

## ON METRIC $\sigma$ -DISCRETE SPACES

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**Abstract.** By studying dimensional types of metric scattered spaces, we consider the wider class of metric  $\sigma$ -discrete spaces. Applying techniques relevant to this wider class, we present new proofs of some embeddable properties of countable metric spaces in such a way that they can be generalized onto uncountable metric scattered spaces. Related topics are also explored, which gives a few new results.

**1. Introduction.** Suppose  $X$  and  $Y$  are topological spaces. The symbol  $X <_E Y$  means that  $X$  is homeomorphic to a subspace of  $Y$ . If  $X <_E Y$ , then we say that  $X$  has a dimensional type smaller than or equal to the dimensional type of  $Y$ . If  $X <_E Y$  and  $Y <_E X$ , then  $X$  and  $Y$  have the same dimensional type, briefly  $X =_E Y$ . If  $X <_E Y$  and the inequality  $Y <_E X$  is not fulfilled, then  $X$  has a smaller dimensional type than  $Y$ . First time the relation  $<_E$  was investigated by M. Fréchet [4]. In [16, p. 24] W. Sierpiński cites alternative names for dimensional types: *type de dimensions*, Fréchet; *Homöie*, Mahlo. Basic properties and definitions relating to dimensional types are also discussed in textbooks [17], [11] and [12]. K. Kuratowski uses the name *topological rank* for dimensional type, [11, p. 112]. The following results are widely known.

In [13] S. Mazurkiewicz and W. Sierpiński proved the following two facts. *There are continuum many non-homeomorphic countable metric and scattered spaces. A countable compact metric space  $X$  is homeomorphic to the ordinal  $\omega^\alpha n + 1$ .* In the second claim  $n = |X^{(\alpha)}|$  is a natural number and  $X^{(\alpha)}$  is the first discrete derivative of  $X$ , where  $\alpha \in \omega_1$ . The countable ordinal  $\omega^\alpha n + 1$  is equipped with the order topology.

B. Knaster and K. Urbanik [9]: *Any countable metric scattered space has a metric scattered compactification.* An alternative proof is given in the monograph [11, Theorem 6, p. 25].

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R. Telgársky [18, Theorem 9]: *Any metric scattered space can be embedded into a sufficiently large ordinal number.* Independently, the same is also proved in [1].

The poset  $(P(\mathbb{Q}), <_E)$ , where  $P(\mathbb{Q})$  is the family of all subsets of the rational numbers  $\mathbb{Q}$ , is described by W. D. Gillam in the paper [5]. The set  $\mathcal{P}(\mathbb{Q})/=_E$  of all equivalence classes  $[X] = \{Y \subseteq \mathbb{Q} : Y =_E X\}$  is partially ordered by the relation  $[X] \leq_d [Y]$  whenever  $X <_E Y$ . In [5], it is shown that the poset  $(\mathcal{P}(\mathbb{Q})/=_E, \leq_d)$  has cardinality  $\omega_1$  and  $[\mathbb{Q}]$  is the only element with  $\omega_1$  many elements below it. Moreover,  $(\mathcal{P}(\mathbb{Q})/=_E, \leq_d)$  lacks infinite anti-chains and infinite strictly descending chains. We propose a new approach to these facts, examining  $k$ -stable dimensional types.

For any space  $X$ , the  $\alpha$ -derivative of  $X$ , which is denoted by  $X^{(\alpha)}$ , is defined inductively:  $X^{(0)} = X$ ;

$$X^{(\alpha+1)} = \{x \in X^{(\alpha)} : x \text{ is not isolated in } X^{(\alpha)}\}; \quad X^{(\alpha)} = \bigcap \{X^{(\beta)} : \beta < \alpha\}$$

for a limit ordinal  $\alpha$ . Thus, each  $X^{(\alpha)}$  is a closed subset of  $X$ . If there exists an ordinal  $\alpha$  such that  $X^{(\alpha)} = \emptyset$ , then  $X$  is called a *scattered* space. The smallest ordinal such that  $X^{(\alpha)} = \emptyset$  is denoted by  $N(X)$  and is called the Cantor–Bendixson *rank* of  $X$ . Other notions of set theory and topology will be used according to textbooks [2] and [10]. In particular, we use the sum of topological spaces like in the book [2, p. 103].

The paper is organized as follows. First, we review some well-known facts concerning ordinal arithmetic. Just after, there are formulated and proved the results about metric  $\sigma$ -discrete spaces. In particular, we show that any metric  $\sigma$ -discrete space is totally paracompact. Then, we adapt the original idea of S. Mazurkiewicz and W. Sierpiński, relying on the use of ordinal arithmetic: In our opinion, this idea is not quite well known and explored; at least we have not found any relevant studies undertaken in this direction, except of [5]. Related topics are discussed in final sections.

**2. Remarks on ordinal arithmetic.** Ordinal arithmetic is comprehensively described in a few textbooks of modern set theory, compare [7, p. 23] or [10, p. 26]. So, we only briefly discuss some aspects, adapting them to our purposes. For example, our definition of the ordinal exponentiation  $\omega^\alpha$  looks unusual, but it improves understanding of several proofs. Topological properties of subsets of ordinals will be considered only with the order topology, i.e. the topology generated by open rays  $\{\beta : \beta < \alpha\}$  and  $\{\beta : \beta > \alpha\}$ , where  $\alpha$  is an ordinal. So, we reconsider schemes of ordinal arithmetic, which were used in the paper by S. Mazurkiewicz and W. Sierpiński [13]. For ordinal numbers, we will use the convention  $\alpha = \{\beta : \beta < \alpha\}$ . If  $\beta \in \alpha$ , we write  $\beta < \alpha$ , except for phrases  $n \in \omega$ , where  $n$  is a finite ordinal and  $\omega$  is the first infinite ordinal. Suppose  $\alpha$  and  $\beta$  are ordinals, then  $\alpha + \beta$  is the unique ordinal  $\gamma$  which is isomorphic to a copy of  $\alpha$  followed by a copy of  $\beta$ . The addition of ordinals is associative, but not commutative. Also  $\beta < \alpha$  implies  $\beta + \gamma \leq \alpha + \gamma$ , for any ordinal  $\gamma$ . The ordinal  $\gamma$  added  $n$ -times is denoted by  $\gamma \cdot n$ . If  $\{\lambda_n : n \in \omega\}$  is a sequence of ordinals, then

$$\sum_{n \in \omega} \lambda_n = \sup\{\lambda_0 + \lambda_1 + \dots + \lambda_n : n \in \omega\}.$$

The following limit ordinals are important because of the above-mentioned Mazurkiewicz–Sierpiński theorem. Put  $\omega^0 = 1$ ,  $\omega^1 = \omega$  and define the countable limit ordinal

$$\omega^\alpha = \sup\{\omega^\beta \cdot n : \beta < \alpha \text{ and } 0 < n \in \omega\},$$

for each countable ordinal  $\alpha$ . If  $\beta < \omega^\alpha$ , then the interval  $(\beta, \omega^\alpha)$  is isomorphic to  $\omega^\alpha = [\emptyset, \omega^\alpha)$  and also these intervals are homeomorphic. If  $\beta < \omega^\alpha$ , then

$$\omega^\alpha + 1 = \beta + \omega^\alpha + 1 =_E \omega^\alpha + \beta + 1.$$

If  $\gamma > \sup \gamma$  is a countable infinite ordinal, then there exist  $n \in \omega$  and an ordinal  $\alpha$  such that  $\omega^\alpha \cdot n + 1 =_E \gamma$ . If  $\gamma$  is a limit ordinal, then there exist  $n \in \omega$  and ordinals  $\alpha$  and  $\beta$  such that the subspace  $\omega^\alpha \cdot n + 1 \setminus \{\beta\} \subseteq \omega^\alpha \cdot n + 1$  is homeomorphic to  $\gamma$ . We omit details of above-mentioned facts. Instead of this, we present the following claim:

*If  $0 < \alpha$ , then  $N(\omega^\alpha) = \alpha$  and  $N(\omega^\alpha + 1) = \alpha + 1$ .*

To prove this claim observe that if  $\alpha = 1$ , then  $\omega + 1$  is homeomorphic to a convergence sequence. So,  $(\omega + 1)^{(1)} = \{\omega\}$  and  $(\omega)^{(1)} = \emptyset$ , hence  $N(\omega) = 1$  and  $N(\omega + 1) = 2$ . Suppose that the assertion holds for all non-zero  $\beta < \alpha$ . If  $\alpha = \beta + 1$ , using the induction assumptions, we get

$$(\omega^\alpha + 1)^{(\beta)} = \{\omega^\beta \cdot n : 0 < n \in \omega\} \cup \{\omega^\alpha\} =_E \omega + 1.$$

Therefore  $(\omega^\alpha + 1)^{(\alpha)} = \{\omega^\alpha\}$  and  $(\omega^\alpha)^{(\alpha)} = \emptyset$ . Hence  $N(\omega^\alpha) = \alpha$  and  $N(\omega^\alpha + 1) = \alpha + 1$ .

Suppose  $\omega^\alpha = \sum_{n \in \omega} \omega^{\beta_n}$ , where  $\alpha = \sup_{n \in \omega} \beta_n$  is a limit ordinal. For any  $\beta < \alpha$ , by the induction assumptions, we have

$$(\omega^\beta)^{(\alpha)} = \emptyset \text{ and } \omega^\alpha \in (\omega^\alpha + 1)^{(\beta)}.$$

Bearing this in mind, we check that

$$(\omega^\alpha)^{(\alpha)} = \bigcup \{(\omega^{\beta_n})^{(\alpha)} : n \in \omega \text{ and } \beta_n < \alpha\} = \emptyset.$$

We still have  $\omega^\alpha \in (\omega^\alpha + 1)^{(\beta_n)}$ , therefore  $(\omega^\alpha + 1)^{(\alpha)} = \{\omega^\alpha\}$ .

**3. On  $\sigma$ -discrete metric spaces.** A metric space is called  $\sigma$ -discrete, if it is a union of countably many discrete subspaces. Any countable metric space, being a countable sum of single points, is  $\sigma$ -discrete. In particular, the space  $\mathbb{Q}$  of all rational numbers is  $\sigma$ -discrete.

LEMMA 1. *Each metric  $\sigma$ -discrete space  $X$  is a union of countably many closed and discrete subspaces.*

*Proof.* Use the Bing theorem [2, 4.4.8] in the following way. Let

$$\mathcal{B} = \bigcup \{\mathcal{B}_n : n \in \omega\}$$

be a  $\sigma$ -discrete base for  $X$ , where each  $\mathcal{B}_n$  is a discrete family. Let  $X_0, X_1, \dots$  be discrete subspaces summing  $X$ . If  $x \in X_k$ , then fix  $V_x^m \in \mathcal{B}_m$  such that  $V_x^m \cap X_k = \{x\}$ . If there is no relevant  $V_x^m$ , then put  $V_x^m = \emptyset$ ,

$$X_{k,m} = X_k \cap \bigcup \{V_x^m : x \in X_k \text{ and } V_x^m \in \mathcal{B}_m\}$$

and then check that sets  $X_{k,m}$  are such as we need. ■

Let  $B(\mathfrak{m}) = \mathfrak{m}^\omega$  be the Baire space of weight  $\mathfrak{m}$ , where  $\mathfrak{m}$  is an infinite cardinal. Since  $0 \in \mathfrak{m}$ , we can put

$$C(\mathfrak{m}) = \{y \in B(\mathfrak{m}) : \text{almost all coordinates of } y \text{ are equal to } 0\}$$

and consider  $C(\mathfrak{m})$  with the topology inherited from  $B(\mathfrak{m})$ . Each Baire space  $B(\mathfrak{m})$  is metric and each  $C(\mathfrak{m})$  is a  $\sigma$ -discrete metric subspace. Note that  $C(\omega_0)$  is a homeomorphic copy of the rational numbers and the Baire space  $B(\omega_0)$  is homeomorphic to the irrational numbers. Therefore is why the next proposition says that spaces  $C(\mathfrak{m})$  are analogues of the rational numbers. A characterization of the rational numbers generalized by the next proposition is usually attributed to G. Cantor, L. E. J. Brouwer or W. Sierpiński. Let us note that *a metric  $\sigma$ -discrete space contains a homeomorphic copy of the rational numbers or it is scattered*. Also: *A nonempty metric  $\sigma$ -discrete space, with all nonempty open subsets of weight  $\mathfrak{m}$ , is homeomorphic to  $C(\mathfrak{m})$ . A metric  $\sigma$ -discrete space of the weight  $\mathfrak{m}$  is homeomorphic to a subspace of  $C(\mathfrak{m})$* . The first claim follows from the fact that a metric  $\sigma$ -discrete space which is not scattered has to contain a subspace dense in itself and therefore, being metric, it contains a copy of the rational numbers. Proofs of the remaining claims can be found in [15] or [14].

**THEOREM 2.** *Any metric scattered space is  $\sigma$ -discrete.*

*Proof.* Applying metrization theorems — for example the Stone theorem, compare [2, 4.4.1] — one obtains the following. A metric locally  $\sigma$ -discrete space is  $\sigma$ -discrete. Then one can check that if a metric space  $X$  is not  $\sigma$ -discrete, then the set

$$\{x \in X : \text{no neighborhood of } x \text{ is } \sigma\text{-discrete}\}$$

is dense in itself. ■

After the lecture presented by the second author at a conference in Hejnice, K. P. Hart offered the following reasoning: Let  $(X, \varrho)$  be a metric scattered space. For every  $x \in X$ , let  $\alpha_x$  be the ordinal such that  $x \in X^{(\alpha_x)}$  and  $x \notin X^{(\alpha_x+1)}$ , and then fix a natural number  $n_x$  such that  $B(x, \frac{1}{n_x}) \cap X^{(\alpha_x)} = \{x\}$ . Finally put

$$D_n = \{x \in X : n_x = n\}.$$

If  $x, y \in D_n$  and  $x \neq y$ , then  $\varrho(x, y) \geq \frac{1}{n}$ . So,  $X$  is the countable union of closed and discrete sets  $D_n$ . Thus, he finds a more elementary reasoning than we have proposed.

Each metric  $\sigma$ -discrete space is paracompact in a stronger sense.

**THEOREM 3.** *Every open cover  $\mathcal{U}$  of a metric  $\sigma$ -discrete space  $X$  has a disjoint open refinement.*

*Proof.* Modifying Engelking’s reasoning 1.3.2 from [3], one can obtain the following. If a normal space is a union of countably many closed and discrete subspaces, then it has a base consisting of closed-open sets. So, any metric  $\sigma$ -discrete space has a base consisting of closed-open sets. Let closed and discrete sets  $X_{k,m} \subseteq X$  be defined as in the proof of Lemma 1. Fix  $k$  and  $m$ . The family

$$\{V_x^m : x \in X_{k,m}\} \subseteq \mathcal{B}_m$$

is discrete. So, we can choose the closed-open sets  $W_x^m \subseteq W \in \mathcal{U}$  and  $W_x^m \subseteq V_x^m$ , for each  $x \in X_{k,m}$ , such that

$$X_{k,m} \subseteq \bigcup \{W_x^m : x \in X_{k,m}\}$$

and the union  $\bigcup \{W_x^m : x \in X_{k,m}\}$  is closed-open. Note that the family  $\{W_x^m : x \in X_{k,m}\}$ , being discrete, consists of pairwise disjoint sets. Enumerate the sets  $X_{k,m}$  as  $\{Y_n : n \in \omega\}$ . Let  $\mathcal{W}_0 = \{W_x^m : x \in Y_0 = X_{k,m}\}$ . If  $X_{k,m} = Y_n$  and the families of closed-open sets  $\mathcal{W}_0, \mathcal{W}_1, \dots, \mathcal{W}_{n-1}$  are already defined such that the unions  $\bigcup \mathcal{W}_0, \bigcup \mathcal{W}_1, \dots, \bigcup \mathcal{W}_{n-1}$  are closed-open sets, then let  $\mathcal{W}_n$  be the family

$$\{W_x^m \setminus \bigcup \{\bigcup \mathcal{W}_i : i < n\} : x \in Y_n \text{ and } x \notin \bigcup \{\bigcup \mathcal{W}_i : i < n\}\}.$$

The union  $\bigcup \{\mathcal{W}_n : n \in \omega\}$  is a needed refinement of  $\mathcal{U}$ . ■

**COROLLARY 4.** *Any metric scattered space can be represented as the sum of subspaces, each one with one-point last derivative.*

*Proof.* Assume that  $X$  is a metric space such that  $N(X) = \alpha$ . Let  $\mathcal{U}$  be an open cover such that if  $A \in \mathcal{U}$ , then there exist  $\beta < \alpha$  such that  $A \subseteq X \setminus X^{(\beta+2)}$  and  $A \cap X^{(\beta+1)}$  is a one-point set. By Theorem 3, this cover has a disjoint open refinement  $\mathcal{V}$ . Let  $\mathcal{V}_0^* \subseteq \mathcal{V}$  consists of open sets with one-point last derivatives. Thus  $N(X \setminus \bigcup \mathcal{V}_0^*) < \alpha$  and we repeat the procedure with  $X \setminus \bigcup \mathcal{V}_0^*$  instead of  $X$ . After all possible repetitions (finitely many) we get families of open sets  $\mathcal{V}_0^*, \mathcal{V}_1^*, \dots, \mathcal{V}_m^*$  such that the sum  $\bigoplus_{A \in \mathcal{V}_0^* \cup \mathcal{V}_1^* \cup \dots \cup \mathcal{V}_m^*} A$ , which is homeomorphic to  $X$ , is such as we need. ■

We get a modification of the Telgársky result, see [18, Theorem 3].

**THEOREM 5.** *Every base of a metric  $\sigma$ -discrete space contains a locally finite open refinement.*

*Proof.* Let  $X$  be a metric  $\sigma$ -discrete space such that  $X = X_0 \cup X_1 \cup \dots$ , where subspaces  $X_n$  are closed, discrete and pairwise disjoint. Fix a base  $\mathcal{B}$ . Afterward, apply the following algorithm. Choose a cover  $\mathcal{U} \subseteq \mathcal{B}$  such that if  $x \in X_0$  and  $x \in A \in \mathcal{U}$ , then  $A \cap X_0 = \{x\}$ . By Theorem 3, the cover  $\mathcal{U}$  has a disjoint open refinement  $\mathcal{W}$ . Choose a refinement  $\mathcal{U}^* \subseteq \mathcal{B}$  and a disjoint open refinement  $\mathcal{W}^*$  such that

$$\mathcal{W}^* \prec \mathcal{U}^* \prec \mathcal{W} \prec \mathcal{U}.$$

Without loss of generality, we can assume that there exists an unique neighborhood  $A_x \in \mathcal{U}^*$  such that  $A_x \cap X_0 = \{x\}$ , for each  $x \in X_0$ . Let  $\mathcal{U}^0$  be the family of all such selected sets  $A_x$ . Thus,  $\mathcal{U}^0$  and

$$\mathcal{V}^0 = \{A \in \mathcal{W}^* : A \cap X_0 \neq \emptyset\}$$

are discrete families. Since  $\mathcal{V}^0 \subseteq \mathcal{W}^*$ , the union of all elements of  $\mathcal{V}^0$  is a closed-open set.

Assume that discrete families  $\mathcal{U}^0, \mathcal{U}^1, \dots, \mathcal{U}^n$  and  $\mathcal{V}^0, \mathcal{V}^1, \dots, \mathcal{V}^n$  are already defined such that the union  $Y = \bigcup \{A \in \mathcal{V}^k : 0 \leq k \leq n\}$  — of closed-open and pairwise disjoint sets — is a closed-open subset of  $X$ . Repeat the above algorithm by substituting  $X \setminus Y$  for  $X$  and  $\{A \in \mathcal{B} : A \subseteq X \setminus Y\}$  for  $\mathcal{B}$  and  $X_{n+1} \setminus Y$  for  $X_0$ . As a result, we get a discrete family  $\mathcal{U}^{n+1} \subseteq \mathcal{B}$  and a discrete family  $\mathcal{V}^{n+1}$  consisting of pairwise disjoint closed-open sets. From the properties of our algorithm we see that  $\mathcal{U} = \mathcal{U}^0 \cup \mathcal{U}^1 \cup \dots \subseteq \mathcal{B}$  is a locally

finite cover of  $X$ . Indeed, the family  $\mathcal{V}^0 \cup \mathcal{V}^1 \cup \dots$  is a disjoint open refinement of  $\mathcal{U}$ . Also, each  $A \in \mathcal{V}^n$  meets no element of  $\mathcal{U}^k$ , for  $k > n$ . Thus, if  $x \in A \in \mathcal{V}^n$ , then there exist open neighborhoods  $B_0, B_1, \dots, B_{n-1}$  of  $x$  such that any  $B_k$  meets at most one element of the discrete family  $\mathcal{U}^k$ . Therefore, the intersection  $A \cap B_0 \cap B_1 \cap \dots \cap B_{n-1}$  meets at most finitely many elements of the cover  $\mathcal{U} \subseteq \mathcal{B}$ . ■

In other words, Theorem 5 says that metric  $\sigma$ -discrete spaces are totally paracompact. R. Telgársky [18] only shows that metric scattered spaces are totally paracompact, so we receive a little stronger result. Simplified versions of Theorem 3 are applied in papers by V. Kannan and M. Rajagopalan [8], A. Arosio and A. V. Ferreira [1] and R. Telgársky [18]. Note that, for similar facts, the phrase *Every finite open cover of ...* is applied in textbooks on dimension theory, for example in [3] or [2].

Let  $\{X_\beta : \beta < \alpha\}$  be a family of scattered spaces such that  $X_\beta^{(\beta)} = \{g_\beta\}$ . If additionally  $\alpha$  is a limit ordinal, then let  $J(\{X_\beta : \beta < \alpha\})$  be the hedgehog space with spininess  $X_\beta$ . The hedgehog space is formed by gluing points  $g_\beta$  into the point  $g$ . The metric is determined such that points of  $X_\beta$  are at the same distance as in  $X_\beta$ , but the distance between points from different spininess is obtained by the addition of distances of these points from  $g$ . Since  $J(\{X_\beta : \beta < \alpha\})^{(\alpha)} = \{g\}$ , this hedgehog space is metric scattered with one-point  $\alpha$ -derivative.

**PROPOSITION 6.** *For any ordinal  $\alpha$  there exists a metric scattered space with one-point  $\alpha$ -derivative.*

*Proof.* If  $\alpha \in \omega_1$ , then the ordinal  $\omega^\alpha + 1$  satisfies the assertion. Suppose that for each  $\beta < \alpha$  there exists a metric scattered space  $Y_\beta$  such that  $Y_\beta^{(\beta)} = \{g_\beta\}$ . If  $\alpha = \beta + 1$ , then put

$$X = Y_\beta \times (\omega + 1) \setminus \{(y, \omega) : y \in Y_\beta \text{ and } y \neq g_\beta\}.$$

When  $X$  is equipped with the topology inherited from the product topology, then  $X$  is a metric space such that  $X^{(\alpha)} = \{(g_\beta, \omega)\}$ . If  $\alpha$  is a limit ordinal, then we construct  $X$  adapting the construction of a hedgehog space, compare [2, 4.1.5]. For  $\beta < \alpha$ , spaces  $Y_\beta$  are homeomorphic to spininess of the hedgehog space  $X$  and the point formed by gluing points  $g_\beta$ , will be the only point in the space  $X$  belonging to its  $\alpha$ -derivative. ■

**COROLLARY 7.** *Let  $\mathfrak{m}$  be an infinite cardinal and  $\alpha$  be an ordinal such that  $\mathfrak{m} \leq \alpha < \mathfrak{m}^+$ . Then there exists a metric scattered space of the cardinality  $\mathfrak{m}$  which has nonempty  $\alpha$ -derivative.*

*Proof.* Each metric scattered space  $X$ , where  $X^{(\alpha)} \neq \emptyset$  and  $\mathfrak{m} \leq \alpha < \mathfrak{m}^+$ , from the above proposition can be constructed to have the cardinality  $\mathfrak{m}$ . ■

**4. On proofs of Mazurkiewicz–Sierpiński and Knaster–Urbanik theorems.** Let us demonstrate how to use the Telgársky idea — modified here as Theorem 3 and Corollary 4, to simplify a proof of the Mazurkiewicz–Sierpiński theorem: *If  $X$  is a countable compact metric space, then  $X$  is homeomorphic to the ordinal  $\omega^\alpha n + 1$ , where  $\alpha < \omega_1$  and  $n \in \omega$  are uniquely determined.* Assume that  $X$  is a countable compact metric space. If the derivative  $X^{(1)}$  is empty, then  $X$  has to be finite since it is compact, hence  $X$  is homeomorphic to the ordinal  $\omega^0 \cdot |X| = 1 \cdot |X|$ . If  $|X^{(1)}| = n$ , where  $0 < n \in \omega$ , then

$X$  has to be the sum of  $n$  copies of a convergent sequence, hence  $X$  is homeomorphic to the ordinal  $\omega \cdot n + 1$ . Assume inductively that if  $N(X) \leq \alpha$ , then  $X$  is homeomorphic to the ordinal  $\omega^\beta \cdot n + 1$ , where  $\beta < \alpha$  and  $n \in \omega$ . Now suppose that  $|X^{(\alpha)}| = 1$ . By Theorem 3 — the difference  $X \setminus X^{(\alpha)}$  is an infinite sum of pairwise disjoint closed-open subsets, each one has empty  $\alpha$ -derivative. The subspace  $X \setminus X^{(\alpha)}$  is homeomorphic to the sum

$$(\omega^{\beta_0} \cdot n_0 + 1) \oplus (\omega^{\beta_1} \cdot n_1 + 1) \oplus \dots,$$

by the induction conditions. If  $\alpha = \gamma + 1$ , then one can assume that every  $\beta_n = \gamma$ . If  $\alpha$  is a limit ordinal, then every  $\beta_n < \alpha$  and  $\lim_{n \rightarrow \infty} \beta_n = \alpha$ . In both cases we see that  $X$  is homeomorphic to  $\omega^\alpha + 1$ . If  $|X^{(\alpha)}| = n \in \omega$ , then  $X$  has a finite open cover  $\mathcal{U}$  such that each  $V \in \mathcal{U}$  meets  $X^{(\alpha)}$  at a single point and the members of  $\mathcal{U}$  are pairwise disjoint. Therefore  $X$  is homeomorphic to the sum of  $(\omega^\alpha + 1)$  taken  $n$ -times and consequently  $X$  is homeomorphic to  $\omega^\alpha \cdot n + 1$ .

Recall that B. Knaster and K. Urbanik [9] proved that any countable metric scattered space is homeomorphic to a subset of a countable ordinal. Therefore, it has a metric scattered compactification, which is a closed subset of some  $\beta + 1$ , where  $\beta < \omega_1$ . A proof that any countable metric scattered space has a countable metric compactification, which is scattered, was also presented in [12, p. 25]. For compact  $X$ , the proof by S. Mazurkiewicz and W. Sierpiński indicates the smallest ordinal number in which  $X$  can be embedded. For any countable metric scattered space a similar indication is not explicitly described. So, let us describe the ordinals which are essential for the induction proof of the Knaster–Urbanik theorem. When  $\alpha$  is a countable ordinal, let  $E(\alpha)$  be the least ordinal such that any countable metric scattered space with one-point  $\alpha$ -derivative can be embedded into  $E(\alpha)$ . Thus  $E(0) = 1$  and  $E(1) = \omega^2 + 1$ , and also  $E(m) = \omega^{2m} + 1$  for any  $m \in \omega$ . In fact, we have the following version of Gillam’s Lemma 8, see [5].

PROPOSITION 8. *If  $m \in \omega$ , then  $E(m) \leq \omega^{2m} + 1$ .*

*Proof.* Suppose  $X$  is a countable metric space such that  $X^{(1)} = \{g\}$ . Let  $\{U_n : n \in \omega\}$  be a decreasing base at the point  $g$ . Then any one-to-one function  $f : X \rightarrow \omega^2 + 1$  such that  $f(g) = \omega^2$  and any image  $f[U_n \setminus U_{n+1}]$  is contained in the interval  $(\omega \cdot n, \omega \cdot (n + 1))$  has to be an embedding of  $X$  into  $\omega^2 + 1$ . Therefore  $E(1) \leq \omega^2 + 1$ .

Assume that if  $Y$  is a countable metric space such that  $Y^{(m-1)} = \{h\}$ , then  $\omega^{2m-2} + 1$  contains a homeomorphic copy of  $Y$  such that the point  $h$  corresponds to the ordinal  $\omega^{2m-2}$ . Suppose  $X$  is a countable metric space such that  $X^{(m)} = \{g\}$ . Choose a family  $\{U_n : n \in \omega\}$  of closed-open sets such that it is a decreasing base at the point  $g$  and each set  $U_n \setminus U_{n+1}$  intersects  $X^{(m-1)}$ . By Corollary 4, each  $U_n \setminus U_{n+1}$  is a union of pairwise disjoint closed-open sets  $Y_{n,k}$  such that

$$Y_{n,k} \cap X^{(m-1)} = \{g_{n,k}\}.$$

Based on induction assumptions, there exist embeddings

$$f_{n,k} : Y_{n,k} \rightarrow \omega^{2m-2} + 1$$

such that each point  $g_{n,k}$  corresponds to the ordinal  $\omega^{2m-2}$ . Line up images  $f_{n,k}[Y_{n,k}]$  such that  $f_{n,i}[Y_{n,i}]$  is followed by  $f_{n,i+1}[Y_{n,i+1}]$ , for  $i \in \omega$ . We get embeddings

$f_n : U_n \setminus U_{n+1} \rightarrow \omega^{2m-1}$ . Again, line up images  $f_n[U_n \setminus U_{n+1}]$  and ordinals  $\{\omega^{2m-1} \cdot k : 0 < k \in \omega\}$  such that  $f_0[U_0 \setminus U_1]$  is followed by  $\{\omega^{2m-1}\}$  followed by  $f_1[U_1 \setminus U_2]$  followed by  $\{\omega^{2m-1} \cdot 2\}$  followed by  $f_2[U_2 \setminus U_3]$  and so on. Except for  $n = 0$ , we have

$$f_n[U_n \setminus U_{n+1}] \subset [\omega^{2m-1} \cdot n + 1, \omega^{2m-1} \cdot (n + 1)] = (\omega^{2m-1} \cdot n, \omega^{2m-1} \cdot (n + 1) + 1).$$

This means that the images  $f_n[U_n \setminus U_{n+1}]$  are contained in pairwise disjoint closed-open intervals. So, we get the embedding  $f : X \rightarrow \omega^{2m} + 1$ , as far as we put  $f(g) = \omega^{2m}$ . Therefore  $E(m) \leq \omega^{2m} + 1$ . ■

**COROLLARY 9.** *If  $m \in \omega$ , then  $E(m) = \omega^{2m} + 1$ .*

*Proof.* Let  $X(1) = \omega^2 + 1 \setminus \{\omega \cdot k : k \in \omega\}$ . So  $X(1)^{(1)} = \{\omega^2\}$ . Without loss of generality, we can assume that  $f : X(1) \rightarrow \omega^2 + 1$  is an embedding such that

$$\beta = f(\omega^2) = \sup f[X(1)].$$

Put  $\beta_1 = \sup f[\omega]$  and  $f[X(1)] \cap [0, \beta_1] = A_1$ . Since  $A_1$  is infinite, we have  $\beta > \beta_1 \geq \omega$ . Inductively assume that ordinals  $\beta_1, \beta_2, \dots, \beta_{n-1}$  and discrete infinite subspaces  $A_1, A_2, \dots, A_{n-1} \subset f[X(1)]$  are already defined and  $\beta_k = \sup A_k \geq \omega \cdot k$ , for  $0 < k < n$ . Choose an infinite and discrete subspace

$$A_n \subseteq f[X(1)] \cap (\beta_{n-1}, \beta)$$

and put  $\beta_n = \sup A_n$ . Assuming inductively that  $\beta_{n-1} \geq \omega \cdot (n - 1)$  we get  $\beta_n \geq \omega \cdot n$ . This implies  $\omega^2 \leq \lim_{n \rightarrow \infty} \beta_n \leq \beta$ . Therefore  $E(1) = \omega^2 + 1$ .

Let  $m > 1$ . Assume that the space  $X(m - 1) \subseteq \omega^{2m-2} + 1$  is already defined such that  $X(m - 1)^{(m-1)} = \{\omega^{2m-2}\}$  and  $X(m - 1)$  cannot be embedded into  $\beta < \omega^{2m-2}$ . Take a countable infinite family  $\mathcal{S}$  consisting of copies  $X(m - 1)$ . Let  $X(m) = \bigcup \mathcal{S} \cup \{g\}$  be equipped with the topology, where  $\bigcup \mathcal{S}$  inherits the sum topology and the point  $g$  has a decreasing base of neighborhoods  $\{U_n : n \in \omega\}$  such that each  $U_n \setminus U_{n+1}$  is the union of an infinite many copies of  $X(m - 1)$ . By the definition,  $X(m)$  can be embedded into  $\omega^{2m} + 1$  such that the point  $g$  corresponds to the ordinal  $\omega^{2m}$ . Without loss of generality, we can assume that  $f : X(m) \rightarrow \omega^{2m} + 1$  is an embedding such that  $\beta = f(g) = \sup f[X(m)]$ . Put  $\beta_0 = \sup f[U_0 \setminus U_1]$  and  $f[X(m)] \cap [0, \beta_0] = A_0$ . By the induction assumptions, we get  $\beta > \beta_0 \geq \omega^{2m-1}$ . Inductively assume that ordinals  $\beta_0, \beta_1, \dots, \beta_{n-1}$  and subspaces  $A_0, A_1, \dots, A_{n-1} \subset f[X(m)]$  are already defined such that

$$\beta > \beta_k = \sup A_k \geq \omega^{2m-1} \cdot (k + 1),$$

for each  $k < n$ . Let  $A_n \subset f[X(m)] \cap (\beta_{n-1}, \beta)$  be an infinite union of copies of  $X(m - 1)$  such that  $\beta > \sup A_n = \beta_n$ . Since  $\beta_{n-1} \geq \omega^{2m-1} \cdot n$  we get  $\beta_n \geq \omega^{2m-1} \cdot (n + 1)$ . This implies  $\omega^{2m} \leq \lim_{n \rightarrow \infty} \beta_n \leq \beta$ . Therefore  $E(m) = \omega^{2m} + 1$ . ■

The spaces  $X(m)$  defined in the above proof can be added the same way as ordinals, except that the result of such addition must be equipped with the inherited topology. However, such an extension rules seem to be a good topic for future research.



PROPOSITION 10. Let  $\alpha = \gamma + m$ , where  $m \in \omega$  and  $\gamma < \omega_1$  is a limit ordinal. Then  $E(\alpha) = \omega^{\gamma+2m+1} + 1$ .

*Proof.* Let the space  $X(\omega)$  be such that  $X(\omega)^\omega = \{g\}$ . Moreover, the point  $g$  has a decreasing base of neighborhoods  $\{U_n : n \in \omega\}$  such that each  $U_n \setminus U_{n+1}$  is an infinite sum of copies of  $X(k)$ , defined in the proof of Corollary 9, where  $k$  runs by infinitely many natural numbers. Without loss of generality, we can assume that  $f : X(\omega) \rightarrow \omega^{\omega+1} + 1$  is an embedding such that  $\beta = f(g) = \sup f[X(\omega)]$ . Put  $\beta_0 = \sup f[U_0 \setminus U_1]$  and  $f[X(\omega)] \cap [0, \beta_0] = A_0$ . By the induction assumptions, we have  $\beta > \beta_0 \geq \omega^\omega$ . Inductively assume that ordinals  $\beta_0, \beta_1, \dots, \beta_{n-1}$  and subspaces  $A_0, A_1, \dots, A_{n-1} \subset f[X(\omega)]$  are already defined and

$$\beta > \beta_{n-1} = \sup A_{n-1} \geq \omega^\omega \cdot n.$$

Let  $A_n \subset f[X(\omega)] \cap (\beta_{n-1}, \beta)$  be an infinite sum of copies of  $X(k)$ , where  $k$  runs by infinitely many natural numbers. We get

$$\beta > \sup A_n = \beta_n > \beta_{n-1} \text{ and } \beta_n \geq \omega^\omega \cdot (n + 1).$$

Assuming inductively that  $\beta_{n-1} \geq \omega^\omega \cdot (n - 1)$  we get  $\beta_n \geq \omega^\omega \cdot n$ . Therefore  $\omega^{\omega+1} \leq \lim_{n \rightarrow \infty} \beta_n \leq \beta$  and  $E(\omega) = \omega^{\omega+1} + 1$ . Similarly, one can prove that  $E(\gamma) = \omega^{\gamma+1} + 1$  for each limit ordinal  $\gamma < \omega_1$ . And also in analogy to the proof of Corollary 9, one can get  $E(\gamma + m) = \omega^{\gamma+2m+1} + 1$ , whenever  $m \in \omega$  and  $\gamma < \omega_1$  is a limit ordinal. ■

COROLLARY 11. If  $0 < \alpha < \omega_1$ , then any countable metric space with nonempty  $\alpha$ -derivative contains a homeomorphic copy of  $\omega^\alpha + 1$ .

*Proof.* Let  $X$  be a countable metric space. Without loss of generality, assume that  $X^{(\alpha)} = \{g\}$ . If  $\alpha = 1$ , then  $X$  contains a convergent sequence, which is homeomorphic to  $\omega + 1$ . Suppose, that the assertion holds for all  $\beta < \alpha$ . Fix a metric  $\rho$  on  $X$ . Choose nonempty closed-open sets  $V_n \subseteq X \setminus \{g\}$  such that  $V_n \subseteq B(g, \frac{1}{n})$ . By the induction assumptions each  $V_n$  contains a homeomorphic copy of  $\omega^{\beta_n} + 1$ , where  $\beta_n < \alpha$ . So, we choose copies of  $\omega^{\beta_n} + 1 \subseteq V_n$  such that  $\omega^\alpha = \sum_{n \in \omega} \omega^{\beta_n}$ . The sum of these copies plus point  $g$  gives a subspace homeomorphic to  $\omega^\alpha + 1$ . ■

Let  $X$  and  $Y$  be countable metric spaces. If  $\gamma < \omega_1$  is a limit ordinal and  $N(X) = \gamma = N(Y)$ , then  $X =_E Y$ . Indeed, by Corollary 4 we get  $X = \bigoplus_{n \in \omega} A_n$  and  $Y = \bigoplus_{n \in \omega} B_n$ , where each set  $A_n$  (also  $B_n$ ) has one-point last derivative. Since  $\sup_{n \in \omega} N(A_n) = \gamma = \sup_{n \in \omega} N(B_n)$ , using Propositions 10 and Corollary 11, we build successively embeddings of  $A_n$  into suitable  $B_{m_n}$ , consequently we get the embedding of  $X$  into  $Y$ . Analogously, an embedding of  $Y$  into  $X$  can be obtained. Denote  $D(\gamma)$  the only dimensional type referred to above. Also for  $\gamma$  being a limit ordinal, let  $G(\gamma)$  denote the dimensional type of a space  $X$  such that  $X^{(\gamma)} = \{g\}$  and the point  $g$  has a decreasing open base  $\{U_n : n \in \omega\}$  such that each  $U_n \setminus U_{n+1}$  has one-point last derivative. But  $I(\gamma)$  denotes the dimensional type of a space  $Y$  such that  $Y^{(\gamma)} = \{g\}$  and the point  $g$  has a decreasing open base  $\{U_n : n \in \omega\}$  such that any  $U_n \setminus U_{n+1}$  has the dimensional type  $D(\gamma)$ . Thus, if  $X^{(\gamma)} = \{g\}$ , then  $X$  has the dimensional type  $G(\gamma)$  or  $I(\gamma)$  or the dimensional type of the sum of two spaces one of them of the dimensional type  $G(\gamma)$  and the other of the dimensional type  $D(\gamma)$ .

**5. More on local embeddable properties.** Let  $\mathcal{A}$  be the poset consisting of dimensional types of countable metric spaces  $X$  with  $1 < N(X) \in \omega$ . Many properties of posets  $(\mathcal{P}(\mathbb{Q})/_{=E, \leq_d})$  and  $(\mathcal{P}(\mathbb{Q}), <_E)$  can be reduced to  $\mathcal{A}$ , as is observed in [5, pp. 69–81]. Now, our purpose is to create a basis for constructing a complete classification of metric scattered spaces. Till now, we do not know a good pattern of such a classification even for countable spaces. Invariants of the poset  $(\mathcal{P}(\mathbb{Q}), <_E)$ , these mentioned in [5], need not have nice wordings for  $(C(\mathfrak{m}), <_E)$ , whenever  $\mathfrak{m}$  is uncountable. It makes that our present description has a dispersed form. Despite this, let us discuss other local embeddable invariants, which are not mentioned in the paper [5], and simplify some of the proofs.

Assume that  $X$  is a countable metric space such that  $X^{(m)} = \{g\}$ , where  $0 < m \in \omega$ . We say that  $X$  has  $m$ -stable dimensional type if no  $Y \subseteq X$  has smaller dimensional type than  $X$ , whenever  $X \setminus Y$  is a closed-open set and  $g \in Y$ . There exist exactly two 1-stable dimensional types, i.e. the dimensional type of the convergent sequence  $G = \omega + 1$  and the dimensional type of the space with the single cluster point which has a base of open neighborhoods  $\{U_n : n \in \omega\}$  such that each difference  $U_n \setminus U_{n+1}$  is infinite and discrete, i.e.  $I = \omega^2 + 1 \setminus \{\omega, \omega \cdot 2, \omega \cdot 3, \dots\}$ . We leave it to the readers to check that there are exactly five 2-stable dimensional types. These are dimensional types of the following spaces:

- $\omega^2 + 1$ ;
- $\omega^3 + 1 \setminus \{\omega^2, \omega^2 \cdot 2, \omega^2 \cdot 3, \dots\}$ ;
- $\sum_\omega I + 1 \subset \omega^3 + 1$ , where the subspace is established as a sequence of  $I$  followed by a copy of  $I$  (infinitely many times) and with 1 at the end;
- $\sum_\omega I \oplus \sum_\omega I \oplus \sum_\omega I \oplus \dots + 1 \subset \omega^4 + 1$ , where the subspace is established as a sequence of  $\sum_\omega I$  followed by a copy of  $\sum_\omega I$  (infinitely many times) with 1 at the end and with the ordinals  $\omega^3, \omega^3 \cdot 2, \omega^3 \cdot 3, \dots$  thrown out;
- $\sum_\omega(\omega^2 \oplus I) + 1 \subset \omega^3 + 1$ , where the operation  $\sum_\omega(\dots) + 1$  is used as above and  $\omega^2 \oplus I \subset \omega^2 \cdot 2 + 1$  is the subspace established as a copy of  $\omega^2$  followed by a copy of  $I$  with  $\omega^2$  thrown out.

Let  $S_n$  be the family of all  $n$ -stable dimensional types, where  $0 < n$ . Thus,  $S_1$  consists of dimensional types of  $G$  and  $I$ . So, if  $N(X) = 2$ , then there exist countable sets  $A$  and  $B$  such that  $X$  is homeomorphic with the sum of  $\bigoplus_{n \in A} I_n$  and  $\bigoplus_{k \in B} G_k$  and  $D$ , where  $I_n =_E I$  and  $G_k =_E G$  occur permanently. In this sum  $D$  may be omitted, if  $A$  is nonempty or  $B$  is infinite.

**THEOREM 12.** *Each family  $S_n$  is finite. Also, if  $2 < N(X) \in \omega$ , then there exists  $m \in \omega$  such that  $X = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_m$ , wherein each  $\mathcal{V}_i$  is the sum of subspaces, each one with the same  $k$ -stable dimensional type.*

*Proof.* For  $n = 1$  and  $n = 2$  the case is settled, as we mentioned above. Assume that the families  $S_1, S_2, \dots, S_{n-1}$  are finite and each countable metric space  $Y$  such that  $N(Y) \leq n$  is a sum of subspaces with  $k$ -stable dimensional types. Without loss of generality, we can assume that  $X^{(n)} = \{g\}$ . By Theorem 3, the subspace  $X \setminus \{g\}$  can be divided into pairwise closed-open sets with Cantor–Bendixson rank  $n$ . From the induction assumptions we get

$$X \setminus \{g\} = \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \dots \oplus \mathcal{W}_m,$$

wherein each  $\mathcal{W}_i$  is the sum of subspaces, each one with the same  $k$ -stable dimensional type. Fix  $q \leq m$  such that  $g \in \text{cl}_X \mathcal{W}_i$ , for  $i \leq q$ , and  $g \notin \text{cl}_X \mathcal{W}_i$ , whenever  $q < i \leq m$ . Also, we can choose a decreasing base  $\{U_n : n \in \omega\}$  of open neighborhoods of the point  $g$  such that each  $U_n \setminus U_{n+1}$  contains a single closed-open set which is relevant for some  $\mathcal{W}_i$  or infinitely many such sets. Thus the sum  $\bigoplus_{i \leq q} \mathcal{W}_i \subseteq X$  has the  $n$ -stable dimensional type, since the dimensional type of  $\bigoplus_{i \leq q} \mathcal{W}_i$  depends only on whether any relevant dimensional type occurs only once in  $U_n \setminus U_{n+1}$  or at least infinitely many times. There are finitely many such opportunities and they produce all members of  $S_n$ . Applying the induction assumptions to  $\bigoplus_{q < i \leq m} \mathcal{W}_i \subseteq X$ , we finish the proof. ■

So, there exist countably many dimensional types of spaces with  $k$ -stable dimensional types. This implies that  $\mathcal{A}$  has to be countable. Indeed, assign to  $X \in \mathcal{A}$  the finite sequence  $(n_0, n_1, \dots, n_m)$ , where each  $n_i \in \omega + 1$  is the size of the collection of  $k$ -stable sets which form the sum  $\mathcal{V}_i$  referred in Theorem 12. Thus, we get a bijection of  $\mathcal{A}$  onto a countable family.

Now, we rearrange the notion of  $k$ -stable dimensional type. Namely, a metric space  $X$  has *stable dimensional type* if  $N(X) = \alpha + 1$  and  $X^{(\alpha)} = \{g\}$  and no  $Y \subseteq X$  has smaller dimensional type than  $X$ , whenever  $X \setminus Y$  is a closed-open set and  $g \in Y$ . Discuss briefly some applications of stable dimensional types in the context of countable metric spaces. If  $\gamma$  is the limit ordinal, then let  $\mathcal{A}(\gamma)$  be the poset consisting of dimensional types of countable metric spaces  $X$  with  $\gamma \leq N(X) < \gamma + \omega$ . Substituting  $D(\gamma), G(\gamma)$  and  $I(\gamma)$  instead of  $D, G$  and  $I$  respectively, we get a counterpart of Theorem 12: *Each family  $\mathcal{A}(\gamma)$  is countable. Also, if  $\gamma \leq N(X) < \gamma + \omega$ , then there exists  $m \in \omega$  such that  $X = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_m$ , wherein each  $\mathcal{V}_i$  is the sum of subspaces, each one with the same stable dimensional type.* This gives tools which absolutely reduce studying cardinal invariants of the poset  $(P(\mathbb{Q}), <_E)$  to well-known purely combinatorial objects. We leave the details to the readers. Although, we hope that such an approach will be explored elsewhere. We do not know whether the size of families  $S_n$  may be well bounded by a polynomial of  $n$ . Also we do not know what are the sizes of uncountable counterparts of  $S_n$ .

**6. Dimensional types of uncountable subspaces of  $\omega_1$ .** Let  $\mathbb{Y}$  be the sum of all countable ordinals. Thus,  $\omega_1$  contains a homeomorphic copy of  $\mathbb{Y}$ . Hence,  $\mathbb{Y}$  is a metric space which has a smaller dimensional type than the non-metric space  $\omega_1$ . The space  $\mathbb{Y}$  is special among the uncountable subspaces of  $\omega_1$ . Namely, if  $X \subseteq \omega_1$  is a metric subspace, then  $X <_E \mathbb{Y}$ . Indeed, take an open cover  $\mathcal{U}$  of  $X$ , which consists of countable sets. Then, use any disjoint open refinement of  $\mathcal{U}$  to construct a required embedding.

**PROPOSITION 13.** *If a subspace  $X \subseteq \omega_1$  contains a homeomorphic copy of any countable ordinal, then  $\mathbb{Y} <_E X$ .*

*Proof.* Let  $\mathbb{F} = \{\mathcal{I}_\alpha : \alpha < \omega_1\}$  be a family of closed and pairwise disjoint intervals of  $\omega_1$  such that each intersection  $X \cap \mathcal{I}_\alpha$  contains a homeomorphic copy of  $\omega^\alpha + 1$ . Then  $X \cap \bigcup \mathbb{F}$  contains a copy of  $\mathbb{Y}$ . ■

Recall that a set  $S \subseteq \omega_1$  is *stationary*, if  $S$  intersects any closed and unbounded subset of  $\omega_1$ , compare [10, p. 78]. The well-known Solovay’s result says that each stationary set can be divided into uncountably many stationary sets, compare [7]. Note that, if  $X \subseteq \omega_1$  is not stationary, then  $X$  is a metric  $\sigma$ -discrete space. Indeed, any complement of a closed unbounded set is a union of pairwise disjoint open intervals of ordinals. Each such interval has to be countable. Therefore  $X$  is contained in a sum of metric spaces. By Theorem 2, it has to be  $\sigma$ -discrete.

PROPOSITION 14. *If  $X \subset \omega_1$  is a discrete subspace, then  $X$  is not stationary.*

*Proof.* If  $X$  is bounded by an ordinal  $\alpha < \omega_1$ , then  $X$  is disjoint from the closed and unbounded interval  $(\alpha, \omega_1)$ , so we can assume that  $X$  is unbounded in  $\omega_1$ . Let  $\{(a_\alpha, b_\alpha) : \alpha \in X\}$  be an uncountable family of pairwise disjoint intervals such that  $X \cap (a_\alpha, b_\alpha) = \{\alpha\}$ , for each  $\alpha \in X$ . Without loss of generality, we can assume that  $\alpha < \beta$  implies  $a_\alpha < b_\alpha \leq a_\beta < b_\beta$ . We deduce that the complement of an open set

$$\bigcup \{(a_\alpha, b_\alpha) : \alpha \in X\} \supset X$$

is unbounded, because it contains  $\{b_\alpha : \alpha \in X\}$ . ■

PROPOSITION 15. *If  $X$  is a stationary set and  $\alpha < \omega_1$ , then  $X^{(\alpha)} \neq \emptyset$ .*

*Proof.* Suppose  $X^{(\alpha)} = \emptyset$ , where  $\alpha < \omega_1$ . Then

$$X = \bigcup \{X^{(\beta)} \setminus X^{(\beta+1)} : \beta < \alpha\}$$

is a union of countably many subspaces with discrete subspace topologies. By Proposition 14 and [10, p. 78], the set  $X$  cannot be stationary. ■

Following M. Ismail and A. Szymański [6], the *discrete metrizability number* of a space  $X$ , denoted by  $dm(X)$ , is the smallest cardinal number  $\kappa$  such that  $X$  can be represented as a union of  $\kappa$  many discrete subspaces. But the *metrizable number*  $m(X)$  is the smallest cardinal number  $\kappa$  such that  $X$  can be represented as a union of  $\kappa$  many metric subspaces. We have the following.

COROLLARY 16. *If  $X \subseteq \omega_1$  is a stationary set, then  $dm(X) = \omega_1 = m(X)$ .*

*Proof.* A stationary set cannot be a union of countably many non-stationary subsets. Hence, we get  $dm(X) = \omega_1 = m(X)$ , using Theorem 2 and Proposition 14. ■

LEMMA 17. *If  $X \subseteq \omega_1$  and  $f : X \rightarrow \omega_1$  is an embedding, then there exists a closed unbounded set  $C$  such that*

$$f[X] \cap C = X \cap C.$$

*Proof.* For countable  $X$ , the set  $\{\alpha : \sup\{X \cup f[X]\} < \alpha\}$  is what we need. Suppose  $X$  is uncountable. Successively by induction choose strictly greater ordinals  $x_{n,\alpha} \in X$  and  $y_{n,\alpha} \in f[X]$  such that

$$x_{\beta,k} < x_{\alpha,n} < y_{\alpha,n} < x_{\alpha,n+1},$$

where  $k, n \in \omega$  and  $\beta < \alpha < \omega_1$ . These ordinals constitute an increasing sequence lexicographically ordered with respect to indices. Cluster points of the set of these ordinals give the required closed unbounded set. ■

Obviously, the above lemma implies that disjoint stationary sets have not comparable dimensional types.

**THEOREM 18.** *If  $X$  is a stationary set, then the poset  $(\mathcal{P}(X)/=_{E}, \leq_d)$  contains uncountable anti-chains and uncountable strictly descending chains.*

*Proof.* Let  $\{S_\alpha : \alpha < \omega_1\} \subseteq X$  be a family of pairwise disjoint stationary set, such a family exists by the above-mentioned result of R. Solovay. By Lemma 17, elements of this family have not comparable dimensional types. Also for the same reasons, the sets  $X_\beta = \bigcup\{S_\alpha : \beta < \alpha\}$  constitute an uncountable strictly descending chain, with respect to the order  $<_E$ . ■

**7. Generalized Knaster–Urbanik Theorem.** Generalizing the above proof of the Knaster–Urbanik Theorem, and using Theorem 3, we get a proof of the following result by R. Telgársky [18], compare [1]:

*Any metric scattered space is homeomorphic to a subset of an ordinal number.*

We offer the following proof of that corollary. If  $X$  is a discrete space, then  $X$  can be embedded into a set of non-limit ordinals, which has to be a subset of some ordinal. Suppose  $X$  is a metric space such that  $X^{(\alpha)} = \{g\}$ , where  $\alpha > 0$ . Assume that any subspace  $Y \subseteq X$  can be embedded into the ordinal  $E(Y)$ , as long as  $Y^{(\beta)}$  has exactly one point and  $\beta < \alpha$ . Without loss of generality, we can assume that  $f_Y : Y \rightarrow E(Y)$  is an embedding such that

$$f_Y[Y^{(\beta)}] = \{\sup E(Y)\}, \text{ where } |Y^{(\beta)}| = 1.$$

Let  $\{U_n : n \in \omega\}$  be a decreasing base of neighborhoods of  $g$  consisting of closed-open sets. By Corollary 4, there exist pairwise disjoint closed-open sets  $Y_{\xi,n} \subseteq U_n \setminus U_{n+1}$  such that for each  $Y_{\xi,n}$  has exactly one-point derivative  $Y_{\xi,n}^{(\beta)}$ , where  $\beta < \alpha$ . We order ordinals  $E(Y_{\xi,n})$  as follows:  $E(Y_{\xi,n})$  followed by  $E(Y_{\nu,n})$ , with respect to the order of first indices, and with 1 at the end. In the next step, we order similarly ordinals  $E(Y_{\xi,n+1})$  and place them, keeping their order, after 1 located at the end of an ordered collection in the previous step. Finally we put the point  $g$ . The union of all  $f_{Y_{\xi,m}}$  contained in the corresponding  $E(Y_{\xi,m})$ , which are ordered as above, gives the required embedding.

If  $\mathfrak{m}$  is an infinite cardinal number, then  $\mathfrak{m}^+$  denotes the least cardinal number greater than  $\mathfrak{m}$ . Thus, the above-mentioned Telgársky result can be formulated more precisely.

**PROPOSITION 19.** *Any metric scattered space of the size  $\mathfrak{m}$  is homeomorphic to a subset of an ordinal  $\alpha < \mathfrak{m}^+$ .*

*Proof.* If a metric scattered space  $X$  has the size  $\mathfrak{m}$  and  $X^{(\alpha)}$  is the last nonempty derivative, then  $\alpha < \mathfrak{m}^+$ . It is enough to see that with the same proof as for the above corollary, the space  $X$  is embeddable in  $\mathfrak{m}^+$ . ■

**8. Non-homeomorphic metric scattered spaces.** Let us start with an improvement of Mazurkiewicz–Sierpiński Theorem [13, Théorème 3], which says that there are continuum many non-homeomorphic countable metric scattered spaces.

PROPOSITION 20. *The ordinal  $\omega^\omega$  contains continuum many non-homeomorphic subspaces.*

*Proof.* For a binary sequence  $(f_1, f_2, \dots)$  define inductively scattered spaces  $X(f_1, f_2, \dots, f_m)$ , with one-point  $m$ -derivative  $\{h_m\}$ . Put  $X(0) = G$  and  $X(1) = I$ , where spaces  $G$  and  $I$  are the same as in Section 5. The cluster points of  $G$  and  $I$  can be denoted by  $g_G$  and  $g_I$ , respectively. If a space  $X(f_1, f_2, \dots, f_n)$  is already defined, then let

$$X(f_1, f_2, \dots, f_n, 0) = X(f_1, f_2, \dots, f_n) \times (G \setminus \{g_G\}) \cup \{(h_n, g_G)\}$$

be a subspace of the product space  $X(f_1, f_2, \dots, f_n) \times G$ , and let

$$X(f_1, f_2, \dots, f_n, 1) = X(f_1, f_2, \dots, f_n) \times (I \setminus \{g_I\}) \cup \{(h_n, g_I)\}$$

be the subspace of the product space  $X(f_1, f_2, \dots, f_n) \times I$ .

If  $f = (f_1, f_2, \dots)$  is an infinite binary sequence, then let  $X_f$  be the sum of spaces  $\{X(f_1, f_2, \dots, f_n) : 0 < n \in \omega\}$ . So, we have  $X_f^{(\omega)} = \emptyset$ . Also, if  $0 < n$  and  $f_n = 0$ , then the difference  $X_f^{(n-1)} \setminus X_f^{(n+1)}$  is a subspace which consists of pairwise disjoint closed-open (with respect to the inherited topology) sets homeomorphic to a convergent sequence. But if  $f_n = 1$ , then the difference  $X_f^{(n-1)} \setminus X_f^{(n+1)}$  has no closed-open subset which is homeomorphic to a convergent sequence. Therefore  $\{X_f : f \in 2^\omega\}$  is a family of non-homeomorphic subspaces of the ordinal  $\omega^\omega$ , what we need. ■

Consider the sum of  $\omega$  many copies of a space  $X$ . We defined the spaces  $G(X)$  and  $I(X)$  by adding a new point  $g$ , with a countable base of neighborhoods, to this sum. Points belonging to the sum have unchanged bases of neighborhoods. The point  $g$  has a decreasing base  $\{U_n : n \in \omega\}$  such that  $U_n \setminus U_{n+1}$  consists of copies of  $X$  as closed-open subsets. So, in  $G(X)$  each  $U_n \setminus U_{n+1}$  consists of a single copy of  $X$ . However, each  $U_n \setminus U_{n+1}$  consists of infinitely many copies of  $X$  in  $I(X)$ . In particular,  $G = G(1)$  and  $I = I(1)$ . Thus, by a modification of the argument used in our proof of Proposition 20, we get the following.

THEOREM 21. *For each infinite cardinal number  $\mathfrak{m}$ , there exist  $2^{\mathfrak{m}}$  many non-homeomorphic metric spaces of the size  $\mathfrak{m}$ , each one with empty  $\mathfrak{m}$ -derivative.*

*Proof.* By Proposition 20, we can assume that  $\mathfrak{m}$  is an uncountable cardinal. For every binary sequence  $f = \{f_\beta : 0 < \beta < \mathfrak{m}\}$  define inductively a scattered space  $Y(f_1, f_2, \dots, f_\beta)$  as follows. Put  $Y(0) = G$  and  $Y(1) = I$ . Suppose that metric scattered spaces  $Y(f_1, f_2, \dots, f_\delta)$  are already defined, for  $\delta < \beta$ . If  $\beta$  is a limit ordinal, then put

$$Y(f_1, f_2, \dots, f_\beta) = J(\{Y(f_1, f_2, \dots, f_\delta) : \delta < \beta\}).$$

If  $\beta$  is a non-limit ordinal, then put

$$Y(f_1, f_2, \dots, f_{\beta-1}, 0) = G(Y(f_1, f_2, \dots, f_{\beta-1}))$$

and

$$Y(f_1, f_2, \dots, f_{\beta-1}, 1) = I(Y(f_1, f_2, \dots, f_{\beta-1})).$$

Finally, let  $Y(f)$  be the sum of spaces  $Y(f_1, f_2, \dots, f_\beta)$ , where  $\beta < \mathfrak{m}$ .

By the definition, if  $\beta < \mathfrak{m}$ , then each space  $Y(f_1, f_2, \dots, f_\beta)$  has the size less than  $\mathfrak{m}$ . We also have  $Y(f_1, f_2, \dots, f_\beta)^{(\mathfrak{m})} = \emptyset$ , hence  $Y(f)^{(\mathfrak{m})} = \emptyset$ . Bearing above in mind and using Proposition 19, one can check that each  $Y(f)$  embeds into  $\mathfrak{m}$ . Since each  $Y(f)$  has the size  $\mathfrak{m}$ , it remains to show that the family  $\{Y(f) : f \in 2^{\mathfrak{m}}\}$  contains a subfamily of size  $2^{\mathfrak{m}}$  consisting of non-homeomorphic metric scattered space. Indeed, if  $\gamma < \mathfrak{m}$  is a non-limit ordinal and  $f(\gamma) \neq g(\gamma)$ , where  $f, g \in 2^{\mathfrak{m}}$ , then the subspaces  $Y(f)^{(\gamma)} \setminus Y(f)^{(\gamma+2)}$  and  $Y(g)^{(\gamma)} \setminus Y(g)^{(\gamma+2)}$  are not homeomorphic, since one of them consists of closed-open subsets homeomorphic to  $I$ , but the second contains no homeomorphic copy of  $I$ . ■

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