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SMOOTH DOUBLE SUBVARIETIES ON SINGULAR VARIETIES, III

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Abstract. Let k be an algebraically closed field, $\operatorname{char} k = 0$. Let C be an irreducible nonsingular curve such that $rC = S \cap F$, $r \in \mathbb{N}$, where S and F are two surfaces and all the singularities of F are of the form $z^3 = x^{3s} - y^{3s}$, $s \in \mathbb{N}$. We prove that C can never pass through such kind of singularities of a surface, unless $r = 3a$, $a \in \mathbb{N}$. We study multiplicity-r structures on varieties $r \in \mathbb{N}$. Let Z be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $rZ = X \cap F$, where X is a normal n-fold, F is a $(N-1)$-fold in $\mathbb{P}^N$, such that $Z \cap \operatorname{Sing}(X) \neq \emptyset$. We study the singularities of X through which Z passes.

Definition 0.1. Let X be an n-dimensional normal variety and P a point of X. Let P be an n-fold isolated singularity (that is, the spectrum of an equicharacteristic local noetherian complete ring of Krull dimension n, without zero divisors, whose closed point P is singular). Let $\pi : \tilde{X} \to X$ be the desingularization of X at P. The *genus* of a normal singularity P is defined to be $\dim_k (R^{n-1}\pi_* \mathcal{O}_{\tilde{X}})_P$. If the genus is 0, the singularity is said to be *rational*. If the genus is 1, it is *elliptic*.

Notation 0.2. Let F be a reduced surface and P a point of F. Let (F, P) be a surface singularity (that is, the spectrum of an equicharacteristic complete local ring of Krull dimension 2 whose closed point P is singular). Let $\pi : \tilde{F} \to F$ be the minimal desingularization of F at P.

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Definition 0.3. Let X be a nonsingular complex surface and $\Delta \subset \mathbb{C}$ a small open disc around the origin.

(1) Let $\Phi : S \to \Delta$ be a surjective proper holomorphic map. If the fibres $S_t = \Phi^{-1}(t)$, $t \neq 0$, are nonsingular connected curves of genus g, we call it *a pencil of curves* of genus g. S_0 is called the *singular fibre.*

(2) Let $\Phi : S \to \Delta$ be a pencil of curves of genus g which has reduced components. Let $P_1, \ldots, P_r \in \mathrm{Supp}(S_0)$ be nonsingular points of S_0. Let $\sigma : S' \to S$ be a finite number of blowing-ups at $P_1, \ldots, P_r$. Let $\psi : M \to X$ be a holomorphic map from an open neighbourhood M of the proper transform of $\mathrm{Supp}(S_0)$ in S' which defines a resolution of X. A normal surface singularity isomorphic to a singularity obtained in this way is called *a Kodaira singularity* of genus g (or associated to Φ), [6, 2.1, 2.2].

Definition 0.4. Let F be a reduced surface with a singular point at O. We shall denote by (V_{a_0,a_1,a_2}, O) the singularity of the form

$$(V_{a_0,a_1,a_2}, O) = \left\{(x_0, x_1, x_2) \in U \subset k^3 \quad \text{such that} \quad {x_0}^{a_0} + {x_1}^{a_1} = {x_2}^{a_2}\right\}$$

with $2 \leq a_0 \leq a_1 \leq a_2$.

Notation 0.5. Let $l = \gcd(a_0, a_1, a_2)$, $l_i = \frac{\gcd(a_j,a_k)}{l}$, $\alpha_i = \frac{a_i}{l_j l_k l}$, $i, j, k \in \{0, 1, 2\}$, $i \neq j$, $i \neq k$, $j \neq k$.

Proposition 0.6. *(V_{a_0,a_1,a_2}, O), $2 \leq a_0 \leq a_1 \leq a_2$, is a Kodaira singularity if and only if $\alpha_0\alpha_1 l_2 \leq \alpha_2$. If this is the case, it is associated to a pencil of curves of genus $\frac{1}{2}\{(a_0 - 1)(a_1 - 1) - \gcd(a_0, a_1) + 1\}$.*

Proof. [4, Proposition 4.4]. ∎

Lemma 0.7. *$(V_{3,3s,3s}, O)$, $s \in \mathbb{N}$, is a Kodaira singularity associated to a pencil of curves of genus $3s - 2$.*

Proof. According to Notation 0.5, $l = 3$, $l_0 = s$, $l_1 = l_2 = 1$, $\alpha_0 = \alpha_1 = \alpha_2 = 1$. By Proposition 0.6, $(V_{3,3s,3s}, O)$, $s \in \mathbb{N}$, is a Kodaira singularity since $\alpha_0\alpha_1 l_2 = 1 \leq \alpha_2 = 1$; it is associated to a pencil of curves of genus $\frac{1}{2}\{(3-1)(3s-1)-\gcd(3, 3s)+1\} = 3s-2$. ∎

Definition 1.0.

(1) We call *maximal cycle* $Z_{\tilde{F}}$ the cycle $Z_{\tilde{F}} = \sum m_i E_i$, defined by the divisorial part of $\mathcal{M}\mathcal{O}_{\tilde{F}}$, where $\mathcal{M}$ is the maximal ideal $\mathrm{Max}\,\mathcal{O}_{F,P}$ of $\mathcal{O}_{F,P}$; the E_i are the irreducible components of dimension 1 of the exceptional fibre $\pi^{-1}(P)$ and the m_i are nonnegative integers. A component E_j such that $m_j = 1$ is called a *reduced component* of the cycle.

(2) Consider positive cycles $Z = \sum r_i E_i$, $r_i \geq 0$, such that $(Z.E_i) \leq 0$, for all i. The unique smallest cycle Z satisfying $(Z.E_i) \leq 0$, for all i, is called *the fundamental cycle* of $\tilde{F}$.

1.1. Study of the fundamental cycle for $(V_{3,3s,3s}, O)$, $s \in \mathbb{N}$. Let $x_0 = z$, $x_1 = y$, $x_2 = x$. We consider the surface singularity (F, O) given by $z^3 = x^{3s} - y^{3s}$. We want to desingularize F at O.

Case $s = 1$: We consider the surface singularity (F, O) given by $z^3 = x^3 - y^3$. We want to desingularize F at O. Applying to $z^3 = x^3 - y^3$ the change $x_{(1)} = \frac{x}{y}$, $z_{(1)} = \frac{z}{y}$, $y_{(1)} = y$, we obtain ${z_{(1)}}^3 = {x_{(1)}}^3 - 1$ in $F_{(1)}$; $\pi_1 : F_{(1)} \longrightarrow F$, $E = {\pi_1}^{-1}(O)$ exceptional divisor given by $y_{(1)} = 0$. The fundamental cycle is $Z = E$. Its self-intersection is $E.E = -3$. We shall write the weighted dual graph of the fundamental cycle as

$$\overset{1}{\bullet}\atop{[1]}$$

1 denotes that E is a reduced component of the fundamental cycle; [1] denotes that the genus of E is 1; $E^2 = -3$. $z^3 = x^3 - y^3$ is a simple elliptic singularity.

Case $s = 2$: We consider the surface singularity (F, O) given by $z^3 = x^6 - y^6$. We want to desingularize F at O. Applying to $z^3 = x^6 - y^6$ the change $x_{(1)} = \frac{x}{y}$, $z_{(1)} = \frac{z}{y}$, $y_{(1)} = y$, we obtain ${z_{(1)}}^3 = {y_{(1)}}^3({x_{(1)}}^6 - 1)$ in $F_{(1)}$; $\pi_1 : F_{(1)} \longrightarrow F$, $E = {\pi_1}^{-1}(O)$ exceptional divisor given by $y_{(1)} = 0$.

Let us apply to ${z_{(1)}}^3 = {y_{(1)}}^3({x_{(1)}}^6 - 1)$ the change $x_{(2)} = x_{(1)}$, $z_{(2)} = \frac{z_{(1)}}{y_{(1)}}$, $y_{(2)} = y_{(1)}$. We obtain ${z_{(2)}}^3 = {x_{(2)}}^6 - 1$ in $F_{(2)}$; $\pi_2 : F_{(2)} \longrightarrow F_{(1)}$, $D = {\pi_2}^{-1}(E)$ exceptional divisor given by $y_{(2)} = 0$.

The fundamental cycle is $Z = D$. Its self-intersection is $D.D = -3$. To know the genus of D we apply Riemann–Hurwitz Formula. We obtain $2g - 2 = 3(0 - 2) + 2.6 = 6$, thus $g = 4$. We shall write the weighted dual graph of the fundamental cycle as

$$\overset{1}{\bullet}\atop{[4]}$$

1 denotes that D is a reduced component of the fundamental cycle; [4] denotes that the genus of D is 4; $D^2 = -3$.

Case $s = 3$: We consider the surface singularity (F, O) given by $z^3 = x^9 - y^9$. We want to desingularize F at O. Applying to $z^3 = x^9 - y^9$ the change $x_{(1)} = \frac{x}{y}$, $z_{(1)} = \frac{z}{y}$, $y_{(1)} = y$, we obtain ${z_{(1)}}^3 = {y_{(1)}}^6({x_{(1)}}^9 - 1)$ in $F_{(1)}$; $\pi_1 : F_{(1)} \longrightarrow F$, $E = {\pi_1}^{-1}(O)$ exceptional divisor given by $y_{(1)} = 0$. The cycle corresponding to the function $x = 0$ is formed by 3 lines G_1, G_2, G_3 intersecting E in one point. The total transform is $E + G_1 + G_2 + G_3$. Let us apply to ${z_{(1)}}^3 = {y_{(1)}}^6({x_{(1)}}^9 - 1)$ the change $x_{(2)} = x_{(1)}$, $z_{(2)} = \frac{z_{(1)}}{y_{(1)}}$, $y_{(2)} = y_{(1)}$. We obtain ${z_{(2)}}^3 = {y_{(2)}}^3({x_{(2)}}^9 - 1)$ in $F_{(2)}$; $\pi_2 : F_{(2)} \longrightarrow F_{(1)}$, $G = {\pi_2}^{-1}(E)$ exceptional divisor given by $y_{(2)} = 0$. The total transform of the cycle corresponding to the function $x_{(1)} = 0$ is $F_1 + F_2 + F_3$; F_1, F_2, F_3 lines intersecting G. $G.G = -3$, $G.F_i = 1$, $1 \le i \le 3$. Let us apply to ${z_{(2)}}^3 = {y_{(2)}}^3({x_{(2)}}^9 - 1)$ the change $x_{(3)} = x_{(2)}$, $z_{(3)} = \frac{z_{(2)}}{y_{(2)}}$, $y_{(3)} = y_{(2)}$. We obtain ${z_{(3)}}^3 = {x_{(3)}}^9 - 1$; $\pi_3 : F_{(3)} \longrightarrow F_{(2)}$, $D = {\pi_3}^{-1}(G)$.

$(\pi_3) \cdot (\pi_2) \cdot (\pi_1)$ desingularizes O. The fundamental cycle is $Z = D$. Its self-intersection is $D.D = -3$. To know the genus of D we apply Riemann–Hurwitz Formula to $(\pi_3) \cdot (\pi_2)$.

We obtain $2g-2=3(0-2)+2.9=12$, thus $g=7$. We shall write the weighted dual graph of the fundamental cycle as

$$\begin{matrix} 1 \\ \bullet \\ [7] \end{matrix}$$

1 denotes that D is a reduced component of the fundamental cycle; [7] denotes that the genus of D is 7; $D^2=-3$.

Case $s\in\mathbb{N}$: We consider the surface singularity (F,O) given by $z^3=x^{3s}-y^{3s}$.

We want to desingularize F at O. Applying to $z^3=x^{3s}-y^{3s}$ the change $x_{(1)}=\frac{x}{y}$, $z_{(1)}=\frac{z}{y}$, $y_{(1)}=y$, we obtain ${z_{(1)}}^3={y_{(1)}}^{3s-3}({x_{(1)}}^6-1)$ in $F_{(1)}$; $\pi_1: F_{(1)}\longrightarrow F$, $E={\pi_1}^{-1}(O)$ exceptional divisor given by $y_{(1)}=0$.

Let us apply to ${z_{(1)}}^3={y_{(1)}}^{3s-3}({x_{(1)}}^{3s}-1)$ the change $x_{(2)}=x_{(1)}$, $z_{(2)}=\frac{z_{(1)}}{y_{(1)}}$, $y_{(2)}=y_{(1)}$. We repeat the process $s-1$ times until we arrive at ${z_{(s)}}^3={x_{(s)}}^{3s}-1$ in $F_{(s)}$; $\pi_s: F_{(s)}\longrightarrow F_{(s-1)}$, $D={\pi_s}^{-1}(E^{(s-1)})$ exceptional divisor. The fundamental cycle is $Z=D$. Its self-intersection is $D.D=-3$. To know the genus of D we apply Riemann–Hurwitz Formula. We obtain $2g-2=3(0-2)+2(3s)=6s-6$, thus $g=3s-2$. We shall write the weighted dual graph of the fundamental cycle as

$$\begin{matrix} 1 \\ \bullet \\ [3s-2] \end{matrix}$$

1 denotes that D is a reduced component of the fundamental cycle; $[3s-2]$ denotes that the genus of D is $3s-2$; $D^2=-3$.

Proposition 1.2. *The families of smooth curves on a normal surface singularity are in one-to-one correspondence with the reduced components of the maximal cycle of its minimal desingularization* π.

Proof. [3, 1.14]. ∎

Proposition 1.3. *For a surface singularity of the type* (V_{a_0,a_1,a_2},O), *the maximal cycle of* π *and the fundamental cycle of its weighted dual graph coincide if and only if* $\alpha_2\geq l_2$.

Proof. [4, Theorem 3.6]. ∎

Corollary 1.4. *For a surface singularity of the type* $(V_{3,3s,3s},O)$, $s\in\mathbb{N}$, *the maximal cycle of* π *coincides with the fundamental cycle of its weighted dual graph.*

Proof. It follows from Proposition 1.3 since $\alpha_2=1\geq l_2=1$. ∎

Theorem 1.5. *Let* C *be an irreducible nonsingular curve such that* $rC=S\cap F$, *where* S *and* F *are surfaces and* $r\in\mathbb{N}$. C *cannot pass through any singular point of the surfaces of the type* $(V_{3,3s,3s},O)$, $s\in\mathbb{N}$, *unless* $r=3a$, $a\in\mathbb{N}$.

Proof. By Proposition 1.2 and Corollary 1.4, if an irreducible nonsingular curve C passes through a singularity of F, then its strict transform must intersect transversally only one exceptional divisor of multiplicity one in the fundamental cycle. Moreover, since our curve C is not a Cartier divisor, we consider rC and consider its total transform which must have intersection 0 with each exceptional divisor. Consider the fundamental cycle of a surface singularity of the type $(V_{3,3s,3s}, O)$, $s \in \mathbb{N}$. If C would pass through such a singularity we should be able to find, for $r \in \mathbb{N}$, a cycle

$$1 \bullet \underset{[3s-2]}{- - - -} \circ\, r$$

This cycle can only happen if $r = 3a$, $a \in \mathbb{N}$. Thus, there are no C passing through a surface singularity of the type $(V_{3,3s,3s}, O)$, $s \in \mathbb{N}$, unless $r = 3a$, $a \in \mathbb{N}$. ∎

Definition 1.6.

(1) Let $(\mathcal{O}_{X,P}, \mathcal{M}_P)$ be the local ring of a point $P \in X$ of a k-scheme. Let $V \subset \mathcal{M}_P$ be a finite-dimensional k-vector space which generates $\mathcal{M}_P$ as an ideal of $\mathcal{O}_{X,P}$. By a *general hyperplane through* P we mean the subscheme H defined in a suitable open neighbourhood U of P by the ideal $(v)\mathcal{O}_X$, where $v \in V$ is a k-point of a certain dense Zariski open set in V, [5, (2.5)]. By a *general linear variety of codimension* r *through* P we mean the subscheme $L \subset U$ defined in a suitable open neighbourhood U of P by the ideal $(v_1, \dots, v_r)\mathcal{O}_X$, where $v_1, \dots, v_r \in V$ are k-points of a certain dense Zariski open set in V.

(2) Let X be a singular n-fold. We say that a point $Q \in \mathrm{Sing}(X)$ is *a general point of* $\mathrm{Sing}(X)$ if, for a general hyperplane H such that $Q \in H$ and for some divisorial resolution $f : V \to X$, the preimage $f^{-1}(Q)$ of Q and the strict transform $f_*^{-1}(X \cap H)$ satisfy $f^{-1}(Q) \subset f_*^{-1}(X \cap H)$, [2, 2.4].

Definition 2. Let X be a singular n-fold embedded in $\mathbb{P}^N$, $\dim \mathrm{Sing}(X) > 0$.

(1) Let H be a hyperplane in $\mathbb{P}^N$ such that $H \cap \mathrm{Sing}(X) \neq \emptyset$. Denote $X \cap H$ by X_0. We say that X_0 is *a general hyperplane section meeting* $\mathrm{Sing}(X)$ if it is irreducible and, for some divisorial resolution $f : V \to X$, the total transform $f^*(X_0)$ is equal to the strict transform $f_*^{-1}(X_0)$.

(2) Let L_{r+1} be a linear variety of codimension $r+1$, $0 \le r \le n-3$, in $\mathbb{P}^N$ such that $L_{r+1} \cap \mathrm{Sing}(X) \neq \emptyset$. Denote $X \cap L_{r+1}$ by W_r, $0 \le r \le n-3$. We say that W_r is *a general linear section meeting* $\mathrm{Sing}(X)$ if it is irreducible and, for some divisorial resolution $f : V \to X$, the total transform $f^*(W_r)$ is equal to the strict transform $f_*^{-1}(W_r)$.

Definition 3. Let X be a n-fold in in $\mathbb{P}^N$.

(1) A point $P \in X$ is called a *linear compound Du Val singularity* or a *lcDV point* if, for a general linear variety W of codimension $n-2$ through P, $P \in W$, $X \cap W$ is a Du Val singularity.

(2) A point $P \in X$ is called a *linear compound* $V3$ *singularity* or a *lcV3 point* if, for a general linear variety W of codimension $n-2$ through P, $P \in W$, $X \cap W$ is a singularity of the type $V_{3,3s,3s}$, $s \in \mathbb{N}$.

Remark 4.

1. $P \in X$ is lcDV if it is locally analytically isomorphic to the hypersurface singularity given by $f + g$, where $g \in k[x_1, \dots, x_N]$ is such that $\operatorname{mult} g > \deg f$ and $f \in k[x_1, x_2, x_3]$ represents a Du Val singularity.

2. $P \in X$ is lcV3 if it is locally analytically isomorphic to the hypersurface singularity of the form $f+g$, where f is a polynomial in $k[x_1, x_2, x_3]$ of the form $c_3(x_3)^3 = c_1(x_1)^{3s} + c_2(x_2)^{3s} + h(x_1, x_2, x_3)$, $s \in \mathbb{N}$, $c_1, c_2, c_3 \in k$, and $g \in k[x_1, \dots, x_N]$, $\operatorname{mult} g > \deg f \geq 6s$. This is so because any monomial w of h would be divisible either by $(x_3)^2$ or $(x_1)^{3s-1}$ or $(x_2)^{3s-1}$ which would make possible to absorb w respectively in either $(x_3)^3$ or $(x_1)^{3s}$ or $(x_2)^{3s}$.

Notation 5. Fix a point $P \in \operatorname{Sing}(X)$. Let L_{r+1} be a general linear variety of codimension $r+1$ in $\mathbb{P}^N$, $0 \leq r \leq n-3$, such that $\operatorname{Sing}(X) \cap L_{r+1} \neq \emptyset$. Let $W_r = X \cap L_{r+1}$.

Proposition 6. *Let X be a quasiprojective scheme over any field. Let R_{l+1} be a general linear variety of codimension $l+1$ in $\mathbb{P}^N$, $0 \leq l < N-1$. Let $T_l = X \cap R_{l+1}$. Let $r \in \mathbb{N}$. If X satisfies Serre's condition S_r, then so does T_l. Thus, if X is a normal variety, then so is T_l.*

Proof. [2, 3.4] ∎

Definition 7. Let us consider the d-uple embedding $\rho_d : \mathbb{P}^N \to \mathbb{P}^M$, $M = \binom{n+d}{d} - 1$. Let Y be a complete intersection nonsingular variety in $\mathbb{P}^N$ defined by l equations $\sum_{i=0}^{M} a_{ij} y_i$, $1 \leq j \leq l$, $1 \leq l \leq N-1$, where all the y_i, $0 \leq i \leq M$, are monomials of degree d in $x_0, \dots, x_N$, for some $l, d \in \mathbb{N}$. We call Y a (d,l) *complete intersection nonsingular variety.*

Proposition 8. *Let X be a normal singular variety of dimension n in $\mathbb{P}^N$. Let Y be a (d,l) complete intersection nonsingular variety such that $Y \cap \operatorname{Sing}(X) \neq \emptyset$. Let us consider the d-uple embedding $\rho_d : \mathbb{P}^N \to \mathbb{P}^M$, $M = \binom{n+d}{d} - 1$. Then, as abstract varieties, $X \cap Y = \rho_d(X) \cap \tilde{Y}$, where $\tilde{Y}$ is a linear variety in $\mathbb{P}^M$ obtained from Y by using ρ_d.*

Proof. Let us consider the d-uple embedding $\rho_d : \mathbb{P}^N \to \mathbb{P}^M$, $M = \binom{n+d}{d} - 1$. Let Y be a complete intersection nonsingular variety defined by equations $\sum_{i=0}^{M} a_{ij} y_i$, $1 \leq j \leq l$, $1 \leq l \leq N-1$, where all the y_i, $0 \leq i \leq M$, are monomials of degree d in $x_0, \dots, x_N$. We see that, as abstract varieties, $X \cap Y = \rho_d(X) \cap \tilde{Y}$, where $\tilde{Y}$ is a linear variety in $\mathbb{P}^M$, since we can identify the y_i with the coordinates of $\mathbb{P}^M$ obtained by using ρ_d. ∎

Definition 9. Let X be a normal singular variety of dimension n in $\mathbb{P}^N$. Let Y be a (d,l) complete intersection nonsingular variety in $\mathbb{P}^N$ such that $Y \cap \operatorname{Sing}(X) \neq \emptyset$. Let us consider the d-uple embedding $\rho_d : \mathbb{P}^N \to \mathbb{P}^M$, $M = \binom{n+d}{d} - 1$. Then, as abstract varieties, $X \cap Y = \rho_d(X) \cap \tilde{Y}$, where $\tilde{Y}$ is a linear variety in $\mathbb{P}^M$ obtained from Y by using ρ_d. We say that Y is *a (d,l) general complete intersection nonsingular variety in $\mathbb{P}^N$ meeting* $\operatorname{Sing} X$ if $\tilde{Y}$ is a general linear variety in $\mathbb{P}^M$ meeting $\operatorname{Sing} \rho_d(X)$, according to Definition 2.

Definition 10. Let X be a n-fold.

(1) A point $P \in X$ is called a (d,l) *complete intersection compound Du Val singularity or a dlcicDV point* if, for a general (d,l) complete intersection nonsingular variety Y of codimension $n-2$ through P, $P \in Y$, $X \cap Y$ is a Du Val singularity.

(2) A point $P \in X$ is called a (d,l) *complete intersection compound* $V3$ *singularity or a dlcicV3 point* if, for a general (d,l) complete intersection nonsingular variety Y of codimension $n-2$ through P, $P \in Y$, $X \cap Y$ is a singularity of type $V_{3,3s,3s}$, $s \in \mathbb{N}$.

11. Let C be an irreducible nonsingular curve such that $2C = V \cap W$, where V and W are two surfaces in $\mathbb{P}^3$ and W has at most rational double points. Let us suppose that C passes through a rational double point P of W. Let $\tilde{W}$ be the minimal desingularization of W at P, $\pi : \tilde{W} \to W$. Let E_k, $1 \leq k \leq n$, be the irreducible components of the exceptional divisor. Write the total transform $\pi^*(2C) = \sum_{j=1}^{n} \beta_j E_j + 2E$, where E is the strict transform of C, $\beta_j \in \mathbb{N}$.

Proposition 12. *Let C be an irreducible nonsingular curve $2C = V \cap W$, where V and W are two surfaces in $\mathbb{P}^3$ and W has only rational double points as singularities. Assume that C passes through a rational double point P of W. P cannot be either of type A_{2r}, $r \in \mathbb{N}$, or type $\mathbb{E}_6$, or $\mathbb{E}_8$. For C to pass through only one singularity of type A_{2r+1}, $r \in \mathbb{N}$, we must have $(\sum_{j=1}^{2r+1} \beta_j E_j)^2 = -(2r+2)$. For C to pass through only one singularity of type $\mathbb{E}_7$, we must have $(\sum_{j=1}^{7} \beta_j E_j)^2 = -6$. For C to pass through only one singularity of type D_n, $n \geq 4$, we must have either $(\sum_{j=1}^{n} \beta_j E_j)^2 = -4$, or, for $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $(\sum_{j=1}^{n} \beta_j E_j)^2 = -n$.*

Proof. [1, Theorem 0.9] ∎

Corollary 12.1. *Let C be an irreducible nonsingular curve $rC = V \cap W$, $r = 2a$, $a \in \mathbb{N}$, where V and W are two surfaces in $\mathbb{P}^3$ and W has only rational double points as singularities. Assume that C passes through a rational double point P of W. P cannot be either of type A_{2r}, $r \in \mathbb{N}$, or type $\mathbb{E}_6$, or $\mathbb{E}_8$. For C to pass through only one singularity of type A_{2r+1}, $r \in \mathbb{N}$, we must have $(\sum_{j=1}^{2r+1} \beta_j E_j)^2 = -(2r+2)$. For C to pass through only one singularity of type $\mathbb{E}_7$, we must have $(\sum_{j=1}^{7} \beta_j E_j)^2 = -6$. For C to pass through only one singularity of type D_n, $n \geq 4$, we must have either $(\sum_{j=1}^{n} \beta_j E_j)^2 = -4$, or, for $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $(\sum_{j=1}^{n} \beta_j E_j)^2 = -n$.*

Proof. It follows from Proposition 12, since, if $2C$ is a Cartier divisor, so it is $2aC$, $a \in \mathbb{N}$. ∎

Proposition 13. *Let Z be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $rZ = X \cap F$, $r = 2a$, $a \in \mathbb{N}$, where X is an n-fold and F is a $(N-1)$-fold in $\mathbb{P}^N$, X normal with a lcDV singularity P and such that $Z \cap \operatorname{Sing}(X) \neq \emptyset$. Let W_{n-3} be a general linear variety as in Notation* 5. *Then Z has empty intersection with a lcDV singularity of X such that $W_{n-3} \cap X = Q$, where Q is a rational surface singularity of types $\mathbb{A}_{2k}$, $k \in \mathbb{N}$, $\mathbb{E}_6$ and $\mathbb{E}_8$. For Z to have non-empty intersection with a lcDV singularity of X such that $W_{n-3} \cap X = Q$, where Q is a rational surface singularity of type $\mathbb{A}_{2k+1}$, $k \in \mathbb{N}$, we must have $(\sum_{j=1}^{2k+1} \beta_j E_j)^2 = -(2k+2)$, where E_j, $1 \leq j \leq 2k+1$, are the irreducible components of the exceptional divisor supported on $\pi^{-1}(Q)$ for $\pi : \tilde{W}_{n-3} \to W_{n-3}$ the minimal resolution of $Q \in W_{n-3} \cap F$. For Q to be of type $\mathbb{E}_7$, we must have $(\sum_{j=1}^{7} \beta_j E_j)^2 = -6$, where E_k, $1 \leq k \leq 7$, are the irreducible components of the exceptional divisor as above. For Q to be of type $\mathbb{D}_n$, $n \geq 4$, we must have either $(\sum_{j=1}^{n} \beta_j E_j)^2 = -4$, or, for $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $(\sum_{j=1}^{n} \beta_j E_j)^2 = -n$, where E_k, $1 \leq k \leq n$, are the irreducible components of the exceptional divisor as above.*

Proof. Let us define recursively W_b. Let H_{b+1}, $0 \leq b \leq n-4$, be a general hyperplane meeting $\mathrm{Sing}(W_b) \cap F$. Let $r = 2a$, $a \in \mathbb{N}$. Given $rZ = X \cap Y$ we intersect it with H_b, $0 \leq b \leq n-3$, as follows: $rZ \cap H_0 \cap \ldots \cap H_{n-3} = F \cap X \cap H_0 \cap \ldots \cap H_{n-3}$. We obtain a nonsingular curve C such that $rC = Y \cap X \cap H_0 \cap \ldots \cap H_{n-3}$ and that $C \cap \mathrm{Sing}(W_{n-3}) \neq \emptyset$. We apply Corollary 12.1 to obtain the result. ∎

Corollary 14. *Let Z be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $rZ = X \cap F$, $r = 2a$, $a \in \mathbb{N}$, where X is an n-fold and F is a $(N-1)$-fold in $\mathbb{P}^N$, X normal with a dlcicDV singularity P and such that $Z \cap \mathrm{Sing}(X) \neq \emptyset$. Let Y be a general (d,l) complete intersection nonsingular variety of codimension $n-2$ as in Definition 7. Then Z has empty intersection with a dlcicDV singularity of X such that $Y \cap X = Q$, where Q is a rational surface singularity of types $\mathbb{A}_{2k}$, $k \in \mathbb{N}$, $\mathbb{E}_6$ and $\mathbb{E}_8$. For Z to have non-empty intersection with a dlcicDV singularity of X such that $Y \cap X = Q$, where Q is a rational surface singularity of type $\mathbb{A}_{2k+1}$, $k \in \mathbb{N}$, we must have $\left(\sum_{j=1}^{2k+1} \beta_j E_j\right)^2 = -(2k+2)$, where E_j, $1 \leq j \leq 2k+1$, are the irreducible components of the exceptional divisor supported on $\pi^{-1}(Q)$ for $\pi : \tilde{Y} \to Y$ the minimal resolution of $Q \in Y \cap F$. For Q to be of type $\mathbb{E}_7$, we must have $\left(\sum_{j=1}^{7} \beta_j E_j\right)^2 = -6$, where E_k, $1 \leq k \leq 7$, are the irreducible components of the exceptional divisor as above. For Q to be of type $\mathbb{D}_n$, $n \geq 4$, we must have either $\left(\sum_{j=1}^{n} \beta_j E_j\right)^2 = -4$, or, for $n = 2k$, $k \in \mathbb{N}$, $k \geq 3$, $\left(\sum_{j=1}^{n} \beta_j E_j\right)^2 = -n$, where E_k, $1 \leq k \leq n$, are the irreducible components of the exceptional divisor as above.*

Proof. Let us consider the d-uple embedding $\rho_d : \mathbb{P}^N \to \mathbb{P}^M$, $M = \binom{n+d}{d} - 1$. Let Y be defined as in Definition 7. Then, as abstract varieties, $X \cap Y = \rho_d(X) \cap \tilde{Y}$, where $\tilde{Y}$ is a linear variety in $\mathbb{P}^M$. To obtain the result we follow the proof of Proposition 13 substituting W_{n-3} by $\tilde{Y}$. ∎

Proposition 15. *Let Z be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $rZ = X \cap F$, $r \in \mathbb{N}$, where X is an n-fold and F is a $(N-1)$-fold in $\mathbb{P}^N$, X normal with a lcV3 singularity P and such that $Z \cap \mathrm{Sing}(X) \neq \emptyset$. Let W_{n-3} be a general linear variety as in Notation 5. Z has empty intersection with a lcV3 singularity of X such that $W_{n-3} \cap X = Q$, where Q is a surface singularity of type $V_{3,3s,3s}$, $s \in \mathbb{N}$, unless $r = 3a$, $a \in \mathbb{N}$.*

Proof. Let us define recursively W_t. Let H_{t+1}, $0 \leq t \leq n-4$, be a general hyperplane meeting $\mathrm{Sing}(W_t) \cap F$. Given $rZ = X \cap F$, $r \in \mathbb{N}$, $r \geq 2$, we intersect it with H_t, $0 \leq t \leq n-3$, as follows: $rZ \cap H_0 \cap \cdots \cap H_{n-3} = F \cap X \cap H_0 \cap \cdots \cap H_{n-3}$. We obtain a nonsingular curve C such that $rC = F \cap X \cap H_0 \cap \cdots \cap H_{n-3}$ and that $C \cap \mathrm{Sing}(W_{n-3}) \neq \emptyset$. We apply Theorem 1.5 to obtain the result. ∎

Corollary 16. *Let Z be a reduced irreducible nonsingular $(n-1)$-dimensional variety such that $rZ = X \cap F$, where X is an n-fold and F is a $(N-1)$-fold in $\mathbb{P}^N$, X normal with a dlcicV3 singularity P and such that $Z \cap \mathrm{Sing}(X) \neq \emptyset$. Let Y be a general (d,l) complete intersection nonsingular variety of codimension $n-2$ as in Definition 7. Then Z has empty intersection with a dlcicV3 singularity of X such that $Y \cap X = Q$, where Q is a surface singularity of type $V_{3,3s,3s}$, $s \in \mathbb{N}$, unless $r = 3a$, $a \in \mathbb{N}$.*

Proof. Let us consider the d-uple embedding $\rho_d : \mathbb{P}^N \to \mathbb{P}^M$, $M = \binom{n+d}{d} - 1$. Let Y be defined as in Definition 7. We see that, as abstract varieties, $X \cap Y = \rho_d(X) \cap \tilde{Y}$, where $\tilde{Y}$ is a linear variety in $\mathbb{P}^M$. To obtain the result we follow the proof of Proposition 15 substituting W_{n-3} by $\tilde{Y}$. ∎

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