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ON QUOTIENTS OF THE SPACE OF ORDERINGS OF THE FIELD $\mathbb{Q}(x)$

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Abstract. In this paper we present a method of obtaining new examples of spaces of orderings by considering quotient structures of the space of orderings $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ — it is, in general, nontrivial to determine whether, for a subgroup $G_0 \subset G_{\mathbb{Q}(x)}$ the derived quotient structure $(X_{\mathbb{Q}(x)}|_{G_0}, G_0)$ is a space of orderings, and we provide some insights into this problem. In particular, we show that if a quotient structure arising from a subgroup of index 2 is a space of orderings, then it necessarily is a profinite one.

1. Preliminaries

1.1. Introduction. The theory of abstract spaces of orderings was developed by Murray Marshall in a series of papers from the late 1970s, and provides an abstract framework for studying orderings of fields and the reduced theory of quadratic forms in general. The monograph [7] will be of frequent use here as far as background, notation, and main results are concerned. Spaces of orderings also occur in a natural way in other, more general settings: as maximal orderings on semi-local rings, as orderings on skew fields, or as orderings on ternary fields. The axioms for spaces of orderings have been also

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generalized in various directions — to quaternionic schemes, to spaces of signatures of higher level, or to abstract real spectra that are used to study orderings on commutative rings.

Since the invention of abstract spaces of orderings there has been a considerable interest in the question of when such a space is realized as a space of orderings of a field. It seems likely that spaces of orderings exist that are not so realized but, so far, no such examples are known. This motivated our interest in profinite spaces, that were introduced in [6], and that have been already studied in our previous paper [3]. In this work we expand methods developed there to investigate a completely new class of spaces of orderings, namely quotient spaces of the space $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ coming from subgroups of $G_{\mathbb{Q}(x)}$ of finite index.

The main result of this paper is a theorem stating that if a quotient structure of $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ arises from a subgroup of index 2 and is a space of orderings then it is profinite. This is interesting because at a first glance it is hard to find any regularity among quotient structures. Necessary and sufficient conditions for a quotient structure of $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ arising from a subgroup of index 2 to be a space of orderings have been given in [4], as well as numerous examples of such spaces. The main result of this paper has been generalized there to the "if and only if" condition. However, the methods used in [4] are completely different from the methods used in our work: we use a geometric approach here, which explains how the profinite space is built.

1.2. Abstract spaces of orderings. We assume the reader is familiar with abstract spaces of orderings as developed, for example, in [7] as well as their associated Witt rings and basic notions such as morphisms, the Harrison topology, subspaces, direct sums, and group extensions. We briefly review the key definitions from that theory that are used in the core arguments of this paper. We view a space of orderings as a pair (X, G) such that X is a nonempty set, G is a subgroup of $\{1, -1\}^X$, which contains the constant function -1, separates points of X, and satisfies two additional axioms. Considering X as a subset of the character group $\chi(G)$, for any pair $a, b \in G$ the value set is defined by

$$D_X(a,b) = \big\{c \in G : \forall x \in X \ \big(c(x) = a(x) \vee c(x) = b(x)\big)\big\},$$

and with this the axioms state

(1) if $x \in \chi(G)$ satisfies $x(-1) = -1$, and if
$$\forall a, b \in \ker(x) \ (D_X(a,b) \subset \ker(x)),$$
then x is in the image of the natural embedding $X \hookrightarrow \chi(G)$, and

(2) $\forall a_1, a_2, a_3, b, c \in G \ \exists d \in G$
$\big[(b \in D_X(a_1, c) \wedge c \in D_X(a_2, a_3)) \Rightarrow (b \in D_X(d, a_3) \wedge d \in D_X(a_1, a_2))\big]$.

Particularly important to us are fans and the structure theory of finite spaces of orders. For any multiplicative group G of exponent 2 with distinguished element -1, we set $X = \{x \in \chi(G) : x(-1) = -1\}$ and call the pair (X, G) a *fan*. A fan is also a space of orderings ([7, Theorem 3.1.1]). If (X, G) is a space of orderings, by a fan in (X, G) we understand a subspace $\mathcal{F}$ such that the space $(\mathcal{F}, G|_{\mathcal{F}})$ is a fan. One easily checks that any one- or two-element subset of a space of orderings forms a fan — thus one-or two-element fans are called trivial fans. For a space of orderings (X,G) we define the connectivity relation $\sim$ as follows: if $x_1,x_2\in X$, then $x_1\sim x_2$ if and only if either $x_1=x_2$ or there exists a four-element fan $\mathcal{F}$ in (X,G) such that $x_1,x_2\in\mathcal{F}$. The equivalence classes with respect to $\sim$ are called the connected components of (X,G). It is known that if (X,G) is a finite space of orders, and $X_1,X_2,\dots$ are its connected components, then $(X,G)=(X_1,G|_{X_1})\sqcup(X_2,G|_{X_2})\sqcup\dots$, where $(X_i,G|_{X_i})$, are either one element spaces or proper group extensions. For details see [7, pages 24–30].

An inverse system of spaces of orderings is a triple consisting of:

(1) a directed set $(I,\succeq)$,
(2) spaces of orderings (X_i,G_i), $i\in I$,
(3) morphisms $F_{ij}:(X_i,G_i)\to(X_j,G_j)$ defined for $i\succeq j$, $i,j\in I$, such that
   (a) $F_{ij}(X_i)=X_j$,
   (b) $F_{ik}=F_{jk}\circ F_{ij}$, for $i\succeq j\succeq k$, $i,j,k\in I$.

Clearly, an inverse system $(I,(X_i,G_i),F_{ij})$ of spaces of orderings automatically defines both a direct system of groups (I,G_i,F_{ij}^*), and an inverse system of character groups (I,X_i,F_{ij}). Further, if we let $G=\varinjlim G_i$, and $X=\varprojlim X_i$, then (X,G) is a space of orderings that is called the inverse limit of the given inverse system and denoted by $\varprojlim(X_i,G_i)$ ([6, Theorem 4.3]). For a fixed $i\in I$ we will denote by π_i the projection $\pi_i:X\to X_i$ such that $\pi_i=F_{ij}\circ\pi_j$, for $j\succeq i$, $j\in I$, and by γ_i the injection $\gamma_i:G_i\to G$ such that $\gamma_i=\gamma_j\circ F_{ij}^*$, for $j\succeq i$, $j\in I$. Since, in fact, $G=\bigcup_{i\in I}G_i$, we will use the same symbol a for an element $a\in G_i$ and its image $a\in\gamma_i(G_i)\subset G$. A space of orderings which is an inverse limit of finite spaces of orderings will be called profinite.

The notions of a quotient structure and quotient spaces are key to this paper. If (X,G) is a space of orderings, and G_0 is a subgroup of G containing the element -1, we denote by X_0 the set $X|_{G_0}$ of all characters from X restricted to G_0. In the case when (X_0,G_0) is a space of orderings, we call it a *quotient space* of (X,G) — otherwise, in general, we call it a *quotient structure*. One of the main goals of this paper is to better understand quotient structures of $(X_{\mathbb{Q}(x)},G_{\mathbb{Q}(x)})$, and for this, we study their finite quotients and investigate if they can be aligned to form an inverse limit.

1.3. Space of orderings of a field. Let k be a formally real field. Denote by X_k the set of all orderings of k, and by G_k the multiplicative group $k^*/(\Sigma k^2)^*$ of all classes of sums of squares of k. G_k is naturally identified with a subgroup of $\{-1,1\}^{X_k}$ via the homomorphism

$$k^*\ni a\mapsto\overline{a}\in\{-1,1\}^{X_k},\ \text{where } \overline{a}(\sigma)=\begin{cases}1, & \text{if } a\in\sigma,\\ -1, & \text{if } a\notin\sigma,\end{cases}\ \text{for } \sigma\in X_k,$$

whose kernel is the set $(\Sigma k^2)^*$ of all nonzero sums of squares of k, and (X_k,G_k) is a space of orderings ([7, Theorem 2.1.4]). For simplicity we shall denote by the same symbol a both an element $a\in k^*$, a class of sums of squares $a\in k^*/(\Sigma k^2)^*$, and a function $a\in\{-1,1\}^{X_k}$. Also, for an abstract space of orderings (X,G) we will usually denote elements of the set X by small letters $x,y,z,\dots$, whilst for a space of orderings (X_k,G_k) of a field k we shall denote orderings from the set X_k by small Greek letters $\sigma,\tau,\upsilon,\dots$.

Recall that an ordering σ of k is called Archimedean if, for every $a \in k$, there exists $n \in \mathbb{N}$ such that $n - a \in \sigma$ — otherwise the ordering σ is non-Archimedean. Orderings on fields are closely related to valuations, and the structure of non-Archimedean orderings of k can be completely described in terms of valuations. For what we need here recall that a valuation $v : k \to \Gamma \cup \{\infty\}$ is said to be compatible with an ordering $\sigma \in X_k$ if $b - a \in \sigma$ implies $v(a) \geq v(b)$ (see [5, Theorem 2.3] for equivalent definitions).

1.4. Lam's Open Problem B. Since the invention of the theory of abstract spaces of orderings it has remained an open question whether all abstract spaces are realizable. It seems likely that the answer to this question is "no", but, to date, no such examples are known. The main motivation to initiate the research that lead to this paper was a search for such an example. It seems that the tool that might be used in determining whether a space of orderings is realizable or not is the following question: is it true that for a space of orderings (X, G) the equality

$$W(X, G) \cap \mathcal{C}(X, 2^n\mathbb{Z}) = I^n(X, G)$$

holds? Here $\mathcal{C}(X, 2^n\mathbb{Z})$ denotes all continuous functions $X \to 2^n\mathbb{Z}$, and $I^n(X, G)$ is the n-th power of the fundamental ideal $I(X, G)$. This question is usually referred to as Lam's Open Problem B. It can be easily verified when $n = 1$ or $n = 2$, and for all spaces of orderings (X, G) such that $\text{stab}(X, G) \leq 3$ [8, Proposition 3.1]. Furthermore, the question has an affirmative answer for realizable spaces of orderings, which has been recently proven by Dickmann and Miraglia in [2] using the celebrated results by Orlov, Vishik and Voevodsky [9], [10].

Therefore, in order to prove that there is a space of orderings that is not realizable, one would like to construct a space of orderings (X, G) which has at least 16-element fans, and a quadratic form $\phi = ((a_1, b_1)) \oplus \ldots \oplus ((a_s, b_s))$, $s \in \mathbb{N}$, such that $\phi(x) \equiv 0 \mod 8$, for all $x \in X$, although

$$\phi \neq c_1((d_1, e_1, f_1)) \oplus \ldots \oplus c_t((d_t, e_t, f_t)),$$

for all possible choices of $t \in \mathbb{N}$, and $d_i, e_i, f_i \in G$, $i \in \{1, \ldots, t\}$.

This observation sparked our interest in methods of "blowing up" existing fans in well understood spaces of orderings, and thus obtaining examples of new spaces, where Lam's problem needs to be verified. We shall describe it in some more detail in later subsections.

1.5. Orderings of the field $\mathbb{Q}(x)$. The space of orderings of the field $\mathbb{Q}(x)$ is both nontrivial and well understood, hence it is a natural candidate for a starting point in the search for new constructions of spaces of orderings. We shall describe it here in some more detail (see, for example, [1, Notation 1.4]).

Each irreducible polynomial $p \in \mathbb{Q}[x]$ with real roots $\alpha_1 < \ldots < \alpha_n$, $n \geq 1$, gives rise to $2n$ orderings of $\mathbb{Q}(x)$, namely σ_j^- and σ_j^+, $j \in \{1, \ldots, n\}$, defined as follows:

(1) for $a \in \mathbb{Q}(x)^*$ and $j \in \{1, \ldots, n\}$, $a \in \sigma_j^-$ if and only if, for some $\epsilon > 0$, a is strictly positive on the interval $(\alpha_j - \epsilon, \alpha_j)$, and
(2) for $a \in \mathbb{Q}(x)^*$ and $j \in \{1, \ldots, n\}$, $a \in \sigma_j^+$ if and only if, for some $\epsilon > 0$, a is strictly positive on the interval $(\alpha_j, \alpha_j + \epsilon)$.

Similarly, we define two orderings ∞^- and ∞^+:

(3) for $a \in \mathbb{Q}(x)^*$, $a \in \infty^-$ if and only if, for some $\xi \in \mathbb{Q}$, a is strictly positive on the interval $(-\infty, \xi)$, and

(4) $a \in \infty^+$ if and only if, for some $\xi \in \mathbb{Q}$, a is strictly positive on the interval $(\xi, +\infty)$.

Finally, for each transcendental number $\zeta \in \mathbb{R}$, we define the ordering ζ^0:

(5) for $a \in \mathbb{Q}(x)^*$, $a \in \zeta^0$ if and only if $a(\zeta) \geq 0$.

These are precisely all the elements of $X_{\mathbb{Q}(x)}$.

For an irreducible polynomial $p \in \mathbb{Q}[x]$ with real roots $\alpha_1 < \ldots < \alpha_n$, $n \geq 1$, consider the p-adic valuation $v_p : \mathbb{Q}(x) \to \mathbb{Z} \cup \{\infty\}$. The orderings compatible with v_p are precisely $\sigma_1^-, \sigma_1^+, \ldots, \sigma_n^-, \sigma_n^+$, and form a subspace of $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ that will be denoted by (X_p, G_p). The valuation ring $\mathcal{O}_p$ of v_p is simply the localization of $\mathbb{Q}[x]$ at (p), its only maximal ideal $\mathfrak{M}_p$ is the principal ideal generated by p, and the residue class field $\overline{\mathbb{Q}(x)}_p$ is isomorphic to $\mathbb{Q}[x]/(p)$. The preordering T_p associated to v_p is $T_p = \Sigma\mathbb{Q}(x)^2[1+\mathfrak{M}_p]$, and $(X_p, G_p) = (X_{T_p}, G_{T_p})$. If we denote by $\mathbb{Q}(x)_p^h$ the Henselization of $\mathbb{Q}(x)$ at v_p, then the inclusion $\mathbb{Q}(x) \hookrightarrow \mathbb{Q}(x)_p^h$ identifies (X_p, G_p) with the space of orderings $(X_{\mathbb{Q}(x)_p^h}, G_{\mathbb{Q}(x)_p^h})$.

Similarly, consider the valuation $v_\infty : \mathbb{Q}(x) \to \mathbb{Z} \cup \{\infty\}$:

$$v_\infty\Big(\frac{f}{g}\Big) = \deg g - \deg f, \quad \text{for } f, g \in \mathbb{Q}[x],$$

(here we assume $\deg 0 = -\infty$). The orderings compatible with v_∞ are ∞^- and ∞^+, they form a subspace that will be denoted by (X_∞, G_∞), the valuation ring of v_∞ is $\mathcal{O}_\infty = \{\frac{f}{g} : \deg f \leq \deg g\}$, with its maximal ideal $\mathfrak{M}_\infty = \{\frac{f}{g} : \deg f < \deg g\}$. Consequently, the residue class field $\overline{\mathbb{Q}(x)}_\infty$ of v_∞ is isomorphic to $\mathbb{Q}$. Analogous statements about the preordering T_∞ associated to v_∞ and the Henselization $\mathbb{Q}(x)_\infty^h$ of $\mathbb{Q}(x)$ also carry over.

All non-Archimedean orderings of $\mathbb{Q}(x)$ are either of the form σ_j^-, σ_j^+, for some irreducible $p \in \mathbb{Q}[x]$ with a real root α_j, or of the form ∞^-, ∞^+. All non-trivial valuations of the field $\mathbb{Q}(x)$ that are trivial on $\mathbb{Q}$ are either v_p, for $p \in \mathbb{Q}[x]$ irreducible, or v_∞. The structure of the space of orderings $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ can be described as follows: for a fixed $p \in \mathbb{Q}[x]$ irreducible with the real roots $\alpha_1, \ldots, \alpha_n$ and $n \geq 2$, the set of orderings $\{\sigma_1^+, \sigma_1^-, \ldots, \sigma_n^+, \sigma_n^-\}$ is a connected component of the cardinality greater than 1 in the sense of the connectivity relation defined before. All connected components of cardinality greater than 1 are of that form. Furthermore, for a fixed $p \in \mathbb{Q}[x]$ irreducible, the space (X_p, G_p) is the group extension $(X_{\mathbb{Q}[x]/(p)}, G_{\mathbb{Q}[x]/(p)}) \times \langle p \rangle$. The orderings of $\mathbb{Q}[x]/(p)$ correspond to the real roots $\alpha_1 < \ldots < \alpha_n$ of p, and will be denoted by $\sigma_1, \ldots, \sigma_n$. We have

$$f + (p) \in \sigma_j \text{ if and only if } f(\alpha_j) > 0,$$

and in the extension p splits σ_j into the orderings σ_j^- and σ_j^+, $j \in \{1, \ldots, n\}$.

The four-element fans in $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ are the sets $\{\sigma_i^-, \sigma_i^+, \sigma_j^-, \sigma_j^+\}$, for $1 \leq i < j \leq n$, and for some irreducible $p \in \mathbb{Q}[x]$ having $n \geq 2$ real roots. As before, a four-element fan $\{\sigma_i^-, \sigma_i^+, \sigma_j^-, \sigma_j^+\}$ will be viewed as the group extension

$$(\{\sigma_i, \sigma_j\}, G_{\mathbb{Q}[x]/(p)}|_{\{\sigma_i, \sigma_j\}}) \times \langle p \rangle,$$

where p splits σ_i and σ_j into $\sigma_i^-, \sigma_i^+, \sigma_j^-$, and σ_j^+.

2. Quotients of $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$

2.1. Regular and non-regular points. We shall describe how the structure of a quotient structure of $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ can be determined, and, in particular, how fans in spaces arising from such structures are constructed. For this we need the notion of what we call regular and non-regular points. To simplify our notation, from now on denote, for a polynomial $g \in \mathbb{Q}[x]$:

$$K_g = \{\beta \in \mathbb{R} : g(\beta) > 0\}.$$

Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$. A point $\alpha \in \mathbb{R}$ on the real line will be called *regular* with respect to the subgroup G_0 if, for every open neighborhood U of α, there exists $g \in G_0$ such that $g(\alpha) > 0$ and $K_g \subset U$. Otherwise a point will be called *non-regular* with respect to $G_0 \subset G_{\mathbb{Q}(x)}$.

We will make a frequent use of the following characterization of regular points:

PROPOSITION 2.1. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, let $\alpha \in \mathbb{R}$. The following conditions are equivalent*:

(1) *α is regular with respect to G_0,*
(2) *the family*

$$\{K_g : g \in G_0,\ g(\alpha) > 0\}$$

forms a basis of neighborhoods of α,
(3) *for every family of polynomials $S \subset \mathbb{Q}[x]$ such that*

$$\{\alpha\} = \bigcap_{f \in S} K_f, \tag{2.1}$$

the following condition is satisfied:

$$\forall f_1, f_2 \in S\ \exists g \in G_0\ \big[(\alpha \in K_g) \wedge (K_g \subset K_{f_1} \cap K_{f_2})\big].$$

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Rightarrow$ (3) are obvious. To show (3) $\Rightarrow$ (1) fix an open neighborhood U of α, and let $I \subset U$ be an open interval such that $\alpha \in I$. For each pair of rational numbers $q_1, q_2 \in I$ with $q_1 < \alpha < q_2$ there is a parabola $f \in \mathbb{Q}[x]$ that zeroes at q_1 and q_2 which is positive at α — thus the condition (2.1) is satisfied for the family S of all such parabolas, and by (3) there exists $g \in G_0$ positive at α with $K_g \subset I \subset U$. ∎

Observe that if $G_0 = G_{\mathbb{Q}(x)}$ then, clearly, all points on the real line are regular: for a fixed point $\alpha \in \mathbb{R}$ and a family $S \subset \mathbb{Q}[x]$ satisfying (2.1), if $f_1, f_2 \in S$ and $\gamma_1 < \alpha < \gamma_2$ are the two of the real roots of f_1, f_2 that are adjacent to α, take some rational numbers q_1, q_2 such that $\gamma_1 < q_1 < \alpha < q_2 < \gamma_2$ and let $g(x) = -(x - q_1)(x - q_2)$ (see Figure 1). In the special cases when f_1, f_2 have no real roots or only one real root, a trivial version of the above argument follows.

In what follows we present a few results that provide some insights into what regular and non-regular points are; some of them are needed in the sequel, some of them are just separate statements.

PROPOSITION 2.2. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, and let $(G_{\mathbb{Q}(x)} : G_0) = n < \infty$. Then the set of all non-regular points with respect to G_0 is not dense.*

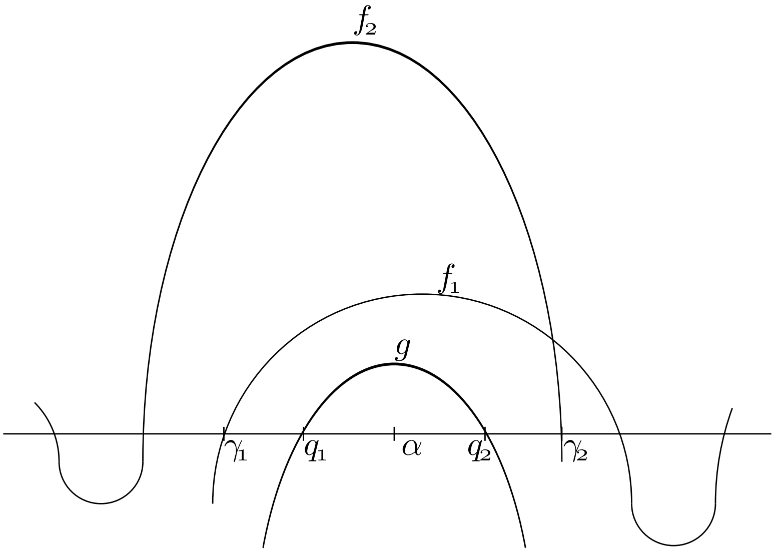

Fig. 1. All points are regular with respect to $G_{\mathbb{Q}(x)}$

Proof. Suppose that the set of non-regular points with respect to G_0 is dense. Fix a non-regular point α_1. Let $S_1 \subset \mathbb{Q}[x]$ be a set satisfying (2.1), and let $f_1^{(1)}, f_2^{(1)} \in S_1$ be such that

$$\forall g \in G_0 \ \big[(\alpha_1 \in K_g) \Rightarrow (K_g \not\subset K_{f_1^{(1)}} \cap K_{f_2^{(1)}})\big].$$

Since the set of non-regular points is dense, there is a non-regular point $\alpha_2 \in K_{f_1^{(1)}} \cap K_{f_2^{(1)}}$, $\alpha_2 \neq \alpha_1$. Let $S_2 \subset \mathbb{Q}[x]$ be a set satisfying (2.1) for α_2, and let $f_1^{(2)}, f_2^{(2)} \in S_2$ be such that

$$\forall g \in G_0 \ \big[(\alpha_2 \in K_g) \Rightarrow (K_g \not\subset K_{f_1^{(2)}} \cap K_{f_2^{(2)}})\big].$$

Clearly $K_{f_1^{(1)}} \cap K_{f_2^{(1)}} \cap K_{f_1^{(2)}} \cap K_{f_2^{(2)}} \neq \emptyset$, as it contains α_2. Thus we can choose a non-regular point $\alpha_3 \in K_{f_1^{(1)}} \cap K_{f_2^{(1)}} \cap K_{f_1^{(2)}} \cap K_{f_2^{(2)}}$, $\alpha_3 \notin \{\alpha_1, \alpha_2\}$, with corresponding set $S_3 \subset \mathbb{Q}[x]$ and polynomials $f_1^{(3)}, f_2^{(3)} \in S_3$. Eventually we will find n non-regular points $\alpha_1, \ldots, \alpha_n$, $\alpha_i \notin \{\alpha_1, \ldots, \alpha_{i-1}\}$, $i \in \{2, \ldots, n\}$, and, for each $i \in \{1, \ldots, n\}$, a set $S_i \subset \mathbb{Q}[x]$ satisfying (2.1) with polynomials $f_1^{(i)}, f_2^{(i)} \in S_i$ such that

$$\forall g \in G_0 \ \big[(\alpha_i \in K_g) \Rightarrow (K_g \not\subset K_{f_1^{(i)}} \cap K_{f_2^{(i)}})\big].$$

Moreover, for $i \in \{1, \ldots, n\}$

$$\alpha_i \in \bigcap_{j=1}^{i} K_{f_1^{(j)}} \cap K_{f_2^{(j)}}.$$

Let, for $i \in \{1, \ldots, n\}$, $\gamma_1^{(i)}$ and $\gamma_2^{(i)}$, $\gamma_1^i < \alpha_i < \gamma_2^{(i)}$, be two adjacent numbers from the combined set of all real roots of polynomials $f_1^{(j)}, f_2^{(j)}$, $j \in \{1, \ldots, i\}$, and the set $\{\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n\}$. Pick two rational numbers $q_1^{(i)}, q_2^{(i)}$ such that

$$\gamma_1^{(i)} < q_1^{(i)} < \alpha_i < q_2^{(i)} < \gamma_2^{(i)},$$

and let $g_i(x) = -(x - q_1^{(i)})(x - q_2^{(i)})$ (see Figure 2). Observe that $\alpha_i \in K_{g_i}$ and $K_{g_i} \subset \bigcap_{j=1}^{i} K_{f_1^{(j)}} \cap K_{f_2^{(j)}}$, so that $g_i \notin G_0$, $i \in \{1, \ldots, n\}$. Therefore, since $(G_{\mathbb{Q}(x)} : G_0) = n$, among the cosets

$$g_1 G_0, \ldots, g_n G_0$$

at least two are equal, say $g_i G_0 = g_j G_0$. Thus $g_i g_j \in G_0$ and, consequently, $-g_i g_j \in G_0$. Say $i > j$. Then $\alpha_i \in K_{-g_i g_j}$ and, at the same time, $K_{-g_i g_j} \subset K_{f_1^{(i)}} \cap K_{f_2^{(i)}}$ — a contradiction. ■

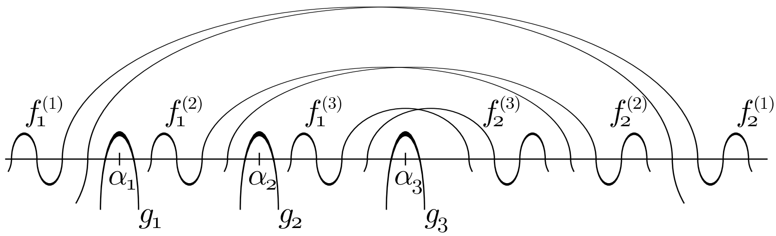

Fig. 2. Construction from the proof of Proposition 2.2 for $n = 3$

In the special case when $(G_{\mathbb{Q}(x)} : G_0) = 2$ we can actually do better than this.

Proposition 2.3. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, and let $(G_{\mathbb{Q}(x)} : G_0) = 2$. Then all non-regular points with respect to G_0 are isolated.*

Proof. Fix a non-regular point α and let $S_1 \subset \mathbb{Q}[x]$ be a set satisfying (2.1) such that for some $f_1^{(1)}, f_2^{(1)} \in S_1$ the following condition is fulfilled:

$$\forall g \in G_0 \ \big[(\alpha \in K_g) \Rightarrow (K_g \not\subset K_{f_1^{(1)}} \cap K_{f_2^{(1)}})\big].$$

It suffices to show that there are no other non-regular points in the set $K_{f_1^{(1)}} \cap K_{f_2^{(1)}}$. For suppose there is a non-regular point $\beta \in K_{f_1^{(1)}} \cap K_{f_2^{(1)}}$, take a family $S_2 \subset \mathbb{Q}[x]$ satisfying (2.1) and polynomials $f_1^{(2)}, f_2^{(2)} \in S_2$ such that

$$\forall g \in G_0 \ \big[(\beta \in K_g) \Rightarrow (K_g \not\subset K_{f_1^{(2)}} \cap K_{f_2^{(2)}})\big].$$

Let γ_1 and γ_2, $\gamma_1 < \alpha < \gamma_2$, be two adjacent numbers from the combined set of all real roots of polynomials $f_1^{(1)}, f_2^{(1)}$ and β. Similarly, let δ_1 and δ_2, $\delta_1 < \beta < \delta_2$, be two adjacent numbers from the combined set of all real roots of polynomials $f_1^{(1)}$, $f_2^{(1)}$, $f_1^{(2)}$, $f_2^{(2)}$ and α. Pick rational numbers q_1, q_2, r_1, r_2 such that $\gamma_1 < q_1 < \alpha < q_2 < \gamma_2$ and $\delta_1 < r_1 < \beta < r_2 < \delta_2$. Now define two polynomials:

$$g_1(x) = -(x - q_1)(x - q_2) \quad \text{and} \quad g_2(x) = -(x - r_1)(x - r_2).$$

Clearly $\alpha \in K_{g_1}$ and $K_{g_1} \subset K_{f_1^{(1)}} \cap K_{f_2^{(1)}}$, so that $g_1 \notin G_0$. Similarly $g_2 \notin G_0$. Thus the cosets $g_1 G_0$ and $g_2 G_0$ are equal, which implies that $g_1 g_2 \in G_0$, and, consequently, $-g_1 g_2 \in G_0$.

Moreover $K_{g_2} \subset K_{f_1^{(1)}} \cap K_{f_2^{(1)}}$, which ensures that $K_{-g_1 g_2} \subset K_{f_1^{(1)}} \cap K_{f_2^{(1)}}$. Since $\alpha \in K_{-g_1 g_2}$ this yields a contradiction. ■

We shall conclude this subsection with the following observation:

PROPOSITION 2.4. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, and let $(G_{\mathbb{Q}(x)} : G_0) = 2$. Let $\alpha \in \mathbb{R}$ be a non-regular point with respect to G_0. Then there exists $\epsilon > 0$ such that either*

$$\forall q \in \mathbb{Q} \cap (\alpha - \epsilon, \alpha)\ (x - q \in G_0) \quad \textit{and} \quad \forall q \in \mathbb{Q} \cap (\alpha, \alpha + \epsilon)\ (x - q \notin G_0),$$

or

$$\forall q \in \mathbb{Q} \cap (\alpha - \epsilon, \alpha)\ (x - q \notin G_0) \quad \textit{and} \quad \forall q \in \mathbb{Q} \cap (\alpha, \alpha + \epsilon)\ (x - q \in G_0).$$

Proof. Let $S \subset \mathbb{Q}[x]$ be a set of polynomials satisfying (2.1) such that for some $f_1, f_2 \in S$

$$\forall g \in G_0\ \big[(\alpha \in K_g) \Rightarrow (K_g \not\subset K_{f_1} \cap K_{f_2})\big].$$

Let γ_1 and γ_2, $\gamma_1 < \alpha < \gamma_2$, be two adjacent real roots of polynomials f_1, f_2, and define

$$\epsilon = \min\{\alpha - \gamma_1, \gamma_2 - \alpha\}.$$

Firstly, observe that

$$\forall q_1 \in \mathbb{Q} \cap (\alpha - \epsilon, \alpha)\ \forall q_2 \in \mathbb{Q} \cap (\alpha, \alpha + \epsilon)\ \big[(x - q_1 \in G_0) \vee (x - q_2 \in G_0)\big].$$

For suppose that for some $q_1 \in \mathbb{Q} \cap (\alpha - \epsilon, \alpha)$ and $q_2 \in \mathbb{Q} \cap (\alpha, \alpha + \epsilon)$ both $x - q_1 \notin G_0$ and $x - q_2 \notin G_0$. Since $(G_{\mathbb{Q}(x)} : G_0) = 2$, the two cosets $(x - q_1)G_0$ and $(x - q_2)G_0$ are equal, which follows that $g(x) = -(x - q_1)(x - q_2) \in G_0$. But then $\alpha \in K_g$ and $K_g \subset K_{f_1} \cap K_{f_2}$, which gives a contradiction.

To finish the proof it suffices to show that if, for some $q_1 \in \mathbb{Q} \cap (\alpha - \epsilon, \alpha)$, $x - q_1 \in G_0$, then, for all $q_2 \in \mathbb{Q} \cap (\alpha, \alpha + \epsilon)$, $x - q_2 \notin G_0$ (the symmetric case with $q_2 \in \mathbb{Q} \cap (\alpha - \epsilon, \alpha)$ and $q_1 \in \mathbb{Q} \cap (\alpha, \alpha + \epsilon)$ is analogous). Indeed, if, for some $q_1 \in \mathbb{Q} \cap (\alpha - \epsilon, \alpha)$, $x - q_1 \in G_0$, and, at the same time, for some $q_2 \in \mathbb{Q} \cap (\alpha, \alpha + \epsilon)$, $x - q_2 \in G_0$, then also $g(x) = -(x - q_1)(x - q_2) \in G_0$ with $\alpha \in K_g$, and $K_g \subset K_{f_1} \cap K_{f_2}$ — a contradiction. ∎

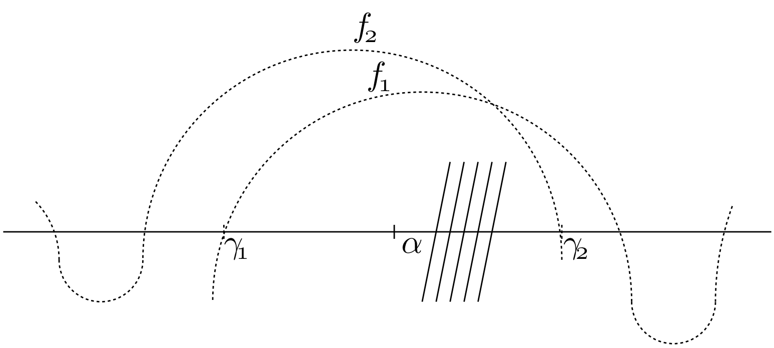

Fig. 3. Proposition 2.4: for a non-regular point α there are rational lines in G_0 lying arbitrarily close to α but only on one side

2.2. Fans in quotient spaces. As our next step towards the description of fans in quotient structures, let us introduce the notion of G_0-transcendental points. Consider the space $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ and let G_0 be a subgroup of the group $G_{\mathbb{Q}(x)}$. A point α on the real line $\mathbb{R}$ will be called G_0*-transcendental* if there is no element $g \in G_0$ such that $g(\alpha) = 0$. A point which is not G_0-transcendental will be called G_0*-algebraic.*

Observe that, clearly, any transcendental number is a G_0-transcendental point regardless of the choice of G_0. Recall that a transcendental number ζ gives rise to only one ordering ζ^0 in $X_{\mathbb{Q}(x)}$, whereas an algebraic number α induces two orderings in $X_{\mathbb{Q}(x)}$ that we denoted by σ^- and σ^+. Now if α is an algebraic number that becomes G_0-transcendental, then $\sigma^-|_{G_0} = \sigma^+|_{G_0}$, and therefore a G_0-transcendental point gives rise to only one ordering in $X_{\mathbb{Q}(x)}|_{G_0}$.

We shall now turn our attention to fans in quotient spaces.

PROPOSITION 2.5. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, and let $(G_{\mathbb{Q}(x)} : G_0) = n < \infty$. If (X_0, G_0) is a space of orderings, where $X_0 = X|_{G_0}$, then all fans in (X_0, G_0) have at most $4n$ elements.*

Proof. Suppose that there is a 2^k-element fan $\mathcal{F}$ in (X_0, G_0) with $2^k > 4n$. Orderings in $\mathcal{F}$ come from at least 2^{k-1} real points, in which case we have $2^{k-1} > n$ G_0-algebraic points $\alpha_1 < \ldots < \alpha_{2^{k-1}}$ that induce 2^k orderings $\sigma_1^-, \sigma_1^+, \ldots, \sigma_{2^{k-1}}^-, \sigma_{2^{k-1}}^+$ constituting the fan $\mathcal{F}$.

Pick rational numbers $q_0, q_1, \ldots, q_{2^{k-1}}$ such that

$$q_0 < \alpha_1 < q_1 < \alpha_2 < q_2 < \ldots < q_{2^{k-1}-1} < \alpha_{2^{k-1}} < q_{2^{k-1}},$$

and, for $i \in \{1, \ldots, 2^{k-1}\}$, define the polynomial $g_i(x) = -(x - q_i)(x - q_{i-1})$. If $g_1 \in G_0$, let $f_1 = g_1$. If not, we move to g_2: if $g_2 \in G_0$, let $f_1 = g_2$, otherwise we proceed to g_3. Eventually we either find a polynomial $f_1 \in G_0$ amongst $g_1, \ldots, g_n$ or see that $g_1, \ldots, g_n \notin G_0$. Note that $n < 2^{k-2}$. Since $(G_{\mathbb{Q}(x)} : G_0) = n$, at least two of the cosets

$$g_1G_0, g_2G_0, \ldots, g_nG_0$$

are equal, say $g_iG_0 = g_jG_0$, which shows that $f_1 = -g_ig_j \in G_0$. Either way we obtain an element $f_1 \in G_0$ that is positive on at least one α_i, $i \in \{1, \ldots, 2^{k-2}\}$, and, since $n < 2^{k-2}$, negative on at least one α_i, $i \in \{1, \ldots, 2^{k-2}\}$, and at all $\alpha_{2^{k-2}+1}, \ldots, \alpha_{2^{k-1}}$.

Now we repeat the procedure starting from $g_{2^{k-2}+1}$ and, as a result, obtain an element $f_2 \in G_0$ that is positive on at least one α_i, $i \in \{2^{k-2} + 1, \ldots, 2^{k-1}\}$, negative on at least one α_i, $i \in \{2^{k-2} + 1, \ldots, 2^{k-1}\}$, and at all $\alpha_1, \ldots, \alpha_{2^{k-2}}$ (see Figure 4).

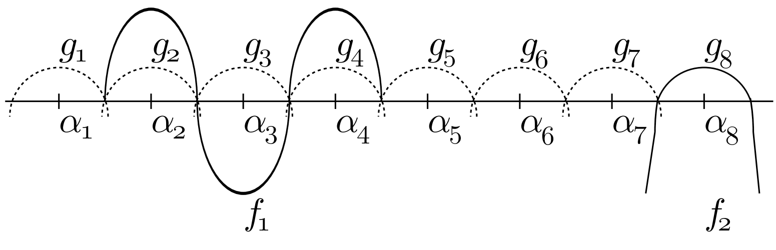

Fig. 4. Proof of Proposition 2.5: for $n = 2$ there are no 16-element fans

Finally, observe that $-f_1f_2 \in D_{X_0}(1, f_1)$, although $-f_1f_2 \notin \{1, f_1\}$. This proves that $\mathcal{F}$ cannot be a fan (see [7, Theorem 3.1.2]) — a contradiction. ∎

In particular Proposition 2.5 yields that there are at most 8-element fans in quotient spaces induced by groups of index 2. The next results provide a detailed description of all fans in that case.

PROPOSITION 2.6. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, and let $(G_{\mathbb{Q}(x)} : G_0) = 2$. If (X_0, G_0) is a space of orderings, where $X_0 = X|_{G_0}$, then all fans in (X_0, G_0) fall into one of the following categories*:

(1) *one- or two-element trivial fans*;
(2) *four-element fans $\{\tau_1, \dots, \tau_4\}$ that satisfy the following conditions*:
   (a) *if τ_i, $i \in \{1, \dots, 4\}$, is an ordering coming from a G_0-transcendental point α_i, then α_i is a non-regular point with respect to G_0;*
   (b) *if τ_i, $i \in \{1, \dots, 4\}$, is an ordering coming from a G_0-algebraic point α_i, $\tau_i = \sigma_i^\epsilon$, where $\epsilon \in \{-, +\}$, but $\sigma_i^\delta \notin \{\tau_1, \dots, \tau_4\}$, where $\delta = -$ if $\epsilon = +$, and $\delta = +$ if $\epsilon = -$, then α_i is a non-regular point with respect to G_0;*
   (c) *if $g \in G_0$, then g is positive with respect to an even number of τ_i, $i \in \{1, \dots, 4\}$,*
(3) 8-*element fans $\{\sigma_1^-, \sigma_1^+, \dots, \sigma_4^-, \sigma_4^+\}$, where σ_i^-, σ_i^+ are two orderings induced by a G_0-algebraic non-regular point α_i, $i \in \{1, \dots, 4\}$, and $\alpha_1, \dots, \alpha_4$ are roots of the same polynomial.*

Proof. The case of one- or two-element fans is obvious. Assume we have a four-element fan $\mathcal{F}$, and an ordering $\zeta^0 \in \mathcal{F}$ coming from a G_0-transcendental point ζ. ζ must be non-regular with respect to G_0, for suppose we take the family of polynomials $S = \{-(x - q_1)(x - q_2) : q_1, q_2 \in \mathbb{Q},\ q_1 < \zeta < q_2\}$ that clearly satisfies (2.1), and two polynomials $f_1, f_2 \in S$ that are positive on ζ, but already negative on all other points giving rise to the remaining orderings from $\mathcal{F}$. Take $g \in G_0$ such that $K_g \subset K_{f_1} \cap K_{f_2}$ — then $g(\zeta^0) = 1$, but $g(\tau) = -1$, for $\tau \in \mathcal{F} \setminus \{\zeta^0\}$, so that $\mathcal{F}$ cannot be a fan, which yields a contradiction. Same argument follows for orderings $\sigma^\epsilon \in \mathcal{F}$, $\epsilon \in \{-, +\}$, coming from a G_0-algebraic point α, such that $\sigma_i^\delta \notin \mathcal{F}$, where

$$\delta = \begin{cases} -, & \text{if } \epsilon = + \\ +, & \text{if } \epsilon = -. \end{cases}$$

The condition (c) is automatically implied by the fact that $\mathcal{F}$ is a 4-element fan (compare [7, Theorem 3.1.3]).

Conversely, assume that $\mathcal{F} = \{\tau_1, \dots, \tau_4\}$ is a set of 4 orderings from X_0 satisfying (a), (b), and (c). In order to prove that $\mathcal{F}$ is a fan it suffices to show that there exist 3 elements $a_1, a_2, a_3 \in G_0$ that have the following sign configuration with respect to $\tau_1, \dots, \tau_4$ (see [7, Theorem 3.1.3]):

	τ_1	τ_2	τ_3	τ_4
a_1	$+$	$+$	$-$	$-$
a_2	$+$	$-$	$+$	$-$
a_3	$-$	$+$	$+$	$-$

(2.2)

There are a few cases to consider. Firstly, assume that all $\tau_1, \dots, \tau_4$ come from G_0-transcendental points, say $\zeta_1, \dots, \zeta_4$. By our assumption, they are all non-regular, thus let S_i be a family satisfying (2.1) and let $f_1^{(i)}, f_2^{(i)} \in S_i$ be polynomials such that

$$\forall g \in G_0 \ \big[(\zeta_i \in K_g) \Rightarrow (K_g \not\subset K_{f_1^{(i)}} \cap K_{f_2^{(i)}})\big],$$

for $i \in \{1, \dots, 4\}$. Next, let $\gamma_1^{(i)}, \gamma_2^{(i)}$, $\gamma_1^{(i)} < \zeta_i < \gamma_2^{(i)}$, be two adjacent numbers from the combined set of real roots of $f_1^{(i)}, f_2^{(i)}$ and the set $\{\zeta_1, \dots, \zeta_4\} \setminus \{\zeta_i\}$, $i \in \{1, \dots, 4\}$. Next we pick rational numbers $q_1^{(i)}, q_2^{(i)}$, $\gamma_1^{(i)} < q_1^{(i)} < \zeta_i < q_2^{(i)} < \gamma_2^{(i)}$, and define $g_i(x) = -(x - q_1^{(i)})(x - q_2^{(i)}) \notin G_0$. As $(G_{\mathbb{Q}(x)} : G_0) = 2$, we see that $-g_1g_2, -g_2g_3, -g_2g_3 \in G_0$. Finally, we define $a_1 = -g_1g_2$, $a_2 = -g_1g_3$, and $a_3 = -g_2g_3$ — a straightforward computation shows that these elements have sign configuration as required by (2.2) (see Figure 5; $\times$ indicates that ζ_i is a G_0-transcendental point).

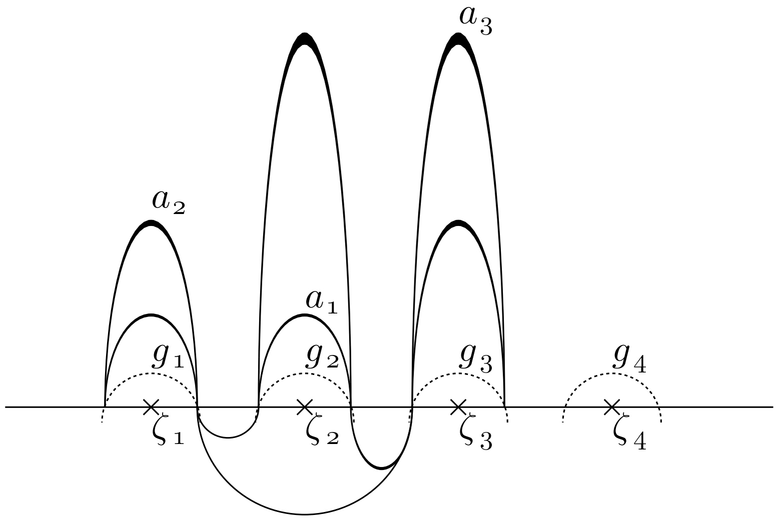

Fig. 5. A four-element fan coming from four G_0-transcendental points

The same argument applies to the case when three of $\tau_1, \dots, \tau_4$ come from G_0-transcendental points, and one of them comes from a G_0-algebraic point. Similarly, the case when two of $\tau_1, \dots, \tau_4$ come from G_0-transcendental points, and two of them come from two different G_0-algebraic points is also covered by the above argument.

Now suppose that two of $\tau_1, \dots, \tau_4$, say τ_1 and τ_2, come from G_0-transcendental points, say ζ_1 and ζ_2, and two of them come from one G_0-algebraic point, say α. Thus $\tau_3 = \sigma^-$ and $\tau_4 = \sigma^+$, according to the notation set above. Arguing as before, let $g_1, g_2 \notin G_0$ be polynomials positive only on τ_1, τ_2, respectively, and negative with respect to the remaining orderings from $\mathcal{F}$. Moreover, denote by f an element of G_0 such that $f(\alpha) = 0$ — we might also require that f is positive with respect to σ^-. Now $g_1g_2, f \in G_0$, and by (c) these elements are positive with respect to an even number of $\tau_1, \dots, \tau_4$, which follows that the elements $a_1 = -g_1g_2$, $a_2 = f$, and $a_3 = -g_1g_2f$ have the sign configuration described by (2.2) (see Figure 6; $\times$ indicates that ζ_i is a G_0-transcendental point, $\bullet$ indicates that α is G_0-algebraic).

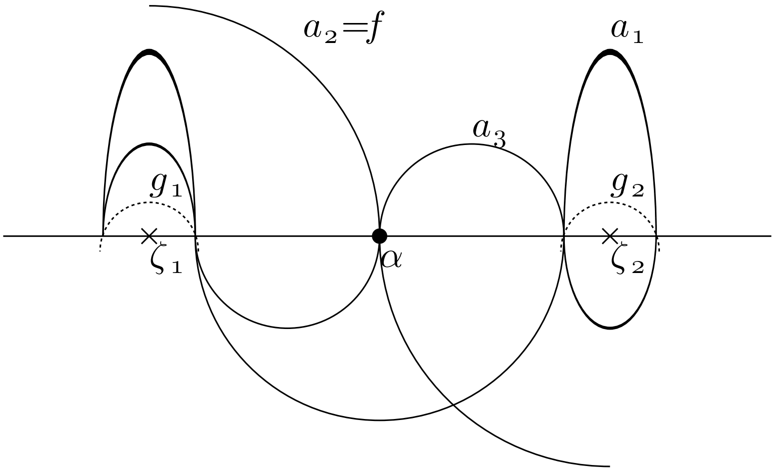

Fig. 6. A four-element fan coming from two G_0-transcendental points and one G_0-algebraic point

The same argument applies to the case when one of $\tau_1, \ldots, \tau_4$ comes from a G_0-transcendental point, and three of them come from two G_0-algebraic points: note that in our reasoning it did not really matter whether one of the ζ_1, ζ_2 was G_0-transcendental, we only used the fact that they were both non-regular with respect to G_0, and, clearly, if three orderings come from two G_0-algebraic points, then one of them must be non-regular as well. Also, the case when one of $\tau_1, \ldots, \tau_4$ comes from a G_0-transcendental point, and three of them come from three G_0-algebraic points is argued in an essentially the same way as the case of four orderings coming from four G_0-transcendental points.

Thus we are down to the case of four orderings coming from G_0-algebraic points. If they come from four different points, we argue as in the case of four orderings coming from four G_0-transcendental points, if they come from three different points, we argue as in the case of two orderings coming from two G_0-transcendental points, and one from a G_0-algebraic point. We are therefore left with one case, namely when four orderings $\tau_1, \ldots, \tau_4$ come from two G_0-algebraic points, call them α_1 and α_2.

Observe that α_1 and α_2 need to be roots of one polynomial $f \in G_0$, for if $g(\alpha_1) = 0$ for some $g \in G_0$, but $g(\alpha_2) \neq 0$, then g would be positive at an odd number of $\tau_1, \ldots, \tau_4$, contrary to the assumption (c). Let $\tau_1 = \sigma_1^-$, $\tau_2 = \sigma_1^+$, $\tau_3 = \sigma_2^-$, and $\tau_4 = \sigma_2^+$. We might as well assume that f is positive with respect to both σ_1^- and σ_2^+ (and, consequently, negative with respect to both σ_1^+ and σ_2^-). Moreover, there is an element $h \in G_0$ such that $h(\alpha_1)h(\alpha_2) < 0$, for otherwise $\sigma_1^+ = \sigma_2^-$. Say $h(\alpha_1) < 0$. Now it suffices to take $a_1 = h$, $a_2 = fh$, and $a_3 = -f$ (see Figure 7).

Finally, we turn to 8-element fans. Repeating the argument from the proof of Proposition 2.5 we see that an 8-element fan might come from at most 4 real points that, therefore, have to be G_0-algebraic.

Firstly, observe that all four G_0-algebraic points inducing an 8-element fan $\mathcal{F}$ have to be non-regular with respect to G_0. For suppose that there is a regular point α giving rise

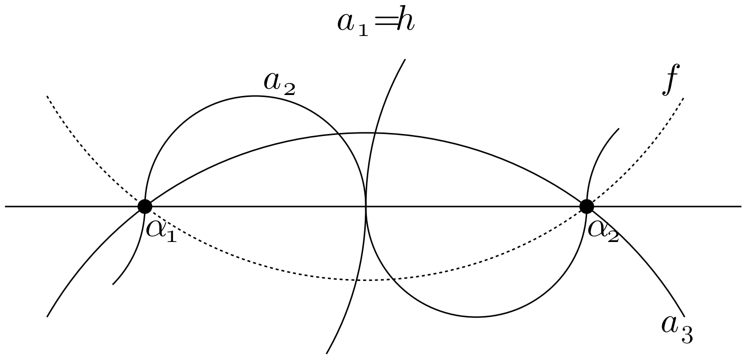

Fig. 7. A four-element fan coming from two G_0-algebraic points

to two orderings $\sigma^-, \sigma^+ \in \mathcal{F}$. Then we can pick an element $g \in G_0$ such that $g(\alpha) > 0$, and, consequently, $g(\sigma^-) = g(\sigma^+) = 1$, but g is negative with respect to the remaining orderings from $\mathcal{F}$. It follows from [7, Theorem 3.1.3] that, for every $a \in G_0$, a is positive with respect to exactly 0, 4 or 8 orderings of an 8-element fan — however, g is positive with respect to only two orderings of $\mathcal{F}$, which yields a contradiction.

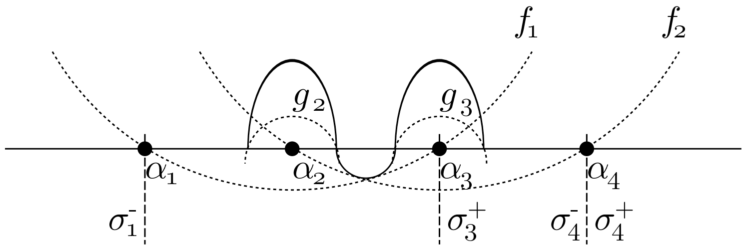

Fig. 8. There are no 8-element fans coming from roots of two polynomials

Secondly, observe that all the points giving rise to orderings in $\mathcal{F}$ must be roots of one polynomial. Indeed, four G_0-algebraic points $\alpha_1, \ldots, \alpha_4$ can be roots of 4, 3, 2 or 1 polynomial. The cases of 4, 3 or 2 polynomials such that 3 of $\alpha_1, \ldots, \alpha_4$ are roots of one of them are impossible, since in each of them the polynomials we consider are positive in an odd number of $\sigma_1^-, \sigma_1^+, \ldots, \sigma_4^-, \sigma_4^+$. Consider the case of 2 polynomials, $f_1, f_2 \in G_0$, each vanishing at two of $\alpha_1, \ldots, \alpha_4$. Two roots of f_1 have to be separated by a root of f_2, for otherwise f_1 (or $-f_1$) would be positive with respect to 6 orderings from $\mathcal{F}$ — say α_1 and α_3 are roots of f_1, and α_2 and α_4 are roots of f_2. Consider the subspace $H_{\mathcal{F}}(f_1)$ of the fan $(\mathcal{F}, G_0|_{\mathcal{F}})$ that contains four orderings. Replacing f_1 with $-f_1$, and interchanging α_2 with α_4, if necessary, we may assume that $H_{\mathcal{F}}(f_1) = \{\sigma_1^-, \sigma_3^+, \sigma_4^-, \sigma_4^+\}$. Since every subspace of a fan is a fan itself (see [7, Theorem 3.1.3]), $H_{\mathcal{F}}(f_1)$ is a 4-element fan. Next, since α_2 and α_3 are non-regular, we can pick polynomials $g_2, g_3 \notin G_0$ that are positive only at α_2 (α_3, respectively) and negative at other $\alpha_1, \ldots, \alpha_4$. Then $-g_2 g_3 \in G_0$, and $-g_1 g_2$ is positive at α_2 and α_3, but negative at α_1 and α_3 — it follows that $-g_1 g_2$ is positive with respect to only one ordering in $H_{\mathcal{F}}(f_1)$, which means that $H_{\mathcal{F}}(f_1)$ cannot be a fan — a contradiction (see Figure 8 for clarification). Therefore only 8-element fans in (X_0, G_0) come from 4 G_0-algebraic non-regular points that are all roots of one polynomial.

Conversely, suppose that $\mathcal{F} = \{\sigma_1^-, \sigma_1^+, \dots, \sigma_4^-, \sigma_4^+\}$ is the set of 8 orderings coming from 4 non-regular roots $\alpha_1, \dots, \alpha_4$ of a single polynomial $f \in G_0$. We shall focus here on the case when $f(\sigma_i^+) = f(\sigma_{i+1}^-)$, $i \in \{1,2,3\}$ — the remaining cases when $f(\sigma_i^+) \neq f(\sigma_{i+1}^-)$, for some $i \in \{1,2,3\}$, that is when f has an odd number of additional roots between α_i and α_{i+1}, for some $i \in \{1,\dots,3\}$, are analogous. By [7, Theorem 3.1.3] $\mathcal{F}$ is a fan if and only if we can exhibit four elements $a_1, \dots, a_4 \in G_0$ that have the following sign configuration with respect to orderings from $\mathcal{F}$:

	σ_1^-	σ_1^+	σ_2^-	σ_2^+	σ_3^-	σ_3^+	σ_4^-	σ_4^+
a_1	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$
a_2	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
a_3	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$
a_4	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$

(2.3)

Replacing f with $-f$, if necessary, we might also assume that $f(\sigma_1^-) = -1$ (see Figure 9).

Since $\alpha_1, \dots, \alpha_4$ are non-regular with respect to G_0, we are allowed to pick polynomials $g_1, \dots, g_4 \notin G_0$ such that g_i is positive at α_i but negative at α_j, $j \neq i$, $i \in \{1,\dots,4\}$. As $(G_{\mathbb{Q}(x)} : G_0) = 2$, clearly $g_1g_2, g_2g_3, g_2g_4 \in G_0$ — now define $a_1 = fg_1g_2$, $a_2 = -f$, $a_3 = fg_2g_3$, and $a_4 = fg_2g_4$. It follows that $a_1, \dots, a_4$ have signs with respect to orderings from $\mathcal{F}$ as required by (2.3) (compare Figure 9). This finishes the proof. ∎

Remarks.

(1) It follows from the proof of the above proposition, that four-element fans might arise from 2 G_0-algebraic points α_1 and α_2 if and only if α_1, α_2 are roots of only one polynomial in G_0, and if there exists an $f \in G_0$ such that $f(\alpha_1)f(\alpha_2) < 0$.

(2) The last part of the proposition explains what we mean by "blowing up" fans: we take n roots of a single polynomial in $\mathbb{Q}[x]$ that give rise to $2n$ orderings in $X_{\mathbb{Q}(x)}$ — if n is big enough, these orderings cannot form a fan in $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$, but we can create a quotient (X_0, G_0) of $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ by removing sufficiently many elements from $G_{\mathbb{Q}(x)}$ so that in the resulting space the $2n$ orderings mentioned above form a fan.

2.3. The profiniteness theorem for quotient spaces. In this subsection we shall prove the main result of this paper, namely that if a quotient structure of $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ coming from a subgroup of $G_{\mathbb{Q}(x)}$ of index 2 is a space of orderings, then it is profinite. We need to make some preparations for the proof of the theorem. For a subgroup G_0 of $G_{\mathbb{Q}(x)}$ we shall call an element $p \in G_0$ G_0-*irreducible* if it does not decompose as a product $p = ab$, where $a, b \in G_0 \setminus \{1, -1\}$.

Proposition 2.7. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, and let $(G_{\mathbb{Q}(x)} : G_0) = 2$. If (X_0, G_0) is a space of orderings, where $X_0 = X|_{G_0}$, and $p_1, p_2 \in G_0$ are two elements that are G_0-irreducible, then p_1, p_2 do not have real roots in common.*

Proof. Suppose, *a contrario*, that $p_1, p_2 \in G_0$ are two different G_0-irreducible elements sharing a common root $\alpha \in \mathbb{R}$. Let $\alpha = \alpha_1, \dots, \alpha_n$ be all the real roots of p_1, p_2. If $n = 2$

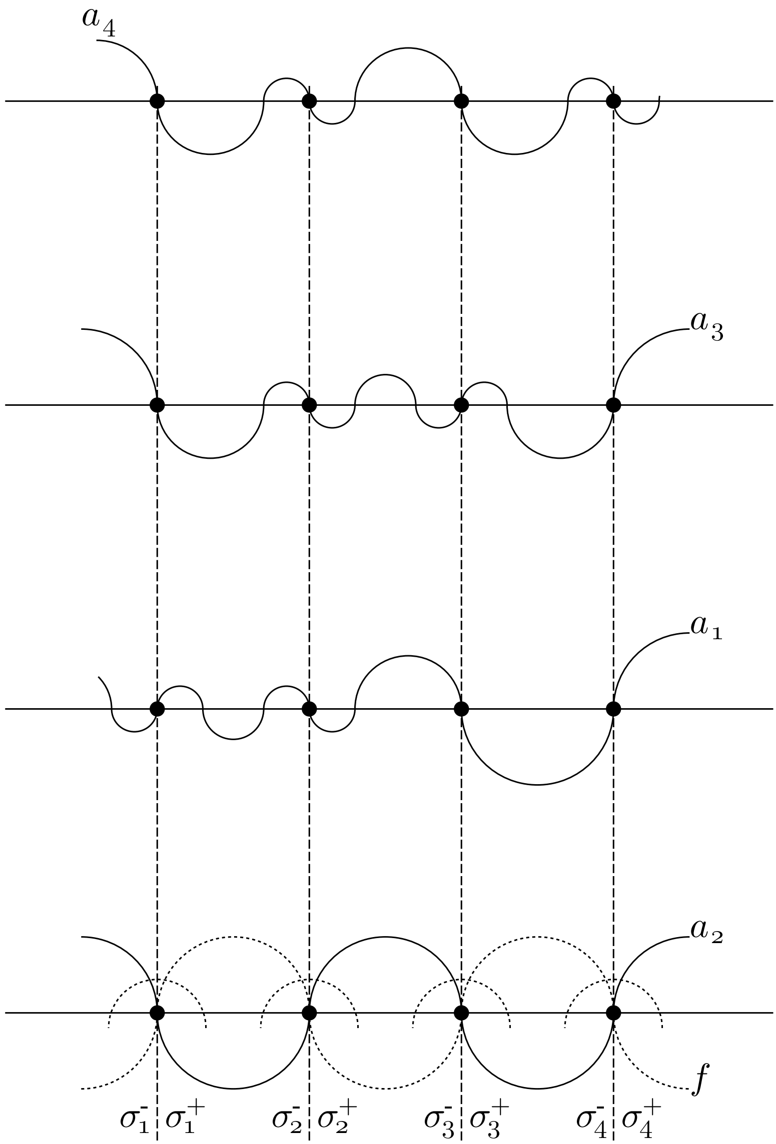

Fig. 9. An example of an 8-element fan

then either $p_1 = p_2$, or one of the p_i's divides the other one, $i \in \{1, 2\}$, in both cases leading to a contradiction. Thus we may assume that $n \geq 3$, so that p_1, p_2 have together at least three real roots.

The roots of p_1, p_2 give rise to $2k$ orderings in X_0, where $k \geq n$: the case when $k = n$ occurs when either all of the $\alpha_0, \alpha_1, \ldots, \alpha_n$ are regular, or when the number of non-regular points among $\alpha_0, \alpha_1, \ldots, \alpha_n$ equals at least three, or when there are only two non-regular points among $\alpha_0, \alpha_1, \ldots, \alpha_n$, say α_i and α_j, but there exists a $p \in G_0$ such that $p(\alpha_i)p(\alpha_j) < 0$, or when there is only one non-regular point. Indeed, it suffices to show that in all three cases, and for any two roots α_i, α_j, $i, j \in \{0, \dots, n\}$, there always exist an $f \in G_0$ such that $f(\alpha_i)p(\alpha_j) < 0$. Surround all the regular points with elements $g_1, \dots, g_p \in G_0$, and all the non-regular points with parabolas $h_1, \dots, h_r \notin G_0$, so that each g_s or h_t is positive only at the root α_s or α_t that it surrounds, but negative at the remaining roots, $s \in \{1, \dots, p\}$, $t \in \{1, \dots, r\}$. Now, if α_i or α_j are regular, then either g_i or g_j is positive at one of them, but negative at the other one, $i, j \in \{1, \dots, p\}$. If both α_i and α_j are non-regular, but there exists another non-regular point $\alpha_{j'}$, we take the element $-h_i h_{j'} \in G_0$ that is positive at α_i but negative at α_j. When there are only two non-regular points, but we can separate them with an element of G_0, we are already done (see Figure 10 for clarification — regular points are denoted by $\bullet$, non-regular ones by $\circ$). Finally, when there is only one non-regular point we can separate it from regular points using the elements g_i surrounding regular roots.

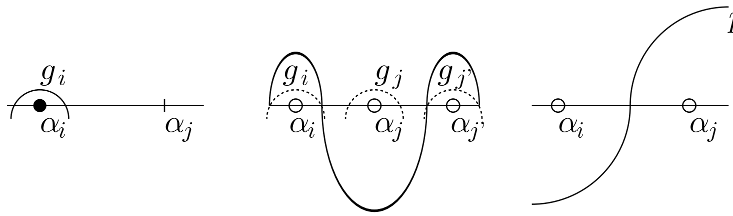

Fig. 10. Proof of Proposition 2.7: various ways of separating G_0-algebraic points

Consider the case when p_1 and p_2 have three roots, and two of them are non-regular points that cannot be separated in (X_0, G_0), say these are α_1, α_2 that we may assume to be the roots of p_1. We have that $|\{\sigma_1^-, \sigma_1^+, \sigma_2^-, \sigma_+^-\}| = 2$ in X_0. Thus p_2, viewed as a function of the set of orderings from X_0, factors as $p_1 \cdot (p_1 p_2)$, which yields a contradiction. Therefore the n roots of p_1, p_2 give rise to $2k$ orderings with $k \geq 3$.

Observe that if $k < n$ we have, in fact, $k = n - 1$: this is the case when there are only two non-regular points among real roots of p_1, p_2, and they cannot be separated by an element $p \in G_0$.

Assume $k = n$. The $2n$, $n \geq 3$, orderings from X_0 form a subspace, call it (Y, H). Firstly, assume that all $\alpha_1, \dots, \alpha_n$ are regular. Surround all of them with the elements $g_1, \dots, g_n \in G_0$ such that g_i is positive only at α_i and negative at α_j, $j \neq i$, $i \in \{1, \dots, n\}$. Clearly $g_1, \dots, g_n \in H$, and p_1, p_2 might be viewed as elements of H. Moreover, none of g_i or p_j, is in the subgroup generated by the remaining $g_{i'}, p_{j'}$, $i \neq i'$, $j \neq j'$, $i \in \{1, \dots, n\}$, $j \in \{1, 2\}$. Thus $|H| = 2^{n+2}$, and, as we already know, $|Y| = 2n$. The only finite space of orderings (Y, H) with $|Y| = 2n$ and $|H| = 2^{n+2}$ has two connected components, one being a 2-element space that might be viewed as a union of two singleton spaces, and the other one being a group extension of a union of $n - 2$ singleton spaces. However, (Y, H) has at least three connected components, one coming from roots of p_1 that are not roots of any other polynomials from G_0 (or some singleton space, if no such roots exist), one coming from roots of p_2 that are not roots of any other polynomials from G_0 (or some singleton space), and one coming from common roots of p_1 and p_2. This yields a contradiction.

Secondly, assume that $\alpha_1, \dots, \alpha_{n-2}$ are regular, but α_{n-1} and α_n are non-regular and can be separated by $p \in G_0$. We surround $\alpha_1, \dots, \alpha_{n-2}$ with $g_1, \dots, g_n \in G_0$ as before, and we surround α_{n-1} and α_n with parabolas $h_1, h_2 \notin G_0$. Then $-h_1h_2 \in G_0$. Moreover, we multiply p by a product of these $-g_i$, $i \in \{1, \dots, n-2\}$, such that $\alpha_i \in K_p$ (we might choose $g_1, \dots, g_{n-2}$ so that $K_{g_1} \cup \dots \cup K_{g_{n-2}}$ is disjoint from the set of all real roots of p). As a result we obtain an element $p' \in G_0$ which is positive at one of the α_{n-1}, α_n, and negative at the remaining ones (see Figure 11). The rest of the proof carries over, as we constructed $n+2$ polynomials in G_0, none of which is in the subgroup generated by the remaining elements. We also note that in view of Proposition 2.8, that is yet to be proved, the process described above simplifies dramatically, as it turns out p has only one root — the proof of Proposition 2.8 relies on what we are proving right now, though.

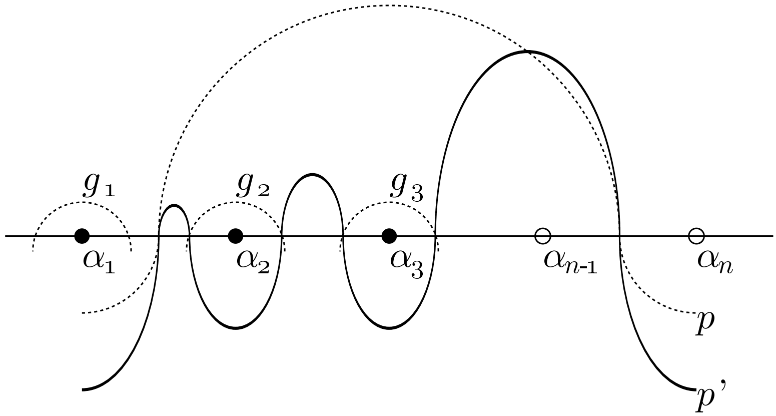

Fig. 11. Proof of Proposition 2.7: construction of p'

Thirdly, assume that there are more than three non-regular points among roots $\alpha_1, \dots, \alpha_n$. Surround all the regular points with elements $g_1, \dots, g_p \in G_0$, and all the non-regular points with parabolas $h_1, \dots, h_r \notin G_0$, with $p + r = n$, so that each g_s or h_t is positive only at the root α_s or α_t that it surrounds, but negative at the remaining roots, $s \in \{1, \dots, p\}$, $t \in \{1, \dots, r\}$. Then the $r-1$ products $-h_1h_2, -h_1h_3, \dots, -h_1h_r$ are elements of G_0, and the group H is generated by $(r-1)+p+2 = n+1$ elements, and we obtain a space (Y, H) with $|Y| = 2n$ and $|H| = 2^{n+1}$. This is only possible when (Y, H) is a group extension of a direct sum of n singleton spaces, so, in particular, (Y, H) has only one connected component — but, as before, this is not the case, which yields a contradiction.

The case when there is only one non-regular point is similar: we can surround it with the product of $-g_i$, $i \in \{1, \dots, n-1\}$, where $\alpha_1, \dots, \alpha_{n-1}$ are all regular, and thus obtain $n+1$ generators of the group H, and the rest of the proof carries over.

Finally, consider the case when $k = n-1$, that is when there are only two non-regular points among real roots of p_1, p_2, and they cannot be separated by an element $p \in G_0$. This is actually the same as the case of only one non-regular point with n replaced by $n-1$, since the two non-regular points give rise to only two orderings. This finishes the proof. ∎

Using similar methods we can also prove the following:

PROPOSITION 2.8. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, and let $(G_{\mathbb{Q}(x)} : G_0) = 2$. If (X_0, G_0) is a space of orderings, where $X_0 = X|_{G_0}$, if α_1, α_2 are two non-regular G_0-algebraic points, and if $p \in G_0$ is a G_0-irreducible element such that $p(\alpha_1)p(\alpha_2) < 0$ (that is, α_1 and α_2 are separated by p), then p has only one real root.*

Proof. Suppose that α_1, α_2 are two non-regular G_0-algebraic points, and that $p \in G_0$ is a G_0-irreducible element such that $p(\alpha_1)p(\alpha_2) < 0$, and has at least two different roots, call them β_1, β_2. There are four cases to consider.

Firstly, assume that $q_1, q_2 \in G_0$ are two different G_0-irreducible elements such that $q_1(\alpha_1) = q_2(\alpha_2) = 0$, and assume that both β_1 and β_2 are regular. Surround β_1 and β_2 with elements $g_1, g_2 \in G_0$, and α_1, α_2 with parabolas $h_1, h_2 \notin G_0$, so that $-h_1h_2 \in G_0$. It follows that the eight orderings induced by $\alpha_1, \alpha_2, \beta_1, \beta_2$ form a subspace of (X_0, G_0), call it (Y, H). Note that in H $g_1g_2 = -h_1h_2$, which follows that H is generated by 5 elements, namely g_1, g_2, q_1, q_2, and p — but the only finite space of orderings (Y, H) with $|Y| = 8$ and $|H| = 2^5$ is the group extension of a direct sum of four singleton spaces, that is, in particular, connected. The subspace (Y, H) that we constructed has, however, five connected components: four singleton spaces coming from α_1 and α_2, and the four-element fan coming from β_1 and β_2.

Secondly, assume that $q_1, q_2 \in G_0$ are two different G_0-irreducible elements such that $q_1(\alpha_1) = q_2(\alpha_2) = 0$, and assume that one of β_1, β_2, say β_1, is non-regular. Surround non-regular points with parabolas $h_1, h_2, h_3 \notin G_0$, and β_2 with $g_1 \in G_0$. Then $-h_1h_2, -h_1h_3 \in G_0$ which leads to a space (Y, H) with $|Y| = 8$ and $|H| = 2^6$ — the only such space has three connected components, two of them being singleton spaces, and one a group extension of a direct sum of three singleton spaces, which yields a contradiction. The case with both β_1 and β_2 being non-regular is analogous.

The remaining two cases coming from the situation when α_1 and α_2 are roots of one G_0-irreducible element of G_0 can be proved in a similar way. ∎

We now arrive to the main theorem of this paper:

THEOREM 2.9. *Let $G_0 \subset G_{\mathbb{Q}(x)}$ be a subgroup of $G_{\mathbb{Q}(x)}$ with $-1 \in G_0$, and let $(G_{\mathbb{Q}(x)} : G_0) = 2$. If (X_0, G_0) is a space of orderings, where $X_0 = X|_{G_0}$, then (X_0, G_0) is profinite.*

Proof. It suffices to show that, for a given finite subset $\{p_1, \ldots, p_m\} \subset G_0$, there exists a finite quotient space (X_1, G_1) of (X_0, G_0) such that

$$p_1, \ldots, p_m \in G_0$$

([6, Remark 5.5]). Thus let $\{p_1, \ldots, p_m\} \subset G_0$, and let $\widetilde{G} = \langle p_1, \ldots, p_m \rangle$ be a subgroup of G_0 generated by the elements $p_1, \ldots, p_m$. Without loss of generality we may assume that $p_1, \ldots, p_m$ are square free polynomials, and replacing, if necessary, the set $\{p_1, \ldots, p_m\}$ with the set of all G_0-irreducible factors of $p_1, \ldots, p_m$, we may also assume that the sets of real roots of polynomials $p_1, \ldots, p_m$ are pairwise disjoint. Surround all regular roots $\beta_1, \ldots, \beta_P$ with elements $g_1, \ldots, g_P \in G_0$, and all non-regular roots $\gamma_1, \ldots, \gamma_R$ with parabolas $h_1, \ldots, h_R \notin G_0$.

We start with constructing finite quotients (X^{p_i}, G^{p_i}), for each p_i, $i \in \{1, \ldots, m\}$, that will later become direct summands of (X_1, G_1). Depending on how real roots of p_i are structured, these constructions vary quite substantially. Fix $p \in \{p_1, \dots, p_m\}$.

Assume the case when all real roots of p are, say, relabeling, if necessary, $\beta_1, \dots, \beta_n$, where $n > 1$. Consider the direct sum of one-element spaces

$$(\{\sigma_1\}, \langle g_1 \rangle) \sqcup (\{\sigma_2\}, \langle g_2 \rangle) \sqcup \ldots \sqcup (\{\sigma_n\}, \langle g_n \rangle).$$

Here by σ_i we mean the unique ordering of this quotient that makes g_i negative, $i \in \{1, \dots, n\}$. By Proposition 2.8 all non-regular roots of all the remaining p_j, $p_j \neq p$, $j \in \{1, \dots, m\}$, lie either within K_p or K_{-p} — replacing p with $-p$, if necessary, we might assume the former is the case. Thus the only roots of p_j, $p_j \neq p$, $j \in \{1, \dots, m\}$, where p is negative, are regular, say these are $\beta_{n+1}, \dots, \beta_s$ surrounded by $g_{n+1}, \dots, g_s$. We multiply p by the product of all $-g_k$, $k \in \{n+1, \dots, s\}$, and as a result we obtain an element $p' \in G_0$ negative only either on $K_{g_i} \cap (-\infty, \alpha_i)$, or on $K_{g_i} \cap (\alpha_i, +\infty)$, for each $i \in \{1, \dots, n\}$, and positive on all remaining K_{g_j}, K_{h_k}, $j \in \{n+1, \dots, P\}$, $k \in \{1, \dots, R\}$. Now take the group extension

$$(X^p, G^p) = \big[(\{\sigma_1\}, \langle g_1 \rangle) \sqcup \ldots \sqcup (\{\sigma_n\}, \langle g_n \rangle)\big] \times \langle p' \rangle.$$

As a result of this extension, each ordering σ_i splits into two orderings on the quotient that can be identified with σ_i^- and σ_i^+, according to $p'(\sigma_i^-) = -1$, $p'(\sigma_i^+) = 1$. See Figure 12.

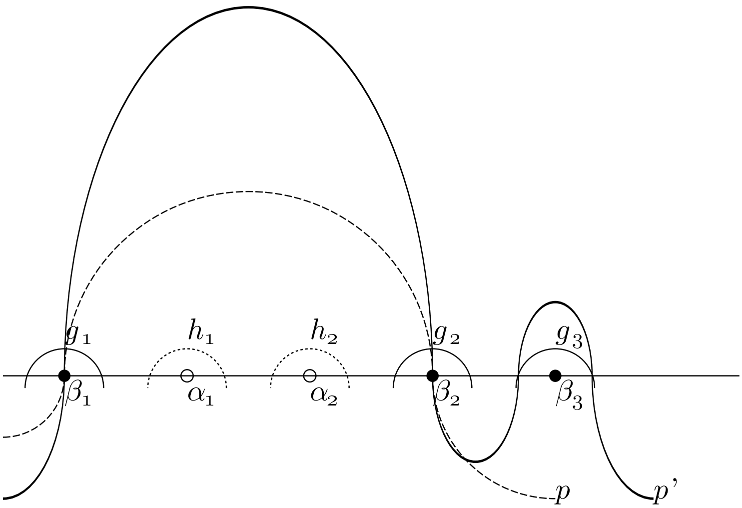

Fig. 12. Proof of Theorem 2.9: all $n > 1$ roots of p are regular

In the special case when p has only one root, say β_1, that is regular, consider the singleton space

$$(\{\sigma_1\}, \langle g_1 \rangle),$$

where σ_1 is the unique ordering of the quotient that makes g_1 negative. As before, we would like to construct an extension of this space, but since now p can separate non-regular points, the construction becomes more delicate. Here it is not always possible to modify p into p' so that the resulting polynomial is positive at all roots of p_j, $p_j \neq p$,

$j \in \{1, \ldots, m\}$ — but since at the end of the proof (X^p, G^p) will become one of the direct summands, it suffices to ensure that $p|_{X^{p_j}} \in G^{p_j}$, p_j, $p_j \neq p$, $j \in \{1, \ldots, m\}$, and this is obvious here since $p|_{X^{p_j}}$ will be either 1 or -1. Thus we simply define

$$(X^p, G^p) = (\{\sigma_1\}, \langle g_1 \rangle) \times \langle p \rangle.$$

Now assume the case when p has s regular roots, say $\beta_1, \ldots, \beta_s$, and t non-regular ones, say $\gamma_1, \ldots, \gamma_t$, with $t > 2$. Here we consider the space

$$(\{\sigma_1\}, \langle g_1 \rangle) \sqcup \ldots \sqcup (\{\sigma_s\}, \langle g_s \rangle) \sqcup (\{\tau_1, \ldots, \tau_t\}, \langle -h_1h_2, -h_1h_3, \ldots, -h_1h_t \rangle),$$

where σ_i is defined as before, and τ_i is the unique ordering that makes all the elements $-h_ih_j$, $j \neq i$, $j \in \{1, \ldots, t\}$, negative. Then we modify p into p' just as before and define

$$\begin{aligned}(X^p, G^p) = \big[&(\{\sigma_1\}, \langle g_1 \rangle) \sqcup \ldots \sqcup (\{\sigma_s\}, \langle g_s \rangle) \\ &\sqcup (\{\tau_1, \ldots, \tau_t\}, \langle -h_1h_2, -h_1h_3, \ldots, -h_1h_t \rangle)\big] \times \langle p' \rangle.\end{aligned}$$

The case when p has, other than some regular roots $\beta_1, \ldots, \beta_s$, exactly two non-regular ones, γ_1 and γ_2 splits into two subcases: if γ_1 and γ_2 cannot be separated, we might treat them as one regular root. If they are, however, split by a polynomial $f \in G_0$ that has only one root, we take

$$(X^p, G^p) = \big[(\{\sigma_1\}, \langle g_1 \rangle) \sqcup \ldots \sqcup (\{\sigma_s\}, \langle g_s \rangle) \sqcup (\{\tau_1, \tau_t\}, \langle -h_1h_2, f \rangle)\big] \times \langle p' \rangle.$$

Finally, if p has only one non-regular root, it can be treated as a regular point. This finishes the proof. ∎

REMARKS.

(1) Observe that the proof in the case when all roots are regular can be applied to show that $(X_{\mathbb{Q}(x)}, G_{\mathbb{Q}(x)})$ is profinite. This relates to our work in [3].

(2) We have at least three reasons why for spaces obtained from subgroups of index 2 of $G_{\mathbb{Q}(x)}$ Lam's Open Problem B is satisfied: firstly, by Proposition 2.3 they are of stability index at most 3, secondly, they are profinite by Theorem 2.9, and thirdly, they are realized by Pythagorean fields as profinite spaces since $G_{\mathbb{Q}(x)}$ is countable (see [3, Theorem 5]).

(3) It remains an open question whether counterexamples to Lam's Open Problem B can be found among quotients coming from subgroups of higher index. Due to the "exponential growth" of complications with recognizing fans in such quotients this seems as an extremely difficult problem.

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