentation character on $L^1(G)$) induces a character on $S(G)$ still denoted by ϕ_0 [ANN, Lemma 2.2].

Samei et al. [SSS, Theorem 3.5] and Choi et al. [CGZ, Theorem 5.3] showed that $S(G)$ is pseudo contractible if and only if G is compact. As pseudo contractibility is a weaker condition than approximate biprojectivity, in the following theorem we extend this result.

THEOREM 2.1. *Let G be a locally compact group. Then $S(G)$ is approximately biprojective if and only if G is compact.*

Proof. Let $S(G)$ be approximately biprojective. Since $S(G)$ has a left approximate identity, Theorem 1.1 shows that $S(G)$ is left ϕ_0-contractible, hence by [NS, Theorem 2.1] there exists $m \in S(G)$ such that $am = \phi_0(a)m$ and $\phi_0(m) = 1$ for every $a \in S(G)$. Since $S(G)$ is dense in $L^1(G)$, it is easy to see that $am = \phi_0(a)m$ and $\phi_0(m) = 1$ for every $a \in L^1(G)$. Using the same argument as in the proof of [H, Theorem IV.5.13], we find that m is a constant function, which shows that G is compact.

The converse is clear by [SSS, Theorem 3.5] or [CGZ, Theorem 5.3]. ∎

The class of non-approximately biprojective Banach algebras is wide enough among the algebras related to locally compact groups. Here we give another class of non-approximately biprojective Banach algebras.

Proposition 2.2. *The triangular Banach algebra*

$$T = \begin{pmatrix} S(G) & S(G) \\ 0 & S(G) \end{pmatrix}$$

is not approximately biprojective for any Segal algebra $S(G)$.

Proof. Since $S(G)$ has a left approximate identity, T has a left approximate identity. As $\Delta(S(G)) \neq \emptyset$, the use of Theorem 1.2 finishes the proof. ∎

Theorem 2.3. *Let G be a SIN group. If $S(G) \otimes_p S(G)$ is approximately biprojective, then G is compact.*

Proof. The main result of [KR] asserts that if G is a SIN group, then $S(G)$ has a central approximate identity, say $(e_\alpha)_{\alpha \in I}$. Since $S(G) \otimes_p S(G)$ is approximately biprojective, there exists a net

$$\rho_\beta : S(G) \otimes_p S(G) \to (S(G) \otimes_p S(G)) \otimes_p (S(G) \otimes_p S(G)), \qquad \beta \in \Theta,$$

of continuous $S(G) \otimes_p S(G)$-bimodule morphisms with $\pi_{S(G)\otimes_p S(G)} \circ \rho_\beta(x) \to x$ for every $x \in S(G) \otimes_p S(G)$. Set $n_\alpha = e_\alpha \otimes e_\alpha$. It is easy to see that for every $x \in S(G) \otimes_p S(G)$ we have $x n_\alpha = n_\alpha x$ and $\phi \otimes \phi(n_\alpha) = \phi \otimes \phi(e_\alpha \otimes e_\alpha) = \phi(e_\alpha)\phi(e_\alpha) \to 1$, where $\phi \in \Delta(S(G))$.

Define $m_\alpha^\beta = \rho_\beta(n_\alpha)$. Then it is easy to see that $x \cdot m_\alpha^\beta = m_\alpha^\beta \cdot x$. Also,

$$\begin{aligned} (2.1) \qquad & \lim_\alpha \lim_\beta \phi \otimes \phi \circ \pi_{S(G)\otimes_p S(G)}(m_\alpha^\beta) - 1 \\ &= \lim_\alpha \lim_\beta \phi \otimes \phi \circ \pi_{S(G)\otimes_p S(G)} \circ \rho_\beta(n_\alpha) - 1 \\ &= \lim_\alpha \phi \otimes \phi(n_\alpha) - 1 = \lim_\alpha \phi(e_\alpha)^2 - 1 = 0. \end{aligned}$$

Set $E = I \times \Theta^I$, where Θ^I is the set of all functions from I into Θ. Consider the product ordering on E, defined as follows:

$$(\alpha, \beta) \leq_E (\alpha', \beta') \Leftrightarrow \alpha \leq_I \alpha', \, \beta \leq_{\Theta^I} \beta' \quad (\alpha, \alpha' \in I, \, \beta, \beta' \in \Theta^I);$$

here $\beta \leq_{\Theta^I} \beta'$ means that $\beta(d) \leq_\Theta \beta'(d)$ for each $d \in I$. Suppose that $\gamma = (\alpha, \beta_\alpha) \in E$ and $m_\gamma = \rho_{\beta_\alpha}(n_\alpha) \in (S(G) \otimes_p S(G)) \otimes_p (S(G) \otimes_p S(G))$. Now using the iterated limit theorem [K, p. 69] in (2.1) we obtain

$$\phi \otimes \phi \circ \pi_{S(G)\otimes_p S(G)}(m_\gamma) \to 1,$$

and similarly $x \cdot m_\gamma = m_\gamma \cdot x$ for every $x \in S(G) \otimes_p S(G)$. By the same argument as in the proof of [SP1, Proposition 2.2] one can show that $S(G)\otimes_p S(G)$ is left $\phi \otimes \phi$-contractible. Hence [NS, Theorem 3.14] shows that $S(G)$ is left ϕ-contractible, and [ANN, Theorem 3.3] implies that G is compact. ∎

Let G be a locally compact group. A *weight* on G is a function $w : G \to \mathbb{R}^+$ such that

$$w(e) = 1 \quad \text{and} \quad w(xy) \le w(x)w(y),$$

where $e \in G$ is the identity and $x, y \in G$. We form the Banach space

$$L^1(G, w) = \{f : G \to \mathbb{C} \mid fw \in L^1(G)\}.$$

Then $L^1(G, w)$, with convolution product, is a Banach algebra, called a *Beurling algebra*. See [DL] for further information on Beurling algebras.

Helemskii [H, Theorem IV.5.13] showed that the group algebra $L^1(G)$ is biprojective if and only if G is compact. In the following theorem we extend this result.

Theorem 2.4. *Let G be a locally compact group and let w be a continuous weight on G. Then $L^1(G, w)$ is approximately biprojective if and only if G is compact, provided that $w(g) \ge 1$ for every $g \in G$.*

Proof. Suppose $L^1(G, w)$ is approximately biprojective. Then by Theorem 1.1, $L^1(G, w)$ is left ϕ-contractible for every $\phi \in \Delta(L^1(G, w))$, in particular for the augmentation character ϕ_0 specified by

$$\phi_0(f) = \int_G f(x)\,dx.$$

By [NS, Theorem 2.1] there exists $m \in L^1(G, w)$ such that $a * m = \phi_0(a)m$ and $\phi_0(m) = 1$ for every $a \in L^1(G, w)$. Pick $f \in L^1(G, w)$ such that $\phi_0(f) = 1$. We have

$$\delta_g * m = \phi_0(f)\delta_g * m = \delta_g * (f * m) = (\delta_g * f) * m = \phi_0(\delta_g * f)m = \phi_0(f)m = m,$$

which shows that m is a constant function in $L^1(G, w)$, so we can assume that $1 \in L^1(G, w)$. Since $w(g) \ge 1$ for every $g \in G$, we have

$$|G| = \int_G 1\,dg \le \int_G w(g)\,dg < \infty.$$

Now apply [HS, Theorem 15.9] to deduce that G is compact.

For the converse, using the same arguments as in [H, Theorem IV.5.13], it is easy to see that $L^1(G, w)$ is biprojective, so $L^1(G, w)$ is approximately biprojective. ∎

Proposition 2.5. *Let G be a locally compact group and let A be a unital Banach algebra with $\Delta(A) \ne \emptyset$. If $A \otimes_p L^1(G)$ is approximately biprojective, then G is compact and A is approximately biprojective. The converse holds if A is biprojective.*

Proof. Suppose that $B = A \otimes_p L^1(G)$ is approximately biprojective. It is easy to see that $(e_A \otimes e_\alpha)$ is an approximate identity for B, where e_A is an identity for A, and (e_α) is a bounded approximate identity for $L^1(G)$. Let $\psi \in \Delta(A)$ and $\phi \in \Delta(L^1(G))$. Then Theorem 1.1 implies that B is left $\psi \otimes \phi$-contractible. By [NS, Theorem 3.14], $L^1(G)$ is left ϕ-contractible, which implies that G is compact (see [NS, Theorem 6.1]).

Let $\rho : G \to \mathbb{C}$ be a group character corresponding to ϕ (see [HS, Theorem 23.7]). It is easy to see that $\rho \in L^\infty(G)$. Since G is compact, $L^\infty(G) \subseteq L^1(G)$. Thus $\rho \in L^1(G)$. Also, since $\rho * f = f * \rho = \phi(f)\rho$ for every $f \in L^1(G)$, one can easily see that ρ is idempotent in $L^1(G)$. Now by a similar argument to that in [Ra, Proposition 2.6], one finds that A is approximately biprojective.

Conversely, it is well-known that $L^1(G)$ is biprojective if and only if G is compact. Now apply [Ra, Proposition 2.4] to complete the proof. ∎

We recall that a Banach algebra A is *left character contractible* if A is left ϕ-contractible for every $\phi \in \Delta(A) \cup \{0\}$; for more information on this notion, see [NS].

PROPOSITION 2.6. *Let G be a locally compact group. Then the following are equivalent:*

(i) *$L^1(G) \otimes_p M(G)$ is biprojective;*
(ii) *$L^1(G) \otimes_p M(G)$ is approximately biprojective;*
(iii) *G is finite.*

Proof. (i)⇒(ii) is clear.

(ii)⇒(iii). Suppose that $L^1(G) \otimes_p M(G)$ is approximately biprojective. Since $M(G)$ is unital, Proposition 2.5 shows that $M(G)$ is approximately biprojective. So by Theorem 1.1, $M(G)$ is left ϕ-contractible for every ϕ in $\Delta(M(G))$. Also $M(G)$ is left 0-contractible. Hence $M(G)$ is left character contractible. Therefore by [NS, Corollary 6.2], G is finite.

(iii)⇒(i) is clear. ∎

PROPOSITION 2.7. *Let G be an amenable locally compact group. If $L^1(G) \otimes_p \mathcal{A}(G)$ is approximately biprojective, then G is finite.*

Proof. It is well-known that $L^1(G)$ has a bounded approximate identity, and by Leptin's theorem, $\mathcal{A}(G)$ has a bounded approximate identity (see [Ru, Theorem 7.1.3]). Therefore $L^1(G) \otimes_p \mathcal{A}(G)$ has a bounded approximate identity. Suppose that $L^1(G) \otimes_p \mathcal{A}(G)$ is approximately biprojective. Then by Theorem 1.1, $L^1(G) \otimes_p \mathcal{A}(G)$ is left $\phi \otimes \psi$-contractible for every $\phi \in \Delta(L^1(G))$ and $\psi \in \Delta(\mathcal{A}(G))$. Now by [NS, Theorem 3.14], $L^1(G)$ is left ϕ-contractible and $\mathcal{A}(G)$ is left ψ-contractible. By [NS, Proposition 6.6], G is discrete and by [NS, Proposition 6.1], G is compact, therefore G must be finite. ∎

PROPOSITION 2.8. *Let G be a locally compact group. If $L^1(G) \oplus_1 \mathcal{A}(G)$ is approximately biprojective, then G is finite.*

Proof. Let $A = L^1(G) \oplus_1 \mathcal{A}(G)$ be approximately biprojective. Then there exists a net $(\rho_\alpha)_\alpha$ of A-bimodule morphisms from A into $A \otimes_p A$ such that $\pi_A \circ \rho_\alpha(a) \to a$ for every $a \in A$. Let $\phi \in \Delta(\mathcal{A}(G))$. Pick $x_0 \in \mathcal{A}(G)$ such that $\phi(x_0) = 1$. Set $m_\alpha = \rho_\alpha(x_0) \in A \otimes_p A$. Since the elements of $\mathcal{A}(G)$ commute with the elements of A, we see that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\phi \circ \pi_A(m_\alpha) \to 1$. By replacing m_α with $m_\alpha/\phi \circ \pi_A(m_\alpha)$ we can assume that $\phi \circ \pi_A(m_\alpha) = 1$. Then by the same argument as in the proof of [SP1, Proposition 2.2] one can show that A is left ϕ-contractible, and so its closed ideal $\mathcal{A}(G)$ is left ϕ-contractible [NS, Proposition 3.8]. Thus [NS, Proposition 6.6] shows that G is discrete. This shows that $L^1(G)$ becomes the unital algebra $\ell^1(G)$ with unit element e. Now working with $e \in \ell^1(G)$ instead of x_0 in the above argument we get $n_\alpha = \rho_\alpha(e) \in A \otimes_p A$ with $a \cdot n_\alpha = n_\alpha \cdot a$ and $\psi \circ \pi_A(n_\alpha) = 1$ for every $a \in A$, where $\psi \in \Delta(\ell^1(G))$. Hence $\ell^1(G)$ is left ψ-contractible. Therefore by [NS, Theorem 6.1], G is finite. ∎

3. ϕ-biflatness. In [SP1], the authors studied ϕ-biflatness of group algebras. In this section we continue the study of ϕ-biflatness of Segal algebras and the second duals of group algebras. We start with a characterization of amenability of a locally compact group.

REMARK 3.1 ([Ru, Exercise 1.1.6]). In order to show that a locally compact group G is amenable, we only need to find a net $(g_\alpha)_\alpha$ in $P(G) = \{f \in L^1(G) \mid f \geq 0, \|f\|_1 = 1\}$ such that $\|\delta_g g_\alpha - g_\alpha\|_1 \to 0$ for all $g \in G$.

We recall that ϕ_0 is the augmentation character on $L^1(G)$; it induces a character on $S(G)$, still denoted by ϕ_0.

THEOREM 3.2. *Suppose that $S(G)$ is a Segal algebra with an approximate identity. Let $S(G)$ be ϕ_0-biflat. Then G is amenable.*

Proof. To show that G is amenable, we construct a net in $L^1(G)$ that satisfies the conditions of Remark 3.1. We do this in two steps.

STEP 1. In this step we show that there exists a net $(b_\lambda)_\lambda$ in $S(G)\otimes_p S(G)$ such that $a \cdot b_\lambda - b_\lambda \cdot a \to 0$ and $\phi_0 \circ \pi_{S(G)}(b_\lambda) \to 1$ for every $a \in S(G)$.

Since $S(G)$ is ϕ_0-biflat, there exists a bounded $S(G)$-bimodule morphism $\rho : S(G) \to (S(G) \otimes_p S(G))^{**}$ such that $\tilde{\phi_0} \circ \pi^{**}_{S(G)} \circ \rho(a) = \phi_0(a)$. Take $m_\alpha = \rho(e_\alpha)$ in $(S(G) \otimes_p S(G))^{**}$, where $(e_\alpha)_{\alpha \in I}$ is an approximate identity for $S(G)$. So we have

$$a \cdot m_\alpha - m_\alpha \cdot a = a \cdot \rho(e_\alpha) - \rho(e_\alpha) \cdot a = \rho(ae_\alpha - e_\alpha a) \to 0 \tag{3.1}$$

and

$$\tilde{\phi_0} \circ \pi^{**}_{S(G)}(m_\alpha) = \tilde{\phi_0} \circ \pi^{**}_{S(G)} \circ \rho(e_\alpha) = \phi_0(e_\alpha) \to 1. \tag{3.2}$$

Take $\epsilon > 0$ and finite sets $F \subseteq S(G)$ and $\Lambda \subseteq (S(G) \otimes_p S(G))^*$. By (3.1) there exists $v(\epsilon, F, \Lambda) \in I$ such that

$$\|a \cdot \rho(e_{v(\epsilon,F,\Lambda)}) - \rho(e_{v(\epsilon,F,\Lambda)}) \cdot a\| < \epsilon/K_0,$$

where $K_0 = \max\{\|f\| \mid f \in \Lambda\}$. But for every $v(\epsilon, F, \Lambda) \in I$, by Goldestine's theorem, there exists a net (b_λ) in $S(G)\otimes_p S(G)$ such that $b_\lambda \xrightarrow{w^*} \rho(e_{v(\epsilon,F,\Lambda)})$. By w^*-continuity of $\pi^{**}_{S(G)}$ we have $\pi_{S(G)}(b_\lambda) \xrightarrow{w^*} \pi^{**}_{S(G)} \circ \rho(e_{v(\epsilon,F,\Lambda)})$, which implies that

$$\phi_0 \circ \pi_{S(G)}(b_\lambda) \to \tilde{\phi_0} \circ \pi^{**}_{S(G)} \circ \rho(e_{v(\epsilon,F,\Lambda)}), \tag{3.3}$$

and for every $f \in \Lambda$ and $a \in F$ we have

$$f \cdot a(b_\lambda) \to \rho(e_{v(\epsilon,F,\Lambda)})(f \cdot a), \qquad a \cdot f(b_\lambda) \to \rho(e_{v(\epsilon,F,\Lambda)})(a \cdot f). \tag{3.4}$$

Using (3.2) one can show that the right hand side of (3.3) tends to 1.

Now for every $f \in \Lambda$ and $a \in F$ using (3.1) and (3.4) we obtain

$$\begin{aligned} |f(a \cdot b_\lambda - b_\lambda \cdot a) &\leq |f(a \cdot b_\lambda) - a \cdot \rho(e_{v(\epsilon,F,\Lambda)})(f) \\ &\quad + a \cdot \rho(e_{v(\epsilon,F,\Lambda)})(f) - \rho(e_{v(\epsilon,F,\Lambda)}) \cdot a(f) \\ &\quad + \rho(e_{v(\epsilon,F,\Lambda)}) \cdot a(f) - f(b_\lambda \cdot a)| \\ &\leq |f \cdot a(b_\lambda) - \rho(e_{v(\epsilon,F,\Lambda)})(f \cdot a)| \\ &\quad + |a \cdot \rho(e_{v(\epsilon,F,\Lambda)})(f) - \rho(e_{v(\epsilon,F,\Lambda)}) \cdot a(f)| \\ &\quad + |\rho(e_{v(\epsilon,F,\Lambda)})(a \cdot f) - a \cdot f(b_\lambda)| \to 0. \end{aligned} \tag{3.5}$$

Consider the directed set $\Delta = \{\gamma = (\epsilon, F, \Lambda)\}$, where $\epsilon > 0$, and F and Λ are finite subsets of $S(G)$ and $S(G) \otimes_p S(G))^*$, respectively. The order in Δ is defined via

$$\gamma = (\epsilon, F, \Lambda) \leq \gamma' = (\epsilon', F', \Lambda') \;\Leftrightarrow\; \epsilon \geq \epsilon',\, F \subseteq F',\, \Lambda \subseteq \Lambda'.$$

Now let $a \in S(G)$ and $f \in (S(G) \otimes_p S(G))^*$. Then there exists a $\gamma = \gamma(\epsilon, F, \Lambda) \in \Lambda$, where $a \in F$ and $f \in \Lambda$ are such that by (3.5), $|f(a \cdot b_\gamma - b_\gamma \cdot a)| \leq \epsilon$, which shows that $a \cdot b_\gamma - b_\gamma \cdot a \to 0$ in the weak topology. Using Mazur's lemma, one can assume that $a \cdot b_\gamma - b_\gamma \cdot a \to 0$ in $(S(G) \otimes_p S(G), \|\cdot\|)$, and also we have shown that $\phi_0 \circ \pi_{S(G)}(b_\gamma) \to 1$, as desired.

Step 2. In this step we show that G is amenable. We start with a bounded linear map $T : S(G) \otimes_p S(G) \to S(G)$ defined by $T(a \otimes b) = \phi_0(b)a$ for $a, b \in S(G)$. Clearly

$$aT(x) = T(a \cdot x), \qquad T(x \cdot a) = \phi_0(a)T(x), \qquad \phi_0 \circ T(x) = \phi_0 \circ \pi_{S(G)}(x),$$

where $a \in S(G)$ and $x \in S(G) \otimes_p S(G)$.

Set $n_\lambda = T(b_\lambda)$, where (b_λ) is a net coming from Step 1. Then for every $a \in S(G)$ we have

$$\begin{aligned}\|an_\lambda - \phi_0(a)n_\lambda\|_S &= \|aT(b_\lambda) - \phi_0(a)T(b_\lambda)\|_S \\ &= \|T(a\cdot b_\lambda - b_\lambda \cdot a)\|_S \to 0,\end{aligned} \tag{3.6}$$

and

$$\phi_0(n_\lambda) = \phi_0 \circ T(b_\lambda) = \phi_0 \circ \pi_{S(G)}(b_\lambda) \to 1.$$

Fix $a_0 \in S(G)$ such that $\phi_0(a_0) = 1$. Since $\int a_0(g^{-1}x)\,dx = \int a_0(x)\,dx$, we have $\phi_0(\delta_g a_0) = \phi_0(a_0) = 1$. Now set $f_\lambda = a_0 n_\lambda$. It follows from (3.6) that

$$\begin{aligned}\|\delta_g f_\lambda - f_\lambda\|_S &\le \|\delta_g a_0 n_\lambda - n_\lambda\|_S + \|n_\lambda - a_0 n_\lambda\|_S \\ &\le \|\delta_g a_0 n_\lambda - \phi_0(\delta_g a_0)n_\lambda\|_S + \|\phi_0(a_0)n_\lambda - a_0 n_\lambda\|_S \\ &\to 0.\end{aligned} \tag{3.7}$$

Since $S(G)$ is a Segal algebra, we have $\|\cdot\|_{L^1} \le \|\cdot\|_S$, so (3.7) holds for L^1-norm instead of S-norm. Note also that $\phi_0(f_\lambda) \to 1$, so since $|\phi_0(f_\lambda)| \le \|f_\lambda\|_{L^1}$ we may assume that $\|f_\lambda\|_{L^1} \ge 1/2$. Define $g_\lambda = |f_\lambda|/\|f_\lambda\|_{L^1}$, which is bounded. Also, we have

$$\|\delta_g g_\lambda - g_\lambda\|_{L^1} \le 2\big\|\delta_g|f_\lambda| - |f_\lambda|\big\|_{L^1} \le 2\|\delta_g f_\lambda - f_\lambda\|_{L^1} \to 0.$$

Since $\|g_\lambda\|_{L^1} = 1$, Remark 3.1 implies that G is amenable. ∎

Let A be a Banach algebra and $\phi \in \Delta(A)$. Then A is called *ϕ-inner amenable* if there exists a bounded net $(e_\alpha)_\alpha$ in A such that $ae_\alpha - e_\alpha a \to 0$ and $\phi(e_\alpha) \to 1$ for every $a \in A$ (see [JMZ]). Note that every Banach algebra with a bounded approximate identity is ϕ-inner amenable.

THEOREM 3.3. *Let A be a ϕ-inner amenable Banach algebra, where $\phi \in \Delta(A)$. If A^{**} is $\tilde{\phi}$-biflat, then A is left ϕ-amenable.*

Proof. Suppose A^{**} is $\tilde{\phi}$-biflat. Then there exists a bounded A^{**}-bimodule morphism $\rho : A^{**} \to (A^{**} \otimes_p A^{**})^{**}$ such that for every $a \in A^{**}$,

$$\tilde{\tilde{\phi}} \circ \pi^{**}_{A^{**}} \circ \rho(a) = \tilde{\phi}(a),$$

where $\tilde{\tilde{\phi}}$ is an extension of $\tilde{\phi}$ on A^{****} as mentioned in the introduction. Suppose that A is ϕ-inner amenable. Thus A has a bounded net, say (e_α), such that $ae_\alpha - e_\alpha a \to 0$ and $\phi(e_\alpha) \to 1$ for every $a \in A$. Now we define $m_\alpha = \rho(e_\alpha)$ for all α. Since ρ is a bounded map, $(m_\alpha)_\alpha$ is bounded. Let M be a w^*-cluster point of (m_α) in $(A^{**} \otimes_p A^{**})^{**}$. Then for every $a \in A$ we have $a \cdot m_\alpha \xrightarrow{w^*} a \cdot M$ and $m_\alpha \cdot a \xrightarrow{w^*} M \cdot a$, therefore

$$a \cdot m_\alpha - m_\alpha \cdot a \xrightarrow{w^*} a \cdot M - M \cdot a \quad (a \in A).$$

On the other hand,

$$a \cdot m_\alpha - m_\alpha \cdot a = a \cdot \rho(e_\alpha) - \rho(e_\alpha) \cdot a = \rho(ae_\alpha - e_\alpha a) \xrightarrow{\|\cdot\|} 0,$$

so $a \cdot M = M \cdot a$ for every $a \in A$.

Also w^*-continuity of $\pi^{**}_{A^{**}}$ implies that $\pi^{**}_{A^{**}}(m_\alpha) \xrightarrow{w^*} \pi^{**}_{A^{**}}(M)$, hence

$$\tilde{\tilde{\phi}} \circ \pi^{**}_{A^{**}}(m_\alpha) = (\pi^{**}_{A^{**}}(m_\alpha))(\tilde{\phi}) \to (\pi^{**}_{A^{**}}(M))(\tilde{\phi}) = \tilde{\tilde{\phi}} \circ \pi^{**}_{A^{**}}(M).$$

On the other hand,

$$\tilde{\tilde{\phi}} \circ \pi^{**}_{A^{**}}(m_\alpha) = \tilde{\tilde{\phi}} \circ \pi^{**}_{A^{**}} \circ \rho(e_\alpha) = \phi(e_\alpha) \to 1,$$

hence

$$\tilde{\tilde{\phi}} \circ \pi^{**}_{A^{**}}(M) = 1.$$

Now take $\epsilon > 0$ and a finite set $F = \{a_1, \ldots, a_r\} \subseteq A$, and set

$$\begin{aligned} V &= \{(a_1 \cdot n - n \cdot a_1, \ldots, a_r \cdot n - n \cdot a_r, \tilde{\phi} \circ \pi_{A^{**}}(n) - 1)\} \\ &\subseteq \prod_{i=1}^{r} (A^{**} \otimes_p A^{**}) \oplus_1 \mathbb{C}, \end{aligned}$$

where $n \in A^{**} \otimes_p A^{**}$ is such that $\|n\| \leq K$ and $K > 0$ is a bound for the bounded net $(m_\alpha)_\alpha$. Then V is a convex set and so the weak and the norm closures of V coincide. But by Goldestine's theorem there exists a bounded net $(n_\alpha) \subseteq A^{**} \otimes_p A^{**}$ such that $n_\alpha \xrightarrow{w^*} M$, and so for every $a \in F$ we have $a \cdot n_\alpha - n_\alpha \cdot a \xrightarrow{w} 0$ and $|\tilde{\phi} \circ \pi_{A^{**}}(n_\alpha) - 1| \to 0$. This shows that $(0, \ldots, 0)$ is a $\|\cdot\|$-cluster point of V. Thus there exists $n_{(F,\epsilon)} \in A^{**} \otimes_p A^{**}$ such that

$$\|a_i \cdot n_{(F,\epsilon)} - n_{(F,\epsilon)} \cdot a_i\| < \epsilon, \quad |\tilde{\phi} \circ \pi_{A^{**}}(n_{(F,\epsilon)}) - 1| < \epsilon \tag{3.8}$$

for every $i \in \{1, \ldots, r\}$. Now we consider the set

$$\Delta = \{(F, \epsilon) \mid F \text{ is a finite subset of } A,\ \epsilon > 0\},$$

with the following order:

$$(F, \epsilon) \leq (F', \epsilon') \ \Leftrightarrow\ F \subseteq F',\ \epsilon \geq \epsilon'.$$

So (3.8) implies that there exists a bounded net $(n_{(F,\epsilon)})_{(F,\epsilon)\in\Delta}$ in $A^{**} \otimes_p A^{**}$ such that

$$a \cdot n_{(F,\epsilon)} - n_{(F,\epsilon)} \cdot a \to 0, \quad \tilde{\phi} \circ \pi_{A^{**}}(n_{(F,\epsilon)}) \to 1$$

for every $a \in A$. By [GLW, Lemma 1.7] there exists a bounded linear map $\psi : A^{**} \otimes_p A^{**} \to (A \otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**} \otimes_p A^{**}$, the following hold:

(i) $\psi(a \otimes b) = a \otimes b$,
(ii) $\psi(m) \cdot a = \psi(m \cdot a)$, $a \cdot \psi(m) = \psi(a \cdot m)$,
(iii) $\pi^{**}_A(\psi(m)) = \pi_{A^{**}}(m)$.

Define $\xi_{(F,\epsilon)} = \psi(n_{(F,\epsilon)})$, which is a net in $(A \otimes_p A)^{**}$ that by the previous properties of ψ satisfies

$$a \cdot \xi_{(F,\epsilon)} - \xi_{(F,\epsilon)} \cdot a \to 0, \quad \tilde{\phi} \circ \pi^{**}_A(\xi_{(F,\epsilon)}) \to 1.$$

Now much as we obtained a net from (m_α) at the beginning of the proof, one can obtain a bounded net $(\gamma_{(F,\epsilon)})_{(F,\epsilon)\in\Delta}$ related to $\xi_{(F,\epsilon)}$ in $A\otimes_p A$ such that

$$a\cdot\gamma_{(F,\epsilon)}-\gamma_{(F,\epsilon)}\cdot a\to 0,\qquad \phi\circ\pi_A(\gamma_{(F,\epsilon)})\to 1.$$

Now define $T: A\otimes_p A\to A$ by $T(a\otimes b)=\phi(b)a$ for a and b in A. It is easy to see that T is a bounded linear map with

$$T(a\cdot m)=aT(m),\qquad T(m\cdot a)=\phi(a)T(m)\qquad (m\in A\otimes_p A).$$

Define $\nu_{(F,\epsilon)}=T(\gamma_{(F,\epsilon)})$. It is easy to see that $\nu_{(F,\epsilon)}$ is a bounded net and

$$a\nu_{(F,\epsilon)}-\phi(a)\nu_{(F,\epsilon)}\to 0,\qquad \phi\circ T(\nu_{(F,\epsilon)})=\phi\circ\pi_A(\gamma_{(F,\epsilon)})\to 1\qquad (a\in A).$$

Therefore by [KLP, Theorem 1.4], A is left ϕ-amenable. ∎

COROLLARY 3.4. *Let G be a locally compact group. If $L^1(G)^{**}$ is $\tilde{\phi}$-biflat, then G is amenable.*

Proof. Since $L^1(G)$ has a bounded approximate identity, it is ϕ-inner amenable. Thus by Theorem 3.3, it is left ϕ-amenable. Now by [ANN, Corollary 3.4], G is amenable. ∎

COROLLARY 3.5. *Let G be a locally compact group and $\phi,\psi\in\Delta(L^1(G))$. If $(M^1(G)\otimes_p L^1(G))^{**}$ is $\widetilde{\phi\otimes\psi}$-biflat, then G is amenable.*

Proof. We note that $M(G)\otimes_p L^1(G)$ has a bounded approximate identity, and so it is ϕ-inner amenable. Now by Theorem 3.3, $M(G)\otimes_p L^1(G)$ is left $\phi\otimes\psi$-amenable, where $\phi,\psi\in\Delta(L^1(G))$. Thus by [KLP, Theorem 3.3], $L^1(G)$ is left ϕ-amenable, hence G is amenable. ∎

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