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# COMBINATORIAL ASPECTS OF THE SEQUENCE OF MILNOR NUMBERS OF DEFORMATIONS

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**Abstract.** We use combinatorics to describe the topology of a singular irreducible plane curve germ $f = 0$ under small perturbation of parameters.

**1. Introduction.** In this paper we present a combinatorial approach to the study of topology of plane complex curves in a neighborhood of a point. This question was studied intensively and is related to research and open problems of for instance [GZ93], [Arn04], [Bod07], [MW14], [BKW14] to name only a few.

Consider a holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ which is defined on some neighborhood of $0 \in \mathbb{C}^2$ and $f(0) = 0$. We study the curve given by the equation $f = 0$. If the set is smooth near 0, the topology is trivial. Therefore we will assume $f$ is a *singularity*, i.e. all its partial derivatives vanish at zero. In this paper we will deal with the case when the set $\nabla f(0) = 0$ is an isolated point on the curve $f = 0$, i.e. the singularity $f$ is *isolated*.

A classical question is:

how does the topology of $f = 0$ change when perturbing the coefficients of $f$?

The result of Milnor [Mil68] describes the topology of the isolated singularity $f$. Take a 3-sphere $S_\epsilon$ centered at 0 with radius $\epsilon$. Consider the link $K_\epsilon$, i.e. the intersection of this sphere with the set $f = 0$. For every $\epsilon$ small enough $F = f/|f| : S_\epsilon \setminus K_\epsilon \to S^1$ is a smooth locally trivial fibration and its fiber has the same homotopy type as a one-point union of a finite collection of circles. This number of circles is constant for positive $\epsilon$ small enough. It is called *the Milnor number* of the isolated singularity $f$ and denoted by $\mu(f)$.

We want to perturb the coefficients of $f$ and see what can happen to the fiber. As in the simple example of $z^3 = 0$ the perturbations $z^3 + \delta$ and $z^3 + \delta z^2$ behave differently.

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A deformation of $f$ is any function germ $f_s$ holomorphic with respect to the variable $z \in \mathbb{C}^2$ and parameters $s \in \mathbb{C}^n$ such that $f_0 = f$ and $f_s(0) = 0$. In fact, for isolated singularities it is sufficient to consider *versal deformations* of $f$ which are of the form

$$f_s = f + \sum_{j=1}^{\mu(f)-1} s_j m_j,$$

where $s \in \mathbb{C}^{\mu(f)-1}$ are sufficiently small and $1, m_1, \ldots, m_{\mu(f)-1}$ are generators of the quotient ring $\mathbb{C}[[x,y]]/(\partial f/\partial x, \partial f/\partial y)$. In the case of $f$ irreducible nondegenerate one can also assume that the generators are monomial, see [AGZV85].

Since the Milnor number says a lot about the topology, we will study the possible changes in Milnor numbers under deformation. This approach gives information on configuration of the space of singularities and its periodicity (which we do not discuss here, see [AGZV85]).

We can interpret any deformation of a given singularity $f$ as a point in $\mathbb{C}^{\mu(f)-1}$ with standard euclidean metric by considering its coefficients. First, note that the Milnor number of a small perturbation of parameters of $f$ has the Milnor number not greater than $f$ does, see [Kou76]. We consider sets $X_j$ of deformations of $f$ that have their Milnor number equal to $j$. Now the main question can be reformulated as a series of questions about $X_j$. Is every $X_j$ non-empty? How many connected components does $X_j$ have? Do closures of $X_j$ and $X_{j+1}$ intersect and how? We will relate to these questions in Section 3 using combinatorial results of Section 2.

## 2. Combinatorial approach.

It is remarkable that for generic singularities (i.e. having coefficients outside an algebraic set, in particular of zero measure) there is a strictly combinatorial formula for the Milnor number in terms of Newton diagrams.

### 2.1. Newton diagram.

A *Newton diagram* $\Gamma_X$ of a set $X \subset \mathbb{Z}^2_{\geq}$ is the convex hull of $X + \mathbb{R}^2_{\geq}$. The boundary of the Newton diagram is a polygonal chain of a vertical half-line, finite number of compact segments and a horizontal half-line. We will identify the Newton diagram with the finite number of segments on its border. Denote by $\mathrm{supp}(f)$ the set of all monomials with nonzero coefficients in $f$ identified with points in $\mathbb{Z}^2$ by $x^k y^m \to (k, m)$. If the set $X$ is equal to $\mathrm{supp}(f)$, then we say that $\Gamma_{\mathrm{supp}(f)}$ is the Newton diagram of the singularity $f$.

Note that we can treat any diagram $\Gamma$ as a graph of a decreasing piecewise linear function $\Gamma : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq} \cup \{\infty\}$ which is constant outside some segment $[0, p(\Gamma)]$, $p(\Gamma) \in \mathbb{N}$. Hence it is easy to introduce a natural order for diagrams. We will write $\Gamma < \Sigma$ if there is a relation $\Gamma(t) \leq \Sigma(t)$ between the functions but the functions are not identical.

When a diagram $\Gamma$ has end-points on both axes with coordinates $(p(\Gamma), 0), (0, q(\Gamma))$, *the Newton number* is equal to

$$\nu(\Gamma) = A(\Gamma) - p(\Gamma) - q(\Gamma) + 1,$$

where $A(\Gamma)$ denotes twice the area bounded by $\Gamma$ and both axes. Otherwise, the Newton number of $\Gamma$ is defined to be equal to the maximum over all Newton numbers of $\Sigma$ such that $\Sigma$ has both end-points on the axes and $\Gamma > \Sigma$.

Take a segment $S$ of the Newton diagram and denote by $f_S$ a function for which only monomials with powers from the set $S$ have non-zero coefficients equal to appropriate coefficients of $f$ (some of them possibly also zero). We say that $f$ is *a non-degenerate singularity* if for every segment $S$ of the Newton diagram the function $f_S$ is an isolated singularity when divided by sufficiently big powers of variables $x$ and $y$.

The equality

$$\nu(\Gamma_{\mathrm{supp}(f)}) = \mu(f) \tag{1}$$

holds if and only if $f$ is a non-degenerate singularity. For any singularity we have $\mu(f) \geq \nu(\Gamma_{\mathrm{supp}(f)})$, see [Kou76].

**2.2. Graph of Newton diagrams.** Let $\mathcal{G}$ be a graph with vertices in the set of all Newton diagrams. The arrows in graph $\mathcal{G}$ connect diagram $\Gamma$ to diagram $\Sigma$ if and only if $\Gamma < \Sigma$. Let the weight of each vertex be equal to its Newton number.

REMARK 2.1. Note that any diagram $\Gamma$ such that $\Gamma(t) \leq 1$ for all $t$ or $\Gamma(t) = 0$ for $t \geq 1$ corresponds to a holomorphic function which is not singular at 0 and the Newton number is 0. The only case we exclude from considerations is when $\Gamma(t) \equiv 0$, i.e. $\Gamma$ being the diagram of a function non-zero at 0.

PROPERTY 2.2. *If $\Gamma_1 < \Gamma_2$, then $\nu(\Gamma_1) \leq \nu(\Gamma_2)$. Moreover, if $\Gamma_1, \Gamma_2$ have the same end-points, the inequality is strict and $\nu(\Gamma_2) - \nu(\Gamma_1)$ is equal to twice the area of the polygon between them.*

We leave the easy proof to the reader. There may not be a strict inequality if end-points differ. It suffices to consider diagrams $\Gamma_1$ which is a chain connecting points $(0,3), (2,1), (6,0)$ and $\Gamma_2$ connecting $(0,3), (2,1), (5,0)$. We have $\Gamma_1 > \Gamma_2$ but $\nu(\Gamma_1) = \nu(\Gamma_2) = 4$, compare also Proposition 2.4.

Denote by $\|P_1, \dots, P_n\|$ the diagram which is a polygonal chain passing through points $P_1, \dots, P_n$.

PROPERTY 2.3. *For all vertices of $\mathcal{G}$ we have*

(i) $\nu(\Gamma) = 0 \iff \Gamma$ *contains* $(0,1)$ *or* $(1,0)$
(ii) $\nu(\Gamma) = 1 \iff \Gamma$ *contains* $(1,1)$
(iii) $\nu(\Gamma) = 2 \iff \Gamma = \|(3,0),(0,2)\|$ *or* $\Gamma = \|(2,0),(0,3)\|$
(iv) *for any $m > 2$ there are infinitely many diagrams $\Gamma$ such that $\mu(\Gamma) = m$.*

*Proof.* To prove (iv) suppose $m > 2$ and consider diagrams $\Gamma_q = \|(0,q),(1,2),(m-1,0)\|$. We have $\nu(\Gamma_q) = m$ for all $q > 2$. ∎

Note that in (ii) it is necessary that $p, q > 1$. Moreover, for every $m \neq 2$ there are infinitely many diagrams with this Newton number.

PROPOSITION 2.4. *Take $\Gamma = \|(0,q),(p,0)\|$ and $4 < m \leq \min(p,q)$. There are diagrams $\Omega, \Omega' < \Gamma$ such that*

$$\nu(\Omega) = \nu(\Omega') = m$$

*and there is no arrow in $\mathcal{G}$ between $\Omega$ and $\Omega'$.*

*Proof.* Consider the diagrams

$$\Omega_{q_1} = \|(0, q_1), (1, 2), (m-1, 0)\| \quad \text{and} \quad \Omega'_{q_2} = \|(0, m-1), (2, 1), (q_2, 0)\|$$

for $2 < q_1 \leq q$, $2 < q_2 \leq p$. $\Omega_{q_1}$ is indeed a diagram because the slope of the segment joining $(1, 2)$ with $(0, q_1)$ is at least equal to 1 whereas the slope of $\|(1, 2), (m-1, 0)\|$ is less than 1. Similarly for $\Omega'_{q_2}$. Furthermore, note that $(2, 1) \notin \operatorname{supp} \Omega_{q_1}$ and $(1, 2) \notin \operatorname{supp} \Omega'_{q_2}$, hence they are incomparable. Moreover, $\nu(\Omega_{q_1}) = 2(m-1) + q - (m-1) - q + 1 = m = \nu(\Omega'_{q_2})$. ■

**2.3. Continued fractions and paths in the graph.** For $(p, q) = 1$, $p < q$ consider the continued fraction

$$\frac{p}{q} = [\alpha_0 : \alpha_1, \ldots, \alpha_{m+1}],$$

where $\alpha_m > 1$, $m \geq 0$.

Consider also its $j$-th convergent

$$\frac{p_j}{q_j} = [\alpha_0 : \alpha_1, \ldots, \alpha_j],$$

where $j = 0, \ldots, m$ and $(p_j, q_j) = 1$. We have

$$p_{j-1} q_j - q_{j-1} p_j = (-1)^j \tag{2}$$

and $\frac{p_j}{p_{j-1}} = \lfloor \alpha_j \rfloor$. For more details see for instance [Sie50].

Remark 2.5. Put $D = P + (-1)^m [p_m, -q_m]$, where $P$ equals $(p, 0)$ if $m$ odd and $(0, q)$ otherwise. From (2) it easily follows that twice the area between the diagrams $\|(0, q), (p, 0)\|$ and $\|(0, q), D, (p, 0)\|$ is equal to 1.

Now fix a one-segment diagram $\Gamma$ with end-points $A, B$ and the slope $\frac{q}{p}$ for $p, q$ coprime. If $m$ is odd put $P$ equal to the end-point $B$ more to the right, otherwise equal to $A$ more to the left. Note that if $\Gamma \cap \mathbb{Z}^2$ consists of $n + 1$ points, then they are of the form

$$P_0 = P, \quad P_1 = P + (-1)^m [p, -q], \ \ldots, \ P_n = P + n(-1)^m [p, -q].$$

Proposition 2.6. *If $\Gamma \cap \mathbb{Z}^2$ consists of two points consider*

$$D_j = P + j(-1)^m [p_m, -q_m] \qquad \text{for} \quad j = 1, \ldots, \lfloor \alpha_{m+1} \rfloor$$

*The diagram $\|A, D_j, B\|$ has the Newton number equal to $\nu(\Gamma) - j$. Moreover,*

$$\|A, D_j, B\| < \|A, D_i, B\| \quad \text{if } j > i.$$

*Proof.* In this case height $q$ and length $p$ of $\Gamma$ are coprime. Note that the end-points of diagrams remain fixed and $D_j$'s are co-linear. Hence it suffices to note that $p - jp_m$ and $q - jq_m$ are coprime and use equality (2) to compute the areas. ■

Let us fix a useful ordering $\prec$ in $\mathbb{Z}^2$ by the relation

$$(\alpha, \beta) \prec (i, j) \quad \text{if and only if} \quad \beta < j \vee (\beta = j \wedge \alpha < i).$$

LEMMA 2.7. *If* $\Gamma \cap \mathbb{Z}^2$ *consists of* $n+1$ *points consider*

$$D_{jl} = P_j + l(-1)^m[p_m, -q_m]$$

*and diagrams* $\Gamma_{jl}$ *spanned by the points* $A, B$ *and* $\{D_{\alpha\beta} :\ (\alpha,\beta) \preceq (j,l)\}$.

*For* $(j,l) \prec (j',l')$ *we have*

$$\Gamma_{jl} > \Gamma_{j'l'}.$$

*Proof.* Note that for fixed $l$ the points $D_{jl}$ lie on a line parallel to the segment $AB$. The claim follows easily. ∎

In particular, if $p_m > 1$, the end-points of diagrams $\Gamma_{jl}$ in Lemma 2.7 remain fixed and

REMARK 2.8. If $p_m > 1$, the last diagram $\Gamma_{n,1}$ is of the form

$$\|A, P + n(-1)^m[p_m, -q_m], B\|$$

and the slopes of its two segments equal $\frac{q_m}{p_m}$ and $\frac{q-q_m}{p-p_m}$.

The diagrams in Lemma 2.7 have other important properties which we omit here, compare [MW14, Lemmas 3.2 and 3.3].

PROPOSITION 2.9. *Consider* $2 \le p < q$, *where* $(p,q) = 1$ *and* $q = kp + r$ *for* $0 < r < p$.

*For the diagram* $\Gamma_{pq} = \|(0,q),(p,0)\|$ *there is a chain in the graph* $\mathcal{G}$ *of Newton diagrams such that the end-points remain fixed and the length of the chain is equal to the number of integer points of the triangle with vertices* $(p,0),(0,q)$ *and* $(r,k(p-r))$ *minus* 1.

*Proof.* First use Proposition 2.6.

Note that Bezout's equality (2) implies $\frac{q_j}{p_j} > \frac{q_{j-1}}{p_{j-1}}$ if $j$ even and the inequality flips for $j$ odd. Recall $p = p_{m+1}, q = q_{m+1}$. Bearing this in mind apply recursively Lemma 2.7 to the segments in consecutive Newton diagrams with slopes $\frac{q_m}{p_m}, \frac{q_m-q_{m-1}}{p_m-p_{m-1}}, \ldots, \frac{q_m-n_mq_{m-1}}{p_m-n_mp_{m-1}}$, where $n_m = \lfloor\alpha_m\rfloor - 1$. According to Lemma 2.7 finally you get a two-segment diagram with slopes $\frac{q_{m-1}}{p_{m-1}}$ and $\frac{q_{m-2}}{p_{m-2}}$. Now treat the segment given by $p_j, q_j$ with higher index $j$ as the new diagram and apply procedure just described. Do this as long as $p_j > 1$. From the inequality (2) all points chosen in this way give an essential change in Newton diagrams.

Do the above construction recursively until $p_i = 1$. In this case take diagrams $\Gamma_{jl}$ as before according to Lemma 2.7 but excluding the ones with index $j = n$ or $l = n - j + 1$. The inequalities between diagrams remain, the end-points do not lie on axes and the last diagram will be of the form

$$\|(0,q),(r,k(p-r)),(p,0)\|. \tag{3}$$

Indeed, this follows from the construction and properties of continued fractions.

Note that we use all points from the triangle bordered by the last diagram and the initial one except points $(p,0)$ and $(0,q)$. Taking $\Gamma_{pq}$ as the beginning of the chain gives the assertion. ∎

REMARK 2.10. Note that since the diagrams used in the above proposition have the same end-points, their Newton numbers form a strictly decreasing sequence.

Proposition 2.11. *Consider a diagram $\|(0, kp), (p, 0)\|$ for $k > 1$. Take the points*

$$P_{ij} = (i, k(p - i) - j)$$

*for $i = 0, \ldots, p - j$ and $j = 1, \ldots, p$. The diagrams $\Sigma_{ij}$ spanned by the set $X_{ij} = \{P_{\lambda\gamma} : (\lambda, \gamma) \preceq (i, j)\}$ form a chain in graph $\mathcal{G}$ of Newton diagrams such that*

$$\nu(\Sigma_{ij}) \leq \nu(\Sigma_{lm})$$

*whenever $(l, m) \prec (i, j)$. Moreover, $\Sigma_{0p} = \|(0, (k - 1)p), (p, 0)\|$ and $\nu(\Sigma_{ij}) = \nu(\Sigma_{lm})$ if and only if $m = p$, $l = 0$, $i = 1$ and $j = p - 1$.*

*Proof.* Just observe that the points $P_{ij}$ for fixed $j$ lie on the line parallel to the segment $(p, 0), (kp, 0)$ and only for $i = 0$ do they lie on the axis. The proof follows easily. ■

Note that the last diagram in Proposition 2.11 for $j = p$ and $i = 0$ is the diagram $\|(0, (k - 1)p), (p, 0)\|$.

Theorem 2.12. *Take $p < q$ coprime. The maximal length of a path in the graph $\mathcal{G}$ joining the smallest element with the diagram $\Gamma_{pq} = \|(q, 0), (0, p)\|$ is equal to*

$$\frac{1}{2}(p + 1)(q + 1) - 1.$$

*The path of maximal length can have*

$$\frac{\nu(\Gamma_{pq})}{2} - \left\lfloor \frac{q}{p} \right\rfloor + p + q - 2$$

*different Newton numbers.*

*Proof.* First we show how to use all points under the diagram $\Gamma_{pq}$ to get the maximal chain. As usual let $q = kp + r$, $0 < r < p$.

Apply Proposition 2.9. You will end up with the diagram (3). Since the end-points of the diagrams remain fixed, due to Property 2.2 all diagrams have different Newton numbers. Apply Proposition 2.11 to the upper segment which has slope $k + 1$ as long as the Newton number changes and the last diagram $L$ will be supported by points $(0, kp + 1)$, $(1, kp - k)$, $(p, 0)$. We will consider two cases dependent on $k$. The choices of points in each case are on lines parallel to the diagram.

If $k = 1$ we will use induction with respect to $p$ as long as $p > 2$. Define

$$\Omega_p = \|(p, 0), (1, p - 1), (0, p + 1)\|.$$

Consider points in the order given below

$$(p - 1, 0), (p - 2, 1), \ldots, (3, p - 4), (2, p - 3), \quad (0, p), (1, p - 2)$$

and diagrams supported by adjoining consecutive points to $\Omega_p$. Note that the last diagram is $\Omega_{p-1}$ and we have adjoined consecutively all points under $\Omega_p$ and over or on $\Omega_{p-1}$. Denote by $(\nu_l)_{l=1,\ldots,p}$ the decreasing sequence of their Newton numbers. The sequence of differences $\nu_k - \nu_{k+1}$ (i.e. jumps of Newton numbers) is of the form

$$p - 2, \underbrace{1, \ldots, 1}_{p-3}, \ 1, 1. \tag{4}$$

Obviously, the last diagram $L$ of the second paragraph of this proof equals $\Omega_p$ when $k=1$. Hence apply induction to $L$ until $p=2$. When $p=2$ we have $\nu(\Omega_2)=1$. Therefore adjoining points $(0,2),(0,1),(1,0)$ will give only one change of Newton number to 0.

If $k>1$ first adjoin the point $(0,kp)$ to $L$ and as in Proposition 2.11 we get the diagram $\|(p,0),(0,kp)\|$ without changing the Newton number. We will now use induction with respect to $p$.

When $p\geq 3$ consider the points

$$T_{ij}=(i,k(p-i)-j)$$

for $j=1,\ldots,k$ and $i=0,\ldots,p-1$. Consider the decreasing sequence of diagrams $\Theta_{ij}$ spanned by the set $Y_{ij}=\{T_{\alpha\beta}:(\alpha,\beta)\preceq(i,j)\}$. Denote by $(\nu_l)_{l=1,\ldots,kp}$ the decreasing sequence of Newton numbers of the diagrams $\Theta_{ij}$ ordered with respect to $\preceq$. The sequence of jumps $\nu_l-\nu_{l+1}$ is of the form

$$p-1,\underbrace{1,\ldots,1}_{p-1},\ \overbrace{p-2,\underbrace{1,\ldots,1}_{p-2},2,\quad\ldots,\quad p-2,\underbrace{1,\ldots,1}_{p-2},2}^{k-2},\ p-2,\underbrace{1,\ldots,1}_{p-1}. \tag{5}$$

The last diagram $\Theta_{p-1,k}$ is equal to the diagram $\|(p-1,0),(0,k(p-1))\|$. Hence we can use the procedure above recursively until $p=2$. Note that every point under the diagram $\Theta_{p,k}$ and over or on the diagram $\Theta_{p-1,k}$ corresponds to a different diagram $\Theta_{ij}$ in the chain.

When $p=2$ we have the case of $A_{2k-1}$ singularity with Newton number $2k-1$. We can consider all diagrams spanned by points $D_j=(1,k-j)$, $C_i=(0,2k-i)$ for $j=1,\ldots,k$ and $i=1,\ldots,2k-1$ in the order $C_1,D_1,C_2,C_3,D_2,C_4,C_5,D_3,\ldots$, i.e. given by the sequence of points $F_k$ such that $F_{3j}=C_{2j}$, $F_{3j+1}=C_{2j+1}$, $F_{3j+2}=D_j$. Consecutive diagrams spanned by $\{F_i : i\leq j\}$ form a strictly decreasing sequence. Therefore the length of the longest chain of deformations of $A_{2k-1}$ diagram is equal to the number $3k-1$ of all points under the diagram $\|(2,0),(0,2k)\|$ discarding point $(0,0)$.

To compute the maximal length of different Newton numbers for $A_{2k-1}$ consider a subsequence of points given as

$$E_j=\begin{cases}(0,2k-j) & j \text{ odd}\\ (1,k-\frac{j}{2}) & j \text{ even}\end{cases}$$

and Newton diagrams $\Gamma_j$ supported by $\{E_i : i\leq j\}$ for $j=1,\ldots,2k-1$. We have $\nu(\Gamma_j)=\nu(\Gamma_{j+1})+1$ for every $j$ and $\Gamma_{2k-1}=\|(2,0),(0,1)\|$ hence $\nu(\Gamma_{2k-1})=0$.

Therefore, we constructed a chain that uses all points under the diagram $\Gamma_{pq}$. Hence there cannot be a longer chain. Moreover, the length of this chain is equal to the number $N$ of integer points in the triangle $(0,0),(p,0),(0,q)$ minus 2 (recall that we do not consider the point $(0,0)$ and we consider $\Gamma_{pq}$ to be the beginning of the chain). Denote by $I$ the number of points in the interior of this triangle and by $B$ the number of points on its sides. Obviously, $B=p+q+1$. From Pick's formula we get $pq=2I+B-2$, thus $2I=(p-1)(q-1)=\nu(\Gamma_{pq})$. Hence $N=I+B=1+\frac{(p+1)(q+1)}{2}$.

If $k=1$, then the points that do not contribute to change in Newton numbers are $(2,0)$ and $(1,0)$. If $k>1$, then the only points that do not change Newton numbers are

$(0, kp)$, the points $(0, 2(k-i))$ for $i = 1, \dots, k-1$ and the point $(0, 1)$. Hence in both cases there are $\frac{1}{2}(p+1)(q+1) - 1 - (k+1) = \frac{1}{2}(p-1)(q-1) + p + q - k - 2$ distinct Newton numbers. This ends the proof. ∎

Remark 2.13. To get the path of maximal length, one can use also inductively Proposition 2.11, but it would give only $\nu(\Gamma_{pq})/2 - \lfloor q/p \rfloor + q$ different Newton numbers.

Remark 2.14. When $p > 4$ the sequence (5) minimizes the consecutive jumps. If $p > 5$, it is unique.

## 3. Deformations of irreducible singularities.

Consider an isolated plane curve singularity $f$. If the singularity $f$ cannot be expressed as $g \cdot h$ with $g, h$ holomorphic vanishing at zero, then we say it is irreducible. Note that the locus of $gh = 0$ is the union of sets $g = 0$ and $h = 0$. Hence irreducible singularities can be considered as building blocks of singularities.

Throughout the rest of this paper we will assume that

$$f \text{ is non-degenerate and irreducible.}$$

In this essential case the Newton diagram of $f$ is exactly one segment

$$\Gamma_{\mathrm{supp}(f)} = \|(0, q), (p, 0)\|.$$

Moreover, the non-zero coordinates $p, q$ are coprime and greater than 1. Without loss of generality assume $p < q$.

Remark 3.1. If $f_s$ is a deformation of $f$, then $\Gamma_{\mathrm{supp}(f_s)} \leq \Gamma_{\mathrm{supp}(f)}$.

Now consider the graph $\mathcal{G}_{p,q}$ which is the restriction of the graph $\mathcal{G}$ to diagrams smaller than or equal to the Newton diagram of $f$. Then all diagrams in $\mathcal{G}_{p,q}$ correspond to diagrams of deformations of $f$ vanishing at 0 (except the case in Remark 2.1). Moreover, in $\mathcal{G}_{p,q}$ there is the smallest element (diagram of a function with both partial derivatives not vanishing at 0) and the biggest element (the diagram of $f$) and $\mathcal{G}_{p,q}$ has finitely many vertices.

Let us introduce some notation. Every element $\Sigma$ of $\mathcal{G}_{p,q}$ corresponds to a constructible set $X(\Sigma)$ of nondegenerate deformations of $f$ with the diagram $\Sigma$. In particular, all deformations from $X(\Sigma)$ have the same Milnor number, compare equality (1). Let us consider sets $X_j^{\mathrm{nd}} \subset \mathbb{C}^{\mu(f)-1}$ of all non-degenerate deformations of $f$ with Milnor number equal to $j$. We have

$$X_j^{\mathrm{nd}} = \bigcup_{\nu(\Sigma)=j} X(\Sigma),$$

where the summation is over a finite set of diagrams. Recall that the set $X_j$ of all deformations of $f$ with Milnor number equal to $j$ is a superset of $X_j^{\mathrm{nd}}$.

Proposition 3.2. *The inequality $\Sigma \leq \Sigma'$ holds if and only if there exists an arc $\gamma : [0,1] \to \mathbb{C}^{\mu(f)-1}$ such that $\gamma((0,1]) \subset X(\Sigma)$ and $\gamma(0) \in X(\Sigma')$.*

*Moreover, if $\Sigma \leq \Sigma'$, for any nondegenerate $g$ with the diagram $\Gamma_{\mathrm{supp}\, g} = \Sigma'$ the arc can be chosen to be a segment with $\gamma(0) = g$.*

*Proof.* Assume that $\Sigma \leq \Sigma'$. Take any $g = f + \sum a_j m_j$ with $a \in X(\Sigma')$. Let $Y = \operatorname{supp}\Sigma \setminus \operatorname{supp} a$. Take $d \in X(\Sigma)$ such that $d_{|\Sigma'} = a_{|\Sigma'}$ and consider

$$f_t = g + t \sum_{m_i \in Y} d_i m_i.$$

Apart from a finite number of values of $t$, the singularity $f_t$ is nondegenerate with the diagram $\Sigma$. Therefore, for some positive $\epsilon$, if $0 < |t| < \epsilon$ then $f_t$ is a nondegenerate deformation of $g$ and $f$ as well.

On the other hand, if there is any arc satisfying the assertion, then $f_{\gamma(t)}$ is a nondegenerate deformation of $f_{\gamma(0)}$ for every $t$. Hence $\Sigma = \Gamma_{\operatorname{supp} f_{\gamma(t)}} \leq \Gamma_{\operatorname{supp} f_{\gamma(0)}} = \Sigma'$. ∎

As a conclusion of Proposition 3.2 we get

Corollary 3.3. *The closure of $X(\Sigma)$ contains $\bigcup_{\Gamma \geq \Sigma} X(\Gamma)$.*

Corollary 3.4. *The singularity $f$ lies in the closure of every $X(\Sigma)$ with $\Sigma < \Gamma_{\operatorname{supp} f}$. Moreover, for any direction $a \in X(\Sigma)$ there is a segment $f + t\sum a_i m_i$, $0 < |t| < \epsilon$, contained in $X(\Sigma)$.*

We know from [MW15] (see also [MW14] for a weaker result) that

Fact 3.5. *If $f$ is irreducible and non-degenerate, then all numbers between $\mu(f)$ and $\mu(f) - \lfloor q/p \rfloor \cdot (p-1)$ are Milnor numbers of non-degenerate deformations of $f$.*

In particular, Fact 3.5 implies that for the diagram $\Gamma_{\operatorname{supp}(f)}$ every number between $\nu(f)$ and $\nu(f) - \lfloor q/p \rfloor (p-1)$ is a Newton number for some diagram $\Sigma < \Gamma$.

Proposition 3.6. *The singularity $f$ lies in the closure of at least $\nu(\Gamma_{pq})/2 + p + q - \lfloor q/p \rfloor - 2$ different sets $X_j$.*

*Proof.* Indeed, use Theorem 2.12 and Corollary 3.4. ∎

Proposition 3.7. *The singularity $f$ lies in the closure of $X_j$ for all $j = \mu(f), \dots, \mu(f) - \lfloor q/p \rfloor (p-1)$.*

*Proof.* Use Fact 3.5 and Corollary 3.4. ∎

Proposition 3.8. *The sets $X_j^{\mathrm{nd}}$ have more than one connected component for $4 < j \leq p$, whereas $X^{\mathrm{nd}}_{\mu(f)-1}$ has only one connected component.*

*Proof.* Indeed, combine Propositions 3.2 and 2.4 to get the first part. To get the second one recall uniqueness in Bezout's equation (2) and Remark 2.5. ∎

Proposition 3.9. *Let $\gamma$ be an arc in the space of nondegenerate deformations of $f$ such that $\gamma(1) = f$ and the Milnor number does not increase along it. It can pass through at most $\frac{1}{2}(p+1)(q+1) - 1$ different sets $X(\Sigma)$.*

*Proof.* Use Theorem 2.12 and Proposition 3.2. ∎

The question whether there exists $j$ such that $f$ does not lie in the closure of $X_j$ remains open in general case (although it has been investigated by many, for good references see [Arn04]).

To end the discussion, we would like to point out that it is not hard to prove that the above results appropriately translate as properties of deformations of any singularity $F$ with diagram $\Gamma_{\mathrm{supp}(F)} \geq \Gamma_{\mathrm{supp}(f)}$.

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