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Uniqueness of the Fréchet algebra topology on certain Fréchet algebras

by

S. R. PATEL (Wadhwan City)

Dedicated to the loving memory of Charles Read

Abstract. In 1978, Dales posed a question about the uniqueness of the (F)-algebra topology for (F)-algebras of power series in k indeterminates. We settle this in the affirmative for Fréchet algebras of power series in k indeterminates. The proof goes via first completely characterizing these algebras; in particular, it is shown that the Beurling–Fréchet algebras of semiweight type do not satisfy a certain equicontinuity condition due to Loy. Some applications to the theory of automatic continuity are also given, in particular to the case of Fréchet algebras of power series in infinitely many indeterminates.

1. Introduction. Throughout the paper, "algebra" will mean a complex, commutative algebra with identity unless otherwise specified. A *Fréchet algebra* is a complete, metrizable locally convex algebra A whose topology τ may be defined by an increasing sequence $(p_m)_{m\geq 1}$ of submultiplicative seminorms. We may refer to τ as "the Fréchet topology of A" in the following. The principal tool for studying Fréchet algebras is the Arens–Michael representation, in which A is given by an inverse limit of Banach algebras A_m (see [12, §5] or [13, §2]).

Let $k \in \mathbb{N}$. We write $\mathcal{F}_k$ for the algebra $\mathbb{C}[[X_1, \ldots, X_k]]$ of all formal power series in k commuting indeterminates $X_1, \ldots, X_k$, with complex coefficients. A fuller description of this algebra is given in [4, §1.6]; we briefly recall some notation, which will be used throughout the paper. Let $J = (j_1, \ldots, j_k) \in \mathbb{Z}^{+k}$. Set

$$|J| = j_1 + \cdots + j_k;$$

ordering and addition in $\mathbb{Z}^{+k}$ will always be componentwise. A generic

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element of $\mathcal{F}_k$ is denoted by

$$\sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J = \sum\{\lambda_{(j_1,\dots,j_k)} X_1^{j_1}\cdots X_k^{j_k} : (j_1,\dots,j_k)\in\mathbb{Z}^{+k}\}.$$

The algebra $\mathcal{F}_k$ is a Fréchet algebra when endowed with the weak topology τ_c defined by the coordinate projections

$$\pi_I : \sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J \mapsto \lambda_I,\ \mathcal{F}_k \to \mathbb{C},$$

for each $I\in\mathbb{Z}^{+k}$. A defining sequence of seminorms for $\mathcal{F}_k$ is (p'_m), where

$$p'_m\Big(\sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J\Big) = \sum_{|J|\le m} |\lambda_J| \quad (m\in\mathbb{N}).$$

A *Fréchet algebra of power series in k variables* (briefly: FrAPS in $\mathcal{F}_k$) is a subalgebra A of $\mathcal{F}_k$ such that A is a Fréchet algebra containing the indeterminates $X_1,\dots,X_k$ and such that the inclusion map $(A,\tau)\hookrightarrow(\mathcal{F}_k,\tau_c)$ is continuous (equivalently, the projections $\pi_I, I\in\mathbb{Z}^{+k}$, are continuous linear functionals on A). It is worth mentioning that in [5, Cor. 11.3 and 11.4], it is shown that the time-honoured definitions of Banach and Fréchet (and, more generally, (F)-) algebras of power series in $\mathcal{F}_1$ contain a redundant clause of the continuity of coordinate projections; this is not the case in the several-variable case by [5, Th. 12.3].

Though Fréchet algebras of power series in k indeterminates have been considered by Loy [10], recently these algebras—and more generally, the power series ideas in general Fréchet algebras—have acquired significance in understanding the structure of Fréchet algebras [1, 4, 5, 13, 14, 15]. Thus it is of interest to investigate the following:

(I) whether one can *completely* characterize these algebras,
(II) whether such algebras have a unique topology as Fréchet algebras.

In this paper we shall be concerned with the solution to the above problems; our argument here is kept short because it uses key ideas involved in the solution to these problems for $k=1$ in [13] (see Th. 3.1 and Cor. 4.3 below). In Section 3, we obtain several results of independent interest. Precisely, we shall classify FrAPS in $\mathcal{F}_k$ which do not satisfy an equicontinuity condition (E): there is a sequence $(\gamma_K)_{K\in\mathbb{N}^k}$ of positive reals such that $(\gamma_K^{-1}\pi_K)$ is equicontinuous [10] (see Th. 3.10 below).

We remark that the uniqueness of the Fréchet topology of $\mathcal{F}_k$ for each $k\in\mathbb{N}$ is established in [4, Th. 4.6.1], and the general case has been open since 1978 [3, Question 11]. We use the structure of the closed ideals and their powers to establish the uniqueness of the Fréchet topology of FrAPS in $\mathcal{F}_k$; this is not known for the larger algebra $\mathcal{F}_\infty = \mathbb{C}[[X_1,X_2,\dots]]$ [15] and FrAPS in $\mathcal{F}_\infty$, and so we cannot apply our approach to establish the uniqueness of the Fréchet topology of FrAPS in $\mathcal{F}_\infty$. What is more, Read [15] showed that in the absence of the uniqueness of the Fréchet topology, the Singer–Wermer conjecture cannot be established in the case of Fréchet algebras, and thus the situation on Fréchet algebras is markedly different from that on Banach algebras. However, we shall give some remarks, establishing the uniqueness of the Fréchet topology of FrAPS in $\mathcal{F}_\infty$ admitting a continuous norm. The answer to Question 11 of [3] does include the uniqueness of the Fréchet topology of FrAPS in $\mathcal{F}_1$, established in [13], as a special case. At this point, we also recall that the uniqueness of the Fréchet topology of commutative semisimple Fréchet algebras is established in [2]. However, our class of Fréchet algebras does contain non-semisimple Fréchet algebras, e.g., Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type (including $\mathcal{F}_k$) which are local Fréchet algebras (see property (5) below).

A Fréchet algebra $(A, (p_m))$ is said to be a *Fréchet algebra with power series generators* $x_1, \ldots, x_k$ if each $y \in A$ is of the form

$$y = \sum_{J \in \mathbb{Z}^{+k}} \lambda_J x^J = \sum \{\lambda_{(j_1,\ldots,j_k)} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} : (j_1, \ldots, j_k) \in \mathbb{Z}^{+k}\},$$

for λ_J complex scalars such that $\sum_{J\in\mathbb{Z}^{+k}} |\lambda_J| p_m(x^J) < \infty$ for all m. Thus if A is a Fréchet algebra with finitely many power series generators $x_1, \ldots, x_k$, then A is a commutative, separable, finitely generated Fréchet algebra generated by $x_1, \ldots, x_k$. To answer (II) above, we will investigate the following two questions on FrAPS A in $\mathcal{F}_k$ as an intermediate step:

(A) When are $X_1, \ldots, X_k$ power series generators for A?
(B) When is A isomorphic to an inverse limit of Banach algebras of power series in k variables?

The solutions to the above problems are given in Th. 3.4 and Th. 3.10, respectively. In Section 4, we also pose some interesting questions in automatic continuity theory.

2. Fréchet algebras. Let M be a closed maximal ideal of a Fréchet algebra A. We shall suppose from now on that $\dim(M/\overline{M^2}) = k$ is finite (it is easy to see that for finitely generated Fréchet algebras this condition is automatically satisfied; see [16, Prop. 2.2] for the Banach case). Then, by [16, remark following Th. 2.3], for each $n \in \mathbb{N}$, the homogeneous monomials of degree n in $t_1, \ldots, t_k \in M$ are representatives of a basis for $\overline{M^n}/\overline{M^{n+1}}$ if and only if

$$\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n} := \frac{(n+k-1)!}{n!(k-1)!}$$

for all n, and so M is not nilpotent. Thus, in a special case, we have the following, with an eye on [13, Lem. 2.1].

Proposition 2.1. *Let $(A,(p_m))$ be a commutative, unital Fréchet algebra with the Arens–Michael isomorphism $A \cong \varprojlim(A_m; d_m)$. Suppose that there exists a fixed $k \in \mathbb{N}$ such that M is a closed maximal ideal of A such that:*

(i) $\bigcap_{n\ge1} \overline{M^n} = \{0\}$;

(ii) $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ *for all* n.

Then there exist $t_1,\dots,t_k \in M$ such that $\overline{M^n} = \overline{M^{n+1}} \oplus \operatorname{span}\{t^I : |I| = n\}$ for each $n \ge 1$.

Assume further that each p_m is a norm. Then, for each sufficiently large m, M_m (closure in A_m) is a non-nilpotent maximal ideal of A_m such that:

(a) $\bigcap_{n\ge1} \overline{M_m^n} = \{0\}$;

(b) $\dim(\overline{M_m^n}/\overline{M_m^{n+1}}) = C_{n+k-1,n}$ *for all* n.

Proof. The first part has already been discussed above. For the second part, follow [14, Prop. 2.3]. ■

Concerning Prop. 2.1, the counter-examples in [14] (for the one-variable case) show that the assumption that each p_m is a norm on A cannot be dropped. The algebra $\mathcal{F}_k$ is a trivial counter-example in the several-variable case. We also remark that if $\dim(M/\overline{M^2}) = 1$, one deduces from [13, Prop. 2.3] that $\dim(\overline{M^n}/\overline{M^{n+1}}) = 1$ for all n, and so we do not require $\dim(\overline{M^n}/\overline{M^{n+1}}) = 1$ for all n as a stronger hypothesis, but then we do require M to be non-nilpotent there. Below, we exhibit an easy counter-example to show that the hypothesis that $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ for all $n \in \mathbb{N}$ is not redundant in the proposition above (many thanks to Professor H. G. Dales for calling my attention to this counter-example).

Let

$$B = \mathcal{F}_2 = \mathbb{C}[[X,Y]]$$

with the usual Fréchet algebra topology τ_c and let J be the ideal generated by $X^2 - Y^3$. Since B is noetherian [17, VII, Cor. p. 139 and Th. 4′], all ideals in B are closed by [19, Th. 5], so J is closed. Hence $A = B/J$ is a noetherian Fréchet algebra, with all the ideals closed. Clearly, M/J is the unique maximal ideal in A, where $M = \ker \pi_0$ $(0 = \{0,0\})$, in B. The elements $X + J$ and $Y + J$ are linearly independent modulo $(M/J)^2$, and so $\dim((M/J)/(M/J)^2) = 2$ since $M/M^2 \cong (M/J)/(M/J)^2$. However $(X+J)^2 \in (M/J)^3$, so $\dim((M/J)^2/(M/J)^3) = 2$ since $XY + J$ and $Y^2 + J$ are linearly independent modulo $(M/J)^3$.

Next, to see that this is a counter-example, we show that $\bigcap_{n\ge1}(M/J)^n \ne J$, the zero element of A. To see this, let us start with an element g of B such that

$$g \in pX^2 + qXY + rY^2 + M^3 + J$$

for some $p, q, r \in \mathbb{C}$, that is,

$$g \in \mathbb{C}X^2 + \mathbb{C}XY + \mathbb{C}Y^2 + M^3 + J,$$

and suppose that $g \in M^4 + J$. Then

$$g \in (X^2 - Y^3)(a + bX + cY) + M^4$$

for some $a, b, c \in \mathbb{C}$ because all other terms in $J = (X^2 - Y^3)B$ are in M^4. Thus there exist $a_1, b_1, c_1 \in \mathbb{C}$ such that

$$pX^2 + qXY + rY^2 = (X^2 - Y^3)(a_1 + b_1X + c_1Y) + M^3.$$

Now, equating the coefficients of XY and Y^2, we see that $q = r = 0$, and equating the coefficients of X^2, we see that $p = a_1$; finally, we equate the coefficients of Y^3 to see that $0 = a_1$. Thus $p = 0$. We conclude that $M^3 + J = M^4 + J$. Similarly, one can see that $M^n + J = M^{n+1} + J$ for each $n \geq 3$, so the only element of B that belongs to $M^n + J$ is actually in $M^{n+1} + J$. Thus $M^3 + J = M^n + J$ for each $n \geq 3$, and so $\bigcap_{n\geq 1}(M/J)^n = (M/J)^3 \neq J$.

3. Fréchet algebras of power series in $\mathcal{F}_k$. We now turn to the problem of describing *all* those commutative Fréchet algebras which may be continuously embedded in $\mathcal{F}_k$ in such a way that they contain the polynomials in $X_1, \ldots, X_k$. The following theorem completely characterizes separable FrAPS in $\mathcal{F}_k$. The method of proof will be used again in the proof of Th. 3.10.

THEOREM 3.1. *Let A be a commutative, unital Fréchet algebra. Suppose that there exists a fixed $k \in \mathbb{N}$ such that A contains a closed maximal ideal M such that:*

(i) $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$;
(ii) $\dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ *for all n.*

Then A is a Fréchet algebra of power series in $\mathcal{F}_k$. The converse holds if the polynomials in $X_1, \ldots, X_k$ are dense in A.

Proof. The proof is the same as that of [13, Th. 3.1]. ∎

We note that in *all* FrAPS A in $\mathcal{F}_k$, the ideal $M = \ker \pi_{0,\ldots,0}$ is a non-nilpotent, closed maximal ideal such that $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$. Two counter-examples in the one-variable case (see [13, Rem. 1(b)]) show that the assumption that the polynomials are dense in A cannot be dropped in the above theorem.

We now turn to answering (A) above. The following lemma, the proof of which we omit, is a several-variable analogue of [13, Lem. 3.2] (for the proof, see [1, Lem. 2.2].

Lemma 3.2. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$ and let*
$$A_1 := \Big\{ y \in A : \sum_{J\in\mathbb{Z}^{+k}} |\lambda_J| p_m(X^J) < \infty \text{ for all } m \Big\}.$$
Then:

(1) *A_1 is continuously embedded in A;*
(2) *A_1 is a Fréchet algebra having power series generators $X_1, \dots, X_k$;*
(3) *A_1 is a Banach algebra provided that A is a Banach algebra.* ∎

We recall that elements $x_1, \dots, x_k$ in a Fréchet algebra A generate a *multi-cyclic basis* if each $y \in A$ can be uniquely expressed as
$$y = \sum_{J\in\mathbb{Z}^{+k}} \lambda_J x^J = \sum \{\lambda_{(j_1,\dots,j_k)}\, x_1^{j_1} \cdots x_k^{j_k} : (j_1, \dots, j_k) \in \mathbb{Z}^{+k}\}$$
with λ_J complex scalars. A seminorm p on a Fréchet algebra A, having power series generators $x_1, \dots, x_k$ generating a multi-cyclic basis for A, is a *power series seminorm* if
$$p\Big(\sum_{J\in\mathbb{Z}^{+k}} \lambda_J x^J\Big) = \sum_{J\in\mathbb{Z}^{+k}} |\lambda_J| p(x^J) \quad (y \in A).$$

Corollary 3.3. *Let A be a Fréchet algebra A having power series generators $x_1, \dots, x_k$. Then $x_1, \dots, x_k$ generate a multi-cyclic basis for A if and only if the topology of A is defined by a sequence of power series seminorms.*

Proof. For the proof of the "only if" part, follow [1, Lem. 2.2]. ∎

Next, we define *Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type*, and list some of their useful properties.

A *semiweight function* on $\mathbb{Z}^{+k}$ is a function $\omega : \mathbb{Z}^{+k} \to \mathbb{R}$ such that
$$\omega(M+N) \le \omega(M)\omega(N), \quad \omega(0) = 1, \quad \omega(N) \ge 0 \quad (M, N \in \mathbb{Z}^{+k});$$
a semiweight function is a *weight function* if $\omega(N) > 0$ for all $N \in \mathbb{Z}^{+k}$. Also, ω is a *proper semiweight* if $\omega(N_0) = 0$ for some $N_0 \in \mathbb{N}^k$. Let $k \in \mathbb{N}$, and let $(A, (p_m))$ be the *Beurling–Fréchet algebra*
$$\ell^1(\mathbb{Z}^{+k}, \Omega) := \Big\{ f = \sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J \in \mathcal{F}_k : \sum_{J\in\mathbb{Z}^{+k}} |\lambda_J| \omega_m(J) < \infty \text{ for all } m \Big\},$$
where $\Omega = (\omega_m)$ is a separating and increasing sequence of semiweight functions on $\mathbb{Z}^{+k}$ defined by $\omega_m(J) = p_m(X^J)$. If Ω is an increasing sequence of weight functions on $\mathbb{Z}^{+k}$, then we define
$$\rho = \sup_m \rho_m, \quad \text{where} \quad \rho_m = \inf_{N\in\mathbb{Z}^{+k}} \omega_m(N)^{1/|N|}.$$
Thus, $\rho = 0$ if and only if $\rho_m = 0$ for each m, if and only if for each m, $\ell^1(\mathbb{Z}^{+k}, \omega_m)$ is a local Banach algebra in the Arens–Michael representation of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ if and only if $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is a local Fréchet algebra; and

$\rho > 0$ if and only if $\rho_m > 0$ for some m if and only if for each $l \geq m$, $\ell^1(\mathbb{Z}^{+k}, \omega_l)$ is a semisimple Banach algebra in the Arens–Michael representation of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ if and only if $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is a semisimple Fréchet algebra.

Suppose that Ω is a separating and increasing sequence of *proper semiweights* on $\mathbb{Z}^{+k}$. Then $\rho = 0$ if and only if $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is a local Fréchet algebra if and only if the completion of $\ell^1(\mathbb{Z}^{+k}, \omega_m)/\ker p_m$ under the induced norm p_m is a local Banach algebra for all m. In this case, $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is either $\mathcal{F}_k$ or a local FrAPS in $\mathcal{F}_k$. We call such a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ an *algebra of semiweight type*. We note that the unique maximal ideal of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is

$$\Big\{f = \sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J \in \ell^1(\mathbb{Z}^{+k}, \Omega) : \lambda_0 = 0\Big\}.$$

For example, if $k = 1$, then, by [1, Th. 2.1], $\ell^1(\mathbb{Z}^{+}, \Omega) = \mathcal{F}$, which is an inverse limit of finite-dimensional algebras, and is also a local algebra. In this case (see [5, p. 131]),

$$\omega_m : \mathbb{Z}^{+} \to [0, \infty), \qquad \omega_m(n) := p_m(X^n) = \begin{cases} 1, & n \leq m, \\ 0, & n > m. \end{cases}$$

If $k = 2$, then, by Th. 3.4 below, $\ell^1(\mathbb{Z}^{+2}, \Omega)$ is either $\mathcal{F}_2$ or A_X or A_Y (all the three algebras are local), where

$$A_X := \Big\{f = \sum_{i,j} \lambda_{i,j} X^i Y^j \in \mathcal{F}_2 : p_m(f) := \sum_{j=0}^{\infty} \sum_{i=0}^{m} |\lambda_{i,j}| < \infty \text{ for all } m\Big\},$$

in which case

$$\omega_m : \mathbb{Z}^{+2} \to [0, \infty), \qquad \omega_m(i,j) := p_m(X^i Y^j) = \begin{cases} 1, & i \leq m,\ j \in \mathbb{Z}^{+}, \\ 0, & i > m,\ j \in \mathbb{Z}^{+}, \end{cases}$$

and where

$$A_Y := \Big\{f = \sum_{i,j} \lambda_{i,j} X^i Y^j \in \mathcal{F}_2 : p_m(f) := \sum_{i=0}^{\infty} \sum_{j=0}^{m} |\lambda_{i,j}| < \infty \text{ for all } m\Big\},$$

in which case

$$\omega_m : \mathbb{Z}^{+2} \to [0, \infty), \qquad \omega_m(i,j) := p_m(X^i Y^j) = \begin{cases} 1, & j \leq m,\ i \in \mathbb{Z}^{+}, \\ 0, & j > m,\ i \in \mathbb{Z}^{+}. \end{cases}$$

For $\ell^1(\mathbb{Z}^{+k}, \Omega) = \mathcal{F}_2$ with

$$p_m(f) := \sum_{0 \leq i+j \leq m} |\lambda_{i,j}|,$$

we define $\Omega = (\omega_m)$, where

$$\omega_m : \mathbb{Z}^{+2} \to [0, \infty), \qquad \omega_m(i,j) := p_m(X^i Y^j) = \begin{cases} 1, & 0 \leq i+j \leq m, \\ 0, & i+j > m \end{cases}$$

(see [5, p. 131]). Clearly, $A_X \cong A_Y$ under the interchange of X and Y.

If $k = 3$, then, again by Th. 3.4 below, $\ell^1(\mathbb{Z}^{+3}, \Omega)$ is either $\mathcal{F}_3$ or A_X or A_Y or A_Z or $A_{X,Y}$ or $A_{X,Z}$ or $A_{Y,Z}$ defined analogously (in fact, $A_X \cong A_Y \cong A_Z$ and $A_{X,Y} \cong A_{X,Z} \cong A_{Y,Z}$). We can extend the above arguments for $k \geq 4$. Thus, for $k \in \mathbb{Z}^+$, we have Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type, with the following properties:

(1) $\mathcal{F}_k$ is the *only* Fréchet algebra of finite type among FrAPS in $\mathcal{F}_k$, by Cor. 3.8 below; other Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ are not Fréchet algebras of finite type (see [8]).

(2) The Arens–Michael representations of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ do not contain BAPS in $\mathcal{F}_k$ (for the time being, we assume that such algebras have unique Fréchet topology, which we shall prove later); for the proof, see Th. 3.10 below or Remarks (b) after Th. 3.10.

(3) The polynomials in k variables are dense in $\ell^1(\mathbb{Z}^{+k}, \Omega)$.

(4) The Fréchet topology τ of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ ($\neq \mathcal{F}_k$), defined by a sequence (p_m), is strictly finer than τ_c of $\mathcal{F}_k$, but surely not equivalent, as otherwise the τ-closure of the algebra of polynomials in k variables (which is $\ell^1(\mathbb{Z}^{+k}, \Omega)$) would be equal to $\mathcal{F}_k$, a contradiction. Hence the rest of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ differ from $\mathcal{F}_k$ (also, from the statement (1) point of view as well).

(5) For $1 \leq r \leq k-1$, the algebras $\mathcal{F}_r$, can be regarded as closed subalgebras of $\ell^1(\mathbb{Z}^{+k}, \Omega)$ via the obvious quotient maps. For example, if $k = 2$, then $\mathbb{C}[[X]] = \mathcal{F}_1$ can be regarded as a closed subalgebra of A_X; the quotient map from A_X obtained by setting $Y = 0$ is denoted by

$$\pi : \sum_{i,j} \lambda_{i,j} X^i Y^j \mapsto \sum_{i=0}^{\infty} \lambda_{i,0} X^i,\ A_X \to \mathcal{F}_1.$$

Hence all Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type are local Fréchet algebras since the closed subalgebras $\mathcal{F}_r, 1 \leq r \leq k$, are local Fréchet algebras, and the unique maximal ideal M is

$$\Big\{ f = \sum_{J \in \mathbb{Z}^{+k}} \lambda_J X^J \in \ell^1(\mathbb{Z}^{+k}, \Omega) : \lambda_0 = 0 \Big\}.$$

Also, for a fixed $k \in \mathbb{N}$, there are finitely many Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type, and these algebras can be properly nested (for example, if $k = 3$, then $A_X \subset A_{X,Y} \subset \mathcal{F}_3$). Further, if $(A, (q_m))$ is a FrAPS in $\mathcal{F}_k$ such that the q_m are proper seminorms on A, then A is continuously embedded in the "least" Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type (note that there might be several such Beurling–Fréchet algebras of semiweight type containing A). Moreover it is clear that if such A contains a Beurling–Fréchet algebra of semiweight type such that

$$\ell^1(\mathbb{Z}^{+k}, \Omega_1) \hookrightarrow A \hookrightarrow \ell^1(\mathbb{Z}^{+k}, \Omega_2)$$

continuously, then, depending on the q_m, A is either $\ell^1(\mathbb{Z}^{+k}, \Omega_1)$ or $\ell^1(\mathbb{Z}^{+k}, \Omega_2)$ or none of these; in the latter case both the inclusions are proper (for example, $A_X \hookrightarrow A_{X,Y} \hookrightarrow \mathcal{F}_3$). We use these facts in the proof of Th. 3.7 below.

We call $\ell^1(\mathbb{Z}^{+k}, \Omega)$ a *Beurling–Fréchet algebra of weight type* if Ω is an increasing sequence of weight functions ω_m on $\mathbb{Z}^{+k}$. In this case, the topology τ is defined by an increasing sequence (p_m) of norms defined in terms of the ω_m, and the corresponding Arens–Michael representation contains Beurling-Banach algebras $\ell^1(\mathbb{Z}^{+k}, \omega_m)$. When it can cause no confusion, we may call $\ell^1(\mathbb{Z}^{+k}, \Omega)$ a Beurling–Fréchet algebra of (semi)weight type if Ω is an increasing sequence of semiweight functions ω_m on $\mathbb{Z}^{+k}$ (which would include both types of algebras: algebras of semiweight type as well as algebras of weight type).

THEOREM 3.4. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$. Suppose that $X_1, \ldots, X_k$ are power series generators for A. Then A is the Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ for an increasing sequence Ω of (semi)weight functions on $\mathbb{Z}^{+k}$.*

Proof. The proof is the same as that of [1, Th. 2.1]. ∎

Next, let A be a FrAPS in $\mathcal{F}_k$. We call a seminorm p on A *closable* if for any p-Cauchy sequence (f_l) in A, $f_l \to 0$ in τ_c implies that $p(f_l) \to 0$. We define p to be of type (E) if given $M \in \mathbb{Z}^{+k}$, there exists $c_M > 0$ such that

$$|\pi_M(f)| \leq c_M p(f)$$

for all $f \in A$ [10]. A seminorm of type (E) is a norm. Also, closability of a norm on a normed algebra of power series in k indeterminates is a necessary and sufficient condition for the completion to be a BAPS in $\mathcal{F}_k$ (see [1, Lem. 3.5]).

We now answer (B) of the introduction. The following proposition, whose proof we omit, is a several-variable analogue of [1, Prop. 3.1].

PROPOSITION 3.5. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$. Let p be a continuous submultiplicative seminorm on A. Let $\ker p = \{f \in A : p(f) = 0\}$. Let A_p be the completion of $A/\ker p$ in the norm $\|f + \ker p\|_p = p(f)$. Then the following are equivalent:*

(i) *p is a norm and A_p is a Banach algebra of power series in $\mathcal{F}_k$.*
(ii) *p is closable and of type* (E). ∎

COROLLARY 3.6. *Let $A = \varprojlim A_m$ be the Arens–Michael representation of a Fréchet algebra of power series in $\mathcal{F}_k$. Assume that each p_m is a norm. Then each A_m is a Banach algebra of power series in $\mathcal{F}_k$ if and only if each p_m is a closable norm of type* (E). ∎

It is readily seen that a Fréchet algebra of power series in $\mathcal{F}_k$ satisfies Loy's condition (E) of [10] if and only if A admits a continuous norm of type (E) if and only if the topology of A is defined by a sequence of norms of type (E).

Next, we give characterizations of Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. We have the following elementary, but crucial, theorem. By identifying the series expansion in $x_1, \ldots, x_k$ with the series expansion in $X_1, \ldots, X_k$, Fréchet algebras with a multi-cyclic basis are realized as Fréchet algebras of power series in $\mathcal{F}_k$, the projections π_J being continuous.

THEOREM 3.7. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$. Then either A is a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type or the Fréchet topology τ of A is defined by a sequence (p_m) of norms.*

Proof. If A is Banach, then certainly the topology τ of A is defined by a norm, and so A is not equal to a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. Now suppose that A is a non-Banach FrAPS in $\mathcal{F}_k$. Let (p_m) be an increasing sequence of seminorms defining the Fréchet topology τ of A, and set

$$G = \{l \in \mathbb{N} : p_l \text{ is a proper seminorm on } A\}.$$

If G is finite the corresponding p_l may be deleted and we have a new sequence of norms, defining the same Fréchet topology τ of A. Otherwise, G is infinite and the corresponding p_l can be taken to define the Fréchet topology τ of A. Then, by Lem. 3.2, there exists a Fréchet subalgebra $(A_1, (q_m))$ of $(A, (p_m))$ continuously embedded in A; A_1 is a Fréchet algebra with power series generators $X_1, \ldots, X_k$. By Th. 3.4, A_1 is a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of (semi)weight type. To rule out the possibility of A_1 being a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of weight type, fix $m \geq 1$. Since $\ker p_m$ is a closed ideal in A, and A is a dense subalgebra of $(\mathcal{F}_k, (p'_m))$, by [12, Lem. B.10], the closure of $\ker p_m$ (say I_m) in the topology τ_c is a closed ideal in $\mathcal{F}_k$ such that $\ker p_m = I_m \cap A$. Since the inclusion map $(A, (p_m)) \hookrightarrow (\mathcal{F}_k, (p'_m))$ is continuous, for each $m \in \mathbb{N}$ there exists $n(m) \in \mathbb{N}$ and a constant $c_m > 0$ such that $p'_m(f) \leq c_m p_{n(m)}(f)$ for all $f \in A$, so we have $\ker p_{n(m)} \subseteq \ker p'_m$. Thus $\ker p_{n(m)} \subseteq I_{n(m)} \subseteq \ker p'_m$ since $I_{n(m)}$ is the closure of $\ker p_{n(m)}$ in the topology τ_c and $\ker p'_m$ is a closed ideal in $\mathcal{F}_k$ containing $\ker p_{n(m)}$. Consequently, we have a subsequence $(p_{n(m)})$ of seminorms on A, generating the same topology on A, such that $\ker p_{n(m)} \subseteq I_{n(m)} \subseteq \ker p'_{m-1}$ since the sequence (p'_m) is increasing, so $\ker p'_m \subseteq \ker p'_{m-1}$. We may from now on denote by (p_m) this subsequence. Recall that $\ker p'_{m-1}$ is finitely generated by all monomials of degree m (see Case 1 below). Since I_m is a finitely generated ideal in $\mathcal{F}_k$ ($\mathcal{F}_k$ being noetherian) and $I_m \subseteq \ker p'_{m-1}$, I_m is finitely generated by the monomials of degree m. Clearly, the monomials of degree m

are in A as A is a FrAPS in $\mathcal{F}_k$. Hence these monomials are in $\ker p_m$, and since $\ker p_m \subseteq I_m$, $\ker p_m$ is finitely generated by the monomials of degree m. Since $(A_1, (q_m))$ is a Fréchet algebra with power series generators $X_1, \ldots, X_k$, where $q_m(\sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J) := \sum_{J\in\mathbb{Z}^{+k}} |\lambda_J| p_m(X^J)$, we have

$$p_m(X^m) = q_m(X^m) = 0 \quad (m \in \mathbb{N}),$$

and so A_1 is indeed a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. Hence A_1 is a local algebra, and therefore A is also a local algebra. Now there are several cases; we consider these cases for $k = 2$ (one can modify the following arguments for any k).

Case 1: For each m, $\ker p_m$ is finitely generated by all monomials of degree m in X and Y, in which case I_m is also finitely generated by all monomials of degree m in X and Y as discussed earlier (and so all monomials of degree m belong to I_m). Since $I_m \subseteq \ker p'_{m-1} = M_m$, and I_m and M_m are generated by the same generators, we have $I_m = M_m$, that is, $\ker p_m = M_m(A)$, where

$$M_m(A) := \{f \in A : o(f) \geq m\}.$$

But then

$$\ker q_m = M_m(A_1) := \{f \in A_1 \subset A : o(f) \geq m\},$$

where A_1 is as in Lem. 3.2. By Th. 3.4, $A_1 = \mathcal{F}_2$. It follows that $A = \mathcal{F}_2$ topologically in view of the open mapping theorem.

Case 2: For each m, $\ker p_m$ is singly generated by the monomial X^m. In this case

$$A_1 = \Big\{f \in \mathcal{F}_2 : q_m(f) = \sum_{j=0}^{\infty}\sum_{i=0}^{m} |\lambda_{i,j}| < \infty \text{ for all } m\Big\} = A_X$$

by Th. 3.4, since $X^m \in \ker q_m$. So, $A_X = A_1 \subset A \subset \mathcal{F}_2$, by Lem. 3.2. But, as we discussed in property (5), $A_X \subset A \subset A_X \subset \mathcal{F}_2$ since p_m is a proper seminorm on A for each m such that $X^m \in \ker p_m$. Thus we have $A = A_X$ topologically in view of the open mapping theorem.

Case 3: For each m, $\ker p_m$ is singly generated by the monomial Y^m. Follow the argument of Case 2. ■

As corollaries, we have the following characterizations of Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type as Fréchet algebras.

Corollary 3.8. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$. Then A is a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type if and only if the Fréchet topology of A is defined by a sequence (p_m) of proper seminorms. In particular, $A = \mathcal{F}_k$ if and only if the Fréchet topology of A is defined by a sequence (p_m) of proper seminorms with finite-codimensional kernels.* ■

In fact, we have the following result on Arens–Michael representation of A.

Corollary 3.9. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$ such that the polynomials are dense in A. Then A is not a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type if and only if $A = \varprojlim A_m$, where each A_m is a Banach algebra of power series in $\mathcal{F}_k$.*

Proof. Suppose that A is not equal to a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. Evidently, by Cor. 3.8, we may suppose that each p_m is a norm on A. Now, by Th. 3.1, A contains a closed maximal ideal $M = \ker \pi_0$ such that

$$\bigcap_{n\ge 1} \overline{M^n} = \{0\} \quad \text{and} \quad \dim(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n} \quad (n \in \mathbb{N}).$$

By Prop. 2.1, for each sufficiently large l, M_l is a non-nilpotent maximal ideal of A_l such that

$$\bigcap_{n\ge 1} \overline{M_l^n} = \{0\} \quad \text{and} \quad \dim(\overline{M_l^n}/\overline{M_l^{n+1}}) = C_{n+k-1,n} \quad (n \in \mathbb{N}).$$

Again, by Th. 3.1, A_l is a BAPS in $\mathcal{F}_k$ for each sufficiently large l. Hence, by passing to a suitable subsequence of (p_m) defining the same Fréchet topology of A, we conclude that each A_l is a BAPS in $\mathcal{F}_k$.

The converse has already been discussed in property (2) above. ∎

The immediate consequence of Cor. 3.9 is: if A is a FrAPS in $\mathcal{F}_k$ such that the polynomials are dense in A and such that A is not a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type, then A satisfies Loy's condition (E) by Cor. 3.6. A somewhat more elaborate version of the same idea enables us to drop the condition on the polynomials in order to get a more general result, given below.

We recall that if the Fréchet topology τ of A is given by a sequence (p_m), then each p_m is of type (E) if and only if A satisfies Loy's condition (E). Also, by a several-variable analogue of [9, Th. 2], A satisfies Loy's condition (E) if and only if A admits a *growth sequence*, i.e. there exists a sequence $(\sigma_K)_{K\in\mathbb{N}^k}$ of positive reals such that $\sigma_K \pi_K(x) \to 0$ for each $x \in A$.

Theorem 3.10. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$. Then A is not a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type if and only if $A = \varprojlim A_m$, where each A_m is a Banach algebra of power series in $\mathcal{F}_k$. In particular, A satisfies Loy's condition* (E) *in this case.*

Proof. The proof is similar to that of [13, Th. 3.6], and so we will merely sketch it. Supposing A is not a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type, we may suppose that each p_m is a norm on A, by Cor. 3.8. We first show that the projections π_J are continuous on (A, p_m) for all J and m, i.e. the inclusion maps $(A, p_m) \hookrightarrow \mathcal{F}_k$ are continuous.

Since $M = \ker \pi_{0,\dots,0}$ is a non-nilpotent, closed maximal ideal of A such that $\bigcap_{n\ge1} \overline{M^n} = \{0\}$, M is a non-nilpotent, maximal ideal of (A, p_m) for each m; also, $\overline{M^{n+1}} \neq \overline{M^n} \neq \{0\}$ (closure in (A,τ)) in (A, p_m) for all m and n. In fact, we may suppose that for each sufficiently large m, M is also closed in (A, p_m) and hence that $\pi_{0,\dots,0}$ is p_m-continuous. Also, by [13, proof of Prop. 2.3], we may suppose that M_m is a non-nilpotent, maximal ideal of A_m and that $\bigcap_{n\ge1} \overline{M_m^n} = \{0\}$ for each m. Assume inductively that π_I is continuous for each $|I| < |J|$, and take (x_n) in (A, p_m) with $p_m(x_n) \to 0$. Then, following the argument given in Th. 3.1, we deduce that some non-zero linear combination of X^I with $|I| = |J|$ lies in $A \cap \overline{M_m^{|J|+1}}$ (which is, in fact, $\overline{M^{|J|+1}}$ in (A, p_m) for each m), a contradiction of the fact that $\overline{M_m^{|J|+1}} \neq \overline{M_m^{|J|}}$.

Next, for each $m \in \mathbb{N}$, one shows that p_m is closable on A by noticing that the inclusion map $(A, p_m) \hookrightarrow \mathcal{F}_k$ can be extended to a continuous homomorphism $\phi : A_m \to \mathcal{F}_k$. So $x_n \to x$ in $(\mathcal{F}_k, \tau_c)$, and hence $p_m(x_n) \to 0$.

Then one shows that ϕ is indeed injective, and so A_m is a Banach algebra of power series in $\mathcal{F}_k$ for each m. In particular, A satisfies Loy's condition (E), by the remark following Cor. 3.6. ∎

Remarks. (a) We again emphasize the fact that the characterizations of FrAPS in $\mathcal{F}_k$ and of the algebra $\mathcal{F}_k$ play an essential role in the proof of Th. 3.10.

(b) The fact that a Fréchet algebra of power series A in $\mathcal{F}_k$ satisfies Loy's condition (E) played an essential role in [10], and it is of interest to obtain a simple characterization of this condition. Here it is easy to see that a Fréchet algebra of power series A in $\mathcal{F}_k$ satisfies Loy's condition (E) if and only if A admits a continuous norm, i.e., there exists a norm p on A such that for some $K > 0$ and integer $m \ge 1$, $p(x) \le K p_m(x)$ $(x \in A)$. The above theorem can be used to prove property (2) as follows: consider $A_Y \subset \mathcal{F}_2$. The topology τ of A_Y is given by a sequence (p_m) of proper power series seminorms, hence A_Y does not satisfy Loy's condition (E), and thus A_Y does not admit a continuous norm. So, $(A_Y)_m$ cannot be a BAPS in $\mathcal{F}_2$.

(c) By the method of proofs, it is clear that the characterizations obtained in Cor. 3.9 and Th. 3.10 are independent of the Arens–Michael representation chosen, in the sense that if (p''_m) is any other sequence of norms defining the Fréchet topology of A, then the proofs are valid with that sequence. Of course, the A''_m, obtained using the sequence (p''_m), may be a different Banach algebra of power series in an Arens–Michael representation of A.

4. Automatic continuity and uniqueness of topology, and open questions. We now establish that every FrAPS in $\mathcal{F}_k$ has a unique Fréchet topology. By [10, Th. 1], it is clear that every FrAPS A in $\mathcal{F}_k$ satisfying Loy's condition (E) has a unique Fréchet topology. Since a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type does not satisfy the condition (E), Th. 3.10 gives the following result from [11] by noticing that [11, Th. 10] holds true if A is considered to be a FrAPS in $\mathcal{F}_k$ in the codomain.

THEOREM 4.1. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$ such that A is not a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. Then a homomorphism $\theta : B \to A$ from a Fréchet algebra B is continuous provided that the range of θ is not one-dimensional.* ∎

Now, we prove the uniqueness of the Fréchet topology for Beurling–Fréchet algebras $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type.

THEOREM 4.2. *Let A be a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. Then A has a unique Fréchet topology.*

Proof. Let τ' be another topology such that (A, τ') is a Fréchet algebra. Suppose that the Fréchet topology τ' of A is defined by a sequence (p''_m) of submultiplicative seminorms on A. First we note that A is a local Fréchet algebra, and hence it has a unique maximal ideal $M = \ker \pi_{0,\dots,0}$ (this is clearly an algebraic property). Since (A, τ') is local, A is a Q-algebra by [18, Cor. 3]. We recall that the original Fréchet topology τ of A is defined by a sequence (p_m) of proper power series seminorms. Since (A, τ) is local, A is a Q-algebra by [18, Cor. 3]. By [18, Th.], A is a Fréchet Q-algebra under a Fréchet topology τ'' stronger than τ' and τ. Note that the Fréchet topology τ'' of A is defined by a sequence (q_m) of submultiplicative seminorms on A, where $q_m(f) := \max\{p''_m(f), p_m(f), |f|_0\}$, $|f|_0 := \sup\{|\lambda| : \lambda \in \sigma(f)\}$. Hence the identity maps $(A, \tau'') \to (A, \tau)$ and $(A, \tau'') \to (A, \tau')$ are continuous. By the open mapping theorem for Fréchet spaces, these maps are homeomorphisms, and so $\tau = \tau' = \tau''$ on A. ∎

REMARK. The uniqueness of the Fréchet topology of $\mathcal{F}_k$ is proved in [4, Th. 4.6.1]. Since $\mathcal{F}_k$ is a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type, the above theorem gives another approach to establish the uniqueness of the Fréchet topology of $\mathcal{F}_k$.

As a corollary of the above two theorems, we have the following result (and Questions) in the theory of automatic continuity. We recall that the continuity of every automorphism of $\mathcal{F}_k$ which is a substitution map is proved in [17, p. 136]. The following gives an answer to [3, Question 11] for FrAPS in $\mathcal{F}_k$.

Corollary 4.3. *Let A be a Fréchet algebra of power series in $\mathcal{F}_k$. Then A has a unique Fréchet topology.* ■

We now list the following questions, which may have some interest in the theory of automatic continuity.

Question 1. Let A be a Beurling–Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. Is every automorphism of A continuous?

The author conjectures that the answer to Question 1 is in the affirmative; also, this question has been studied for special cases (see [6, 7]). More generally, we have the following

Question 2. Is every homomorphism $\theta : B \to \mathcal{F}_k$ $(k > 1)$ from a Fréchet algebra B continuous?

The above question has recently been partially settled in [5]: (i) if $\theta : B \to \mathcal{F}_k$ is a discontinuous homomorphism from an (F)-algebra B (i.e., a complete, metrizable topological algebra) into $\mathcal{F}_k$ such that $\theta(B)$ is dense in $(\mathcal{F}_k, \tau_c)$ and such that the separating ideal of θ has finite codimension in $\mathcal{F}_k$, then θ is an epimorphism (see [5, Th. 12.1]), and (ii) there exists a Banach algebra $(A, \|\cdot\|)$ such that $\mathbb{C}[X_1, X_2] \subset A \subset \mathcal{F}_2$, but the embedding $(A, \|\cdot\|) \hookrightarrow (\mathcal{F}_2, \tau_c)$ is not continuous (see [5, Th. 12.3]); surprisingly, A is (isometrically isomorphic to) a Banach algebra of power series in $\mathcal{F}_1$ (see [5, Th. 10.1(i)]).

As shown in [15], $\mathcal{F}_\infty$ admits two inequivalent Fréchet algebra topologies. It is clear that the unique maximal ideal $M = \ker \pi_0$ $(0 = (0, 0, \dots))$ is closed under both topologies, since $\mathcal{F}_\infty$ is a local Fréchet Q-algebra. Thus it is of interest to know whether every FrAPS in $\mathcal{F}_\infty$ (except $\mathcal{F}_\infty$ itself) has a unique Fréchet topology. In other words, we have the following natural question.

Question 3. Is there any other proper, unital subalgebra of $\mathcal{F}_\infty$ with two inequivalent Fréchet topologies? In particular, is there any other proper subalgebra of $\mathcal{F}_\infty$ which is closed under the topology imposed by Charles Read on $\mathcal{F}_\infty$ and which is also FrAPS in $\mathcal{F}_\infty$ in its "usual" topology?

To answer the latter part of the above question, the "natural" extension of Beurling–Fréchet algebras of semiweight type (i.e., $\ell^1((\mathbb{Z}^+)^{<\omega}, \Omega)$) would be an easy target. Also, we have the following curious question.

Question 4. Does there exist a Fréchet algebra with infinitely many inequivalent Fréchet algebra topologies?

For $m \in \mathbb{N}$, set

$$\mathcal{U}_m = \Big\{ f = \sum \{\alpha_r X^r : r \in (\mathbb{Z}^+)^{<\omega}\} \in \mathcal{F}_\infty : p_m(f) := \sum |\alpha_r| m^{|r|} < \infty \Big\},$$

and then set

$$\mathcal{U} = \bigcap \{\mathcal{U}_m : m \in \mathbb{N}\}.$$

It is clear that each $(\mathcal{U}_m, p_m)$ is a unital Banach subalgebra of $\mathcal{F}_\infty$ and that $(\mathcal{U}, (p_m))$ is a unital Fréchet subalgebra of $\mathcal{F}_\infty$. Being semisimple Fréchet algebras, the test algebra $\mathcal{U}$ for the still unsolved "Michael problem" (and $\mathcal{U}_m$ for each m appearing in the Arens–Michael representation of $\mathcal{U}$), $\ell^1((\mathbb{Z}^+)^{<\omega})$ and the algebra $\ell^1(S_c)$, where S_c denotes the free semigroup on c generators, have unique Fréchet topologies (this also follows from [5, Th. 10.1 and 10.5] and [13, Cor. 4.2]). In this regard, we first give the following result (for the proof, see either [9, Th. 1] or [10, Th. 2]) by noticing that a FrAPS A in $\mathcal{F}_\infty$ satisfies Loy's condition (E) if there is a sequence $(\gamma_K : K \in \mathbb{N}^k, k \in \mathbb{N})$ of positive reals such that $(\gamma_K^{-1}\pi_K)$ is equicontinuous.

Theorem 4.4. *Let A be a Fréchet algebra of power series in $\mathcal{F}_\infty$ satisfying Loy's condition* (E) *above, and let $\phi : B \to A$ be a homomorphism from a Fréchet algebra B into A such that $X_1 \in \phi(B)$. Then ϕ is continuous. In particular, every automorphism of A is continuous, and A has a unique Fréchet algebra topology.* ∎

We remark that the Fréchet topology of A in the above theorem is defined by a sequence of norms. We have the following "natural" generalization of [5, Th. 10.1].

Theorem 4.5. *Let A be a Fréchet algebra of power series in $\mathcal{F}_\infty$ with topology defined by a sequence (p_m) of norms. Suppose that A is a graded subalgebra of $\mathcal{F}_\infty$. Then there is a continuous embedding θ of A into $(\mathcal{F}, \tau_c)$ such that $\theta(X_1) = X$, and so A is* (*isometrically isomorphic to*) *a Fréchet algebra of power series in $\mathcal{F}_1$. In particular, A has a unique Fréchet topology.* ∎

We cannot drop the norm assumption on (p_m) in the above theorem: $(\mathcal{F}_\infty, \tau_c)$ is a counterexample, since it can be continuously embedded in $(\mathcal{F}_2, \tau_c)$, by [5, Th. 9.1], but, by [5, Th. 2.6 (or Th. 11.8)], there is no embedding of $\mathcal{F}_2$ into $\mathcal{F}$.

Finally, we remark that we have recently established the uniqueness of an (F)-algebra topology for the (F)-algebra of power series in the indeterminate X (see [5, Cor. 11.7]). Then the natural question is to extend this result to the several-variable case, in order to settle the Dales question completely. But our approach fails, because the clause of the continuity of coordinate projections cannot be dropped from the definitions of Banach and Fréchet algebras of power series in k indeterminates (see [5, Th. 12.3]). The approach here is based on a sequence (p_m) of submultiplicative seminorms, and so these results cannot be extended to *all* (F)-algebras of power series in k indeterminates.

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S. R. Patel
Department of Mathematics
C. U. Shah University
Wadhwan City, Gujarat, India
E-mail: srpatel.math@gmail.com
coolpatel1@yahoo.com