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FUNDAMENTAL UNITS FOR ORDERS OF UNIT RANK 1 AND GENERATED BY A UNIT

STÉPHANE R. LOUBOUTIN
Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373
13453 Marseille, France
E-mail: stephane.louboutin@univ-amu.fr

Abstract. Let ε be an algebraic unit for which the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is equal to 1. Assume that ε is not a complex root of unity. It is natural to wonder whether ε is a fundamental unit of this order. It turns out that the answer is in general yes, and that a fundamental unit of this order can be explicitly given (as an explicit polynomial in ε) in the rare cases when the answer is no. This paper is a self-contained exposition of the solution to this problem, solution which was up to now scattered in many papers in the literature.

1. Notation. Let $\Pi_\alpha(X) = X^n - a_{n-1}X^{n-1} + \ldots + (-1)^{n-1}a_1X + (-1)^n a_0 \in \mathbb{Z}[X]$ be the minimal polynomial of an algebraic integer α. It is monic and $\mathbb{Q}$-irreducible. Let $d_\alpha > 0$ be the absolute value of its discriminant $D_\alpha \neq 0$. Let $\beta \in \mathbb{Z}[\alpha]$ and assume that $\mathbb{Q}(\beta) = \mathbb{Q}(\alpha)$, i.e. that $\deg \Pi_\beta(X) = \deg \Pi_\alpha(X)$. Then

$$D_\beta = (\mathbb{Z}[\alpha] : \mathbb{Z}[\beta])^2 D_\alpha \text{ and hence } d_\beta = (\mathbb{Z}[\alpha] : \mathbb{Z}[\beta])^2 d_\alpha. \tag{1}$$

In particular, $\mathbb{Z}[\beta] = \mathbb{Z}[\alpha]$ if and only if $d_\beta = d_\alpha$.

Now, assume that ε is an algebraic unit, i.e. assume that $a_0 \in \{\pm 1\}$. Since $\mathbb{Z}[\varepsilon] = \mathbb{Z}[-\varepsilon] = \mathbb{Z}[1/\varepsilon] = \mathbb{Z}[-1/\varepsilon]$ and $D_\varepsilon = D_{-\varepsilon} = D_{1/\varepsilon} = D_{-1/\varepsilon}$, we may assume that $|a_1| \leq a_{n-1}$. By an appropriate choice of the root of $\Pi_\varepsilon(X)$, we may also assume that $|\varepsilon| \geq 1$. If ε is real, we may even assume that $\varepsilon \geq 1$.

If ε is an algebraic unit for which the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is equal to 1 then either (i) ε is a totally real quadratic unit, or (ii) ε is a non-totally real cubic unit or (iii) ε is a totally imaginary quartic unit. In these three situations, if we assume that ε is not a complex root of unity, it is natural to ask whether ε is a fundamental unit of the order $\mathbb{Z}[\varepsilon]$. And in case it is not, it is natural to construct one from it.

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2. The real quadratic case. We introduce in this very simple situation three tools and a method we will use to solve the more difficult cubic and quartic cases.

Let ε be an algebraic quadratic unit which is not a complex root of unity and for which the rank of the group of units of the quadratic order $\mathbb{Z}[\varepsilon]$ is equal to 1. Hence, ε is totally real. We may and we will assume that $\varepsilon > 1$.

LEMMA 1. *The smallest quadratic unit greater than* 1 *is* $\eta_0 := (1+\sqrt{5})/2$.

LEMMA 2. *Let* ε *be an algebraic integer. If* $\varepsilon = \pm\eta^n$ *with* $n \in \mathbb{Z}$ *and* $\eta \in \mathbb{Z}[\varepsilon]$*, then* $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$*. Hence,* $D_\eta = D_\varepsilon$ *and* $d_\eta = d_\varepsilon$.

Proof. Notice that $\mathbb{Z}[\eta] \subseteq \mathbb{Z}[\varepsilon] = \mathbb{Z}[\pm\eta^n] \subseteq \mathbb{Z}[\eta]$ and use (1). ∎

LEMMA 3. *Let* α *be a real quadratic unit. Then*

$$\left(|\alpha| - |\alpha|^{-1}\right)^2 \leq d_\alpha \leq \left(|\alpha| + |\alpha|^{-1}\right)^2. \tag{2}$$

Proof. We have $d_\alpha = (\alpha - \alpha')^2$, where $\alpha' = \pm 1/\alpha$ is the conjugate of α. ∎

THEOREM 4. *A real quadratic unit* $\varepsilon > 1$ *is always the fundamental unit of the quadratic order* $\mathbb{Z}[\varepsilon]$*, except if* $\varepsilon = (3+\sqrt{5})/2$*, in which case* $\varepsilon = \eta^2$*, where* $1 < \eta = (1+\sqrt{5})/2 = \varepsilon - 1 \in \mathbb{Z}[\varepsilon]$ *is the fundamental unit of* $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$*, and* $d_\varepsilon = d_\eta = 5$.

Proof. Assume that $\varepsilon > 1$ is not the fundamental unit of $\mathbb{Z}[\varepsilon]$. Then there exist a quadratic unit $1 < \eta \in \mathbb{Z}[\varepsilon]$ and $n \geq 2$ such that $\varepsilon = \eta^n$, which yields $d_\varepsilon = d_\eta$ (Lemma 2). Using (2) we obtain

$$0 < \eta - \eta^{-1} = \frac{\eta^2 - \eta^{-2}}{\eta + \eta^{-1}} \leq \frac{\eta^n - \eta^{-n}}{\eta + \eta^{-1}} = \frac{\varepsilon - \varepsilon^{-1}}{\eta + \eta^{-1}} \leq \sqrt{d_\varepsilon / d_\eta} = 1.$$

But $0 < \eta - \eta^{-1} \leq 1$ implies $1 < \eta \leq (1+\sqrt{5})/2$ and $\eta = \eta_0$, by Lemma 1. We now obtain $1 = \eta_0 - \eta_0^{-1} = (\eta_0^2 - \eta_0^{-2})/(\eta_0 + \eta_0^{-1}) \leq (\eta_0^n - \eta_0^{-n})/(\eta_0 + \eta_0^{-1}) \leq 1$, which yields $n = 2$. ∎

3. The non-totally real cubic case. The aim of this section is to prove Theorem 8. It was first proved in [Nag]. However, while working on class numbers of some cubic number fields, we come up in [Lou06] with a completely different proof of Nagell's result. Our proof was based on lower bounds on absolute discriminants of non-totally real algebraic cubic units (see Theorem 9), and then simplified in [Lou10].

DEFINITION 5. A *cubic polynomial of type* (T) is a monic cubic polynomial $P(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ which is $\mathbb{Q}$-irreducible ($\Leftrightarrow b \neq a$ and $b \neq -a-2$), of negative discriminant $D_{P(X)} < 0$ and whose only real root ε_P satisfies $\varepsilon_P > 1$ ($\Leftrightarrow P(1) < 0 \Leftrightarrow b \leq a-1$).

Let ε be an algebraic cubic unit for which the rank of the group of units of the cubic order $\mathbb{Z}[\varepsilon]$ is equal to 1. Hence, ε is not totally real. We may and we will assume that ε is real and that $\varepsilon > 1$, i.e. that $\Pi_\varepsilon(X)$ is a cubic polynomial of type (T):

LEMMA 6. *Let* $\varepsilon_P > 1$ *be the only real root of a cubic polynomial* $P(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ *of type* (T)*. Then*

$$0 \leq a < \varepsilon_P + 2 \text{ and } |b| \leq \sqrt{4a+4}\,. \tag{3}$$

Proof. Let $\varepsilon_P^{-1/2}e^{i\phi}$ and $\varepsilon_P^{-1/2}e^{-i\phi}$ be the non-real complex roots of $P(X)$. Then $a = \varepsilon_P + 2\varepsilon_P^{-1/2}\cos\phi \geq \varepsilon_P - 2 > -1$ and $b = 2\varepsilon_P^{1/2}\cos\phi + \varepsilon_P^{-1}$. Hence,

$$4a - b^2 = 4\varepsilon_P \sin^2\phi + 4\varepsilon_P^{-1/2}\cos\phi - \varepsilon_P^{-2} > -5 \tag{4}$$

and (3) is true. ∎

Notice that (3) makes it easy to list all the cubic polynomials of type (T) whose real roots are less than or equal to a given upper bound B. Taking $B = 2$, we obtain:

LEMMA 7. *The real root $\eta_0 = 1.32471\ldots$ of $\Pi(X) = X^3 - X - 1$ is the smallest real but non-totally real cubic unit greater than* 1.

3.1. Statement of the result for the cubic case

THEOREM 8. *Let $\varepsilon > 1$ be a real cubic algebraic unit of negative discriminant $-d_\varepsilon < 0$. Let $\eta > 1$ be the fundamental unit of the cubic order $\mathbb{Z}[\varepsilon]$. Then $\varepsilon = \eta$, except in the following cases*:

1. *The infinite family of exceptions for which $\Pi_\varepsilon(X) = X^3 - M^2X^2 - 2MX - 1$, $M \geq 1$, in which case $\varepsilon = \eta^2$ where $\eta = \varepsilon^2 - M^2\varepsilon - M \in \mathbb{Z}[\varepsilon]$ is the real root of $X^3 - MX^2 - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\varepsilon = d_\eta = 4M^3 + 27$.*
2. *The 8 following sporadic exceptions*:

(a) i. $\Pi_\varepsilon(X) = X^3 - 2X^2 + X - 1$, *in which case $\varepsilon = \eta^2$ where $\eta = \varepsilon^2 - \varepsilon \in \mathbb{Z}[\varepsilon]$.*

ii. $\Pi_\varepsilon(X) = X^3 - 3X^2 + 2X - 1$, *in which case $\varepsilon = \eta^3$ where $\eta = \varepsilon - 1 \in \mathbb{Z}[\varepsilon]$.*

iii. $\Pi_\varepsilon(X) = X^3 - 2X^2 - 3X - 1$, *in which case $\varepsilon = \eta^4$ where $\eta = \varepsilon^2 - 2\varepsilon - 2 \in \mathbb{Z}[\varepsilon]$,*

iv. $\Pi_\varepsilon(X) = X^3 - 5X^2 + 4X - 1$, *in which case $\varepsilon = \eta^5$ where $\eta = \varepsilon^2 - 4\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$.*

v. $\Pi_\varepsilon(X) = X^3 - 12X^2 - 7X - 1$, *in which case $\varepsilon = \eta^9$, $\eta = -3\varepsilon^2 + 37\varepsilon + 10 \in \mathbb{Z}[\varepsilon]$.*

In these five cases, $\eta > 1$ is the real root of $\Pi_\eta(X) = X^3 - X - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\eta = d_\varepsilon = 23$.

(b) i. $\Pi_\varepsilon(X) = X^3 - 4X^2 + 3X - 1$, *in which case $\varepsilon = \eta^3$ where $\eta = \varepsilon^2 - 3\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$.*

ii. $\Pi_\varepsilon(X) = X^3 - 6X^2 - 5X - 1$, *in which case $\varepsilon = \eta^5$, $\eta = -2\varepsilon^2 + 13\varepsilon + 5 \in \mathbb{Z}[\varepsilon]$.*

In these two cases, $\eta > 1$ is the real root of $\Pi_\eta(X) = X^3 - X^2 - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\eta = d_\varepsilon = 31$.

(c) $\Pi_\varepsilon(X) = X^3 - 7X^2 + 5X - 1$, *in which case $\varepsilon = \eta^3$, where $1 < \eta = -\varepsilon^2 + 7\varepsilon - 3 \in \mathbb{Z}[\varepsilon]$ is the real root of $\Pi_\eta(X) = X^3 - X^2 - X - 1$, $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\eta = d_\varepsilon = 44$.*

3.2. Sketch of proof. Let $\varepsilon > 1$ be a real but non-totally real cubic unit. Then ε is not a fundamental unit of the order $\mathbb{Z}[\varepsilon]$ if and only if there exists $p \geq 2$ a prime and $\eta \in \mathbb{Z}[\varepsilon]$ such that $\varepsilon = \eta^p$ (the main feature that makes it easier to deal with this cubic case than with the totally imaginary quartic case dealt with below is that -1 and $+1$ are the only complex roots of unity in $\mathbb{Z}[\varepsilon]$). Now, if $\varepsilon = \eta^n$ for some $\eta \in \mathbb{Z}[\varepsilon]$, then $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ and $d_\varepsilon = d_\eta$, by Lemma 2.

1. Assume that $\varepsilon = \eta^n$ for some non-totally real cubic unit $1 < \eta \in \mathbb{Z}[\varepsilon]$ and some $n \geq 3$. Using $\eta \geq \eta_0 = 1.32471\ldots$ (Lemma 7) and a double bound (5) for d_ε similar to (2), we will obtain in Corollary 11 that $1 < \eta \leq 4.5$ and $n \leq 10$.

2. In contrast with the quadratic case, this double bound (5) does not prevent ε from being infinitely many often a square in $\mathbb{Z}[\varepsilon]$. Hence, we characterize in Lemma 12 when this is indeed the case.

3. Finally, to determine all the $1 < \varepsilon$'s that admit a p-th root in $\mathbb{Z}[\varepsilon]$ for some $p \in \{3,5,7\}$, and to determine this p-th root $\eta > 1$, we make a list of all cubic polynomials $\Pi_\eta(X) = X^3 - AX^2 + BX - 1 \in \mathbb{Z}[X]$ of type (T) with $0 \leq A \leq 6 < 4.5 + 2$, by (3), for which there exist $p \in \{3,5,7\}$ such that $\mathbb{Z}[\eta] = \mathbb{Z}[\eta^p]$, i.e. such that $D_\eta = D_{\eta^p}$, where $\Pi_{\eta^p}(X) = X^3 - aX^2 + bX - 1$ is computed as the resultant of $\Pi_\eta(Y)$ and $X - Y^n$, considered as polynomials of the variable Y. A Maple Program 1 settling this step is given below. We found 6 such occurrences, of discriminants -23, -31 or -44. Taking also into account the points 2 and 3 of Lemma 12, both of discriminant -23, and singling out the only case of point 1 of Lemma 12 of discriminant in $\{-23,-31,-44\}$, namely the case $M = 1$ of discriminant -31, we obtain Table 1, which completes the proof of Theorem 8:

p	$\Pi_\eta(X)$	$\Pi_{\eta^p}(X)$	$D_\eta = D_{\eta^p}$
2	$X^3 - X - 1$	$\mathbf{X^3 - 2X^2 + X - 1}$	-23
2	$\mathbf{X^3 - 2X^2 + X - 1}$	$X^3 - 2X^2 - 3X - 1$	-23
3	$X^3 - X - 1$	$\mathbf{X^3 - 3X^2 + 2X - 1}$	-23
3	$\mathbf{X^3 - 3X^2 + 2X - 1}$	$X^3 - 12X^2 - 7X - 1$	-23
5	$X^3 - X - 1$	$X^3 - 5X^2 + 4X - 1$	-23
2	$X^3 - X^2 - 1$	$X^3 - X^2 - 2X - 1$	-31
3	$X^3 - X^2 - 1$	$X^3 - 4X^2 + 3X - 1$	-31
5	$X^3 - X^2 - 1$	$X^3 - 6X^2 - 5X - 1$	-31
3	$X^3 - X^2 - X - 1$	$X^3 - 7X^2 + 5X - 1$	-44

Table 1

Program 1:

```
for A from 0 to 6 by 1 do
borneB := isqrt(4A + 4):
for B from −borneB to min(borneB,A − 1) by 1 do
p := x^3 − A · x^2 + B · x − 1;
if irreduc(p) then
Dp := discrim(p, x);
if Dp < 0 then
for n in [3,5,7] do
q := resultant(subs(x = y, p), x − y^n, y); Dq := discrim(q, x);
if Dq = Dp then print(n, sort(p, x), sort(q, x), Dp) end if
end do
end if
end if
end do
end do:
```

3.3. Bounds on discriminants

THEOREM 9. *Let α be a real cubic algebraic unit of negative discriminant. Then*

$$\max(|\alpha|^{3/2}, |\alpha|^{-3/2})/2 \le d_\alpha \le 4(|\alpha| + |\alpha|^{-1})^3 \le 32\max(|\alpha|^3, |\alpha|^{-3}). \tag{5}$$

Proof. Since (5) remains unchanged if we change α into $-\alpha$, $1/\alpha$ and $-1/\alpha$, we may assume that $\alpha > 1$, i.e. that $\Pi_\alpha(X)$ is of type (T). Let $\beta = \alpha^{-1/2}e^{i\phi}$ and $\bar{\beta} = \alpha^{-1/2}e^{-i\phi}$ be the non-real complex roots of $\Pi_\alpha(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$. Then

$$d_\alpha = -(\alpha-\beta)^2(\alpha-\bar{\beta})^2(\beta-\bar{\beta})^2 = 4(\alpha^{3/2} - 2\cos\phi + \alpha^{-3/2})^2\sin^2\phi. \tag{6}$$

Hence, setting $X_\alpha = \alpha^{3/2} + \alpha^{-3/2} \ge 2$, we have

$$d_\alpha = X_\alpha^2 + 4 - (X_\alpha\cos\phi + 2\sin^2\phi)^2 - 4\cos^2\phi \le X_\alpha^2 + 4 = \alpha^3 + \alpha^{-3} + 6 \le (\alpha + \alpha^{-1})^3.$$

Let us now prove the lower bound on d_α. By (6) we have

$$d_\alpha \ge 4(\alpha^{3/4} - \alpha^{-3/4})^4 \sin^2\phi.$$

Assume that $\alpha > 16.2$.

First, assume that $\sin^2\phi \ge 2\alpha^{-3/2}$. Then we obtain $d_\alpha \ge 7\alpha^{3/2}$.

Second, assume that $\sin^2\phi < 2\alpha^{-3/2}$. By (4), we have

$$-1 < -4\alpha^{-1/2} - \alpha^{-2} \le 4a - b^2 = 4\alpha\sin^2\phi + 4\alpha^{-1/2}\cos\phi - \alpha^{-2} < 12\alpha^{-1/2} < 3.$$

Since $4a - b^2 \equiv 0$ or $3 \pmod 4$, we obtain $4a = b^2$ and $\cos\phi < 0$ (otherwise $4a - b^2 \ge 4\alpha^{-1/2}\sin^2\phi + 4\alpha^{-1/2}\cos^2\phi - \alpha^{-2} > 0$). Hence,

$$0 = 4a - b^2 = 4\alpha\sin^2\phi - 4\alpha^{-1/2}\sqrt{1 - \sin^2\phi} - \alpha^{-2}.$$

Therefore, $\sin^2\phi = \alpha^{-3/2} - \alpha^{-3}/4$, $\cos\phi = -1 + \alpha^{-3/2}/2$ and (6) yields

$$d_\alpha = 4(\alpha^{3/2} + 2)^2(\alpha^{-3/2} - \alpha^{-3}/4) > 4\alpha^{3/2}.$$

Therefore, $d_\alpha \ge 4\alpha^{3/2}$ for $\alpha > 16.2$.

Finally, if $1 < \alpha \le 16.2$, then $\Pi_\alpha(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ is of type (T) with $0 \le a \le 18$, by (3). Using (3) we obtain that there are 211 such cubic polynomials. By computing approximations to the real root $\alpha > 1$ of each of these 211 cubic polynomials, we check that the lower bound on d_α given in (5) holds for each of these 211 cubic polynomials. ■

REMARK 10. We can reformulate the lower bound on d_α in (5) as follows: let γ be a non-real cubic algebraic unit of negative discriminant satisfying $|\gamma| > 1$. Then $|\Im(\gamma)| \gg |\gamma|^{-1/2}$ (explicitly). We wish we understood beforehand why such a lower bound must hold.

COROLLARY 11. *Let $\varepsilon > 1$ be a real cubic algebraic unit of negative discriminant. If $\varepsilon = \eta^n$ for some $1 < \eta \in \mathbb{Z}[\varepsilon]$ and some $n \ge 3$, then $\eta \le 4.5$ and $n \le 10$. In particular, by* (3), *if $\Pi_\eta(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ is of type* (T), *then $0 \le a \le 6$ and $|b| \le \sqrt{4a+4}$.*

Proof. By (5) we have

$$\eta^{9/2}/2 \le \eta^{3n/2}/2 = \varepsilon^{3/2}/2 \le d_\varepsilon = d_\eta \le 4(\eta + \eta^{-1})^3,$$

that implies $\eta \leq 4.5$. Moreover, we have $\eta \geq \eta_0 = 1.32471\dots$ (Lemma 7). Hence, by (5), we have

$$1 = d_\eta/d_\varepsilon \leq \frac{4(\eta+\eta^{-1})^3}{\varepsilon^{3/2}/2} = 8\Big(\frac{\eta+\eta^{-1}}{\eta^{n/2}}\Big)^3 \leq 8\Big(\frac{\eta_0+\eta_0^{-1}}{\eta_0^{n/2}}\Big)^3,$$

that implies $n < 11$. ∎

3.4. Being a square

LEMMA 12. *Let $\varepsilon > 1$ be a real cubic algebraic unit of negative discriminant. Then $\varepsilon = \eta^2$ for some $1 < \eta \in \mathbb{Z}[\varepsilon]$ if and only if we are in one of the following three cases*:

1. $\Pi_\varepsilon(X) = X^3 - M^2X^2 - 2MX - 1$ *with* $M \geq 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = \varepsilon^2 - M^2\varepsilon - M \in \mathbb{Z}[\varepsilon]$, $\Pi_\eta(X) = X^3 - MX^2 - 1$ *and* $d_\eta = d_\varepsilon = 4M^3 + 27$.

2. $\Pi_\varepsilon(X) = X^3 - 2X^2 - 3X - 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = -\varepsilon^2 + 3\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$, $\Pi_\eta(X) = X^3 - 2X^2 + X - 1$ *and* $d_\eta = d_\varepsilon = 23$.

3. $\Pi_\varepsilon(X) = X^3 - 2X^2 + X - 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = \varepsilon^2 - \varepsilon \in \mathbb{Z}[\varepsilon]$, $\Pi_\eta(X) = X^3 - X - 1$ *and* $d_\eta = d_\varepsilon = 23$.

Proof. Assume that $\varepsilon = \eta^2$ for some $1 < \eta \in \mathbb{Z}[\varepsilon]$ with $\Pi_\eta(X) = X^3 - aX^2 + bX - 1 \in \mathbb{Z}[X]$ of type (T). Then $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$ and $d_\varepsilon = d_\eta$ (Lemma 2). Clearly, the index $(\mathbb{Z}[\eta] : \mathbb{Z}[\eta^2])$ is equal to $|ab-1|$, where $\Pi_\eta(X) = X^3 - aX^2 + bX - 1$. Hence, we must have $|ab-1| = 1$, and we will have $\eta = (\varepsilon^2 - (a^2-b)\varepsilon - a)/(1-ab)$ and $\Pi_\varepsilon(X) = \Pi_{\eta^2}(X) = X^3 - (a^2-2b)X^2 + (b^2-2a)X - 1$.

First, assume that $ab = 2$. Then $a = 2$ and $b = 1$ (for $a \geq 0$ and $b \leq a-1$), $\Pi_\eta(X) = X^3 - 2X^2 + X - 1$ and $\Pi_\varepsilon(X) = X^3 - 2X^2 - 3X - 1$.

Second, assume that $ab = 0$. If $a = 0$, then $b \leq a - 1 = -1$ and $d_\eta = 4b^3 + 27 > 0$ yields $b = -1$, $\Pi_\eta(X) = X^3 - X - 1$, and $\Pi_\varepsilon(X) = X^3 - 2X^2 + X - 1$. If $a \neq 0$, then $b = 0$, $\Pi_\eta(X) = X^3 - aX^2 - 1$, $d_\eta = 4a^3 + 27$ and $\Pi_\varepsilon(X) = X^3 - a^2X^2 - 2aX - 1$. ∎

4. The totally imaginary quartic case. The aim of this section is to prove Theorem 17. Indeed, after having found a completely different proof of Nagell's result we thought it should now be possible to settle this third case where the rank of the group of units of the order $\mathbb{Z}[\varepsilon]$ is equal to 1. In [Lou08] we partially solved this problem and conjectured Theorem 17. We could not prove it because we could not come up with lower bounds on discriminants of totally imaginary quartic algebraic units (see Theorem 24). Such a lower bound was then obtained in [PL] and its proof was simplified in [Lou10].

Let ε be an algebraic quartic unit which is not a complex root of unity, and for which the rank of the group of units of the quadratic order $\mathbb{Z}[\varepsilon]$ is equal to 1. Hence, ε is totally imaginary and $|\varepsilon| \neq 1$ (use [Was, Lemma 1.6]). Notice that if ε_1, $\varepsilon_2 = \overline{\varepsilon_1}$, ε_3 and $\varepsilon_4 = \overline{\varepsilon_3}$ are the four complex conjugates of ε, then $1 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = |\varepsilon_1|^2|\varepsilon_3|^2$. By changing ε into $-\varepsilon$, $1/\varepsilon$, or $-1/\varepsilon$ if necessary, we may and we will assume that its minimal polynomial $\Pi_\varepsilon(X)$ is of type (T):

DEFINITION 13. A *quartic polynomial of type* (T) is a $\mathbb{Q}$-irreducible monic quartic polynomial $P(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ which satisfies $|c| \leq a$ and which has no real root (see Lemma 14 for a characterization). It is of positive discriminant $D_{P(X)}$.

LEMMA 14. *Let ε_P be any complex root of a quartic polynomial $P(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ of type* (T). *Then*

$$-1 \leq b \leq |\varepsilon_P|^2 + 1/|\varepsilon_P|^2 + 4 \quad \textit{and} \quad |c| \leq a \leq \sqrt{4b+5}. \tag{7}$$

Proof. Let $\rho e^{i\phi}$, $\rho e^{-i\phi}$, $\rho^{-1}e^{i\psi}$ and $\rho^{-1}e^{-i\psi}$ be the four complex roots of $P(X)$. Then $a = 2\rho\cos\phi + 2\rho^{-1}\cos\psi$ and

$$b = \rho^2 + \rho^{-2} + 4(\cos\phi)(\cos\psi). \tag{8}$$

Hence,

$$4b - a^2 = 4(\sin\phi)^2\rho^2 + 4(\sin\psi)^2\rho^{-2} + 8(\cos\phi)(\cos\psi) > -8. \tag{9}$$

Since $4b - a^2 \equiv 0$ or $3 \pmod 4$, we have $4b - a^2 \geq -5$. ∎

LEMMA 15. *Let $P(X) = X^4 - aX^3 + bX^2 - cX + d \in \mathbb{Q}[X]$ be $\mathbb{Q}$-irreducible. Then $P(X)$ has no real root if and only if $D_{P(X)} > 0$ and either $A := 3a^2 - 8b < 0$ or $B := 3a^4 - 16a^2b + 16ac + 16b^2 - 64d < 0$. Assume moreover that $d = 1$ and let $\eta = \rho e^{i\alpha}$, $\bar\eta$, $\eta' = \rho^{-1}e^{i\beta}$ and $\bar\eta'$ be these four non-real roots. Then $\rho^2 + 1/\rho^2 \geq 2$, $2\cos(\alpha+\beta)$ and $2\cos(\alpha-\beta)$ are the roots of*

$$R(X) = X^3 - bX^2 + (ac-4)X - (a^2 - 4b + c^2) \in \mathbb{Q}[X],$$

of positive discriminant $D_{R(X)} = D_{P(X)} = d_\eta$.

Proof. Write $P(X) = (X-\alpha_1)(X-\alpha_2)(X-\alpha_3)(X-\alpha_4)$ in $\mathbb{C}[X]$.

1. Set $\beta_1 = (\alpha_1+\alpha_2-\alpha_3-\alpha_4)^2$, $\beta_2 = (\alpha_1-\alpha_2+\alpha_3-\alpha_4)^2$ and $\beta_3 = (\alpha_1-\alpha_2-\alpha_3+\alpha_4)^2$. Then $Q(X) := (X-\beta_1)(X-\beta_2)(X-\beta_3) = X^3 - AX^2 + BX - C$, where A and B are given in the statement of Lemma 15 and $C = (a^3 - 4ab + 8c)^2$. Moreover, $D_{Q(X)} = 2^{12}D_{P(X)}$.

(i) If $P(X)$ has two real roots, say α_1 and α_2, and two non-real roots, say α_3 and $\alpha_4 = \bar\alpha_3$, then $\beta_1 > 0$, whereas β_2 and $\beta_3 = \bar\beta_2$ are non-real. Hence $D_{Q(X)} < 0$.

(ii) If $P(X)$ has four non-real roots, say α_1, $\alpha_2 = \bar\alpha_1$, α_3 and $\alpha_4 = \bar\alpha_3$, then $Q(X)$ has three real roots $\beta_1 > 0$, $\beta_2 < 0$ and $\beta_3 < 0$. Hence $D_{Q(X)} > 0$ and $Q'(X) = 3X^2 - 2AX + B$ has a negative real root $\gamma \in (\beta_2, \beta_3)$, which implies $A < 0$ or $B < 0$.

(iii) If $P(X)$ has four real roots, say α_1, α_2, α_3 and α_4, then $Q(X)$ has three real roots $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 > 0$. Hence $D_{Q(X)} > 0$ and $Q'(X) = 3X^2 - 2AX + B$ has two positive real roots γ_1 and γ_2, which implies $A \geq 0$ and $B \geq 0$.

The proof of the first part is complete.

2. Set $\gamma_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $\gamma_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$ and $\gamma_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$. Then $R(X) := (X-\gamma_1)(X-\gamma_2)(X-\gamma_3) = X^3 - bX^2 + (ac-4d)X - (a^2d - 4bd + c^2)$, and $D_{R(X)} = D_{P(X)}$. In our situation, $d = 1$ and we may assume that $\alpha_1 = \eta$, $\alpha_2 = \bar\eta$, $\alpha_3 = \eta'$ and $\alpha_4 = \bar\eta'$. Hence, $\gamma_1 = |\eta|^2 + 1/|\eta|^2 \geq 2$, $\gamma_2 = 2\Re(\eta\eta') = 2\cos(\alpha+\beta)$ and $\gamma_3 = 2\Re(\eta\bar\eta') = 2\cos(\alpha-\beta)$. ∎

Notice that (7) and the first part of Lemma 15 make it easy to list all the quartic polynomials of type (T) whose roots are of absolute values less than or equal to a given upper bound B. Using the second part of Lemma 15 we can compute these absolute values. Taking $B = 2$, we obtain:

LEMMA 16. *Let η be a totally imaginary quartic unit. If $|\eta| > 1$ then $|\eta| \geq |\eta_0| = 1.18375\ldots$, where $\Pi_{\eta_0}(X) = X^4 - X^3 + 1$.*

4.1. Statement of the result for the quartic case

THEOREM 17. *Let ε be a totally imaginary quartic unit with $\Pi_\varepsilon(X)$ of type* (T). *Assume that ε is not a complex root of unity. Let η be a fundamental unit of $\mathbb{Z}[\varepsilon]$. We can choose $\eta = \varepsilon$, except in the following cases*:

1. *The* infinite family of exceptions *for which* $\Pi_\varepsilon(X) = X^4 - 2bX^3 + (b^2+2)X^2 - (2b-1)X + 1$, $b \geq 3$, *in which cases* $\varepsilon = -1/\eta^2$ *where* $\eta = \varepsilon^3 - 2b\varepsilon^2 + (b^2+1)\varepsilon - (b-1) \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - X^3 + bX^2 + 1$ *of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 16b^4 - 4b^3 - 128b^2 + 144b + 229$.

2. *The* 14 following sporadic exceptions:

(a) i. $\Pi_\varepsilon(X) = X^4 - 3X^3 + 2X^2 + 1$, *in which case* $\varepsilon = -\eta^{-2}$ *where* $\eta = -\varepsilon^2 + \varepsilon + 1 \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1$ *of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\varepsilon = d_\eta = 117$.

ii. $\Pi_\varepsilon(X) = X^4 - 3X^3 + 5X^2 - 3X + 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = -\varepsilon^3 + 2\varepsilon^2 - 2\varepsilon \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - X^3 - X^2 + X + 1$ *of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\varepsilon = d_\eta = 117$.

iii. $\Pi_\varepsilon(X) = X^4 - 5X^3 + 8X^2 - 4X + 1$, *in which case* $\zeta_3 = \varepsilon^3 - 4\varepsilon^2 + 5\varepsilon - 2 \in \mathbb{Z}[\varepsilon]$ *and* $\varepsilon = \zeta_3\eta^3$ *where* $\eta = -\varepsilon^2 + 3\varepsilon - 1 \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1$ *is of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 117$.

(b) $\Pi_\varepsilon(X) = X^4 - 5X^3 + 9X^2 - 5X + 1$, *in which case* $\varepsilon = -\eta^2$ *where* $\eta = -\varepsilon^3 + 4\varepsilon^2 - 6\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - X^3 + 3X^2 - X + 1$ *of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 189$.

(c) i. $\Pi_\varepsilon(X) = X^4 - X^3 + 2X^2 + 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = -\varepsilon^3 + \varepsilon^2 - \varepsilon \in \mathbb{Z}[\varepsilon]$.

ii. $\Pi_\varepsilon(X) = X^4 - 3X^3 + 3X^2 - X + 1$, *in which case* $\varepsilon = 1/\eta^3$, $\eta = -\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$.

iii. $\Pi_\varepsilon(X) = X^4 - 4X^3 + 6X^2 - 3X + 1$, *in which case* $\varepsilon = -1/\eta^4$ *where* $\eta = \varepsilon^3 - 3\varepsilon^2 + 3\varepsilon \in \mathbb{Z}[\varepsilon]$.

iv. $\Pi_\varepsilon(X) = X^4 - 5X^3 + 5X^2 + 3X + 1$, *in which case* $\varepsilon = -\eta^6$ *where* $\eta = \varepsilon^2 - 2\varepsilon - 1 \in \mathbb{Z}[\varepsilon]$.

v. $\Pi_\varepsilon(X) = X^4 - 7X^3 + 14X^2 - 6X + 1$, *in which case* $\varepsilon = -1/\eta^7$ *where* $\eta = \varepsilon^2 - 4\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$.

In these five cases, $\Pi_\eta(X) = X^4 - X^3 + 1$ *is of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$, $d_\eta = d_\varepsilon = 229$.

(d) i. $\Pi_\varepsilon(X) = X^4 - 2X^3 + 3X^2 - X + 1$, *in which case* $\varepsilon = -1/\eta^2$ *where* $\eta = \varepsilon^3 - 2\varepsilon^2 + 2\varepsilon \in \mathbb{Z}[\varepsilon]$.

ii. $\Pi_\varepsilon(X) = X^4 - 3X^3 + X^2 + 2X + 1$, *in which case* $\varepsilon = 1/\eta^3$, $\eta = \varepsilon^2 - 2\varepsilon \in \mathbb{Z}[\varepsilon]$.

iii. $\Pi_\varepsilon(X) = X^4 - 5X^3 + 7X^2 - 2X + 1$, *in which case* $\varepsilon = -\eta^4$, $\eta = \varepsilon^2 - 2\varepsilon \in \mathbb{Z}[\varepsilon]$.

In these three cases, $\Pi_\eta(X) = X^4 - X^3 + X^2 + 1$ *is of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 257$.

(e) $\Pi_\varepsilon(X) = X^4 - 4X^3 + 7X^2 - 4X + 1$, *in which case* $\zeta_4 = -\varepsilon^3 + 4\varepsilon^2 - 6\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$ *and* $\varepsilon = \zeta_4\eta^2$, *where* $\eta = -\varepsilon^3 + 3\varepsilon^2 - 4\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + X^2 + 1$ *of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 272$.

(f) $\Pi_\varepsilon(X) = X^4 - 13X^3 + 43X^2 - 5X + 1$, *in which case* $\varepsilon = -\eta^3$ *where* $\eta = -\varepsilon^3 + 6\varepsilon^2 + 3\varepsilon \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + 4X^2 - X + 1$ *of type* (T), $\mathbb{Z}[\eta] = \mathbb{Z}[\varepsilon]$ *and* $d_\eta = d_\varepsilon = 1229$.

4.2. Sketch of proof. Let ε be a totally imaginary quartic unit which is not a complex root of unity. Compared with the non-totally real cubic case, this quartic case is more tricky.

The first problem is that the totally imaginary quartic order $\mathbb{Z}[\varepsilon]$ may contain complex roots of unity. Let $\mu(\varepsilon)$ denote the cyclic group of order $2N \geq 2$ of complex roots of unity contained in $\mathbb{Z}[\varepsilon]$. Since the cyclotomic field of conductor $2N$ and degree $\phi(2N)$ is contained in the quartic field $\mathbb{Q}(\varepsilon)$, we deduce that $\phi(2N)$ divides 4, hence that $2N \in \{2, 4, 6, 8, 10, 12\}$.

We will devote Section 4.4 to the case that $2N \in \{8, 10, 12\}$ and will settle our problem in this situation.

Hence we may and we now assume that $2N \in \{2, 4, 6\}$.

We want to determine when $\varepsilon = \zeta\eta^p$ for some $\zeta \in \mu(\varepsilon)$, some $\eta \in \mathbb{Z}[\varepsilon]$ and some prime $p \geq 2$. We may and we will assume that η is also a totally imaginary quartic unit which is not a complex root of unity (if η is not totally imaginary then it is a real quadratic unit and $\zeta \neq \pm 1$, and we have $\varepsilon = \zeta'\eta'^p$ with $\eta' = \zeta\eta \in \mathbb{Z}[\varepsilon]$ a totally imaginary quartic unit and $\zeta' = \zeta^{1-p} \in \mu(\varepsilon)$).

Clearly there are three subcases.

1. For $p = 2$ we determine when $\varepsilon = \pm\eta^2$ (Lemma 18), and when $\varepsilon = \zeta_4\eta^2$ where ζ_4 is a complex root of unity of order 4 in $\mathbb{Z}[\varepsilon]$ (Lemma 20).
2. For $p = 3$ we determine when $\varepsilon = \eta^3$ (next subcase), and when $\varepsilon = \zeta_3\eta^3$ where ζ_3 is a complex root of unity of order 3 in $\mathbb{Z}[\varepsilon]$ (Lemma 20).
3. For $p \geq 3$ we determine when $\varepsilon = \eta^p$ for some $\eta \in \mathbb{Z}[\varepsilon]$. Using $|\eta| \geq |\eta_0| = 1.18375\ldots$ (Lemma 16) and a double bound (10) for d_ε similar to (2) and (5), we will obtain in Corollary 27 $p \in \{3, 5, 7, 11, 13\}$.

p	$\Pi_\eta(X)$	$Q(X)$	D
2	$X^4 - X^3 - X^2 + X + 1$	$\Pi_{\eta^2}(X) = X^4 - 3X^3 + 5X^2 - 3X + 1$	117
2	$X^4 - 2X^3 + 2X^2 - X + 1$	$\Pi_{-1/\eta^2}(X) = X^4 - 3X^3 + 2X^2 + 1$	117
3	$X^4 - 2X^3 + 2X^2 - X + 1$	$\Pi_{\zeta_3\eta^3}(X) = X^4 - 5X^3 + 8X^2 - 4X + 1$	117
2	$X^4 - X^3 + 3X^2 - X + 1$	$\Pi_{-\eta^2}(X) = X^4 - 5X^3 + 9X^2 - 5X + 1$	189
3	$X^4 - X^3 + 1$	$\Pi_{1/\eta^3}(X) = \mathbf{X^4 - 3X^3 + 3X^2 - X + 1}$	229
2	$\mathbf{X^4 - 3X^3 + 3X^2 - X + 1}$	$\Pi_{-1/\eta^2}(X) = X^4 - 5X^3 + 5X^2 + 3X + 1$	229
2	$X^4 - X^3 + 1$	$\Pi_{\eta^2}(X) = \mathbf{X^4 - X^3 + 2X^2 + 1}$	229
2	$\mathbf{X^4 - X^3 + 2X^2 + 1}$	$\Pi_{-1/\eta^2}(X) = X^4 - 4X^3 + 6X^2 - 3X + 1$	229
3	$\mathbf{X^4 - X^3 + 2X^2 + 1}$	$\Pi_{-\eta^3}(X) = X^4 - 5X^3 + 5X^2 + 3X + 1$	229
7	$X^4 - X^3 + 1$	$\Pi_{-1/\eta^7}(X) = X^4 - 7X^3 + 14X^2 - 6X + 1$	229
2	$X^4 - X^3 + X^2 + 1$	$\Pi_{-1/\eta^2}(X) = \mathbf{X^4 - 2X^3 + 3X^2 - X + 1}$	257
2	$\mathbf{X^4 - 2X^3 + 3X^2 - X + 1}$	$\Pi_{-1/\eta^2}(X) = X^4 - 5X^3 + 7X^2 - 2X + 1$	257
3	$X^4 - X^3 + X^2 + 1$	$\Pi_{1/\eta^3}(X) = X^4 - 3X^3 + X^2 + 2X + 1$	257
2	$X^4 - 2X^3 + X^2 + 1$	$\Pi_{\zeta_4\eta^2}(X) = X^4 - 4X^3 + 7X^2 - 4X + 1$	272
3	$X^4 - 2X^3 + 4X^2 - X + 1$	$\Pi_{-\eta^3}(X) = X^4 - 13X^3 + 43X^2 - 5X + 1$	1229

Table 2

In the cubic case, if $\varepsilon = \eta^p > 1$ and $\Pi_\varepsilon(X)$ is of type (T), then so is $\Pi_\eta(X)$. This is no longer true in the present quartic case. For example, if $\Pi_\varepsilon(X) = X^4 - 3X^3 + X^2 + 2X + 1$, of type (T), and $\eta = -\varepsilon^2 + \varepsilon + 1 \in \mathbb{Z}[\varepsilon]$, then $\varepsilon = \eta^3$ and $\Pi_\eta(X) = X^4 + X^2 - X + 1$ is not of type (T). Since we want η to be of type (T) in all our statements, we may have to present our results in using $-\eta$, $1/\eta$ or $-1/\eta$ instead, as in Lemma 18.

Putting together the results of Lemma 18, Lemma 20 and Corollary 27, we obtain Table 2 on the previous page (similar to Table 1 of Section 3.2, where we single out the cases $b = 1$ and $b = 2$ of point 1 of Lemma 18, of discriminants $D = 257$ and $D = 229$), which completes the proof of Theorem 17.

4.3. Being a square

Lemma 18. *Let ε be a totally imaginary quartic algebraic unit which is not a complex root of unity, with $\Pi_\varepsilon(X)$ of type* (T). *Then $\pm\varepsilon$ is a square in $\mathbb{Z}[\varepsilon]$ if and only if we are in one of the seven following cases*:

1. $\Pi_\varepsilon(X) = X^4 - 2bX^3 + (b^2+2)X^2 - (2b-1)X + 1$, $b \geq 1$, *in which cases* $\varepsilon = -1/\eta^2$ *where* $\eta = \varepsilon^3 - 2b\varepsilon^2 + (b^2+1)\varepsilon - (b-1) \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - X^3 + bX^2 + 1$ *of type* (T), *and* $d_\varepsilon = d_\eta = 16b^4 - 4b^3 - 128b^2 + 114b + 229$.
2. $\Pi_\varepsilon(X) = X^4 - X^3 + 2X^2 + 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = -\varepsilon^3 + \varepsilon^2 - \varepsilon \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - X^3 + 1$ *of type* (T), *and* $d_\varepsilon = d_\eta = 229$.
3. $\Pi_\varepsilon(X) = X^4 - 3X^3 + 2X^2 + 1$, *in which case* $\varepsilon = -\eta^{-2}$ *where* $\eta = -\varepsilon^2 + \varepsilon + 1 \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1$ *of type* (T), *and* $d_\varepsilon = d_\eta = 117$.
4. $\Pi_\varepsilon(X) = X^4 - 3X^3 + 5X^2 - 3X + 1$, *in which case* $\varepsilon = \eta^2$ *where* $\eta = -\varepsilon^3 + 2\varepsilon^2 - 2\varepsilon \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - X^3 - X^2 + X + 1$ *of type* (T), *and* $d_\varepsilon = d_\eta = 117$.
5. $\Pi_\varepsilon(X) = X^4 - 5X^3 + 9X^2 - 5X + 1$, *in which case* $\varepsilon = -\eta^2$ *where* $\eta = -\varepsilon^3 + 4\varepsilon^2 - 6\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - X^3 + 3X^2 - X + 1$ *of type* (T), *and* $d_\varepsilon = d_\eta = 189$.
6. $\Pi_\varepsilon(X) = X^4 - 5X^3 + 5X^2 + 3X + 1$, *in which case* $\varepsilon = -\eta^{-2}$ *where* $\eta = -\varepsilon^2 + 2\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 3X^3 + 3X^2 - X + 1$ *of type* (T), $d_\eta = d_\varepsilon = 229$.
7. $\Pi_\varepsilon(X) = X^4 - 5X^3 + 7X^2 - 2X + 1$, *in which case* $\varepsilon = -\eta^{-2}$ *where* $\eta = \varepsilon^3 - 4\varepsilon^2 + 4\varepsilon \in \mathbb{Z}[\varepsilon]$ *is a root of* $\Pi_\eta(X) = X^4 - 2X^3 + 3X^2 - X + 1$ *of type* (T), *and* $d_\eta = d_\varepsilon = 257$.

Proof. Assume that $\varepsilon = \pm\eta^2$ or $\pm\eta^{-2}$ for some $\eta \in \mathbb{Z}[\varepsilon]$, with $\Pi_\eta(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ of type (T). Hence,

$$-1 \leq b \quad \text{and} \quad |c| \leq a \leq \sqrt{4b+5}.$$

The index $(\mathbb{Z}[\eta] : \mathbb{Z}[\eta^2])$ is equal to $|a^2 + c^2 - abc|$. Hence, we must have

$$|a^2 + c^2 - abc| = 1,$$

and we will have $\Pi_{\eta^2}(X) = X^4 - AX^3 + BX^2 - CX + 1$, where $A = a^2 - 2b$, $B = b^2 - 2ac + 2$ and $C = c^2 - 2b$.

Assume that $c = 0$. Then $1 = |a^2 + c^2 - abc| = a^2$, hence $a = 1$ and we are in the first or the second case.

Assume that $c \neq 0$. Then $a \geq 1$ and $a^2 + c^2 - abc = \pm 1$ yield $|b| \leq f(|c|)$, where $f(x) = \frac{a}{x} + \frac{x}{a} + \frac{1}{ax}$ is convex. Hence,

$$|b| \leq g(a) := \max(f(1), f(a)) = \max(a + 2/a, 2 + 1/a^2) = a + 2/a.$$

Since g is convex we obtain $|b| \leq \max(g(1), g(\sqrt{4b+5})) = \max(3, \frac{4b+7}{\sqrt{4b+5}})$. Hence, $b \leq 5$. There are 9 triplets (a, b, c) satisfying $-1 \leq b \leq 5$ and $1 \leq |c| \leq a \leq \sqrt{4b+5}$ for which $|a^2 + c^2 - abc| = 1$. Getting rid of the three of them for which $\Pi_\eta(X) = X^4 - aX^3 + bX^2 - cX + 1$ is of negative discriminant and of the one of them for which η is a fifth complex root of unity, namely $(a, b, c) = (1, 1, 1)$, we fall in one of the five remaining last cases.

Finally, by choosing between the four units $\varepsilon = \pm\eta^2$ or $\varepsilon = \pm\eta^{-2}$ the ones for which $\Pi_\varepsilon(X) = X^4 - AX^3 + BX^2 - CX + 1$ satisfies $|C| \leq A$, we complete the proof of this lemma. ■

4.4. The case where $\mu(\varepsilon)$ is of order 8, 10 or 12

LEMMA 19. *Let ε be a totally imaginary quartic algebraic unit which is not a complex root of unity. If $\zeta_{2N} \in \mathbb{Z}[\varepsilon]$, with $2N \in \{8, 10, 12\}$, then ε is a fundamental unit of the order $\mathbb{Z}[\varepsilon]$.*

Proof. We have $\mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\zeta_{2N}]$ ($\mathbb{Z}[\zeta_{2N}]$ is the ring of algebraic integers of $\mathbb{Q}(\zeta_{2N})$). Hence, $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\zeta_{2N}]$ and $d_\varepsilon = d_{\zeta_{2N}}$. Let η_{2N} be a fundamental unit of $\mathbb{Z}[\zeta_{2N}]$. Then $\eta_{2N} \in \mathbb{Z}[\varepsilon]$ and $\varepsilon = \zeta_{2N}^m \eta_{2N}^n$, with $m \in \mathbb{Z}$ and $0 \neq n \in \mathbb{Z}$. We want to prove that $|n| = 1$. We prove that $|n| \geq 2$ implies $d_\varepsilon > d_{\zeta_{2N}}$. Set $\rho = |\eta_{2N}|$. Let σ_t be the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\zeta_{2N})$ such that $\sigma_t(\zeta_{2N}) = \zeta_{2N}^t$, where $\gcd(t, 2N) = 1$ and $t \not\equiv \pm 1 \pmod{2N}$, i.e. σ_t is neither the identity nor the complex conjugation. Then $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \sigma_t(\varepsilon)$, $\varepsilon_3 = \overline{\varepsilon_1}$ and $\varepsilon_4 = \overline{\varepsilon_2}$ are the four complex conjugates of ε. Since $|\varepsilon_1|^2|\varepsilon_2|^2 = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 = N_{\mathbb{Q}(\zeta_{2N})/\mathbb{Q}}(\varepsilon) = 1$, we have $|\varepsilon_2| = 1/|\varepsilon_1| = 1/\rho^n$ and

$$\begin{aligned} d_\varepsilon &= \big((\varepsilon_1 - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)\big)^2 \\ &= 16\Im^2(\varepsilon_1)\Im^2(\varepsilon_2)|\varepsilon_1 - \varepsilon_2|^4|\varepsilon_1 - \overline{\varepsilon_2}|^4 \geq 16\Im^2(\varepsilon)\Im^2(\sigma_t(\varepsilon))|\rho^n - 1/\rho^n|^8. \end{aligned}$$

Notice that if η_{2N} is real, then $16\Im^2(\varepsilon)\Im^2(\sigma_t(\varepsilon)) = 16\sin^2(\frac{\pi m}{N})\sin^2(\frac{t\pi m}{N})$.

1. If $2N = 8$, we may take $\eta_8 = 1 + \sqrt{2} = \rho$ and $t = 3$ and we obtain

$$d_\varepsilon \geq 16\sin^2\Big(\frac{\pi m}{4}\Big)\sin^2\Big(\frac{3\pi m}{4}\Big)(\rho^n - \rho^{-n})^8 \geq 4(\rho^2 - \rho^{-2})^8 = 2^{22} > 256 = d_{\zeta_8}$$

(since ε is totally imaginary, we have $m \not\equiv 0 \pmod 4$).

2. If $2N = 10$, we may take $\eta_{10} = (1 + \sqrt{5})/2 = \rho$ and $t = 3$ and we obtain

$$d_\varepsilon \geq 16\sin^2\Big(\frac{\pi m}{5}\Big)\sin^2\Big(\frac{3\pi m}{5}\Big)(\rho^n - \rho^{-n})^8 \geq 5(\rho^2 - \rho^{-2})^8 = 5^5 > 125 = d_{\zeta_{10}}$$

(since ε is totally imaginary, we have $m \not\equiv 0 \pmod 5$).

3. If $2N = 12$, then $\varepsilon_0 = 2 + \sqrt{3} = |1 + \zeta_{12}|^2 = N_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}(\sqrt{3})}(1 + \zeta_{12})$ is the fundamental unit of $\mathbb{Z}[\sqrt{3}]$. Hence, we may take $\eta_{12} = 1 + \zeta_{12}$, $\rho = \varepsilon_0^{1/2}$ and $t = 5$. Noticing that $\eta_{12} = \zeta_{24}(\zeta_{24} + \zeta_{24}^{-1}) = \zeta_{24}\rho$ and $\sigma(\eta_{12}) = 1 + \zeta_{12}^5 = \frac{\zeta_{12}^3}{1+\zeta_{12}} = \zeta_{12}^3/\eta_{12} = \zeta_{24}^5/\rho$, we obtain $16\Im^2(\varepsilon)\Im^2(\sigma_t(\varepsilon)) = 16\sin^2\big(\frac{(2m+n)\pi}{12}\big)\sin^2\big(\frac{5(2m+n)\pi}{12}\big) \in \{1, 4, 9, 16\}$ and

$$d_\varepsilon \geq (\varepsilon_0^{n/2} - \varepsilon_0^{-n/2})^8 \geq (\varepsilon_0 - 1/\varepsilon_0)^8 = 144^2 > 144 = d_{\zeta_{12}}$$

(since ε is totally imaginary, we have $2m + n \not\equiv 0 \pmod{12}$). ■

4.5. The case where $\varepsilon = \zeta_4\eta^2$ or $\varepsilon = \zeta_3\eta^3$

LEMMA 20. *Let ε be a totally imaginary quartic unit which is not a complex root of unity such that $\Pi_\varepsilon(X) = X^4 - aX^3 + bX^2 - cX + 1$ satisfies $|c| \leq a$.*

1. *Assume that $\varepsilon = \zeta_3\eta^3$ for some totally imaginary quartic unit $\eta \in \mathbb{Z}[\varepsilon]$ and some complex root of unity $\zeta_3 \in \mathbb{Z}[\varepsilon]$ of order 3. Then $\Pi_\varepsilon(X) = X^4 - 5X^3 + 8X^2 - 4X + 1$, in which case $\varepsilon = \zeta_3\eta^3$, where $\eta = -\varepsilon^2 + 3\varepsilon - 1 \in \mathbb{Z}[\varepsilon]$ and $\zeta_3 = \varepsilon^3 - 4\varepsilon^2 + 5\varepsilon - 2 \in \mathbb{Z}[\varepsilon]$. Moreover, $\Pi_\eta(X) = X^4 - 2X^3 + 2X^2 - X + 1$ is of type (T) and $d_\varepsilon = d_\eta = 117$ and $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$.*

2. *Assume that $\varepsilon = \zeta_4\eta^2$ for some totally imaginary quartic unit $\eta \in \mathbb{Z}[\varepsilon]$ and some complex root of unity $\zeta_4 \in \mathbb{Z}[\varepsilon]$ of order 4. Then $\Pi_\varepsilon(X) = X^4 - 4X^3 + 7X^2 - 4X + 1$, in which case $\varepsilon = \zeta_4\eta^2$, where $\eta = -\varepsilon^3 + 3\varepsilon^2 - 4\varepsilon + 1 \in \mathbb{Z}[\varepsilon]$ and $\zeta_4 = -\varepsilon^3 + 4\varepsilon^2 - 6\varepsilon + 2 \in \mathbb{Z}[\varepsilon]$. Moreover, $\Pi_\eta(X) = X^4 - 2X^3 + X^2 + 1$ is of type (T), $d_\varepsilon = d_\eta = 272$ and $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\eta]$.*

Proof. Set $\mathbb{K} := \mathbb{Q}(\zeta_3)$ and $\mathbb{A} := \mathbb{Z}[\zeta_3]$. Since η is quadratic over $\mathbb{K}$, there exist α and β in $\mathbb{A}$ such that $\eta^2 - \alpha\eta + \beta = 0$. Clearly, $\beta \in \mathbb{A}^*$ and $\alpha \neq 0$. Moreover, $\mathbb{A}[\eta] \subseteq \mathbb{A}[\varepsilon] = \mathbb{A}[\zeta_3\eta^3] = \mathbb{A}[\eta^3] \subseteq \mathbb{A}[\eta]$ yields $\mathbb{A}[\eta] = \mathbb{A}[\eta^3]$. Since $\eta^3 = (\alpha^2 - \beta)\eta - \alpha\beta$ and since $\eta^3 \notin \mathbb{A}$ (otherwise η would be a complex root of unity), we obtain $\alpha^2 - \beta \in \mathbb{A}^*$. Now, $1 \leq |\alpha|^2 \leq |\alpha^2 - \beta| + |\beta| = 2$ yields $|\alpha|^2 = 1$ (there is no element of norm 2 in $\mathbb{A}$) and $\alpha \in \mathbb{A}^*$. Hence, α, β and $\alpha^2 - \beta$ are in $\mathbb{A}^*$. Setting $\beta = -\alpha^2\gamma$ with $\gamma \in \mathbb{A}^*$, we have $1 + \gamma \in \mathbb{A}^* = \{\pm 1, \pm\zeta_3, \pm\zeta_3^2\}$. Hence, $\gamma \in \{\zeta_3, \zeta_3^2\}$ and $\eta^2 - \alpha\eta - \alpha^2\gamma = 0$ yields that $\varepsilon = \zeta_3\eta^3$ is a root of $P(X) = X^2 - \delta\zeta_3(3\gamma + 1)X - \zeta_3^2 \in \mathbb{K}[X]$ (use $\alpha^6 = \gamma^3 = 1$), where $\delta = \alpha^3 \in \{\pm 1\}$. Hence,

$$\Pi_\varepsilon(X) = P(X)\overline{P(X)} = \begin{cases} X^4 + 4\delta X^3 + 8X^2 + 5\delta X + 1 & \text{if } \gamma = \zeta_3, \\ X^4 - 5\delta X^3 + 8X^2 - 4\delta X + 1 & \text{if } \gamma = \zeta_3^2. \end{cases}$$

Set $\mathbb{K} := \mathbb{Q}(\zeta_4)$ and $\mathbb{A} := \mathbb{Z}[\zeta_4]$. Since η is quadratic over $\mathbb{K}$, there exist α and β in $\mathbb{A}$ such that $\eta^2 - \alpha\eta + \beta = 0$. Clearly, $\beta \in \mathbb{A}^*$ and $\alpha \neq 0$. Moreover, $\mathbb{A}[\eta] \subseteq \mathbb{A}[\varepsilon] = \mathbb{A}[\zeta_4\eta^2] = \mathbb{A}[\eta^2] \subseteq \mathbb{A}[\eta]$ yields $\mathbb{A}[\eta] = \mathbb{A}[\eta^2]$. Since $\eta^2 = \alpha\eta - \beta$ and since $\eta^2 \notin \mathbb{A}$ we obtain $\alpha \in \mathbb{A}^*$. Hence, α and β are in $\mathbb{A}^*$. Setting $\beta = -\alpha^2\gamma$ with $\gamma \in \mathbb{A}^*$, we obtain $\eta^2 - \alpha\eta - \alpha^2\gamma = 0$, with $\alpha, \gamma \in \mathbb{A}^* = \{\pm 1, \pm\zeta_4\}$. It follows that $\varepsilon = \zeta_4\eta^2$ is a root of $P(X) = X^2 - \delta\zeta_4(1 + 2\gamma)X - \gamma^2 \in \mathbb{K}[X]$ (use $\alpha^4 = 1$), where $\delta = \alpha^2 \in \{\pm 1\}$. Hence,

$$\Pi_\varepsilon(X) = P(X)\overline{P(X)} = \begin{cases} X^4 - X^2 + 1 & \text{if } \gamma = -1, \\ X^4 + 4\delta X^3 + 7X^2 + 4\delta X + 1 & \text{if } \gamma = \zeta_4, \\ X^4 - 4\delta X^3 + 7X^2 - 4\delta X + 1 & \text{if } \gamma = -\zeta_4, \\ X^4 + 7X^2 + 1 & \text{if } \gamma = 1. \end{cases}$$

If $\gamma = -1$ then ε is a complex root of unity of order 12, a contradiction. If $\gamma = 1$ then $\zeta_4 \in \{\pm(\varepsilon^3 + 8\varepsilon)/3\}$ is not in $\mathbb{Z}[\varepsilon]$. ∎

REMARK 21. Whereas there are only finitely many cases for which the quartic order $\mathbb{Z}[\varepsilon]$ contains a complex root of unity of order 8, 10 or 12, by Lemma 19, it happens infinitely often that is contains a complex root of unity of order 3 or 4. For example, if $\Pi_\varepsilon(X) = X^4 - 2AX^3 + (A^2 - 1)X^2 + AX + 1$, $A \geq 0$, then $\zeta_3 = -\varepsilon^2 + A\varepsilon \in \mathbb{Z}[\varepsilon]$. If $\Pi_\varepsilon(X) = X^4 - 2AX^3 + A^2X^2 + 1$, $A \geq 0$, then $\zeta_4 = \varepsilon^2 - A\varepsilon \in \mathbb{Z}[\varepsilon]$.

4.6. When does $\mathbb{Z}[\varepsilon]$ contain a third or fourth root of unity? Even though we do not need the results of the present subsection to settle our problem, we have the following result which essentially was proved in [Lou11]:

PROPOSITION 22. *Let ε be a totally imaginary quartic algebraic unit, of minimal polynomial $\Pi_\varepsilon(X) = X^4 - aX^3 + bX^2 - cX + d \in \mathbb{Z}[X]$ with $|c| \leq a$.*

1. *The order $\mathbb{Z}[\varepsilon]$ contains a third root of unity ζ_3 if and only if we are in one of the following four situations*:

(a) $\Pi_\varepsilon(X) = X^4-(2A-B)X^3+(A^2+3-AB)X^2-(2A-B)X+1$ *with $B \in \{\pm 1\}$ and $A \geq 1$, in which case we may take* $\zeta_3 = B\big(-\varepsilon^3+(2A-B)\varepsilon^2-(A^2+2-AB)\varepsilon+A-B\big)$.
(b) $\Pi_\varepsilon(X) = X^4-(2A-B)X^3+(A^2-1-AB)X^2+(2A-B)X+1$ *with $B \in \{\pm 1\}$ and $A \geq 1$, in which case we may take* $\zeta_3 = B\big(\varepsilon^3-(2A-B)\varepsilon^2+(A^2-AB)\varepsilon+A-B\big)$.
(c) $\Pi_\varepsilon(X) = X^4-2AX^3+(A^2-\delta)X^2+\delta AX+1$ *with $\delta \in \{\pm 1\}$ and $A \geq 0$, in which case we may take* $\zeta_3 = \delta(-\varepsilon^2+A\varepsilon)$.
(d) $\Pi_\varepsilon(X) = X^4-3X^3+2X^2+1$ *or* $\Pi_\varepsilon(X) = X^4-5X^3+8X^2-4X+1$.

2. *The order $\mathbb{Z}[\varepsilon]$ contains a fourth root of unity ζ_4 if and only if we are in one of the following four situations*:

(a) $\Pi_\varepsilon(X) = X^4-2AX^3+(A^2+3)X^2-2AX+1$ *with $A \geq 0$, in which case* $d_\varepsilon = 16(A^4-6A^2+25)$ *and we may take* $\zeta_4 = \varepsilon^3-2A\varepsilon^2+(A^2+2)\varepsilon-A$.
(b) $\Pi_\varepsilon(X) = X^4-2AX^3+(A^2-1)X^2+2AX+1$ *with $A \geq 0$, in which case* $d_\varepsilon = 16(A^4+10A^2+9)$ *and we may take* $\zeta_4 = \varepsilon^3-2A\varepsilon^2+A^2\varepsilon+A$.
(c) $\Pi_\varepsilon(X) = X^4-2AX^3+A^2X^2+1$ *with $A \geq 0$, in which case* $d_\varepsilon = 16(A^4+16)$ *and we may take* $\zeta_4 = \varepsilon^2-A\varepsilon$.
(d) $\Pi_\varepsilon(X) = X^4-4X^3+5X^2-2X+1$, *in which case $d_\varepsilon = 144$ and we may take* $\zeta_4 = \varepsilon^3-3\varepsilon^2+3\varepsilon-1$.

Proof. Assume that a primitive complex root of unity ζ_N of order $N = 3$ or $N = 4$ is in $\mathbb{Z}[\varepsilon]$. Let $\Pi_{\varepsilon,k}(X) = X^2 - \alpha X + \beta \in \mathbb{Z}[\zeta_N]$ be the minimal polynomial of ε over the imaginary quadratic number field $k = \mathbb{Q}(\zeta_N)$, with $\alpha = A + B\zeta_N \in \mathbb{Z}[\zeta_N]$ and β a unit of k, hence a complex root of unity. By considering $\bar{\varepsilon}$ if necessary, we may assume that $\beta \in \{1, -1, \pm\zeta_3\}$ for $\zeta_N = \zeta_3$ and $\beta \in \{1, -1, \zeta_4\}$ for $\zeta_N = \zeta_4$. Set $T = \zeta_N + \bar{\zeta}_N \in \{0, -1\}$. Then $\Pi_\varepsilon(X) = \Pi_{\varepsilon,k}(X)\overline{\Pi_{\varepsilon,k}(X)} = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ with $a = \alpha + \bar{\alpha} = 2A + TB$, $b = \alpha\bar{\alpha} + \beta + \bar{\beta} = A^2 + B^2 + TAB + \beta + \bar{\beta}$ and $c = \alpha\bar{\beta} + \bar{\alpha}\beta$. Now, we distinguish four cases:

1. Assume that $\beta = +1$. Then $a = 2A + TB = c$, $b = A^2 + B^2 + TAB + 2$ and $\Pi_{\varepsilon,k}(\varepsilon) = \Pi_\varepsilon(\varepsilon) = 0$ yield $B\zeta_N = \varepsilon + 1/\varepsilon - A = -\varepsilon^3 + a\varepsilon^2 + (1-b)\varepsilon + c - A \in B\mathbb{Z}[\varepsilon]$. Hence, $B \in \{\pm 1\}$.

If $N = 3$, then $T = -1$, $a = 2A - B = c$, $b = A^2 - AB + 3$ and $B\zeta_N = -\varepsilon^3 + (2A-B)\varepsilon^2 - (A^2+2-AB)\varepsilon + c - A \in B\mathbb{Z}[\varepsilon]$.

If $N = 4$, then $T = 0$, $a = 2A$, $b = A^2 + 3$, $c = a = 2A$ and $B\zeta_N = -\varepsilon^3 + 2A\varepsilon^2 - (A^2+2)\varepsilon + A$.

2. Assume that $\beta = -1$. The proof is similar to the previous one, the only difference being that now $a = 2A + TB = -c$, $b = A^2 + B^2 + TAB - 2$ and $B\zeta_N = \varepsilon - 1/\varepsilon - A = \varepsilon^3 - a\varepsilon^2 + (1+b)\varepsilon - c - A \in B\mathbb{Z}[\varepsilon]$.

3. Assume that $\beta = \delta\zeta_3$, $\delta \in \{\pm 1\}$. Then $\zeta_N = \zeta_3$, $T = -1$, $a = 2A - B$, $b = A^2 + B^2 - AB - \delta$, $c = \delta(2B - A)$ and $|c| \le a$ if and only if $|B| \le A$. Now, $\Pi_{\varepsilon,k}(\varepsilon) = \Pi_\varepsilon(\varepsilon) = 0$ yield $\zeta_3 = \frac{\varepsilon^2 - A\varepsilon}{B\varepsilon - \delta} = \frac{\pm B}{AB - \delta}\varepsilon^3 + \ldots \in \mathbb{Z}[\varepsilon]$, and, more precisely, $\zeta_3 = \delta(-\varepsilon^2 + A\varepsilon)$ for $B = 0$. If $B = 0$, we are in case 1(c). Assume now that $B \neq 0$. Then $AB - \delta$ divides B, hence divides AB, hence divides 1. Therefore $|AB - \delta| = 1$, which implies $A = 2$, $B = \delta$ (use $0 \neq |B| \le A$) and $\Pi_\varepsilon(X) = X^4 - (4 - \delta)X^3 + (5 - 3\delta)X^2 + \delta(2 - 2\delta)X + 1$, $\delta \in \{\pm 1\}$, and we are in case 1(d).

4. Assume that $\beta = \zeta_4$. Then $\zeta_N = \zeta_4$, $T = 0$, $a = 2A$, $b = A^2 + B^2$, $c = 2B$ and $|c| \le a$ if and only if $|B| \le A$. Now, $\Pi_{\varepsilon,k}(\varepsilon) = \Pi_\varepsilon(\varepsilon) = 0$ yield $\zeta_4 = \frac{\varepsilon^2 - A\varepsilon}{B\varepsilon - 1} = \frac{B}{AB - 1}\varepsilon^3 + \ldots \in \mathbb{Z}[\varepsilon]$, and, more precisely, $\zeta_4 = -\varepsilon^2 + A\varepsilon$ if $B = 0$. If $B = 0$, we are in case 2(c). Assume now that $B \neq 0$. Then $AB - 1$ divides B. Therefore $|AB - 1| = 1$, which implies $A = 2$, $B = 1$ (use $0 \neq |B| \le A$) and we are in case 2(d).

The proof is complete. ■

Corollary 23. *Let ε be a totally imaginary quartic algebraic unit, of minimal polynomial $\Pi_\varepsilon(X) = X^4 - aX^3 + bX^2 - cX + d \in \mathbb{Z}[X]$ with $|c| \le a$.*

1. *The order $\mathbb{Z}[\varepsilon]$ contains a complex root of unity ζ_8 of order 8 if and only if $\Pi_\varepsilon(X) = X^4 + 1$, in which case ε is a primitive complex 8th root of unity.*

2. *The order $\mathbb{Z}[\varepsilon]$ contains a complex root of unity ζ_{12} of order 12 if and only if $\Pi_\varepsilon(X) = X^4 - X^2 + 1 = \Phi_{12}(X)$, in which case ε is a primitive complex 12th root of unity, or $\Pi_\varepsilon(X) = X^4 - 4X^3 + 5X^2 - 2X + 1 = \Phi_{12}(X - 1)$, in which case $d_\varepsilon = 144$ and ε is a fundamental unit of the order $\mathbb{Z}[\varepsilon]$.*

Proof. If $\zeta_8 \in \mathbb{Z}[\varepsilon]$, then $\mathbb{Q}(\zeta_8) = \mathbb{Q}(\varepsilon)$ (both fields being of degree 4), hence $\mathbb{Z}[\varepsilon] \subseteq \mathbb{Z}[\zeta_8]$ (the latter order being the ring of algebraic integers of $\mathbb{Q}(\zeta_8)$), hence $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\zeta_8]$ and $d_\varepsilon = 256$. Now one can easily check that in the four cases of the second point we have $d_\varepsilon = 256$ if and only if we are in case 2(c) with $A = 0$. The proof of the second assertion is similar, except that now we have $\mathbb{Z}[\varepsilon] = \mathbb{Z}[\zeta_{12}]$ and $d_\varepsilon = 144$. ■

4.7. Bounds on discriminants

Theorem 24. *Let α be a totally imaginary quartic algebraic unit. Then*

$$7\max(|\alpha|^4, |\alpha|^{-4})/10 \le d_\alpha \le 16(|\alpha| + |\alpha|^{-1})^8 \le 256\max(|\alpha|^8, |\alpha|^{-8}). \qquad (10)$$

Proof. Since both terms of (10) remain unchanged if we change α into $-\alpha$, $1/\alpha$ and $-1/\alpha$, we may assume that $\Pi_\alpha(X) = X^4 - aX^3 + bX^2 - cX + 1 \in \mathbb{Z}[X]$ satisfies $|c| \le a$. By taking an appropriate choice of the root of $\Pi_\alpha(X)$, we may also assume that $\rho := |\alpha| \ge 1$.

Let $\alpha = \rho e^{i\phi}$, $\bar\alpha$, $\alpha' = \rho^{-1}e^{i\psi}$ and $\bar\alpha'$ be the four complex roots of $\Pi_\alpha(X)$ (use $|\alpha|^2|\alpha'|^2 = 1$). Then

$$\begin{aligned} d_\alpha &= \big((\alpha - \bar\alpha)(\alpha - \alpha')(\alpha - \bar\alpha')(\bar\alpha - \alpha')(\bar\alpha - \bar\alpha')(\alpha' - \bar\alpha')\big)^2 \\ &= 16(\sin\phi)^2(\sin\psi)^2\big|\rho - \rho^{-1}e^{i(\psi-\phi)}\big|^4\big|\rho - \rho^{-1}e^{i(\psi+\phi)}\big|^4 \le 16(\rho + \rho^{-1})^8. \end{aligned}$$

Assume that $\rho \ge \sqrt{3}$ and $a \ge 37$.

Then $|2\rho^{-1}\cos\phi + 2\rho\cos\psi| = |c| \le a = 2\rho\cos\phi + 2\rho^{-1}\cos\psi$ implies $\cos\phi \ge |\cos\psi|$ and

$$d_\alpha \ge \big(4(\rho - \rho^{-1})^4\sin^2\phi\big)^2.$$

First, if $\sin^2\phi \geq \frac{3}{4}\rho^{-2}$, then $d_\alpha \geq \big(3(1-\rho^{-2})\big)^2\rho^4 \geq 4\rho^4$ (use $\rho \geq \sqrt{3}$).

Second, assume that $\sin^2\phi < \frac{3}{4}\rho^{-2}$. Since $\rho \geq 1$, we have

$$-8 < 4b - a^2 < 3 + 4\sin^2\psi + 8\sqrt{1-\sin^2\psi} \leq 11,$$

by (9) (α is totally imaginary, hence $\sin\phi \neq 0$ and $\sin\psi \neq 0$, and $\sqrt{1-t} \leq 1 - t/2$ for $0 \leq t = \sin^2\psi \leq 1$). Since $4b - a^2 \equiv 0$ or $3 \pmod 4$, we obtain $J := 4b - a^2 \in \{-5,-4,-1,0,3,4,7,8\}$ and we are in the situation of Lemma 25. By Lemma 26 we have $d_\alpha \geq 4\max(|\alpha|^4, |\alpha|^{-4})$.

Finally, since $|c| \leq a \leq \sqrt{4b+5}$ and $-1 \leq b \leq \rho^2+\rho^{-2}+4$, it is easy to list all possible polynomials $\Pi_\alpha(X)$ for which $1 \leq \rho \leq \sqrt{3}$ or for which $a \leq 36$ and $b = (a^2+J)/4$ with $J \in \{-5,-4,-1,0,3,4,7,8\}$ and also to check that the lower bound for d_α in (10) holds for these polynomials, by using Lemma 15. ∎

LEMMA 25. *Fix $J \in \{-5,-4,-1,0,3,4,7,8\}$. For $a \in \mathbb{Z}$ with $a^2 \equiv -J \pmod 4$, i.e. with a even if $J \equiv 0 \pmod 4$ and a odd if $J \equiv 3 \pmod 4$, set*

$$\Pi_J(X) = X^4 - aX^3 + \frac{a^2+J}{4}X^2 - cX + 1 \in \mathbb{Z}[X].$$

Assume $|c| \leq a$, that $a \geq 15$ and that $\Pi_J(X)$ is $\mathbb{Q}$-irreducible with no real roots, hence is of positive discriminant $D(a,J,c)$ (a quartic polynomial in c). Set

$$B := \begin{cases} a-1 & \text{if } J = 8\\ Ja/8 & \text{if } J \in \{-4,0,4\}\\ (Ja-1)/8 & \text{if } J \in \{-5,-1,3,7\}.\end{cases}$$

Then $-a \leq c \leq B$ and $D(a,J,c) \geq F(a,J) := \min(D(a,J,-a), D(a,J,B))$.

Proof. Assume that $a = 2A$ is even. Then $J = 4j$ and $\Pi_J(A) = jA^2 - cA + 1 > 0$ (for $\Pi_J(X)$ has no real root) yields $c \leq jA = Ja/8$. Moreover, since $X^4 - 2AX^3 + (A^2+2)X^2 - 2AX + 1 = (X^2 - AX + 1)^2$ is not irreducible, we obtain that $c \neq a = Ja/8$ for $J = 8$. Hence, $c \leq B$.

Now, assume that a is odd. Then $0 < 16\Pi_J(a/2) = Ja^2 - 8ac + 16 \equiv 3 \pmod 4$ yields $Ja^2 - 8ac + 16 \geq 3$. Hence, $8c \leq Ja + 13/a < Ja + 1$ for $a \geq 15$. Hence, $c \leq (Ja-1)/8 = B$.

Numerical investigations suggest that $D(a,J,c)$ as a function of c has four real roots close to $-a$, $Ja/8$, a and $a^3/54 + Ja/24$. Set $e = 0$ if J is even and $e = 1$ if J is odd, and

$$\Delta(a,J,c) = -27(c+a)\Big(c - \frac{Ja-e}{8}\Big)\Big(c - \Big(a - 1 + \frac{e}{8}\Big)\Big)\Big(c - \frac{a^3}{54} - \frac{Ja}{24} - 1\Big).$$

Using any software of symbolic computation, we check that $D(a,J,c) = \Delta(a,J,c) + P(a,J,c)$, where $P(a,J,c) = \alpha(a,J)c^2 + \beta(a,J)c + \gamma(a,J)$ is a quadratic polynomial in c whose leading coefficient

$$\alpha(a,J) = \begin{cases} -\dfrac{a^3}{2} + \Big(3 + \dfrac{J^2}{64}\Big)a^2 + \Big(27 - \dfrac{9J}{8}\Big)a - \dfrac{J^3}{16} + 36J - 27, & J \text{ even}\\ -\dfrac{a^3}{2} + \Big(3 + \dfrac{J^2}{64}\Big)a^2 + \Big(\dfrac{189}{8} - \dfrac{45J}{64}\Big)a - \dfrac{J^3}{16} + 36J - \dfrac{1539}{64}, & J \text{ odd}\end{cases}$$

is less than or equal to 0 for $J \in \{-5,-4,-1,0,3,4,7,8\}$ and $a \geq 14$.

Clearly, we have $\Delta(a,J,c)\geq 0$ for $-a\leq c\leq B$ (notice that $a^3/54-Ja/24-1\geq a^3/54-a/3-1\geq a-1\geq B$ for $a\geq 9$). Hence, $D(a,J,c)\geq P(a,J,c)$. Since $\alpha(a,J)\leq 0$, we have

$$P(a,J,c)\geq \min(P(a,J,-a),P(a,J,B))\quad \text{for}\quad -a\leq c\leq B.$$

Finally, we have $\Delta(a,J,-a)=\Delta(a,J,B)=0$. Hence, $D(a,J,-a)=P(a,J,-a)$ and $D(a,J,B)=P(a,J,B)$. The desired result follows. ∎

LEMMA 26. *Assume that* $\Pi_\alpha(X)=X^4-aX^3+\frac{a^2+J}{4}X^2-cX+1\in\mathbb{Z}[X]$ *is a quartic polynomial of type* (T) *with* $J\in\{-5,-4,-1,0,3,4,7,8\}$, $a\in\mathbb{Z}$ *and* $a\equiv J\,(\mathrm{mod}\,2)$. *Assume that* $a\geq 37$. *Then* $d_\alpha\geq 4\max(|\alpha|^4,|\alpha|^{-4})$.

Proof. Set $M=\max(|\alpha|,|\alpha|^{-1})$. Using (8), we obtain

$$(a^2+J-16)/8\leq M^2\leq (a^2+J+16)/4. \tag{11}$$

By Lemma 25, for $J\in\{-4,0,4,8\}$ we have

$$\begin{aligned}D(a,J,c)&\geq \min(D(a,J,-a),D(a,J,B))\\&\geq D(a,4,a/2)=9((a^2-8)^2+192)/16\end{aligned}$$

(check (i) that the quartic polynomials with positive leading coefficient $D(a,J,-a)-D(a,4,a/2)=\frac{(J+5)(J+11)}{16}a^4+\dots$ are non-negative for $J\in\{-4,0,4,8\}$ and $a\geq 1$, (ii) that $D(a,8,a-1)-D(a,4,a/2)\geq 0$ for $a\geq 0$ and (iii) that the quartic polynomials with non-negative leading coefficient $D(a,J,Ja/8)-D(a,4,a/2)=\frac{(16-J^2)(112-J^2)}{4096}a^4+\dots$ are non-negative for $J\in\{-4,0,4\}$ and $a\in\mathbb{Z}$).

Using (11), we have $M^2\geq (a^2-20)/8>8$ and $a^2-8\geq 4(M^2-8)>0$. Hence, $d_\alpha\geq 9((M^2-8)^2+12)\geq 4M^4$.

In the same way, for $J\in\{-5,-1,3,7\}$ we have

$$\begin{aligned}D(a,J,c)&\geq \min(D(a,J,-a),D(a,J,(Ja-1)/8))\\&\geq D(a,-5,-a)=9((a^2+19)^2-192)/16\end{aligned}$$

(check (i) that the quartic polynomials with non-negative leading coefficient $D(a,J,-a)-D(a,-5,-a)=\frac{(J+5)(J+11)}{16}a^4+\dots$ are non-negative for $J\in\{-5,-1,3,7\}$ and $a\geq 1$ and (ii) that the quintic polynomials with positive leading coefficient $D(a,J,(Ja-1)/8)-D(a,-5,-a)=\left(1-\frac{J^2}{64}\right)\frac{a^5}{16}+\dots$ are positive for $J\in\{-5,-1,3,7\}$ and $a\geq 36$).

Using (11), we have $M^2\geq (a^2-21)/8>1$ and $4(M^2-1)\leq a^2+19$. Hence, $d_\alpha\geq 9((M^2-1)^2-12)\geq 4M^4$. ∎

COROLLARY 27. *Let* ε *be a totally imaginary quartic algebraic unit which is not a complex root of unity. If* $\varepsilon=\eta^n$ *for some* $\eta\in\mathbb{Z}[\varepsilon]$ *and some* $n\in\mathbb{Z}$, *then* $\max(|\eta|,|\eta|^{-1})<2.787$ *and* $|n|\leq 13$. *In particular, by* (7), *if* $\Pi_\eta(X)=X^4-aX^3+bX^2-cX+1\in\mathbb{Z}[X]$ *is of type* (T), *then* $-1\leq b\leq 11$ *and* $|c|\leq a\leq\sqrt{4b+5}$.

Proof. We may assume that $|\varepsilon|\geq 1$, $|\eta|\geq 1$ and $n\geq 3$. Notice that η is necessarily a totally imaginary quartic algebraic unit which is not a complex root of unity. By (10), we have

$$7|\eta|^{12}/10\leq 7|\eta|^{4n}/10=7|\varepsilon|^4/10\leq d_\varepsilon=d_\eta\leq 16\left(|\eta|+|\eta|^{-1}\right)^8,$$

that implies $|\eta| \leq 2.787$. Moreover, we have $|\eta| \geq |\eta_0| = 1.18375\ldots$ (Lemma 16). Hence, by (10), we have

$$1 = d_\eta/d_\varepsilon \leq \frac{16(|\eta| + |\eta|^{-1})^8}{7|\varepsilon|^4/10} = \frac{160}{7}\Big(\frac{|\eta| + |\eta|^{-1}}{|\eta|^{n/2}}\Big)^8 \leq \frac{160}{7}\Big(\frac{|\eta_0| + |\eta_0|^{-1}}{|\eta_0|^{n/2}}\Big)^8,$$

that implies $n < 14$. ∎

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