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THE BÜCHI SEQUENCES AND HILBERT'S TENTH PROBLEM

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Abstract. In this short survey paper we state the Büchi conjecture and discuss its relations with the Hilbert Tenth Problem. We give some generalizations of the conjecture, and include some numerical examples.

1. The Büchi sequences. We begin with two examples of sequences of integers, such that the sequences of the second differences of their squares are constant equal to 2. Such sequences are called Büchi sequences.

EXAMPLE 1. A trivial Büchi sequence:

$$\begin{array}{rcccccccccccccc}
a_n: & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & & 10 \\
a_n^2: & 16 & & 25 & & 36 & & 49 & & 64 & & 81 & & 100 \\
\Delta^1(a_n^2) = a_{n+1}^2 - a_n^2: & & 9 & & 11 & & 13 & & 15 & & 17 & & 19 & \\
\Delta^2(a_n^2) = \Delta^1(a_{n+1}^2) - \Delta^1(a_n^2): & & & 2 & & 2 & & 2 & & 2 & & 2 & &
\end{array}$$

Every sequence which, up to signs of its terms, is an increasing sequence of consecutive integers, is a Büchi sequence. We call it a trivial Büchi sequence.

EXAMPLE 2. A nontrivial Büchi sequence:

$$\begin{array}{rccccccc}
a_n: & 6 & & 23 & & 32 & & 39 \\
a_n^2: & 36 & & 529 & & 1024 & & 1521 \\
\Delta^1(a_n^2): & & 493 & & 495 & & 497 & \\
\Delta^2(a_n^2): & & & 2 & & 2 & &
\end{array}$$

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There are infinitely many nontrivial Büchi sequences of length 4, namely put

$$\begin{aligned} a_1(t) &= 2t^3 + 6t^2 + t - 3, \\ a_2(t) &= 2t^3 + 8t^2 + 9t + 4, \\ a_3(t) &= 2t^3 + 10t^2 + 15t + 5, \\ a_4(t) &= 2t^3 + 12t^2 + 19t + 6. \end{aligned} \tag{1}$$

One can verify that, for every $t \in \mathbb{Z}$, $(a_1(t), a_2(t), a_3(t), a_4(t))$ is a Büchi sequence. It is nontrivial for $t \neq 0$.

For $t = 1$ we get the sequence of Example 2.

REMARK 1. It is easy to see that in (1) we have $a_4(t) = a_1(t+1)$. Consequently, for the infinite sequence

$$a_n: \quad a_1(1),\ a_2(1),\ a_3(1),\ a_4(1) = a_1(2),\ a_2(2),\ a_3(2),\ a_4(2) = a_1(3),\ a_2(3), \ldots$$

the sequence of second differences of its squares is

$$\Delta^2(a_n^2): \quad 2,\ 2,\ b_1,\ 2,\ 2,\ b_2,\ 2,\ 2,\ \ldots$$

where $b_1, b_2, \ldots$ are large positive integers, e.g. $b_1 = 2882,\ b_2 = 9602$.

No nontrivial Büchi sequence of length 5 is known.

BÜCHI'S CONJECTURE. *For some $n \geq 5$ every Büchi sequence of length n is trivial.*

2. Hilbert's Tenth Problem. The Büchi Conjecture is related to Hilbert's Tenth Problem.

HILBERT'S TENTH PROBLEM. *Does there exist an algorithm which decides whether there is an integer solution of any polynomial equation with integral coefficients $f(x_1, \ldots, x_n) = 0$?*

THEOREM (M. Davis, H. Putnam, J. Robinson, Yu. Matiyasevich). *Such an algorithm does not exist.*

One can replace the ring of integers by any other ring and state a similar problem. For example, for the field of rational numbers the Hilbert's Tenth Problem is still open.

We may also replace the set of all polynomial equations by its proper subset, or by a set of systems of polynomial equations.

For example, for the set of all systems of linear equations over any field there is an algorithm, given in Linear Algebra, which decides the solvability of every such system.

The solvability in integers of a polynomial equation can be reduced to the solvability of a polynomial equation of degree at most 4. Thus by the DPRM theorem there is no algorithm which decides the solvability of all polynomial equations of degree at most 4.

We explain this on an example.

EXAMPLE 3. We consider the equation

$$x^5 + 2y^2 + 7 = 0. \tag{2}$$

Put

$$x_2 = x_1^2,\ x_3 = x_2^2,\ x_4 = x_1 x_3,\ x_6 = x_5^2 \quad \text{and} \quad x_4 + 2x_6 + 7 = 0. \tag{3}$$

From (3) we have $x_4 = x_1^5$ and $x_6 = x_5^2$. Therefore $(x_1, x_2, x_3, x_4, x_5, x_6)$ is a solution of (3) if and only if $(x, y) = (x_1, x_5)$ is a solution of (2).

Consequently (2) is solvable in $\mathbb{Z}$ if and only if

$$(x_1^2 - x_2)^2 + (x_2^2 - x_3)^2 + (x_4 - x_1 x_3)^2 + (x_5^2 - x_6)^2 + (x_4 + 2x_6 + 7)^2 = 0$$

is solvable in $\mathbb{Z}$. The last polynomial is of degree 4, but have more variables than the polynomial (2).

3. Relations between Büchi's Conjecture and Hilbert's Tenth Problem. By definition, $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ is a Büchi sequence if

$$\Delta^2(a_j^2) = a_j^2 - 2a_{j+1}^2 + a_{j+2}^2 = 2 \quad \text{for} \quad j = 1, 2, \ldots, n-2. \tag{4}$$

This sequence is trivial if and only if $\varepsilon_{j+1} a_{j+1} = \varepsilon_j a_j + 1$ for $j = 1, 2, \ldots, n-1$ and some $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{-1, 1\}$.

LEMMA 1. *A Büchi sequence $a_1, a_2, \ldots, a_n$ is trivial if and only if $\varepsilon_2 a_2 = \varepsilon_1 a_1 + 1$ for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.*

Proof. Assume that $\varepsilon_2 a_2 = \varepsilon_1 a_1 + 1$ for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. We prove by induction on k, $1 \leq k \leq n-1$, that there exist $\varepsilon_k, \varepsilon_{k+1} \in \{-1, 1\}$ such that

$$\varepsilon_{k+1} a_{k+1} = \varepsilon_k a_k + 1. \tag{5}$$

For $k = 1$ (5) holds by assumption. Assume that (5) holds for some $k < n-1$. We shall prove (5) with k replaced by $k+1$.

From (4) it follows that

$$\begin{aligned} a_{k+2}^2 &= 2 + 2a_{k+1}^2 - a_k^2 = 2 + 2a_{k+1}^2 - (\varepsilon_{k+1} a_{k+1} - 1)^2 \\ &= a_{k+1}^2 + 2\varepsilon_{k+1} a_{k+1} + 1 = (\varepsilon_{k+1} a_{k+1} + 1)^2. \end{aligned}$$

Hence $\varepsilon_{k+2} a_{k+2} = \varepsilon_{k+1} a_{k+1} + 1$ for some $\varepsilon_{k+2} \in \{-1, 1\}$.

By induction the lemma follows. ■

Applying Lemma 1 the Büchi conjecture can be stated equivalently as follows.

BÜCHI'S CONJECTURE. *There exists $n \geq 5$ such that for every sequence of integers $a_1, a_2, \ldots, a_n$ satisfying (4) we have $\varepsilon_2 a_2 = \varepsilon_1 a_1 + 1$ for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.*

THEOREM (J. R. Büchi). *If for some $n \geq 5$ every Büchi sequence of length n is trivial, then the solvability in integers of any polynomial equation with integral coefficients is equivalent to the solvability in integers of an appropriate system of quadratic diagonal equations*

$$\sum_j a_{ij} y_j^2 = b_i, \qquad i = 1, \ldots, k, \quad a_{ij}, b_i \in \mathbb{Z}. \tag{6}$$

Proof. Similarly as in Example 3 above, it is easy to see that the solvability in integers of any polynomial equation with integral coefficients is equivalent to the solvability in integers of an equation of the form

$$\sum_{i,j,k} (x_i x_j - x_k)^2 + \sum_{l,m} (x_l - x_m^2)^2 + \Big(\sum_r a_r x_r\Big)^2 = 0,$$

where all a_r are integers.

Therefore to prove the Büchi theorem it is sufficient to prove, assuming the Büchi conjecture, that the solvabilty in integers of any equation of the form

$$x_i x_j = x_k, \quad x_l = x_m^2, \quad \text{and} \quad \sum_r a_r x_r = 0$$

is equivalent to the solvability in integers of a system of quadratic diagonal equations of the form (6).

We give the details of the proof for the first equation only, so let $uv = w$, where u, v, w are variables. Fix $n \geq 5$ satisfying the Büchi conjecture.

We consider the following existential formulas with systems of equations of the form (6).

$S(x_1, x_2) := \exists\, x_3, x_4, \ldots, x_n$ such that (4), with every a_j replaced by x_j, holds.

Then, by Büchi's conjecture, $S(x_1, x_2)$ implies that $\varepsilon_2 x_2 = \varepsilon_1 x_1 + 1$ for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.

$T(x, y) := \exists\, z\ S(y, z) \,\&\, (z^2 = y^2 + 2x + 1)$, where x is a quadratic diagonal form, and y, z are variables.

Thus $T(x, y)$ implies that $\varepsilon_2 z = \varepsilon_1 y + 1$ for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, and $(\varepsilon_1 y + 1)^2 = y^2 + 2x + 1$, that is $\varepsilon_1 y = x$, so $y^2 = x^2$.

Now, $uv = w$ implies that $u^2 v^2 = w^2$, hence

$$(u^2 + v^2)^2 = u^4 + v^4 + 2w^2. \tag{7}$$

Using new variables r, s, t we see that

$T(u^2 + v^2, r)$ means that $(u^2 + v^2)^2 = r^2$,

$T(u^2, s)$ means that $u^4 = s^2$,

$T(v^2, t)$ means that $v^4 = t^2$.

Consequently $u^2 v^2 = w^2$ is equivalent to $r^2 - s^2 - t^2 - 2w^2 = 0$, that is to the existential formula with polynomials of the form (6):

$$\exists\, r, s, t\ T(u^2 + v^2, r) \,\&\, T(u^2, s) \,\&\, T(v^2, t) \,\&\, (r^2 - s^2 - t^2 - 2w^2 = 0). \ \blacksquare$$

COROLLARY 1. *The DPRM Theorem and Büchi's conjecture imply that there does not exist an algorithm which decides the solvability in integers of every system of quadratic diagonal equations with integer coefficients.*

4. Modifications of Büchi's conjecture. We cannot prove the original Büchi's conjecture, but we can modify it:

a) Replacing $\mathbb{Z}$ by some other ring, for example, by $\mathbb{Q}$.

b) Considering sequences of positive integers, such that the sequence of their squares has constant second differences, not necessarily equal to 2.

In the table below we give the information on our knowledge on the existence of nontrivial sequences of the length n with constant second difference Δ^2 of the sequence of its squares. In the case $\Delta^2 \neq 2$ the sequence is called trivial if its terms with appropriate signs form an arithmetic progression. We give separately an information on symmetric sequences.

	$\Delta^2 = 2$		Δ^2 arbitrary	Author(s)
n	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{Z} \sim \mathbb{Q}$	
4	∞	$\Rightarrow \infty$	$\Rightarrow \infty$	D. A. Buell
5	?	∞	$\Rightarrow \infty$	E. J. Barbeau
6		∞ sym ? nonsym	$\Rightarrow \infty$ sym ∞ nonsym	 BB
7		? sym	∞ nonsym from $n = 8$ 19 other nonsym ? sym	D. Allison D. Allison, A. Bremner, BB
8			∞ sym ? nonsym	D. Allison
10			No sym	E. González-Jiménez, X. Xarles

Notation: ∞ means that there are infinitely many such sequences, $\Rightarrow$ means that this follows from the statement on the left. BB is an abbreviation of J. Browkin & J. Brzeziński.

We conjecture that in every case marked by "?" the corresponding Büchi sequences do not exist.

Other results

- In $\mathbb{R}$ there are infinite nontrivial Büchi sequences, e.g. $a_n := \sqrt{n^2 + c}$, $c \in \mathbb{R}^*$. Also the sequence of Example 2, can be extended to an infinite Büchi sequence in $\mathbb{R}$.
- The same holds for $\mathbb{Q}_p$. In $\mathbb{Z}_p$ there are Büchi sequences of an arbitrary finite length, but there is no infinite one, see [Br].

Examples:

n		Δ^2
5	$\frac{1}{9}(11, 50, 71, 88, 103)$	2
6	$\frac{1}{60}(241, 209, 191, 191, 209, 241)$	2
6	$(54, 229, 316, 381, 434, 479)$	-2110
7	$(53, 173, 217, 233, 227, 197, 127)$	-9960
8	$(17, 53, 67, 73, 73, 67, 53, 17)$	-840

5. Methods of algebraic geometry. The Büchi conjecture can be also interpreted as a problem in arithmetic algebraic geometry.

For $n \geq 5$ there is given a projective algebraic variety in $\mathbb{P}^n$:

$$X_n \;:\; a_j^2 - 2a_{j+1}^2 + a_{j+2}^2 = 2a_0^2, \quad \text{where} \quad 1 \leq j \leq n-2. \tag{8}$$

On the variety there are 2^{n-1} lines L_ε, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and $\varepsilon_j \in \{-1, 1\}$, given by

$$\varepsilon_{j+1} a_{j+1} = \varepsilon_j a_j + a_0 \quad \text{for} \quad 1 \leq j \leq n-1.$$

The Büchi conjecture claims that there are no other points $(a_0, a_1, \ldots, a_n) \in X_n$ such that $a_0 = 1$ and $a_1\ a_2, \ldots, a_n \in \mathbb{Z}$.

If we replace in (8) the second difference 2 by an arbitrary Δ we get

$$a_j^2 - 2a_{j+1}^2 + a_{j+2}^2 = \Delta a_0^2. \tag{9}$$

Eliminating Δa_0^2 from consecutive equations in (9) we obtain a projective algebraic variety in $\mathbb{P}^{n-1}$:

$$X_n' \ : \ a_i^2 - 3a_{i+1}^2 + 3a_{i+2}^2 - a_{i+3}^2 = 0, \quad i = 1, 2, \ldots, n-4. \tag{10}$$

We look for rational points on the varieties X_n and X_n' for small values of n, for the details see [BB2].

The case $n = 5$

The substitution

$$\begin{aligned} a_1 &= X_0X_1 - 2X_2^2, \\ a_2 &= X_0X_2 - X_1X_2, \\ a_3 &= X_0X_3, \\ a_4 &= X_0X_2 + X_1X_2, \\ a_5 &= X_0X_1 + 2X_2^2 \end{aligned}$$

gives a birational mapping $\varphi : X_5 \to Y_5$, where Y_5 is a projective variety in $\mathbb{P}^3$.

$$Y_5 \ : \ X_0^2X_1^2 + 4X_2^4 - 4X_0^2X_2^2 - 4X_1^2X_2^2 + 3X_0^2X_3^2 = 0. \tag{11}$$

If we put here $X_0 = 1$ and assume that X_2 is fixed, then (11) is an equation of a quadric in X_1, X_3 with a rational point $(X_0, X_1, X_2, X_3) = (1, 1, 1, 1)$. Consequently, Y_5 has a rational parametrization over $\mathbb{Q}$, which can be written explicitly. Then we get infinitely many points on Y_5, hence on X_5, with rational coordinates, which correspond to nontrivial Büchi sequences with rational terms.

The case $n = 6$

Similarly as above the substitution:

$$\begin{aligned} a_1 &= X_0X_1 - 5X_2X_3, \\ a_2 &= X_0X_2 - 3X_1X_3, \\ a_3 &= X_0X_3 - X_1X_2, \\ a_4 &= X_0X_3 + X_1X_2, \\ a_5 &= X_0X_2 + 3X_1X_3. \\ a_6 &= X_0X_1 + 5X_2X_3 \end{aligned}$$

gives a birational mapping $\varphi : X_6$ in $\mathbb{P}^6$ on the variety Y_6, in $\mathbb{P}_3$:

$$Y_6 \ : \ X_0^2X_1^2 - 3X_0^2X_2^2 + 2X_1^2X_2^2 + 25X_2^2X_3^2 - 27X_1^2X_3^2 = 0.$$

It is a K3 surface. On this surface there are elliptic curves defined over $\mathbb{Q}$ with positive ranks, so there are infinitely many nontrivial Büchi sequences of length 6 with rational terms.

In all known numerical examples these sequences are symmetric.

The case $n = 7$

The substitution

$$\begin{aligned}
a_1 &= X_0X_1 - 3X_2X_3,\\
a_2 &= X_0X_2 - 2X_1X_3,\\
a_3 &= X_0X_3 - X_1X_2,\\
a_4 &= X_0X_4,\\
a_5 &= X_0X_3 + X_1X_2,\\
a_6 &= X_0X_2 + 2X_1X_3,\\
a_7 &= X_0X_1 + 3X_2X_3
\end{aligned}$$

gives a birational equivalence of X_7' with the variety Y_7 given by two equations:

$$X_0^2X_1^2 + 9X_2^2X_3^2 - 3X_0^2X_2^2 - 12X_1^2X_3^2 + 3X_1^2X_3^2 + 3X_1^2X_2^2 - X_0^2X_4^2 = 0,$$
$$X_0^2X_2^2 + 4X_1^2X_3^2 - 4X_0^2X_3^2 - 4X_1^2X_2^2 + 3X_0^2X_4^2 = 0.$$

If we put $X_0 = 1$, the last equation is a conic in X_2, X_3, which can be parametrized. This leads to a biquadratic equation in X_4. Some of its solutions can be found numerically. This leads to some examples of nontrivial generalized Büchi sequences of length 7. There is also an approach of Bremner, leading to other examples, see [Bre].

P. Vojta deduced the Büchi conjecture (for some n not given explicitly) from the Bombieri conjecture, which is a particular case of the Lang conjecture, see [Voj].

Bombieri's Conjecture. *If X is a smooth projective variety of general type defined over an algebraic number field k, then there exists a proper algebraic subset Z of X, such that the set $X(k) \setminus Z(k)$ is finite.*

There are many papers devoted to the Büchi conjecture, some of them are included in References below, see also References in [PPV].

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