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UNIVERSALITY RESULTS ON HURWITZ ZETA-FUNCTIONS

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Abstract. In the paper, we give a survey of the results on the approximation of analytic functions by shifts of Hurwitz zeta-functions. Theorems of such a kind are called universality theorems. Continuous, discrete and joint universality theorems of Hurwitz zeta-functions are discussed.

1. Introduction. Let $s = \sigma + it$ be a complex variable, and α, $0 < \alpha \leq 1$, be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \tag{1}$$

and is analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s, \alpha)$ was introduced and studied by A. Hurwitz in the paper [H]. He obtained meromorphic continuation for $\zeta(s, \alpha)$ and

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proved the functional equation: for $\sigma < 0$

$$\zeta(s,\alpha) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\Big(\sin\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\cos 2\pi m\alpha}{m^{1-s}} + \cos\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\sin 2\pi m\alpha}{m^{1-s}}\Big),$$

where, as usual, $\Gamma(s)$ denotes the Euler gamma function.

The Hurwitz zeta-function is a direct generalization of the Riemann zeta-function $\zeta(s)$ which is defined in the half plane $\sigma > 1$ by the series

$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s}\,. \tag{2}$$

From (1) and (2), we see that $\zeta(s,1) = \zeta(s)$ and $\zeta(s,\frac{1}{2}) = (2^s-1)\zeta(s)$. Some properties of $\zeta(s)$, e.g., that of having a simple pole at $s=1$, generalize to $\zeta(s,\alpha)$. The main difference is that the function $\zeta(s)$ has the Euler product over primes

$$\zeta(s) = \prod_p\Big(1-\frac{1}{p^s}\Big)^{-1}, \quad \sigma > 1,$$

while $\zeta(s,\alpha)$, except for the values $\alpha = 1, \frac{1}{2}$, cannot be expressed as a product over primes [H].

On the other hand, the function $\zeta(s,\alpha)$ is a special case of the more general Lerch zeta-function

$$L(\lambda,\alpha,s) = \sum_{m=0}^{\infty}\frac{e^{2\pi i m\lambda}}{(m+\alpha)^s}\,, \quad \sigma > 1,$$

where λ is a real parameter. Obviously, for $\lambda \in \mathbb{Z}$, the functions $\zeta(s,\alpha)$ and $L(\lambda,\alpha,s)$ coincide. Also, the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{a})$,

$$\zeta(s,\alpha;\mathfrak{a}) = \sum_{m=0}^{\infty}\frac{a_m}{(m+\alpha)^s}\,, \quad \sigma > 1,$$

where $\mathfrak{a} = \{a_m : m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}\}$ is a periodic sequence of complex numbers with minimal period $q \in \mathbb{N}$, is a generalization of $\zeta(s,\alpha)$. The function $\zeta(s,\alpha;\mathfrak{a})$ was introduced in [L2]. In view of the periodicity of $\mathfrak{a}$, we have

$$\zeta(s,\alpha;\mathfrak{a}) = \frac{1}{q^s}\sum_{m=0}^{q-1}a_m\zeta\Big(s,\frac{m+\alpha}{q}\Big).$$

The function $\zeta(s,\alpha)$ with rational parameter $\alpha = \frac{a}{q}$ is related to the theory of Dirichlet L-functions $L(s,\chi)$,

$$L(s,\chi) = \sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}\,, \quad \sigma > 1,$$

with character χ modulo q because

$$L(s,\chi) = \frac{1}{q^s}\sum_{a=1}^{q}\chi(a)\zeta\Big(s,\frac{a}{q}\Big).$$

On the other hand, denoting by $\varphi(q)$ the Euler totient function, we have for $(a,q)=1$

$$\zeta\Big(s,\frac{a}{q}\Big) = \frac{q^s}{\varphi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\overline{\chi}(a)L(s,\chi). \tag{3}$$

In this paper, we give a survey of the problem on the approximation of analytic functions by shifts of the Hurwitz zeta-function $\zeta(s+i\tau,\alpha)$, $\tau\in\mathbb{R}$, or $\zeta(s+ikh,\alpha)$, $k\in\mathbb{N}_0$, where h is a fixed positive number. In the case of shifts $\zeta(s+i\tau,\alpha)$, $\tau\in\mathbb{R}$, we have a continuous universality property of $\zeta(s,\alpha)$, and in the case of shifts $\zeta(s+ikh,\alpha)$, $k\in\mathbb{N}_0$, the approximation property of $\zeta(s,\alpha)$ is called a discrete universality.

The universality of zeta-functions was discovered by S. M. Voronin who proved [V] the universality of the Riemann zeta-function. More precisely, he obtained the following statement.

Theorem 1. *Suppose that $0<r<\frac{1}{4}$, and that the function $f(s)$ is continuous and non-vanishing on the disc $|s|\le r$, and analytic in the interior of this disc. Then, for every $\varepsilon>0$, there exists a real number $\tau=\tau(\varepsilon)$ such that*

$$\max_{|s|\le r}\left|\zeta\Big(s+\frac{3}{4}+i\tau\Big)-f(s)\right|<\varepsilon.$$

For a modern version of the Voronin theorem, we need some notation. Let $D=\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\}$ be the right-hand side of the critical strip. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$, $K\in\mathcal{K}$, the class of continuous non-vanishing functions on K which are analytic in the interior of K. Moreover, let meas A denote the measure of a Lebesgue-measurable set $A\subset\mathbb{R}$. Then the following theorem is a generalization of Theorem 1, see, for example, [L1].

Theorem 2. *Suppose that $K\in\mathcal{K}$ and $f(s)\in H_0(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\Big\}>0.$$

The Voronin theorem can be considered as a multidimensional generalization of a Bohr–Courant result [BC] on the denseness of the set $\{\zeta(\sigma+i\tau):t\in\mathbb{R}\}$ for every fixed σ, $\frac{1}{2}<\sigma<1$.

The proof of Theorem 1 in [V] is based on the approximation in the mean of $\zeta(s)$ by a finite Euler's product, it also uses a version of the Riemann theorem on rearrangement of series in Hilbert space as well as the Kronecker approximation theorem.

The proof of Theorem 2 is probabilistic, based on properties of weakly convergent probability measures on the space of analytic functions, and was proposed by B. Bagchi [B]. A very good extensive survey on universality of zeta-functions is given in the paper [M].

2. Continuous universality of the function $\zeta(s,\alpha)$. The first universality results for the Hurwitz zeta-function were obtained by S. M. Gonek in his thesis [Go, Chapter IV].

Theorem 2.1 ([Go]). *Let K be a compact simply connected subset of the strip D. Let α be rational or transcendental number, $\alpha\ne\frac{1}{2}$, $\alpha\ne1$. If $f(s)$ is continuous on K and analytic in the interior of K, then, for any $\varepsilon>0$, there exists a $\tau\in\mathbb{R}$ such that*

$$\max_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon.$$

In the case of rational α, the identity (3) and a joint universality theorem for Dirichlet L-functions are applied. The latter theorem is the first result on the so-called hybrid universality, therefore, we state it because of its influence upon further investigations. Denote by $\|u\|$ the distance from u to the nearest integer.

THEOREM 2.2 ([Go]). *Let $q \in \mathbb{N}$, and let K be a simply connected set contained in D. Suppose that, for each prime $p\,|\,q$, we have $0 \le \theta_p < 1$, and that, for each character χ modulo q, $f_\chi(s)$ is continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there is a $\tau \in \mathbb{R}$ such that for all $p\,|\,q$ we have*

$$\left\| \frac{-\tau \log p}{2\pi} - \theta_p \right\| < \varepsilon$$

and, for all χ modulo q,

$$\max_{s\in K} \left| L(s+i\tau,\chi) - e^{f_\chi(s)} \right| < \varepsilon.$$

Theorem 2.2 connects the Kronecker theorem in the theory of Diophantine approximation with the Voronin theorem on joint universality of Dirichlet L-functions. Theorem 2.2 was improved and generalized by J. Kaczorowski and M. Kulas [KK], and by Ł. Pańkowski [P].

A proof of the case of transcendental α in Theorem 2.1 is based on the fundamental lemma of Gonek [Go, Lemma 2.2]. Let Λ be the set of terms of a monotone increasing sequence of real numbers tending to $+\infty$, and

$$N_\Lambda(x) = \sum_{\lambda \le x,\, \lambda\in\Lambda} 1.$$

LEMMA 2.3 ([Go] Fundamental lemma). *Suppose that*

$$N_\Lambda(x) \ll e^x,$$

and that, for any fixed $c > 0$,

$$\left| N_\Lambda\Big(x \pm \frac{c}{x^2}\Big) - N_\Lambda(x) \right| \gg \frac{e^x}{x^3}\,.$$

Let K be a simply connected compact set in the strip $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 < 1$, and let $f(s)$ be a function continuous on K and analytic in the interior of K. Then, for each $\mu > 0$, there exists a constant $\rho_0 = \rho_0(\sigma_1, \sigma_2, K, \Lambda, f, \mu)$ such that if $\rho \ge \rho_0$, then there are numbers $\theta_\lambda \in \mathbb{R}$ for which

$$\max_{s\in K} \left| f(s) - \sum_{\mu < e^\lambda \le \rho,\, \lambda\in\Lambda} e^{2\pi i \theta_\lambda} e^{-\lambda s} \right| \ll \mu^{-1/2}.$$

The constant implied in $\ll$ depends only on σ_1, σ_2, K and Λ.

In the case of the function $\zeta(s,\alpha)$, we can take $\Lambda = \{\log(m+\alpha) : m \in \mathbb{N}_0\}$, so

$$N_\Lambda(x) = \sum_{\log(m+\alpha)\le x} 1 = e^x + O(1)$$

and

$$\left| N_\Lambda\Big(x + \frac{c}{x^2}\Big) - N_\Lambda(x) \right| \gg \frac{e^x}{x^2}\,.$$

For an arbitrary μ we can apply Lemma 2.3 to the function

$$\sum_{0\le m\le \mu} \frac{1}{(m+\alpha)^s}$$

in place of $f(s)$. Methods of Diophantine analysis provide an upper bound for the sum

$$\sum_{0\le m\le\mu} \frac{1}{(m+\alpha)^s} + \sum_{\mu<m\le\rho} \frac{e^{2\pi i\theta_m}}{(m+\alpha)^s} - \sum_{0\le m\le\rho} \frac{1}{(m+\alpha)^{s+i\tau}}\,.$$

Combining this with the outcome of Lemma 2.3 and approximation of $\zeta(s,\alpha)$ by a finite sum leads to the assertion of Theorem 2.1.

We observe that S. M. Gonek uses the notion of a simply connected set on the complex plane which is an equivalent of a set with connected complement.

Another proof of Theorem 2.1 was given by B. Bagchi in his thesis [B]. Bagchi's method is based on probabilistic limit theorems on weakly convergent probability measures in the space of analytic functions $H(D)$ endowed with the topology of uniform convergence on compacta. We briefly discuss the case of rational α. Consider the infinite-dimensional torus

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\} \stackrel{\text{def}}{=} \gamma$ for every prime p. With the product topology and pointwise multiplication, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\mathbb{P}$ be the set of all prime numbers. Then we see from the definition of Ω that every element $\omega \in \Omega$ is a function $\omega : \mathbb{P} \to \gamma$. The formula

$$\omega(m) = \prod_{p^k \,|\, m,\, p^{k+1} \nmid m} \omega^k(p)$$

extends the function ω to the set $\mathbb{N}$.

Now we return to the function $\zeta(s,\alpha)$. Since $\alpha \neq 1, \frac{1}{2}$, there exist numbers $a, q \in \mathbb{N}$, $1 \le a \le q$, $(a,q) = 1$, $q \ge 3$, such that $\alpha = \frac{a}{q}$. Then, for $\sigma > 1$,

$$\zeta(s,\alpha) = q^s \sum_{m=1,\, m \equiv a \,(\mathrm{mod}\, q)}^{\infty} \frac{1}{m^s} \stackrel{\text{def}}{=} f_1(s) f_2(s).$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define two random elements of $H(D)$ by the formulae

$$f_1(s,\omega) = \overline{\omega(q)} q^s \tag{4}$$

and

$$f_2(s,\omega) = \sum_{m=1,\, m \equiv a \,(\mathrm{mod}\, q)}^{\infty} \frac{\omega(m)}{m^s}\,. \tag{5}$$

Let $P_{(f_1,f_2)}$ be the distribution of the random element (f_1, f_2), i.e.,

$$P_{(f_1,f_2)}(A) = m_H\bigl\{\omega \in \Omega : (f_1(s,\omega), f_2(s,\omega)) \in A\bigr\}, \quad A \in \mathcal{B}(H^2(D)).$$

Then it is shown that shifting the functions $f_1(s)$ and $f_2(s)$ vertically has an effect similar to that of the randomizations (4) and (5). More precisely,

$$\frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0,T] : (f_1(s+i\tau), f_2(s+i\tau)) \in A\bigr\}, \quad A \in \mathcal{B}(H^2(D)),$$

converges weakly to $P_{(f_1,f_2)}$ as $T \to \infty$. This implies that

$$\frac{1}{T} \operatorname{meas}\{\tau \in [0,T] : \zeta(s+i\tau,\alpha) \in A\}, \quad A \in \mathcal{B}(H(D)), \tag{6}$$

converges weakly to the distribution $P_{f_1 f_2}$ of the random element $f_1(s,\omega) f_2(s,\omega)$.

The next step is to show that the support of $P_{f_1 f_2}$ is the whole of $H(D)$. Using joint probabilistic results for Dirichlet L-functions and their connection to $f_2(s)$, we infer that the support of $f_2(s,\omega)$ is the whole of $H(D)$. Since $(a,q)=1$, the random variable $\omega(q)$ is independent of each random variable $\omega(m)$ with $m \equiv a \pmod q$. Thus, $f_1(s,\omega)$ is independent of $f_2(s,\omega)$, and $f_1(s,\omega)$ is not degenerate at the point 0. This implies that the range of $f_1(s,\omega)f_2(s,\omega)$, i.e. the support of $P_{f_1f_2}$, is the entire $H(D)$.

The further part of the proof is very simple, and involves the Mergelyan theorem on the approximation of analytic functions by polynomials. Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $f(s)$ be a function continuous on K and analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s)-p(s)| < \frac{\varepsilon}{2}\,. \tag{7}$$

The polynomial $p(s)$ is an element of the support of the measure $P_{f_1f_2}$. Let

$$G = \Big\{ g \in H(D) : \sup_{s\in K} |g(s)-p(s)| < \frac{\varepsilon}{2} \Big\}. \tag{8}$$

Then G is an open neighbourhood of $p(s)$, therefore, $P_{f_1f_2}(G) > 0$. Moreover, by properties of the weak convergence of probability measures, we have

$$\liminf_{T\to\infty} \frac{1}{T}\,\text{meas}\{\tau \in [0,T] : \zeta(s+i\tau,\alpha) \in G\} \geq P_{f_1f_2}(G) > 0.$$

Thus, (7) and (8) prove the assertion in a more general form: the set of shifts $\zeta(s+i\tau,\alpha)$ approximating a given analytic function is infinite and even has a positive lower density.

Bagchi's proof of Theorem 2.1 in the case of transcendental α uses the linear independence over the field of rational numbers $\mathbb{Q}$ of the set

$$\{\log(m+\alpha) : m \in \mathbb{N}_0\}.$$

In this case, in place of the torus Ω, the torus

$$\Omega_1 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$, is considered. Then he proves a limit theorem on the weak convergence of (6), as $T \to \infty$, to the distribution of the $H(D)$-valued random element

$$\sum_{m=0}^{\infty} \frac{\omega_1(m)}{(m+\alpha)^s}\,, \tag{9}$$

where $\omega_1(m)$ is the m-th component of $\omega_1 \in \Omega_1$. The details can be found in [LG], Chapters 5 and 6.

We note that B. Bagchi stated his analogue of Theorem 2.1 for compact simply connected and locally path-connected sets $K \subset D$.

The modern version of Theorem 2.1 is the following. Denote by $H(K)$, $K \in \mathcal{K}$, the class of continuous functions on K which are analytic in the interior of K.

Theorem 2.4. *Suppose that α is transcendental or rational, $\alpha \neq \frac{1}{2}$, $\alpha \neq 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\,\text{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\Big\} > 0.$$

3. Discrete universality of the function $\zeta(s,\alpha)$. The first result on the discrete universality of the function $\zeta(s,\alpha)$ belongs to B. Bagchi. Using the joint discrete universality of Dirichlet L-functions, he obtained in [B] the following theorem.

THEOREM 3.1. *Suppose that α is a rational number, $\alpha \neq \frac{1}{2}$, $\alpha \neq 1$. Let K be a compact simply connected and locally path-connected subset of D, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\Big\} > 0.$$

The case of transcendental α is more complicated, and a certain relation between the numbers h and π is needed. The next theorem was obtained for more general periodic Hurwitz zeta-functions [LM].

THEOREM 3.2. *Suppose that α is a transcendental number, $K \in \mathcal{K}$ and $f(s) \in H(K)$. Let $h > 0$ be such that $\exp\big(\frac{2\pi}{h}\big)$ is a rational number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\Big\} > 0.$$

Define the set

$$\mathrm{L}(\alpha,h,\pi) = \Big\{(\log(m+\alpha) : m \in \mathbb{N}_0), \frac{\pi}{h}\Big\}, \quad h > 0.$$

In [L7], the following extension of Theorem 3.2 was obtained.

THEOREM 3.3. *Suppose that the set $\mathrm{L}(\alpha,h,\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let K and $f(s)$ be the same as in Theorem 3.2. Then the assertion of Theorem 3.2 is true.*

It is not difficult to see that if the numbers α and $\exp\big(\frac{\pi}{h}\big)$ are algebraically independent over $\mathbb{Q}$, then the set $\mathrm{L}(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. This allows us to construct the numbers α and h satisfying the hypothesis of Theorem 3.3.

It is known [G] that the numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent over $\mathbb{Q}$. Therefore, the numbers $\alpha = 2^{-\sqrt[3]{2}}$ and $h = \frac{\pi}{\sqrt[3]{4}\log 2}$ satisfy the hypothesis of Theorem 3.3.

The second example is based on the theorem of Nesterenko [Ne] that the numbers π and e^{π} are algebraically independent over $\mathbb{Q}$. Thus, we can take $\alpha = \frac{1}{\pi}$ and rational h.

A proof of Theorem 3.3 is probabilistic, and is based on the following limit theorem.

THEOREM 3.4. *Suppose that the set $\mathrm{L}(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Then*

$$\frac{1}{N+1} \#\big\{0 \le k \le N : \zeta(s+ikh,\alpha) \in A\big\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the distribution of the random element (9) *as $N \to \infty$. Moreover, the support of this distribution is the whole of $H(D)$.*

The key lemma for the proof of Theorem 3.4 is a limit theorem on the torus Ω_1. Let m_H be the probability Haar measure on $(\Omega_1, \mathcal{B}(\Omega_1))$.

LEMMA 3.5. *Suppose that the set $\mathrm{L}(\alpha,h,\pi)$ is linearly independent over $\mathbb{Q}$. Then*

$$\frac{1}{N+1} \#\big\{0 \le k \le N : ((m+\alpha)^{-ikh} : m \in \mathbb{N}_0) \in A\big\}, \quad A \in \mathcal{B}(\Omega_1),$$

converges weakly to m_H as $N \to \infty$.

Lemma 3.5 is proved by the method of the Fourier transforms and in the proof the linear independence of the set $\mathrm{L}(\alpha, h, \pi)$ is essentially used.

4. Joint universality of Hurwitz zeta-functions. Let $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$ be a collection of Hurwitz zeta-functions. In this section, we discuss the simultaneous approximation of a collection of analytic functions by shifts $\zeta(s+i\tau, \alpha_1), \dots, \zeta(s+i\tau, \alpha_r)$, $\tau \in \mathbb{R}$, or $\zeta(s+ikh_1, \alpha_1), \dots, \zeta(s+ikh_r, \alpha_r)$, $k \in \mathbb{N}_0$ and $h_1 > 0, \dots, h_r > 0$. Obviously, in this case, a kind of independence of the functions $\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)$ is necessary. The simplest way of ensuring this is a requirement that the parameters $\alpha_1, \dots, \alpha_r$ would be algebraically independent over $\mathbb{Q}$. We recall that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$ if there is no polynomial $p(x_1, \dots, x_r) \not\equiv 0$ with rational coefficients such that $p(\alpha_1, \dots, \alpha_r) = 0$.

The following joint continuous universality theorem is a corollary of more general statements for Lerch zeta-functions and periodic Hurwitz zeta-functions [L2], [LM1], [LM2], [Na].

Theorem 4.1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \sup_{1\le j\le r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

In [L3], the algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$ was replaced by a weaker hypothesis that the set

$$\mathrm{L}(\alpha_1, \dots, \alpha_r) = \{\log(m+\alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r\}$$

is linearly independent over $\mathbb{Q}$.

H. Mishou [Mi2] has obtained a very interesting result on the joint universality of two Hurwitz zeta-functions with algebraically dependent parameters α_1 and α_2.

Theorem 4.2 ([Mi2]). *Suppose that α_1 and α_2 are transcendental numbers, $\alpha_1 \neq \alpha_2$ and $\alpha_2 \in \mathbb{Q}(\alpha_1)$. For $j = 1, 2$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0, T] : \sup_{1\le j\le 2} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\Big\} > 0.$$

Theorem 4.2 was generalized by A. Dubickas [D] for an arbitrary finite number r of transcendental numbers $\alpha_1, \dots, \alpha_r$. Let

$$\mathrm{L}(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\},$$

and, for $q \in \mathbb{N}_0$,

$$\mathrm{L}_q(\alpha) = \{\log(m+q+\alpha) : m \in \mathbb{N}_0\}.$$

Theorem 4.3 ([D]). *Suppose that $\alpha_1, \dots, \alpha_r$ are transcendental numbers such that, for some $q_1, \dots, q_r \in \mathbb{N}_0$, the set*

$$D_{q_1}(\alpha_1) \cup \ldots \cup D_{q_r}(\alpha_r)$$

is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let K_j and $f_j(s)$ be as in Theorem 4.1. Then the same assertion as that of Theorem 4.1 holds.

We note that the above-mentioned paper [L3] corresponds to the case $q_1 = \ldots = q_r = 0$. Next we present an example from [D] of Hurwitz zeta-functions which are not jointly universal. We recall that the largest cardinality of an algebraically independent subset of $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)$ over $\mathbb{Q}$ is called the transcendence degree of the field extension $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)/\mathbb{Q}$. If $\alpha_1, \ldots, \alpha_r$ are algebraically independent, then this transcendence degree is equal to r. Suppose that $r \geq 3$ and $2 \leq j \leq r-1$, and

$$\alpha_1 = \alpha, \quad \alpha_2 = \frac{\alpha}{j}, \quad \alpha_3 = \frac{\alpha+1}{j}, \quad \ldots, \quad \alpha_{j+1} = \frac{\alpha+j-1}{j},$$

and that α and α_k, $k = j+1, \ldots, r$, are algebraically independent over $\mathbb{Q}$. Then the transcendence degree of $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)$ is equal to $r - j \leq r-2$. Therefore, in this case the first $j+1$ Hurwitz zeta-functions are linearly dependent

$$j^s \zeta(s, \alpha_1) = \zeta(s, \alpha_2) + \ldots + \zeta(s, \alpha_{j+1}).$$

From this it follows that the Hurwitz zeta-functions $\zeta(s, \alpha_1), \ldots, \zeta(s, \alpha_r)$ cannot be jointly universal. In the case of Theorem 4.3, the transcendence degree of the field extension $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)/\mathbb{Q}$ is precisely $r-1$.

The first joint discrete universality theorem for Hurwitz zeta-functions was proved in [SS]. To state it, we need some notation. For $q \in \mathbb{N}$, denote by $\chi_1, \ldots, \chi_{\varphi(q)}$ distinct Dirichlet characters modulo q, where $\varphi(q)$ is Euler's totient function, and define the $\varphi(q) \times \varphi(q)$-matrix M by

$$M = \big(\overline{\chi}_j(a)/\varphi(q)\big)_{1 \leq j \leq \varphi(q),\, 1 \leq a \leq q,\, (a,q)=1}.$$

Moreover, for some functions $f_a(s)$, $1 \leq a \leq q$, $(a,q) = 1$, we put

$$\underline{f} = (q^{-s} f_a(s))^T_{1 \leq a \leq q,\, (a,q)=1}.$$

Then [SS] contains the following theorem.

THEOREM 4.4. *Let $K \in \mathcal{K}$, $q \in \mathbb{N}$, and, for each $1 \leq a \leq q$, $(a,q)=1$, let $f_a(s) \in H(K)$. If all components of $M^{-1}\underline{f}$ are non-vanishing on K, then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\Big\{0 \leq k \leq N : \max_{1 \leq a \leq q,\, (a,q)=1} \max_{s \in K} \Big|\zeta\Big(s + ikh, \frac{a}{q}\Big) - f_a(s)\Big| < \varepsilon\Big\} > 0,$$

where $h = \frac{2\pi m}{\log q}$ with any $m \in \mathbb{N}$ if $q \geq 2$, and $h \neq 0$ otherwise.

For $h > 0$ define

$$\mathrm{L}(\alpha_1, \ldots, \alpha_r; h, \pi) = \Big\{\big(\log(m + \alpha_j) : m \in \mathbb{N}_0,\, j = 1, \ldots, r\big), \frac{\pi}{h}\Big\}.$$

Then we have the following theorems [L8].

THEOREM 4.5. *Suppose that the set $\mathrm{L}(\alpha_1, \ldots, \alpha_2; h, \pi)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \to \infty} \frac{1}{N+1} \#\big\{0 \leq k \leq N : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s + ikh, \alpha) - f_j(s)| < \varepsilon\big\} > 0.$$

Theorem 4.5 can be generalized in the following direction. For $h_1 > 0, \ldots, h_r > 0$, let

$$\mathrm{L}(\alpha_1, \ldots, \alpha_r; h_1, \ldots, h_r; \pi) = \big\{(h_j \log(m + \alpha_j) : m \in \mathbb{N}_0,\, j = 1, \ldots, r), \pi\big\}.$$

THEOREM 4.6. *Suppose that the set* $\mathrm{L}(\alpha_1,\dots,\alpha_2;h_1,\dots,h_r;\pi)$ *is linearly independent over* $\mathbb{Q}$. *For* $j=1,\dots,r$, *let* $K_j\in\mathcal{K}$ *and* $f_j(s)\in H(K_j)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\le k\le N:\sup_{1\le j\le r}\sup_{s\in K_j}|\zeta(s+ikh_j,\alpha_j)-f_j(s)|<\varepsilon\Big\}>0.$$

Clearly, Theorem 4.5 is a consequence of Theorem 4.6 with $h_1=\ldots=h_r=h$.

For example, in view of the Nesterenko theorem, the set $\mathrm{L}(\pi^{-1},2\pi^{-1};h_1,h_2;\pi)$ with rational h_1 and h_2, is linearly independent over $\mathbb{Q}$.

For the proof of Theorem 4.6, the probabilistic approach is applied. Let

$$\underline{\Omega}=\widehat{\Omega}_1\times\ldots\times\widehat{\Omega}_r,$$

where $\widehat{\Omega}_j=\Omega_1$ for $j=1,\dots,r$. This gives the probability space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}),m_H)$, where m_H is the probability Haar measure on $(\underline{\Omega},\mathcal{B}(\underline{\Omega}))$. Let $\omega_j(m)$ denote the projection of $\omega_j\in\Omega_j$ to the coordinate space γ_m, $m\in\mathbb{N}_0$, $j=1,\dots,r$. For elements of $\underline{\Omega}$, we use the notation $\omega=(\omega_1,\dots,\omega_r)$, $\omega_j\in\widehat{\Omega}_j$, $j=1,\dots,r$. Let $\underline{\alpha}=(\alpha_1,\dots,\alpha_r)$. On the probability space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}),m_H)$, define the $H^r(D)$-valued random element $\underline{\zeta}(s,\underline{\alpha},\omega)$ by the formula

$$\underline{\zeta}(s,\underline{\alpha},\omega)=\big(\zeta(s,\alpha_1,\omega_1),\dots,\zeta(s,\alpha_r,\omega_r)\big),$$

where

$$\zeta(s,\alpha_j,\omega_j)=\sum_{m=0}^{\infty}\frac{\omega_j(m)}{(m+\alpha_j)^s},\quad j=1,\dots,r.$$

Let $P_{\underline{\zeta}}$ denote the distribution of the random element $\underline{\zeta}(s,\underline{\alpha},\omega)$, i.e.,

$$P_{\underline{\zeta}}(A)=m_H(\omega\in\underline{\Omega}:\underline{\zeta}(s,\underline{\alpha},\omega)\in A),\quad A\in\mathcal{B}(H^r(D)).$$

Then Theorem 4.6 is derived from the following statement.

THEOREM 4.7. *Suppose that the set* $\mathrm{L}(\alpha_1,\dots,\alpha_2;h_1,\dots,h_r;\pi)$ *is linearly independent over* $\mathbb{Q}$. *Then*

$$\frac{1}{N+1}\#\big\{0\le k\le N:(\zeta(s+ikh_1,\alpha_1),\dots,\zeta(s+ikh_r,\alpha_r))\in A\big\},\quad A\in\mathcal{B}(H^r(D)),$$

converges weakly to $P_{\underline{\zeta}}$ *as* $N\to\infty$. *Moreover, the support of* $P_{\underline{\zeta}}$ *is the whole of* $H^r(D)$.

5. Mixed joint universality. In this section, we present joint universality theorems for the Riemann zeta-function and Hurwitz zeta-function. We call such theorems mixed joint universality theorems.

A joint continuous universality theorem of the above kind was obtained by H. Mishou [Mi1].

THEOREM 5.1 ([Mi1]). *Suppose that the number* α *is transcendental. Let* $K_1,K_2\in\mathcal{K}$ *and* $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\Big\{\tau\in[0,T]:\sup_{s\in K_1}|\zeta(s+i\tau)-f_1(s)|<\varepsilon,\ \sup_{s\in K_2}|\zeta(s+i\tau,\alpha)-f_2(s)|<\varepsilon\Big\}>0.$$

Denote by $\mathbb{P}$ the set of all prime numbers, and define the sets

$$\mathrm{L}(\mathbb{P},\alpha,h,\pi)=\Big\{(\log p:p\in\mathbb{P}),(\log(m+\alpha):m\in\mathbb{N}_0),\frac{\pi}{h}\Big\}$$

and

$$\mathrm{L}(\mathbb{P},\alpha,h_1,h_2,\pi)=\big\{(h_1\log p: p\in\mathbb{P}),(h_2\log(m+\alpha): m\in\mathbb{N}_0),\pi\big\},$$

where h, h_1 and h_2 are fixed positive numbers. Then discrete versions of Theorem 5.1 are of the form [BL].

THEOREM 5.2 ([BL]). *Suppose that the set* $\mathrm{L}(\mathbb{P},\alpha,h,\pi)$ *is linearly independent over* $\mathbb{Q}$. *Let* $K_1,K_2\in\mathcal{K}$, *and let* $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\le k\le N: \sup_{s\in K_1}|\zeta(s+ikh)-f_1(s)|<\varepsilon, \sup_{s\in K_2}|\zeta(s+ikh,\alpha)-f_2(s)|<\varepsilon\Big\}>0.$$

In view of the algebraic independence of the numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$, we can take, for example, $\alpha=(2^{\sqrt[3]{2}})^{-1}$ and $h=\pi(\sqrt[3]{4}\log 2)^{-1}$.

THEOREM 5.3. *Suppose that the set* $\mathrm{L}(\mathbb{P},\alpha,h_1,h_2,\pi)$ *is linearly independent over* $\mathbb{Q}$. *Let* $K_1,K_2\in\mathcal{K}$, *and let* $f_1(s)\in H_0(K_1)$, $f_2(s)\in H(K_2)$. *Then, for every* $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\le k\le N: \sup_{s\in K_1}|\zeta(s+ikh_1)-f_1(s)|<\varepsilon, \sup_{s\in K_2}|\zeta(s+ikh_2,\alpha)-f_2(s)|<\varepsilon\Big\}>0.$$

All theorems of this section are obtained by using limit theorems on the weak convergence of probability measures in the spaces of analytic functions.

Universality theorems for composite functions of Hurwitz zeta-functions can be found in [L4], [L5], [L6].

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